Article

On the General Solutions of Some Non-Homogeneous Div-Curl Systems with Riemann–Liouville and Caputo Fractional Derivatives

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Abstract: In this work, we investigate analytically the solutions of a nonlinear div-curl system with fractional derivatives of the Riemann–Liouville or Caputo types. To this end, the fractional-order vector operators of divergence, curl and gradient are identified as components of the fractional Dirac operator in quaternionic form. As one of the most important results of this manuscript, we derive general solutions of some non-homogeneous div-curl systems that consider the presence of fractional-order derivatives of the Riemann–Liouville or Caputo types. A fractional analogous to the Teodorescu transform is presented in this work, and we employ some properties of its component operators, developed in this work to establish a generalization of the Helmholtz decomposition theorem in fractional space. Additionally, right inverses of the fractional-order curl, divergence and gradient vector operators are obtained using Riemann–Liouville and Caputo fractional operators. Finally, some consequences of these results are provided as applications at the end of this work.

Keywords: fractional div-curl systems; Helmholtz decomposition theorem; Riemann–Liouville derivative; Caputo derivative; fractional vector operators

MSC: 34Axx; 91Bxx

1. Introduction

From an analytical point of view, the literature offers a wide range of reports that focus on the extension of integer-order methods and results for the fractional case. From a more particular point of view, the fractional generalization of the classical vector calculus operators (that is, the gradient, divergence, curl and Laplacian operators) has been also an active topic of research, which has been developed from different approaches. Some of the first attempts to extend these operators to the fractional scenario are described in [1,2] using the Nishimoto fractional derivative. These operators were used later on in [3] to provide a physical interpretation for the fractional advection-dispersion equation for flow in heterogeneous porous media. In 2008, Vasily E. Tarasov described different approaches to formulate a fractional form of vector calculus with physical applications in [4] (see also references therein). More recently, a new generalization of the Helmholtz decomposition theorem for both fractional time and space was proposed in [5,6] using the discrete Grünwald–Letnikov fractional derivative. Another related work is [7], where the authors investigate the dynamic creation of fractionalized half-quantum vortices in Bose–Einstein condensates of sodium atoms.

In this work, we consider fractional derivatives of the Riemann–Liouville and the Caputo types and provide extensions of the definitions of the main differential operators from vector calculus using these fractional operators. In such a way, we present fractional forms of the divergence, the rotational and the gradient operators. Moreover, we also consider generalized forms of the Dirac and the Laplace operators using fractional derivatives. Our goal in this work is to extend quaternionic analysis to consider fractional forms of

the classical differential operators. Analogues of the properties satisfied by the classical operators will be mathematically established in this work. For instance, we will show that

$$^{\mathrm{RL}}\mathcal{D}_{a+}^{\alpha}[w] = -^{\mathrm{RL}}\operatorname{div}_{a+}^{\alpha}\vec{w} +^{\mathrm{RL}}\operatorname{grad}_{a+}^{\alpha}w_0 +^{\mathrm{RL}}\operatorname{curl}_{a+}^{\alpha}\vec{w}, \tag{1}$$

$$^{\mathrm{C}}\mathcal{D}_{a+}^{\alpha}[w] = -^{\mathrm{C}}\operatorname{div}_{a+}^{\alpha}\vec{w} +^{\mathrm{C}}\operatorname{grad}_{a+}^{\alpha}w_0 +^{\mathrm{C}}\operatorname{curl}_{a+}^{\alpha}\vec{w}. \tag{2}$$

In other words, when we apply the fractional Dirac operator to a quaternion valued function $w = w_0 + \vec{w}$, this expression can be decomposed in terms of a fractional divergence, a fractional gradient and a fractional rotational operator. Based on this, we will also provide explicit expressions of general weak solutions for some fractional forms of the div-curl system, considering various analytical hypotheses. More precisely, we will prove that if we restrict ourselves to the class of functions with fractional divergence zero and whose Riemann–Liouville fractional integral has zero normal trace, then the fractional Teodorescu transform represents a solution of the above-mentioned div-curl systems (see Theorems 2 and 3).

This manuscript is organized as follows. In Section 2, we recall some important definitions from the literature, including those of the left and right Riemann–Liouville fractional derivatives, the left and right Caputo fractional derivatives and the two-parameter Mittag–Leffler function. A useful characterization of the functions with summable fractional derivatives is also recalled. In Section 2.2, the Riemann–Liouville and the Caputo Dirac and Laplace operators are introduced. Moreover, some fundamental solutions for the fractional Dirac operators are recalled in Section 2.3. In Section 3, we introduce fractional extensions of the divergence, rotational and gradient differential operators. Some properties among these operators are established, and a useful factorization theorem for the fractional Laplace operators is proven. It is worth mentioning that this factorization is not new; however, we were able to derive it only using the identities preserving the fractional gradient, divergence and rotational operators. Among the most important results, we establish that the fractional Teodorescu transform is a right inverse of the fractional Dirac operator under suitable analytical conditions, and we prove a fractional form of the Divergence Theorem.

Section 4 is devoted to establishing the existence of weak solutions for Riemann–Liouville and Caputo fractional div-curl systems. The explicit form of the operators involved in the solution, as well as some of their properties, allow the solution to be re-expressed as the sum of the fractional gradient of a scalar potential plus a fractional curl of a vector potential; we can say that our solutions preserve a Helmholtz-type decomposition (see Propositions 6 and 7). As a consequence, right inverses of the fractional rotational and divergence operators are provided in a subclass of the fractional divergence-free vector fields. In turn, Section 5 provides some consequences of the factorization results proven in Section 2 to the construction of fractional hyper-conjugate pairs. A theorem providing necessary and sufficient conditions for the existence of Caputo fractional hyper-conjugate pairs is proven in this stage, along with a result of the existence of a right inverse for the fractional gradient. Finally, this manuscript closes with a section of concluding remarks.

2. Background

2.1. Fractional Calculus

The present section is devoted to recalling some useful definitions from fractional calculus. Throughout, we assume that $a, b, \alpha \in \mathbb{R}$ satisfy $\alpha > 0$ and $a < b$. Meanwhile, we suppose that $f : \mathbb{R} \to \mathbb{R}$ is a sufficiently smooth function, with the property that f is identically equal to zero outside of the interval $[a, b]$.

Definition 1. *The* left *and* right Riemann–Liouville fractional integrals *of f of order α with respect to the interval $[a, b]$ (whenever they exist) are the functions $I_{a+}^{\alpha}[f]$ and $I_{b-}^{\alpha}[f]$, defined, respectively, by (see [8])*

$$I_{a+}^{\alpha}[f](x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)}{(x-t)^{1-\alpha}} \, dt, \quad x > a, \tag{3}$$

$$I_{b-}^{\alpha}[f](x) = \frac{1}{\Gamma(\alpha)} \int_x^b \frac{f(t)}{(t-x)^{1-\alpha}} \, dt, \quad x < b. \tag{4}$$

Let $m = [\alpha] + 1 \in \mathbb{Z}$. *The* left Riemann–Liouville *and the* left Caputo *fractional derivatives of order α with respect to the interval $[a, b]$ are, respectively, defined as follows:*

$$^{RL}D_{a+}^{\alpha}[f](x) = D^m I_{a+}^{m-\alpha}[f](x) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dx^m} \int_a^x \frac{f(t)}{(x-t)^{\alpha-m+1}} \, dt, \quad x > a, \tag{5}$$

$$^{C}D_{a+}^{\alpha}[f](x) = I_{a+}^{m-\alpha}D^m[f](x) = \frac{1}{\Gamma(m-\alpha)} \int_a^x \frac{f^{(m)}(t)}{(x-t)^{\alpha-m+1}} \, dt, \quad x > a. \tag{6}$$

Finally, we define the right Riemann–Liouville *and the* right Caputo *fractional derivatives of order α with respect to the point a, respectively, as the functions*

$$^{RL}D_{b-}^{\alpha}[f](x) = D^m I_{b-}^{m-\alpha}[f](x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \frac{d^m}{dx^m} \int_x^b \frac{f(t)}{(t-x)^{\alpha-m+1}} \, dt, \quad x < b, \tag{7}$$

$$^{C}D_{b-}^{\alpha}[f](x) = I_{b-}^{m-\alpha}D^m[f](x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^b \frac{f^{(m)}(t)}{(t-x)^{\alpha-m+1}} \, dt, \quad x < b. \tag{8}$$

For the sake of convenience, we will employ the notation $D_{a\pm}^{\alpha}$ when we present properties satisfied by both fractional derivatives $^{RL}D_{a\pm}^{\alpha}$ and $^{C}D_{a\pm}^{\alpha}$. Throughout, we let $I_{a+}^{\alpha}(L_1)$ denote the class of all functions f that are represented by the fractional integral (3) of some integrable function, i.e., $f = I_{a+}^{\alpha}[g]$, for some $g \in L_1(a, b)$. Using this notation, the following result provides a characterization of these functions.

Theorem 1 (Samko et al. [9]). *Let $\alpha > 0$ and $m = [\alpha] + 1$. Then, the function f belongs to $I_{a+}^{\alpha}(L_1)$ if and only if $I_{a+}^{m-\alpha}[f] \in AC^m([a, b])$, and $(I_{a+}^{m-\alpha}[f])^k(a) = 0$, for each $k \in \{0, 1, \ldots, m-1\}$.*

Definition 2. *If $(I_{a+}^{m-\alpha}[f])^{(k)}(a) = 0$, for each $k \in \{0, 1 \ldots, m-1\}$, then it follows that $f^{(k)}(a) = 0$ holds, for each $k \in \{0, \ldots, m-1\}$ (see [8,9]). In light of this fact, we will say that f has a summable fractional derivative $D_{a+}^{\alpha}[f]$ of order α on $[a, b]$ if $I_{a+}^{m-\alpha}[f] \in AC^m([a, b])$.*

In the following discussion, suppose that $\alpha > 0$, f admits a summable fractional derivative of order $\alpha > 0$ on $[a, b]$, and let $m = [\alpha] + 1$. Then, the following composition rules are satisfied:

$$I_{a+}^{\alpha}\,^{RL}D_{a+}^{\alpha}[f](x) = f(x) - \sum_{k=0}^{m-1} \frac{(x-a)^{\alpha-k-1}}{\Gamma(\alpha-k)} (I_{a+}^{m-\alpha}[f])^{m-k-1}(a), \tag{9}$$

$$I_{a+}^{\alpha}\,^{C}D_{a+}^{\alpha}[f](x) = f(x) - \sum_{k=0}^{m-1} \frac{f^{(k)}(a)}{k!} (x-a)^k. \tag{10}$$

On the other hand, we know that both fractional derivatives $^{RL}D_{a+}^{\alpha}$ and $^{C}D_{a+}^{\alpha}$ satisfy the one-sided invertibility property $D_{a+}^{\alpha}I_{a+}^{\alpha}[f] = f$. It is worth noting that this identity is a particular case of the property $D_{a+}^{\alpha}I_{a+}^{\beta}[f] = D_{a+}^{\alpha-\beta}[f]$, which holds for each $\alpha, \beta \in \mathbb{R}$ satisfying $\alpha \geq \beta > 0$.

It is important to recall also that the following semi-group property for the composition of fractional derivatives is not generally satisfied:

$$^{C}D_{a+}^{\alpha}\,^{C}D_{a+}^{\beta}[f] = {}^{C}D_{a+}^{\beta}\,^{C}D_{a+}^{\alpha}[f] = {}^{C}D_{a+}^{\alpha+\beta}[f]. \tag{11}$$

However, if $f^{(k)}(a) = 0$ for $k = 0, 1, \ldots, \max\{[\alpha] + 1, [\beta] + 1\} - 1$, then (11) does hold; see Section 2.2.6 [8]. An analogous condition for the semi-group property in the context

of the Riemann–Liouville derivative is found in Section 2.3.6 [8]. Finally, the following relation between the Riemann–Liouville and Caputo fractional derivatives is valid:

$$^{C}D_{a^+}^{\alpha}[f](x) = {}^{RL}D_{a^+}^{\alpha}[f](x) - \sum_{j=0}^{m-1} \frac{(x-a)^{j-\alpha}}{\Gamma(j-\alpha+1)} D_x^{(j)}[f](a). \tag{12}$$

Definition 3 (Gorenflo et al. [10])**.** *Let $\mu, \nu \in \mathbb{R}$ be such that $\mu > 0$. We define the two-parameter Mittag–Leffler function $E_{\mu,\nu}: \mathbb{C} \to \mathbb{C}$ with parameters μ and ν in terms of the following power series:*

$$E_{\mu,\nu}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\mu n + \nu)}, \quad \forall z \in \mathbb{C}. \tag{13}$$

2.2. Fractional Quaternionic Analysis

In this section, we will mention some recent results in fractional Clifford analysis. The Dirac operator in Clifford analysis, also known as the Moisil–Teodorescu differential operator, represents the cornerstone of the analysis in higher dimensions. A remarkable number of systems of differential equations have been analyzed using this operator or a perturbation of it, and the monographs [11–13] of the authors Gürlebeck and Sprößig contain many examples of the applications that have been made over the years. See also [14], where the authors introduced the fractional Dirac operator with Caputo derivatives as well as the basic tool of a fractional function theory in more dimensions.

More precisely, this section is devoted to the collection of some recent results of the authors Ferreira et al. [15–17], by whom fundamental solutions of the fractional Laplacian were found, where the derivatives are of Riemann–Liouville and Caputo types, as well as of the fractional Dirac operators.

For the remainder, let $a_i, b_i \in \mathbb{R}$ satisfy $a_i < b_i$, for each $i = 1, 2, 3$. We will suppose that $\Omega = \Pi_{i=1}^{3}(a_i, b_i)$ is a bounded open rectangular domain in $\mathbb{R}^3$, and let $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, with $\alpha_i \in (0, 1]$, for all $i = 1, 2, 3$.

Definition 4 (Ferreira et al. [15–17])**.** *The* fractional Riemann–Liouville *and* fractional Caputo Dirac operators *are represented by ${}^{RL}\mathcal{D}_{a^+}^{\alpha}$ and ${}^{C}\mathcal{D}_{a^+}^{\alpha}$, respectively, and they are defined as*

$$^{RL}\mathcal{D}_{a^+}^{\alpha} = \sum_{i=1}^{3} e_i \, {}^{RL}\partial_{x_i,a_i}^{\frac{1+\alpha_i}{2}}, \tag{14}$$

$$^{C}\mathcal{D}_{a^+}^{\alpha} = \sum_{i=1}^{3} e_i \, {}^{C}\partial_{x_i,a_i}^{\frac{1+\alpha_i}{2}}. \tag{15}$$

Here, ${}^{RL}\partial_{x_i,a_i}^{\frac{1+\alpha_i}{2}}$ and ${}^{C}\partial_{x_i,a_i}^{\frac{1+\alpha_i}{2}}$ are, respectively, the Riemann–Liouville and the Caputo fractional derivative operators of order $(1 + \alpha_i)/2$ with respect to the variable $x_i \in (a_i, b_i)$, for each $i = 1, 2, 3$. We define the fractional Laplace operators ${}^{RL}\Delta_{a^+}^{\alpha}$ and ${}^{RL}\Delta_{a^+}^{\alpha}$, respectively, by

$$^{RL}\Delta_{a^+}^{\alpha} = \sum_{i=1}^{3} {}^{RL}\partial_{x_i,a_i}^{1+\alpha_i}, \tag{16}$$

$$^{C}\Delta_{a^+}^{\alpha} = \sum_{i=1}^{3} {}^{C}\partial_{x_i,a_i}^{1+\alpha_i}, \tag{17}$$

where ${}^{RL}\partial_{x_i,a_i}^{1+\alpha_i}$ and ${}^{C}\partial_{x_i,a_i}^{1+\alpha_i}$ are, respectively, the Riemann–Liouville and Caputo fractional derivatives of order $1 + \alpha_i$ with respect to the variable $x_i \in (a_i, b_i)$, for each $i = 1, 2, 3$.

2.3. Fundamental Solutions

The purpose of this subsection is to determine fundamental solutions for the fractional Dirac operator and use their properties in the investigation of the solutions of fractional

div-curl systems. Beforehand, it is worth recalling that a family of fundamental solutions for the fractional Laplace operators $^{RL}\Delta^\alpha_{a^+}$ and $^C\Delta^\alpha_{a^+}$, and a family of fundamental solutions for the fractional Dirac operators $^{RL}\mathcal{D}^\alpha_{a^+}$ and $^C\mathcal{D}^\alpha_{a^+}$, were obtained in [15,16] for the Riemann–Liouville and Caputo case, respectively. In the case of Riemann–Liouville fractional operators, the authors employed some properties of the Mittag–Leffler function and the Laplace transform in two dimensions.

We will begin this section by recalling some relevant results derived in [15]. To this end, let u be an eigenfunction of the fractional Laplace operator, i.e., suppose that $^{RL}\Delta^\alpha_{a^+} u = \lambda u$ for some $\lambda \in \mathbb{C}$, and assume that $u(\vec{x})$ admits a summable fractional derivative $^{RL}\partial^{\frac{1+\alpha_1}{2}}_{x_1,a_1^+}$ in the variable x_1, and that it belongs to $I^{1+\alpha_i}_{a_i^+}(L_1)$ in the variables x_2 and x_3. In what follows, we will consider the following integral and differential conditions of Cauchy type:

$$\begin{cases} f_0(x_2,x_3) = I^{1-\alpha_1}_{a_1^+}[u](a_1,x_2,x_3), \\ f_1(x_2,x_3) = \partial^{\alpha_1}_{x_1,a_1^+}[u](a_1,x_2,x_3). \end{cases} \tag{18}$$

Lemma 1 (Ferreira et al. [15,17]). *A family of eigenfunctions of the fractional Laplace operator* $^{RL}\Delta^\alpha_{a^+}$ *is given by the function*

$$\begin{aligned} u_\lambda(\vec{x}) &= (x_1-a_1)^{\alpha_1-1} E_{1+\alpha_1,\alpha_1}\left(-(x_1-a_1)^{1+\alpha_1}\left(^{RL}\partial^{1+\alpha_2}_{x_2,a_2^+} + {}^{RL}\partial^{1+\alpha_3}_{x_3,a_3^+} - \lambda\right)\right) f_0(x_2,x_3) \\ &+ (x_1-a_1)^{\alpha_1} E_{1+\alpha_1,1+\alpha_1}\left(-(x_1-a_1)^{1+\alpha_1}\left(^{RL}\partial^{1+\alpha_2}_{x_2,a_2^+} + {}^{RL}\partial^{1+\alpha_3}_{x_3,a_3^+} - \lambda\right)\right) f_1(x_2,x_3). \end{aligned} \tag{19}$$

Meanwhile, a family of fundamental solutions of the fractional Dirac operator $^{RL}\mathcal{D}^\alpha_{a^+}$ *is obtained by considering* $\lambda \equiv 0$ *in* (19)*. More precisely, this family of solutions is given by*

$$^{RL}\mathcal{E}^\alpha_{a^+}(\vec{x}) = -{}^{RL}\mathcal{D}^\alpha_{a^+}[u_0](\vec{x}), \tag{20}$$

where u_0 *is a fundamental solution of* $^{RL}\Delta^\alpha_{a^+}$*, i.e.,*

$$\begin{aligned} u_0(\vec{x}) &= (x_1-a_1)^{\alpha_1-1} E_{1+\alpha_1,\alpha_1}\left(-(x_1-a_1)^{1+\alpha_1}\left(^{RL}\partial^{1+\alpha_2}_{x_2,a_2^+} + {}^{RL}\partial^{1+\alpha_3}_{x_3,a_3^+}\right)\right) f_0(x_2,x_3) \\ &+ (x_1-a_1)^{\alpha_1} E_{1+\alpha_1,1+\alpha_1}\left(-(x_1-a_1)^{1+\alpha_1}\left(^{RL}\partial^{1+\alpha_2}_{x_2,a_2^+} + {}^{RL}\partial^{1+\alpha_3}_{x_3,a_3^+}\right)\right) f_1(x_2,x_3). \end{aligned} \tag{21}$$

Here, f_0 *and* f_1 *satisfy the conditions* (18)*.*

Let v be an eigenfunction of the fractional Laplace operator, i.e., suppose that $^C\Delta^\alpha_{a^+} v = \lambda v$, for some $\lambda \in \mathbb{C}$. Assume that $v(\vec{x})$ admits a summable fractional derivative $^{RL}\partial^{\frac{1+\alpha_1}{2}}_{x_1,a_1^+}$ in the variable x_1, and that it belongs to $I^{1+\alpha_i}_{a_i^+}(L_1)$ in the variables x_2 and x_3. By Theorem 1, $v(x_1,a_2,x_3) = v(x_1,x_2,a_3) = 0$. In what follows, we will consider the following Cauchy conditions:

$$\begin{cases} g_0(x_2,x_3) = v(a_1,x_2,x_3), \\ g_1(x_2,x_3) = v'_{x_1}(a_1,x_2,x_3). \end{cases} \tag{22}$$

As a consequence, $g_0(a_2,x_3) = g_0(x_2,a_3) = g_1(a_2,x_3) = g_1(x_2,a_3) = 0$.

Lemma 2 (Ferreira et al. [16,17]). *A family of eigenfunctions for the fractional Laplace operator ${}^C\Delta^\alpha_{a+}$ is given by the function*

$$
\begin{aligned}
v_\lambda(\vec{x}) = {} & E_{1+\alpha_1,1}\left(-(x_1-a_1)^{1+\alpha_1}\left({}^C\partial^{1+\alpha_2}_{x_2,a_2^+} + {}^C\partial^{1+\alpha_3}_{x_3,a_3^+} - \lambda\right)\right)g_0(x_2,x_3) \\
& + (x_1-a_1)E_{1+\alpha_1,2}\left(-(x_1-a_1)^{1+\alpha_1}\left({}^C\partial^{1+\alpha_2}_{x_2,a_2^+} + {}^C\partial^{1+\alpha_3}_{x_3,a_3^+} - \lambda\right)\right)g_1(x_2,x_3).
\end{aligned}
\tag{23}
$$

Meanwhile, a family of fundamental solutions of the fractional Dirac operator ${}^C\mathcal{D}^\alpha_{a+}$ is obtained by considering $\lambda \equiv 0$ in (23). More precisely, this family of solutions is given by

$$
{}^C\mathcal{E}^\alpha_{a+}(\vec{x}) = -{}^C\mathcal{D}^\alpha_{a+}[v_0](\vec{x}),
\tag{24}
$$

where v_0 is a fundamental solution of ${}^C\Delta^\alpha_{a+}$, i.e.,

$$
\begin{aligned}
v_0(\vec{x}) = {} & E_{1+\alpha_1,1}\left(-(x_1-a_1)^{1+\alpha_1}\left({}^C\partial^{1+\alpha_2}_{x_2,a_2^+} + {}^C\partial^{1+\alpha_3}_{x_3,a_3^+}\right)\right)g_0(x_2,x_3) \\
& + (x_1-a_1)E_{1+\alpha_1,2}\left(-(x_1-a_1)^{1+\alpha_1}\left({}^C\partial^{1+\alpha_2}_{x_2,a_2^+} + {}^C\partial^{1+\alpha_3}_{x_3,a_3^+}\right)\right)g_1(x_2,x_3),
\end{aligned}
\tag{25}
$$

where $g_0(x_2,x_3)$ and $g_1(x_2,x_3)$ satisfy (22).

3. Fractional Vector Calculus

For the remainder of this section, we will study the fractional divergence, gradient and rotational operators as parts of a decomposition of the fractional Dirac operator in three dimensions. More precisely, recall that if $w = w_0 + \vec{w}$ is a quaternionic-valued function, then the following decomposition in quaternionic form is satisfied:

$$
Dw = -\operatorname{div}\vec{w} + \operatorname{grad} w_0 + \operatorname{curl}\vec{w}.
\tag{26}
$$

Here, $D = \sum_{i=1}^3 e_i\partial_i$ is the classical Dirac operator, which is also called the Moisil–Teodorescu differential operator. For more details about quaternionic analysis, see [11,18,19]. Our goal in this section is to provide an extension of this decomposition (27) using fractional operators of the Riemann–Liouville and Caputo types.

Let $w = w_0 + \sum_{i=1}^3 e_i w_i$ be a quaternionic-valued function in $AC(\Omega)$, whose scalar part is denoted by $\mathrm{Sc}[w] = w_0$ and its vector part by $\mathrm{Vec}[w] = \vec{w} = \sum_{i=1}^3 e_i w_i$. Then, the action of the operator ${}^{\mathrm{RL}}\mathcal{D}^\alpha_{a+}$ on w reduces to

$$
\begin{aligned}
{}^{\mathrm{RL}}\mathcal{D}^\alpha_{a+}[w] = {} & \sum_{i=1}^3 e_i \,{}^{\mathrm{RL}}\partial^{\frac{1+\alpha_i}{2}}_{x_i,a_i}[w] \\
= {} & -\left(\sum_{i=1}^3 {}^{\mathrm{RL}}\partial^{\frac{1+\alpha_i}{2}}_{x_i,a_i}[w_i]\right) + \left(\sum_{i=1}^3 e_i \,{}^{\mathrm{RL}}\partial^{\frac{1+\alpha_i}{2}}_{x_i,a_i}[w_0]\right) + e_1\left({}^{\mathrm{RL}}\partial^{\frac{1+\alpha_2}{2}}_{x_2,a_2}[w_3] - {}^{\mathrm{RL}}\partial^{\frac{1+\alpha_3}{2}}_{x_3,a_3}[w_2]\right) \\
& + e_2\left({}^{\mathrm{RL}}\partial^{\frac{1+\alpha_3}{2}}_{x_3,a_3}[w_1] - {}^{\mathrm{RL}}\partial^{\frac{1+\alpha_1}{2}}_{x_1,a_1}[w_3]\right) + e_3\left({}^{\mathrm{RL}}\partial^{\frac{1+\alpha_1}{2}}_{x_1,a_1}[w_2] - {}^{\mathrm{RL}}\partial^{\frac{1+\alpha_2}{2}}_{x_2,a_2}[w_1]\right).
\end{aligned}
\tag{27}
$$

The above decomposition of Equation (27) originates a fractional version of the classical divergence, rotational and gradient differential operators from vector calculus. These operators are, respectively, the scalar component of (27), the vector term acting over $\vec{w}$ and the vector term of the equation acting over w_0. These facts are the motivation to analyze the following fractional differential operators.

Definition 5. *Define the fractional divergence, curl and gradient operators in the Riemann–Liouville sense by*

$$^{RL}\operatorname{div}^{\alpha}_{a+}\vec{w} = \sum_{i=1}^{3} {}^{RL}\partial^{\frac{1+\alpha_i}{2}}_{x_i,a_i}[w_i], \tag{28}$$

$$^{RL}\operatorname{curl}^{\alpha}_{a+}\vec{w} = e_1\left({}^{RL}\partial^{\frac{1+\alpha_2}{2}}_{x_2,a_2}[w_3] - {}^{RL}\partial^{\frac{1+\alpha_3}{2}}_{x_3,a_3}[w_2]\right) + e_2\left({}^{RL}\partial^{\frac{1+\alpha_3}{2}}_{x_3,a_3}[w_1] - {}^{RL}\partial^{\frac{1+\alpha_1}{2}}_{x_1,a_1}[w_3]\right) \tag{29}$$

$$+ e_3\left({}^{RL}\partial^{\frac{1+\alpha_2}{2}}_{x_1,a_1}[w_2] - {}^{RL}\partial^{\frac{1+\alpha_2}{2}}_{x_2,a_2}[w_1]\right),$$

$$^{RL}\operatorname{grad}^{\alpha}_{a+}[w_0] = \sum_{i=1}^{3} e_i\,{}^{RL}\partial^{\frac{1+\alpha_i}{2}}_{x_i,a_i}[w_0]. \tag{30}$$

It is important to point out that the fractional operators (28)–(30) reduce, respectively, to the classical div, curl, and grad operators from vector calculus when $\alpha_i = 1$, for each $i = 1, 2, 3$. Moreover, if $\alpha^* > 0$ and $\alpha_i = \alpha^*$, for each $i = 1, 2, 3$, then the above fractional operators coincide with the divergence, curl and gradient operators defined in [1,2] up to a constant factor. See also [4] and references therein for a historic account of the efforts to formulate a fractional form of vector calculus. Unlike the classical vector calculus operators, these fractional operators are non-local. Consequently, the fractional divergence, curl and gradient depend on the domain Ω.

Notice now that (27) can be rewritten as the following decomposition

$$^{RL}\mathcal{D}^{\alpha}_{a+}[w] = -{}^{RL}\operatorname{div}^{\alpha}_{a+}\vec{w} + {}^{RL}\operatorname{grad}^{\alpha}_{a+}w_0 + {}^{RL}\operatorname{curl}^{\alpha}_{a+}\vec{w}. \tag{31}$$

Since the specific form of the Riemann–Liouville fractional derivative does not affect the above decomposition, we analogously obtain the following decomposition in terms of Caputo fractional derivatives:

$$^{C}\mathcal{D}^{\alpha}_{a+}[w] = -{}^{C}\operatorname{div}^{\alpha}_{a+}\vec{w} + {}^{C}\operatorname{grad}^{\alpha}_{a+}w_0 + {}^{C}\operatorname{curl}^{\alpha}_{a+}\vec{w}. \tag{32}$$

Here, the operators ${}^{C}\operatorname{div}^{\alpha}_{a+}$, ${}^{C}\operatorname{curl}^{\alpha}_{a+}$ and ${}^{C}\operatorname{grad}^{\alpha}_{a+}$ are defined as in (28)–(30), respectively, using Caputo fractional derivatives instead of Riemann–Liouville operators.

We define now a class of functions in $AC^1(\Omega)$ where we can apply the semi-group property (11).

Definition 6. *We set $\mathcal{Z}^{\alpha}_{a+}(\Omega) = \{f \in AC^1(\Omega)\colon f(a_1, x_2, x_3) = f(x_1, a_2, x_3) = f(x_1, x_2, a_3) = 0\}$.*

Proposition 1. *If $f = f_0 + \vec{f} \in \mathcal{Z}^{\alpha}_{a+}(\Omega)$, then*

(i) $^{RL}\operatorname{div}^{\alpha}_{a+}\,{}^{RL}\operatorname{curl}^{\alpha}_{a+}[\vec{f}] = 0$,
(ii) $^{RL}\operatorname{curl}^{\alpha}_{a+}\,{}^{RL}\operatorname{grad}^{\alpha}_{a+}[f_0] = 0$,
(iii) $^{RL}\operatorname{div}^{\alpha}_{a+}\,{}^{RL}\operatorname{grad}^{\alpha}_{a+}[f_0] = {}^{RL}\Delta^{\alpha}_{a+}[f_0]$,
(iv) $^{RL}\operatorname{grad}^{\alpha}_{a+}\,{}^{RL}\operatorname{div}^{\alpha}_{a+}[\vec{f}] - {}^{RL}\operatorname{curl}^{\alpha}_{a+}\,{}^{RL}\operatorname{curl}^{\alpha}_{a+}[\vec{f}] = {}^{RL}\Delta^{\alpha}_{a+}[\vec{f}]$.

Moreover, the identities (i)–(iv) *also hold for the Caputo fractional operators.*

Proof. The results readily follow from the identities

$$^{RL}\partial^{\frac{1+\alpha_i}{2}}_{x_i,a_i}\,{}^{RL}\partial^{\frac{1+\alpha_i}{2}}_{x_i,a_i} = {}^{RL}\partial^{1+\alpha_i}_{x_i,a_i}, \quad \forall i = 1, 2, 3. \tag{33}$$

$$^{RL}\partial^{\frac{1+\alpha_i}{2}}_{x_i,a_i}\,{}^{RL}\partial^{\frac{1+\alpha_j}{2}}_{x_j,a_j} = {}^{RL}\partial^{\frac{1+\alpha_j}{2}}_{x_j,a_j}\,{}^{RL}\partial^{\frac{1+\alpha_i}{2}}_{x_i,a_i}, \quad \forall i, j = 1, 2, 3, \tag{34}$$

which are trivially satisfied in $\mathcal{Z}^{\alpha}_{a+}(\Omega)$. The identities with Caputo fractional operators are established in a similar fashion. $\square$

Similar identities using Caputo fractional derivatives were proven in [4] when all the orders of the fractional derivatives are equal, i.e., when there exists $\alpha^* > 0$ such that $\alpha_i = \alpha^*$, for each $i = 1, 2, 3$. On the other hand, a direct consequence of Proposition 1 is that the fractional Dirac operator $^{\mathrm{RL}}\mathcal{D}_{a+}^{\alpha}$ factorizes the fractional Laplace operator $^{\mathrm{RL}}\Delta_{a+}^{\alpha}$. A more general factorization for functions taking values in $\mathcal{C}l_{0,3}$ was proven in Section 4 [16]. In light of these remarks, the following result is a direct consequence of Proposition 1.

Corollary 1. *If $f = f_0 + \vec{f} \in \mathcal{Z}_{a+}^{\alpha}(\Omega)$, then the following factorizations of the fractional Laplace operators are satisfied:*

$$^{RL}\Delta_{a+}^{\alpha}[f] = -\,^{RL}\mathcal{D}_{a+}^{\alpha}\,^{RL}\mathcal{D}_{a+}^{\alpha}[f], \tag{35}$$

$$^{C}\Delta_{a+}^{\alpha}[f] = -\,^{C}\mathcal{D}_{a+}^{\alpha}\,^{C}\mathcal{D}_{a+}^{\alpha}[f]. \tag{36}$$

We turn our attention now to the right Caputo fractional Dirac operator, which is given by the expression

$$^{C}\mathcal{D}_{b-}^{\alpha} = \sum_{i=1}^{3} e_i\,^{C}\partial_{x_i, b_i^-}^{\frac{1+\alpha_i}{2}}. \tag{37}$$

The following *fractional Stokes formula* was proven in Theorem 10 [17]:

$$\int_{\Omega} \left(-([h]^{C}\mathcal{D}_{b-}^{\alpha})(\vec{y})f(\vec{y}) + h(\vec{y})(^{\mathrm{RL}}\mathcal{D}_{a+}^{\alpha}[f])(\vec{y}) \right) d\vec{y} = \int_{\partial\Omega} h(\vec{y})\eta(\vec{y}) ds_{\vec{y}} I_{a+}^{\alpha}[f](\vec{y}). \tag{38}$$

Here, we require that $f, h \in \mathrm{AC}^1(\Omega) \cap \mathrm{AC}(\overline{\Omega})$ and $I_{a+}^{\alpha}[f] = \sum_{i=1}^{3} I_{a_i^+}^{\frac{1-\alpha_i}{2}}[f]$. It is worth noting here that the operator $^{C}\mathcal{D}_{b-}^{\alpha}$ acts on the right, while $^{\mathrm{RL}}\mathcal{D}_{a+}^{\alpha}$ acts on the left. Intuitively, the last formula shows that the left Riemann–Liouville and right Caputo fractional Dirac operators act by 'intertwining' to obtain the fractional analogue of the Stokes formula.

The following result is a fractional form of the well-known Divergence Theorem.

Proposition 2 (Fractional Divergence Theorem). *If $\vec{f} \in AC^1(\Omega) \cap AC(\overline{\Omega})$, then*

$$\int_{\Omega}\,^{RL}\mathrm{div}_{a+}^{\alpha}[\vec{f}](\vec{y}) \, d\vec{y} = \int_{\partial\Omega} \eta(\vec{y}) \cdot I_{a+}^{\alpha}[\vec{f}](\vec{y}) \, ds_{\vec{y}}, \tag{39}$$

$$\int_{\Omega}\,^{RL}\mathrm{curl}_{a+}^{\alpha}[\vec{f}](\vec{y}) \, d\vec{y} = \int_{\partial\Omega} \eta(\vec{y}) \times I_{a+}^{\alpha}[\vec{f}](\vec{y}) \, ds_{\vec{y}}. \tag{40}$$

Proof. Taking $h \equiv 1$ in (38) and using that $[1]^{C}\mathcal{D}_{b-}^{\alpha} = 0$ yields that

$$\int_{\Omega}\,^{\mathrm{RL}}\mathcal{D}_{a+}^{\alpha}[\vec{f}](\vec{y}) \, d\vec{y} = \int_{\partial\Omega} \eta(\vec{x}) I_{a+}^{\alpha}[\vec{f}](\vec{y}) \, ds_{\vec{y}}. \tag{41}$$

Due to the decomposition (31) and because $I_{a+}^{\alpha}[\vec{f}]$ is purely vectorial, we can readily calculate their scalar and vector parts, respectively. As a consequence, we readily achieve formulas (39) and (40), respectively. □

Proposition 3. *Let $\vec{f} \in \mathcal{Z}_{a+}^{\alpha}(\Omega)$, $c_0 \in AC(\Omega)$ and $0 < \alpha_i < 1$, for all $i = 1, 2, 3$. Then, the following identities hold:*

$$^{RL}\mathrm{div}_{a+}^{\alpha}[c_0\vec{f}] = \vec{f} \cdot \,^{RL}\mathrm{grad}_{a+}^{\alpha}[c_0], \tag{42}$$

$$^{RL}\mathrm{curl}_{a+}^{\alpha}[c_0\vec{f}] = \vec{f} \times \,^{RL}\mathrm{grad}_{a+}^{\alpha}[c_0]. \tag{43}$$

Proof. We will only calculate $^{\mathrm{RL}}\partial_{x_1, a_1^+}^{\frac{1+\alpha_1}{2}}[f_i c_0]$ using integration by parts and Leibniz' rule, the determination of $^{\mathrm{RL}}\partial_{x_2, a_2^+}^{\frac{1+\alpha_2}{2}}[f_i c_0]$ and $^{\mathrm{RL}}\partial_{x_3, a_3^+}^{\frac{1+\alpha_3}{2}}[f_i c_0]$ being similar. Beforehand, note that

$m_i = 1$, for all $i = 1, 2, 3$. Recall now that $D_{a^+}^\alpha$ is a left inverse of $I_{a^+}^\alpha$; and by hypothesis $f_i(a_1, x_2, x_3) = 0$, for all $i = 1, 2, 3$. It follows then that

$$
\begin{aligned}
{}^{\mathrm{RL}}\partial_{x_1, a_1^+}^{\frac{1+\alpha_1}{2}}[f_i c_0](\vec{x}) &= \frac{\partial}{\partial x_1} \frac{1}{\Gamma(\frac{1-\alpha_1}{2})} \int_{a_1}^{x_1} \frac{f_i(t, x_2, x_3) c_0(t, x_2, x_3)}{(x_1 - t)^{\frac{1+\alpha_1}{2}}}\, dt \\
&= \frac{\partial}{\partial x_1} \left(f_i(t, x_2, x_3) I_{a_1^+}^{\frac{1-\alpha_1}{2}}[c_0](t, x_2, x_3) \Big|_{a_1}^{x_1} - I_{a_1^+}^1 \left[I_{a_1^+}^{\frac{1-\alpha_1}{2}}[c_0] \frac{\partial}{\partial x_1} f_i \right] \right) \\
&= \frac{\partial}{\partial x_1} \left(f_i(x_1, x_2, x_3) I_{a_1^+}^{\frac{1-\alpha_1}{2}}[c_0](x_1, x_2, x_3) \right) - I_{a_1^+}^{\frac{1-\alpha_1}{2}}[c_0] \frac{\partial}{\partial x_1} f_i \\
&= f_i(\vec{x})\, {}^{\mathrm{RL}}\partial_{x_1, a_1^+}^{\frac{1+\alpha_1}{2}}[c_0](\vec{x}).
\end{aligned}
\tag{44}
$$

Analogously, ${}^{\mathrm{RL}}\partial_{x_j, a_j^+}^{\frac{1+\alpha_j}{2}}[f_i c_0] = f_i\, {}^{\mathrm{RL}}\partial_{x_j, a_j^+}^{\frac{1+\alpha_j}{2}}[c_0]$, for each $j = 2, 3$ and $i = 1, 2, 3$. As a consequence,

$$
{}^{\mathrm{RL}}\operatorname{div}_{a^+}^\alpha[c_0 \vec{f}] = \sum_{i=1}^3 {}^{\mathrm{RL}}\partial_{x_i, a_i^+}^{\frac{1+\alpha_i}{2}}[c_0 f_i] = \sum_{i=1}^3 f_i\, {}^{\mathrm{RL}}\partial_{x_i, a_i^+}^{\frac{1+\alpha_i}{2}}[c_0] = \vec{f} \cdot {}^{\mathrm{RL}}\operatorname{grad}_{a^+}^\alpha[c_0].
\tag{45}
$$

This establishes the first identity of the conclusion. The proof of the second equation is analogous. □

Before closing this section, it is natural to compare qualitatively the results obtained in traditional vector calculus against those in the fractional case. In classical vector calculus, the following product rules are satisfied:

$$
\operatorname{div}(c_0 \vec{f}) = \vec{f} \cdot \operatorname{grad} c_0 + c_0 \operatorname{div} \vec{f},
\tag{46}
$$

$$
\operatorname{curl}(v_0 \vec{f}) = \vec{f} \times \operatorname{grad} c_0 + c_0 \operatorname{curl} \vec{f}.
\tag{47}
$$

On the other hand, in the fully fractional case considered in Proposition 3, when we restrict $\vec{f}$ to the class of functions $\mathcal{Z}_{a^+}^\alpha(\Omega)$, the first part of these identities is also satisfied, except that the second terms on the right-hand sides of (46) and (47) are not present anymore. Notice that it is not difficult to construct a family of functions belonging to $\mathcal{Z}_{a^+}^\alpha(\Omega)$, for instance $\vec{f}(x) = (x_1 - a_1)^{\gamma_1}(x_2 - a_2)^{\gamma_2}(x_3 - a_3)^{\gamma_3}\vec{g}(x)$ for all $\vec{g} \in \mathrm{AC}^1(\Omega)$ and $\gamma_i \geq 0$ for all $i = 1, 2, 3$.

4. Fractional Div-Curl Systems

4.1. Properties of the Fractional Teodorescu Transform

As a derivation of the fractional Borel–Pompeiu formula [17], the authors defined the Caputo-type Teodorescu transform in a very similar way to the following definition, the difference being that the kernel is now a fundamental solution of the fractional Dirac operator defined in (20).

Definition 7. *Let $\vec{x} = (x_1, x_2, x_3)$ with $x_i > a_i$, for all $i = 1, 2, 3$. Define the* Riemann–Liouville *and* Caputo *fractional Teodorescu transform by*

$$
{}^{\mathrm{RL}}T_{a^+}^\alpha[w](\vec{x}) = \int_\Omega {}^{\mathrm{RL}}\mathcal{E}_{a^+}^\alpha(\vec{x} + a - \vec{y}) w(\vec{y})\, d\vec{y},
\tag{48}
$$

$$
{}^{C}T_{a^+}^\alpha[w](\vec{x}) = \int_\Omega {}^{C}\mathcal{E}_{a^+}^\alpha(\vec{x} + a - \vec{y}) w(\vec{y})\, d\vec{y}.
\tag{49}
$$

Here, we follow the nomenclature and conventions of Lemma 1. Moreover, the derivatives ${}^{\mathrm{RL}}\mathcal{E}_{a^+}^\alpha = -{}^{\mathrm{RL}}\mathcal{D}_{a^+}^\alpha[u_0]$ and ${}^{C}\mathcal{E}_{a^+}^\alpha = -{}^{C}\mathcal{D}_{a^+}^\alpha[v_0]$ that appear in the kernel of (48) and (49), respecrively, are with respect to the variable $\vec{x}$.

In the following, it will be convenient to employ the translation operator, which is defined by $\mathcal{T}_{\vec{z}} f(\vec{y}) = f(\vec{y} + \vec{z})$. Similarly, we will use the reflection operator given by $\mathcal{R}_{\vec{y}} f(\vec{y}) = f(-\vec{y})$. An important relation between $\mathcal{D}_{a^+}^\alpha$ and $\mathcal{T}_\theta$ is

$$^{\mathrm{RL}}\mathcal{D}_{(-\theta+a)^+}^\alpha [\mathcal{T}_\theta f](\vec{x}) = {}^{\mathrm{RL}}\mathcal{D}_{a^+}^\alpha [f](\vec{x} + \theta), \tag{50}$$

where the derivative is taken with respect to the variable $\vec{x}$. The proof is analogous to Theorem 11 [17], which is for the Caputo case, but we will give it here for completeness.

Proposition 4. *Let $\alpha_i \in (0, 1)$, for each $i = 1, 2, 3$, and let $\alpha^* = \min_{1 \le i \le 3} \{\alpha_i\}$. The fractional Teodorescu transform $^{\mathrm{RL}}T_{a^+}^\alpha$ is a right inverse of the fractional Dirac operator $^{\mathrm{RL}}\mathcal{D}_{a^+}^\alpha$ in $L^p(\Omega)$, for all $p \in \mathbb{R}$ satisfying $1 < p < \dfrac{2}{1 - \alpha^*}$.*

Proof. Observe firstly that the fundamental solution $^{\mathrm{RL}}\mathcal{E}_{a^+}^\alpha(\vec{x})$ of the fractional Dirac operator $^{\mathrm{RL}}\mathcal{D}_{a^+}^\alpha$ is defined only for $x_i > a_i$, for each $i = 1, 2, 3$. Moreover, it satisfies the identity $^{\mathrm{RL}}\mathcal{D}_{a^+}^\alpha {}^{\mathrm{RL}}\mathcal{E}_{a^+}^\alpha(\vec{x}) = \delta(\vec{x} - a)$ or, equivalently, $^{\mathrm{RL}}\Delta_{a^+}^\alpha [u_0](\vec{x}) = -{}^{\mathrm{RL}}\mathcal{D}_{a^+}^\alpha {}^{\mathrm{RL}}\mathcal{D}_{a^+}^\alpha [u_0](\vec{x}) = \delta(\vec{x} - a)$. In the following, the derivatives $^{\mathrm{RL}}\mathcal{D}_{\vec{y}^+}^\alpha$ and $^{\mathrm{RL}}\mathcal{D}_{a^+}^\alpha$ are with respect to the variable $\vec{x}$. Using (50) with $\theta = a - \vec{y}$ yields

$$\begin{aligned}
^{\mathrm{RL}}\mathcal{D}_{a^+}^\alpha {}^{\mathrm{RL}}T_{a^+}^\alpha [w](\vec{x}) &= \int_\Omega {}^{\mathrm{RL}}\mathcal{D}_{\vec{y}^+}^\alpha {}^{\mathrm{RL}}\mathcal{E}_{a^+}^\alpha(\vec{x} + a - \vec{y}) w(\vec{y}) \, d\vec{y} \\
&= -\int_\Omega {}^{\mathrm{RL}}\mathcal{D}_{\vec{y}^+}^\alpha \left(\mathcal{T}_{a-\vec{y}} \big({}^{\mathrm{RL}}\mathcal{D}_{a^+}^\alpha [u_0](\vec{x}) \big) \right) w(\vec{y}) \, d\vec{y} \\
&= -\int_\Omega \mathcal{T}_{a-\vec{y}} \big({}^{\mathrm{RL}}\mathcal{D}_{a^+}^\alpha {}^{\mathrm{RL}}\mathcal{D}_{a^+}^\alpha [u_0](\vec{x}) \big) w(\vec{y}) \, dy \\
&= \int_\Omega \mathcal{T}_{a-\vec{y}} \delta(\vec{x} - a) w(\vec{y}) \, dy \\
&= \int_\Omega \delta(\vec{x} - \vec{y}) \, w(\vec{y}) \, d\vec{y} = w(\vec{x}),
\end{aligned} \tag{51}$$

which we wished to prove. □

In the following, we will employ a key decomposition of the classical Teodorescu operator used in [20–23] for different kinds of bounded or unbounded domains in $\mathbb{R}^3$. For the fractional version, we denote the component operators of the fractional Teodorescu transform as follows:

$$^{\mathrm{RL}}T_{a^+}^\alpha [w_0 + \vec{w}] := {}^{\mathrm{RL}}T_{0,a^+}^\alpha [\vec{w}] + {}^{\mathrm{RL}}\overrightarrow{T}_{1,a^+}^\alpha [w_0] + {}^{\mathrm{RL}}\overrightarrow{T}_{2,a^+}^\alpha [\vec{w}]. \tag{52}$$

The first term on the right-hand side of Equation (52) is the scalar part, while the last two summands represent the vector part, and it has been split into two components for the sake of convenience. These three terms are given by

$$^{\mathrm{RL}}T_{0,a^+}^\alpha [\vec{w}](\vec{x}) = -\int_\Omega {}^{\mathrm{RL}}\mathcal{E}_{a^+}^\alpha(\vec{x} + a - \vec{y}) \cdot \vec{w}(\vec{y}) \, d\vec{y}, \tag{53}$$

$$^{\mathrm{RL}}\overrightarrow{T}_{1,a^+}^\alpha [w_0](\vec{x}) = \int_\Omega {}^{\mathrm{RL}}\mathcal{E}_{a^+}^\alpha(\vec{x} + a - \vec{y}) w_0(\vec{y}) \, d\vec{y}, \tag{54}$$

$$^{\mathrm{RL}}\overrightarrow{T}_{2,a^+}^\alpha [\vec{w}](\vec{x}) = \int_\Omega {}^{\mathrm{RL}}\mathcal{E}_{a^+}^\alpha(\vec{x} + a - \vec{y}) \times \vec{w}(\vec{y}) \, d\vec{y}. \tag{55}$$

We will see later in Corollary 2 how some of these component operators themselves represent right inverse operators of the fractional divergence and rotational operators, under certain conditions. Moreover, $^{\mathrm{RL}}T_{a^+}^\alpha$, as a good generalization of the classical Teodorescu operator in quaternionic analysis, preserves many of its properties. To this end, we use the above decomposition (52), $^{\mathrm{RL}}T_{a^+}^\alpha [\vec{g}] = {}^{\mathrm{RL}}T_{0,a^+}^\alpha [\vec{g}] + {}^{\mathrm{RL}}\overrightarrow{T}_{2,a^+}^\alpha [\vec{g}]$, to see necessary and sufficient conditions to guarantee that both its scalar part and its vector part belong

to the kernel of the fractional Laplacian ${}^{RL}\Delta_{a+}^{\alpha}$. In order to apply the factorization of Corollary 1 to the fractional Teodorescu transform, we will need to guarantee that the condition ${}^{RL}T_{a+}^{\alpha}[\vec{g}] \in \mathcal{Z}_{a+}^{\alpha}(\Omega)$ can be satisfied.

Proposition 5. *If ${}^{RL}T_{a+}^{\alpha}[\vec{g}] \in \mathcal{Z}_{a+}^{\alpha}(\Omega)$, then the following hold for the fractional Teodorescu transform ${}^{RL}T_{a+}^{\alpha}[\vec{g}]$:*

(i) *The scalar part of ${}^{RL}T_{a+}^{\alpha}[\vec{g}]$, ${}^{RL}T_{0,a+}^{\alpha}[\vec{g}]$, belongs to the kernel of ${}^{RL}\Delta_{a+}^{\alpha}$ if and only if ${}^{RL}\operatorname{div}_{a+}^{\alpha} \vec{g} = 0$.*

(ii) *The vector part of ${}^{RL}T_{a+}^{\alpha}[\vec{g}]$, ${}^{RL}\overrightarrow{T}_{2,a+}^{\alpha}[\vec{g}]$, belongs to the kernel of ${}^{RL}\Delta_{a+}^{\alpha}$ if and only if ${}^{RL}\operatorname{curl}_{a+}^{\alpha} \vec{g} = 0$.*

Moreover, the statements (i) *and* (ii) *also hold for the Caputo fractional Teodorescu transform.*

Proof. Using the factorization in Corollary 1, Proposition 4 and the decomposition (31), it readily follows that

$$ {}^{RL}\Delta_{a+}^{\alpha} \, {}^{RL}T_{a+}^{\alpha}[\vec{g}] = -{}^{RL}\mathcal{D}_{a+}^{\alpha} \, {}^{RL}\mathcal{D}_{a+}^{\alpha} \, {}^{RL}T_{a+}^{\alpha}[\vec{g}] = {}^{RL}\operatorname{div}_{a+}^{\alpha} \vec{g} - {}^{RL}\operatorname{curl}_{a+}^{\alpha} \vec{g}. \tag{56} $$

Taking its scalar part or vector part, respectively, we obtain the desired result. □

4.2. Riemann–Liouville System

In the following, we will study a fractional form of the classical div-curl system and construct its solution. More precisely, we fix the domain Ω, and consider the fractional system

$$ \begin{cases} {}^{RL}\operatorname{div}_{a+}^{\alpha} \vec{w} = g_0, \\ {}^{RL}\operatorname{curl}_{a+}^{\alpha} \vec{w} = \vec{g}, \end{cases} \tag{57} $$

where $g_0 \in L^p(\Omega, \mathbb{R})$ and $\vec{g} \in L^p(\Omega, \mathbb{R}^3)$, for some $1 < p < 2/(1 - \alpha^*)$. Notice that if the solution is such that $\vec{w} \in \mathcal{Z}_{a+}^{\alpha}(\Omega)$, then $\vec{g}$ satisfies ${}^{RL}\operatorname{div}_{a+}^{\alpha} \vec{g} = 0$, i.e., $\vec{g}$ is a 'fractional divergence-free' vector field. Let $f_0 \in AC(\Omega)$. The following relations will be fundamental in the sequel:

$$ \begin{aligned} {}^{RL}\operatorname{grad}_{a+}^{\alpha}[f_0](\theta - \vec{y}) &= -{}^{RL}\operatorname{grad}_{(\theta-a)-}^{\alpha}[\mathcal{T}_\theta \mathcal{R}_{\vec{y}}[f_0]](\vec{y}), \\ {}^{C}\operatorname{grad}_{a+}^{\alpha}[f_0](\theta - \vec{y}) &= -{}^{C}\operatorname{grad}_{(\theta-a)-}^{\alpha}[\mathcal{T}_\theta \mathcal{R}_{\vec{y}}[f_0]](\vec{y}). \end{aligned} \tag{58} $$

Here, the derivatives are taken with respect to the variable $\vec{y}$. In the sequel and for the sake of convenience, we will employ Ker to denote the kernel of operators. As in the previous section, we will let $\Omega = \Pi_{i=1}^{3}(a_i, b_i)$ be a bounded open rectangular domain in $\mathbb{R}^3$, and assume that $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, with $\alpha_i \in (0, 1)$, for each $i = 1, 2, 3$. Using this notation, we have the following result.

Theorem 2. *Let $g = g_0 + \vec{g} \in L^p(\Omega)$ with $1 < p < 2/(1 - \alpha^*)$. If ${}^{RL}\operatorname{div}_{a+}^{\alpha}[\vec{g}] = 0$ and the normal trace of $I_{a+}^{\alpha}[\vec{g}]$ vanishes, then a general weak solution of the fractional Riemann–Liouville div-curl system (57) is given by*

$$ \vec{w} = -{}^{RL}\overrightarrow{T}_{1,a+}^{\alpha}[g_0] + {}^{RL}\overrightarrow{T}_{2,a+}^{\alpha}[\vec{g}] + {}^{RL}\operatorname{grad}_{a+}^{\alpha}[h], \tag{59} $$

where $h \in \operatorname{Ker}({}^{RL}\Delta_{a+}^{\alpha}) \cap \mathcal{Z}_{a+}^{\alpha}(\Omega)$ is an arbitrary scalar function.

Proof. By Proposition 4 and the decomposition (31), the fractional Teodorescu transform ${}^{RL}T_{a+}^{\alpha}[-g_0 + \vec{g}]$ is a quaternionic solution of the system (57). To obtain a pure-vector solution, note that the decomposition (52) yields

$$ {}^{RL}T_{a+}^{\alpha}[-g_0 + \vec{g}] = {}^{RL}T_{0,a+}^{\alpha}[\vec{g}] - {}^{RL}\overrightarrow{T}_{1,a+}^{\alpha}[g_0] + {}^{RL}\overrightarrow{T}_{2,a+}^{\alpha}[\vec{g}]. \tag{60} $$

Taking $h(\vec{y}) = \mathcal{T}_{\vec{x}+a}\mathcal{R}_{\vec{y}}u_0(\vec{y})$ and $f(\vec{y}) = \vec{g}(\vec{y})$ in the Stokes formula (38) (all derivatives with respect to the variable $\vec{y}$) and letting u_0 be a fundamental solution of $^{\mathrm{RL}}\Delta_{a+}^{\alpha}$ given in Lemma 1, we obtain

$$\int_{\partial\Omega} \mathcal{T}_{\vec{x}+a}\mathcal{R}_{\vec{y}}[u_0](\vec{y})\eta(\vec{y})I_{a+}^{\alpha}[\vec{g}]\,ds_{\vec{y}}$$
$$= \int_{\Omega}\left(\left(-[\mathcal{T}_{\vec{x}+a}\mathcal{R}_{\vec{y}}[u_0]]^{\mathrm{C}}\mathcal{D}_{b-}^{\alpha}\right)(\vec{y})\vec{g}(\vec{y}) + \mathcal{T}_{\vec{x}+a}\mathcal{R}_{\vec{y}}[u_0](\vec{y})^{\mathrm{RL}}\mathcal{D}_{a+}^{\alpha}[\vec{g}](\vec{y})\right)d\vec{y}. \tag{61}$$

Calculate the scalar part of (61) and use the hypotheses $-\mathrm{Sc}\,\mathcal{D}_{a+}^{\alpha}[\vec{g}] = {}^{\mathrm{RL}}\mathrm{div}_{a+}^{\alpha}[\vec{g}] = 0$ in Ω and $I_{a+}^{\alpha}[\vec{g}]|\cdot\eta = 0$ on $\partial\Omega$ to reach

$$\int_{\Omega}{}^{\mathrm{C}}\mathrm{grad}_{b-}^{\alpha}\left[\mathcal{T}_{\vec{x}+a}\mathcal{R}_{\vec{y}}[u_0]\right](\vec{y})\cdot\vec{g}(\vec{y})\,d\vec{y} = 0. \tag{62}$$

By differentiating now under the integral sign and using the traditional Leibniz' rule, we readily obtain the following fundamental relation:

$$^{\mathrm{RL}}\mathrm{grad}_{a+,\vec{x}}^{\alpha}[u_0](\vec{x}+a-\vec{y}) = -^{\mathrm{RL}}\mathrm{grad}_{a+,\vec{y}}^{\alpha}[u_0](\vec{x}+a-\vec{y}). \tag{63}$$

Here, the second sub-index indicates whether we are taking derivatives with respect to the variable $\vec{x}$ or $\vec{y}$. Recall that $^{\mathrm{RL}}\mathcal{E}_{a+}^{\alpha} = -^{\mathrm{RL}}\mathrm{grad}_{a+,\vec{x}}^{\alpha}[u_0]$ with respect to the variable $\vec{x}$. On the other hand, due to $u_0 \in \mathcal{Z}_{a+}^{\alpha}(\Omega)$ and relations (12), (63) and (58), we obtain

$$^{\mathrm{RL}}\mathcal{E}_{a+}^{\alpha}(\vec{x}+a-\vec{y}) = {}^{\mathrm{C}}\mathrm{grad}_{a+,\vec{y}}^{\alpha}[u_0](\vec{x}+a-\vec{y}) = -^{\mathrm{C}}\mathrm{grad}_{x-,\vec{y}}^{\alpha}[\mathcal{T}_{\vec{x}+a}\mathcal{R}_{\vec{y}}[u_0]](\vec{y}). \tag{64}$$

However, we know that $^{\mathrm{RL}}\mathcal{E}_{a+}^{\alpha}(\vec{x}+a-\vec{y})$ is only defined when $x_i > y_i$, for all $i = 1, 2, 3$. As a consequence, we can readily replace the operator $^{\mathrm{C}}\mathrm{grad}_{b-}^{\alpha}$ by $^{\mathrm{C}}\mathrm{grad}_{x-}^{\alpha}$ in (62). Finally, by employing (64) and (62), we readily reach

$$^{\mathrm{RL}}T_{0,a+}^{\alpha}[\vec{g}](\vec{x}) = -\int_{\Omega}{}^{\mathrm{RL}}\mathcal{E}_{a+}^{\alpha}(\vec{x}+a-\vec{y})\cdot\vec{g}(\vec{y})\,d\vec{y} = 0, \quad \text{in } \Omega. \tag{65}$$

This means that $^{\mathrm{RL}}T_{a+}^{\alpha}[\vec{g}] = -^{\mathrm{RL}}\overrightarrow{T}_{1,a+}^{\alpha}[g_0] + {}^{\mathrm{RL}}\overrightarrow{T}_{2,a+}^{\alpha}[\vec{g}]$ is purely vectorial and, moreover,

$$\left(-^{\mathrm{RL}}\mathrm{div}_{a+}^{\alpha} + {}^{\mathrm{RL}}\mathrm{curl}_{a+}^{\alpha}\right){}^{\mathrm{RL}}T_{a+}^{\alpha}[-g_0+\vec{g}] = {}^{\mathrm{RL}}\mathcal{D}_{a+}^{\alpha}{}^{\mathrm{RL}}T_{a+}^{\alpha}[-g_0+\vec{g}] = -g_0 + \vec{g}. \tag{66}$$

Setting the scalar and vector parts equal to each other, we obtain that $^{\mathrm{RL}}T_{a+}^{\alpha}[\vec{g}] = -^{\mathrm{RL}}\overrightarrow{T}_{1,a+}^{\alpha}[g_0] + {}^{\mathrm{RL}}\overrightarrow{T}_{2,a+}^{\alpha}[\vec{g}]$ is a solution of the fractional div-curl system (57). Finally, the fact that the solution is not unique is a consequence of the identities (ii) and (iii) of Proposition 1. $\square$

In the limit $\alpha_i \to 1^-$, the fractional div-curl system (57) reduces to the well-known integer-order system from vector calculus, and the hypothesis $^{\mathrm{RL}}\mathrm{div}_{a+}^{\alpha}[\vec{g}] = 0$ reduces to the evident requirement that $\vec{g}$ be a divergence-free vector field. Moreover, $I_{a_i^+}^{\frac{1-\alpha_i}{2}}[u_0\vec{g}] \to u_0\vec{g}$. This means that it is sufficient to require that $\vec{g}$ has zero normal trace. Taking $g_0 \equiv 0$ or $\vec{g} \equiv 0$ in (57), we readily obtain

Corollary 2. *Under the same assumptions of Theorem 2, $^{RL}\overrightarrow{T}_{2,a+}^{\alpha}$ is a right inverse operator of $^{RL}\mathrm{curl}_{a+}^{\alpha}$ in the class of functions considered in Theorem 2. Meanwhile, $-^{RL}\overrightarrow{T}_{1,a+}^{\alpha}$ is always a right inverse operator of $^{RL}\mathrm{div}_{a+}^{\alpha}$ in $L^p(\Omega)$.*

Another important analogy with the classical vector calculus is that the solution of the non-linear fractional system (57) also admits a Helmholtz-type fractional decomposition as follows.

Proposition 6. *Under the same assumptions of Theorem 2, the general weak solution of the fractional Riemann–Liouville div-curl system (57) admits a fractional Helmholtz decomposition as follows*

$$\vec{w} = {}^{RL}\operatorname{grad}^{\alpha}_{\vec{x}-} \varphi_0 - {}^{RL}\operatorname{curl}^{\alpha}_{\vec{x}-} \vec{\varphi}, \quad in \ \Omega, \tag{67}$$

where the scalar potential φ_0 and the vector potential $\vec{\varphi}$ are given by

$$\varphi_0(\vec{x}) = \int_{\Omega} \mathcal{T}_{\vec{x}+a} \mathcal{R}_{\vec{y}}[u_0](\vec{y}) g_0(\vec{y}) \, d\vec{y}, \quad \vec{\varphi}(\vec{x}) = \int_{\Omega} \mathcal{T}_{\vec{x}+a} \mathcal{R}_{\vec{y}}[u_0](\vec{y}) \times \vec{g}(\vec{y}) \, d\vec{y}.$$

Proof. Due to ${}^{RL}\mathcal{E}^{\alpha}_{a+} = -{}^{RL}\operatorname{grad}^{\alpha}_{a+}[u_0]$ and by relation (58), we can write

$$^{RL}\overrightarrow{T}^{\alpha}_{1,a+}[g_0](\vec{x}) = \int_{\Omega} {}^{RL}\operatorname{grad}^{\alpha}_{\vec{x}-} \mathcal{T}_{\vec{x}+a} \mathcal{R}_{\vec{y}}[u_0](\vec{y}) g_0(\vec{y}) \, d\vec{y},$$

$$^{RL}\overrightarrow{T}^{\alpha}_{2,a+}[\vec{g}](\vec{x}) = \int_{\Omega} {}^{RL}\operatorname{grad}^{\alpha}_{\vec{x}-} \mathcal{T}_{\vec{x}+a} \mathcal{R}_{\vec{y}}[u_0](\vec{y}) \times \vec{g}(\vec{y}) \, d\vec{y}.$$

Finally, since the fractional gradient involved in the above re-expressions of the component operators ${}^{RL}\overrightarrow{T}^{\alpha}_{1,a+}$ and ${}^{RL}\overrightarrow{T}^{\alpha}_{2,a+}$ is taken with respect to the variable $\vec{x}$, we readily obtain (67). □

4.3. Caputo System

For the corresponding fractional div-curl system in the sense of Caputo derivatives, we will follow the same approach as that used with the Riemann–Liouville div-curl system (57). Let us consider the system

$$\begin{cases} {}^{C}\operatorname{div}^{\alpha}_{a+} \vec{w} = g_0, \\ {}^{C}\operatorname{curl}^{\alpha}_{a+} \vec{w} = \vec{g}, \end{cases} \tag{68}$$

where $g_0 \in L^p(\Omega, \mathbb{R})$ and $\vec{g} \in L^p(\Omega, \mathbb{R}^3)$, for some $1 < p < 2/(1 - p^*)$. The cornerstone in our analysis will be again the fractional Teodorescu transform associated with Caputo derivatives, which means that its kernel is a fundamental solution of the fractional Dirac operator ${}^{C}\mathcal{D}^{\alpha}_{a+}$. As in the case of the Riemann–Liouville fractional Teodorescu operator, we define the decomposition

$$^{C}T^{\alpha}_{a+}[w_0 + \vec{w}] := {}^{C}T^{\alpha}_{0,a+}[\vec{w}] + {}^{C}\overrightarrow{T}^{\alpha}_{1,a+}[w_0] + {}^{C}\overrightarrow{T}^{\alpha}_{2,a+}[\vec{w}], \tag{69}$$

where ${}^{C}T^{\alpha}_{0,a+}$, ${}^{C}\overrightarrow{T}^{\alpha}_{1,a+}$ and ${}^{C}\overrightarrow{T}^{\alpha}_{2,a+}$ are given by (53), (54) and (55), respectively, but using now the Caputo kernel ${}^{C}\mathcal{E}^{\alpha}_{a+}$ in the integrand, instead of the Riemann–Liouville kernel ${}^{RL}\mathcal{E}^{\alpha}_{a+}$.

Analogously to the proof of Theorem 2, it follows that ${}^{C}T^{\alpha}_{a+}[-g_0 + \vec{g}]$ is a quaternionic solution of the fractional div-curl system (68). This is a direct consequence of the fact that ${}^{C}T^{\alpha}_{a+}$ is a right inverse of ${}^{C}\mathcal{D}^{\alpha}_{a+}$ in L^p(see [17], Theorem 11). Apply the decomposition (69) to the quaternion-valued function $g = g_0 + \vec{g}$, where g_0 and $\vec{g}$ are the known data provided by the fractional div-curl system (68). In this way, we notice that

$$^{C}T^{\alpha}_{a+}[-g_0 + \vec{g}] = {}^{C}T^{\alpha}_{0,a+}[\vec{g}] - {}^{C}\overrightarrow{T}^{\alpha}_{1,a+}[g_0] + {}^{C}\overrightarrow{T}^{\alpha}_{2,a+}[\vec{g}].$$

To obtain a purely vectorial solution, we will impose suitable conditions over $\vec{g}$ in order to guarantee that the scalar part of ${}^{C}T^{\alpha}_{a+}[-g_0 + \vec{g}]$ vanishes in Ω, i.e., that ${}^{C}T^{\alpha}_{0,a+}[\vec{g}] \equiv 0$ is satisfied in Ω. We will see that the fractional divergence-free functions of the Riemann–Liouville type, whose Riemann–Liouville fractional integral has zero normal trace, belong to the kernel of the operator ${}^{C}T^{\alpha}_{0,a+}$, in the same way as seen in Theorem 2 for the operator ${}^{RL}T^{\alpha}_{0,a+}$. Nevertheless, the upcoming proof is relatively more straightforward.

Theorem 3. *Let* $g = g_0 + \vec{g} \in L^p(\Omega)$ *with* $1 < p < 2/(1 - \alpha^*)$. *If* $^{RL}\operatorname{div}_{a+}^{\alpha}[\vec{g}] = 0$ *and the normal trace of* $I_{a+}^{\alpha}[\vec{g}]$ *vanishes, then a general weak solution of the fractional Caputo div-curl system* (68) *is given by*

$$\vec{w} = -^C \overrightarrow{T}_{1,a+}^{\alpha}[g_0] + ^C \overrightarrow{T}_{2,a+}^{\alpha}[\vec{g}] + ^C \operatorname{grad}_{a+}^{\alpha}[h], \tag{70}$$

where $h \in \operatorname{Ker}(^C\Delta_{a+}^{\alpha}) \cap \mathcal{Z}_{a+}^{\alpha}(\Omega)$ *is an arbitrary scalar function.*

Proof. It suffices to prove that $^C T_{0,a+}^{\alpha}[\vec{g}] = 0$ in Ω, in light of the previous discussion. Take $h(\vec{y}) = \mathcal{T}_{\vec{x}+a}\mathcal{R}_{\vec{y}}v_0(\vec{y})$ and $f(\vec{y}) = \vec{g}(\vec{y})$ in the Stokes formula (38), and let v_0 be the fundamental solution of $^C\Delta_{a+}^{\alpha}$ (25). It follows that

$$\int_{\partial\Omega} \mathcal{T}_{\vec{x}+a}\mathcal{R}_{\vec{y}}[v_0](\vec{y})\eta(\vec{y})I_{a+}^{\alpha}[\vec{g}]\,ds_{\vec{y}}$$
$$= \int_{\Omega} \left(\left(-[\mathcal{T}_{\vec{x}+a}\mathcal{R}_{\vec{y}}[v_0]]^C\mathcal{D}_{b-}^{\alpha} \right)(\vec{y})\vec{g}(\vec{y}) + \mathcal{T}_{\vec{x}+a}\mathcal{R}_{\vec{y}}[v_0](\vec{y})^{RL}\mathcal{D}_{a+}^{\alpha}[\vec{g}](\vec{y}) \right) d\vec{y}. \tag{71}$$

Taking now the scalar part of (71) and using the hypotheses, we readily obtain

$$\int_{\Omega} {}^C \operatorname{grad}_{b-}^{\alpha}\left[\mathcal{T}_{\vec{x}+a}\mathcal{R}_{\vec{y}}[v_0]\right](\vec{y}) \cdot \vec{g}(\vec{y})\,d\vec{y} = 0. \tag{72}$$

As a consequence of (58), we reach $^C\mathcal{E}_{a+}^{\alpha}(\vec{x} + a - \vec{y}) = -^C \operatorname{grad}_{x-}^{\alpha}[\mathcal{T}_{\vec{x}+a}\mathcal{R}_{\vec{y}}[v_0]](\vec{y})$. However, we know that $^C\mathcal{E}_{a+}^{\alpha}(\vec{x} + a - \vec{y})$ is only defined for $x_i > y_i$, for all $i = 1, 2, 3$. This implies that we can replace the operator $^C \operatorname{grad}_{b-}^{\alpha}$ with $^C \operatorname{grad}_{x-}^{\alpha}$ in (72). It is easy to see then that

$$^C T_{0,a+}^{\alpha}[\vec{g}](\vec{x}) = -\int_{\Omega} {}^C\mathcal{E}_{a+}^{\alpha}(\vec{x} + a - \vec{y}) \cdot \vec{g}(\vec{y})\,d\vec{y} = 0, \quad \text{in } \Omega, \tag{73}$$

whence the conclusion readily follows. $\square$

Corollary 3. *Under the same assumptions of Theorem 3,* $^C \overrightarrow{T}_{2,a+}^{\alpha}$ *is a right inverse operator of* $^C \operatorname{curl}_{a+}^{\alpha}$ *in the class of functions considered in Theorem 3. Moreover,* $-^C \overrightarrow{T}_{1,a+}^{\alpha}$ *is a right inverse operator of* $^C \operatorname{div}_{a+}^{\alpha}$ *in* $L^p(\Omega)$.

Analogously to the Riemann–Liouville case, the fractional Caputo div-curl system (68) also admits a fractional Helmholtz decomposition, but now the potential is in terms of v_0 defined in (25), which is a fundamental solution of the fractional Laplace operator of Caputo type.

Proposition 7. *Under the same assumptions of Theorem 3, the general weak solution of the fractional Caputo div-curl system* (68) *admits a fractional Helmholtz decomposition as follows*

$$\vec{w} = {}^C \operatorname{grad}_{\vec{x}-}^{\alpha} \psi_0 - {}^C \operatorname{curl}_{\vec{x}-}^{\alpha} \vec{\psi}, \quad \text{in } \Omega, \tag{74}$$

where the scalar potential ψ_0 *and the vector potential* $\vec{\psi}$ *are given by*

$$\psi_0(\vec{x}) = \int_{\Omega} \mathcal{T}_{\vec{x}+a}\mathcal{R}_{\vec{y}}[v_0](\vec{y})g_0(\vec{y})\,d\vec{y}, \quad \vec{\psi}(\vec{x}) = \int_{\Omega} \mathcal{T}_{\vec{x}+a}\mathcal{R}_{\vec{y}}[v_0](\vec{y}) \times \vec{g}(\vec{y})\,d\vec{y}.$$

Proof. The proof is analogous to that of Proposition 6 for the Riemann–Liouville case. $\square$

5. Application

Let $\Omega = \Pi_{i=1}^{3}(a_i, b_i)$ be as in the previous sections, and assume $0 < \alpha_i < 1$, for all $i = 1, 2, 3$. The present section provides some consequences of the factorization provided by Corollary 1 to the construction of fractional hyper-conjugate pairs. In addition, we

will give an explicit expression of a right inverse of the fractional gradient of Caputo type considering different derivative orders.

Definition 8. *Let $w = w_0 + \vec{w} \in AC(\Omega)$. We say that $(w_0, \vec{w})$ is a Riemann–Liouville fractional hyper-conjugate pair if $w \in \mathrm{Ker}({}^{RL}\mathcal{D}^\alpha_{a+})$. Analogously, $(w_0, \vec{w})$ is a Caputo fractional hyper-conjugate pair if $w \in \mathrm{Ker}({}^C\mathcal{D}^\alpha_{a+})$.*

The following result is a straightforward consequence of the factorization of the fractional Laplace operator in the class $\mathcal{Z}^\alpha_{a+}(\Omega)$ provided by Corollary 1. For this reason, we omit the proof.

Corollary 4. *Let $w = \sum_{i=0}^3 w_i$, and suppose that $w \in \mathrm{Ker}({}^{RL}\mathcal{D}^\alpha_{a+}) \cap \mathcal{Z}^\alpha_{a+}(\Omega)$ (respectively, $w \in \mathrm{Ker}({}^C\mathcal{D}^\alpha_{a+}) \cap \mathcal{Z}^\alpha_{a+}(\Omega)$). Then, $w_i \in \mathrm{Ker}({}^{RL}\Delta^\alpha_{a+})$ (respectively, $w_i \in \mathrm{Ker}({}^C\Delta^\alpha_{a+})$), for all $i = 0, 1, 2, 3$.*

By Definition 8, it is obvious that $(w_0, \vec{w})$ forms a Riemann–Liouville fractional hyper-conjugate pair if and only if the following fractional div-curl system is satisfied:

$$
\begin{aligned}
{}^{RL}\operatorname{div}^\alpha_{a+} \vec{w} &= 0, \\
{}^{RL}\operatorname{curl}^\alpha_{a+} \vec{w} &= -{}^{RL}\operatorname{grad}^\alpha_{a+} w_0.
\end{aligned}
\tag{75}
$$

Similarly, $(w_0, \vec{w})$ is a Caputo fractional hyper-conjugate pair if and only if

$$
\begin{aligned}
{}^{C}\operatorname{div}^\alpha_{a+} \vec{w} &= 0, \\
{}^{C}\operatorname{curl}^\alpha_{a+} \vec{w} &= -{}^{C}\operatorname{grad}^\alpha_{a+} w_0.
\end{aligned}
\tag{76}
$$

The above systems (75) and (76) can be considered fractional generalizations of the Moisil–Teodorescu system studied for the first time in [24].

Let us define the following integral operator in terms of the Riemann–Liouville fractional integrals $I^{\frac{1+\alpha_i}{2}}_{a_i^+}$ as

$$
\mathcal{A}^\alpha_{a+}[\vec{f}](\vec{x}) = I^{\frac{1+\alpha_1}{2}}_{a_1^+}[f_1](x_1, a_2, a_3) + I^{\frac{1+\alpha_2}{2}}_{a_2^+}[f_2](x_1, x_2, a_3) + I^{\frac{1+\alpha_3}{2}}_{a_3^+}[f_3](x_1, x_2, x_3).
\tag{77}
$$

As the following result shows, it turns out that $\mathcal{A}^\alpha_{a+}$ behaves as a right-inverse operator of ${}^C\operatorname{grad}^\alpha_{a+}$ in the class of functions satisfying ${}^C\operatorname{curl}^\alpha_{a+} \vec{f} = 0$. For this reason, $\mathcal{A}^\alpha_{a+}$ is called the *fractional anti-gradient operator*.

Proposition 8. *If ${}^C\operatorname{curl}^\alpha_{a+} \vec{f} = 0$, then ${}^C\operatorname{grad}^\alpha_{a+} \mathcal{A}^\alpha_{a+}[\vec{f}] = \vec{f}$.*

Proof. Using the characterization of Caputo fractional hyper-conjugate pairs given under Corollary 4 and differentiating under the integral sign, we readily obtain

$$
\begin{aligned}
{}^C\partial^{\frac{1+\alpha_1}{2}}_{x_1, a_1^+} \mathcal{A}^\alpha_{a+}[\vec{f}](\vec{x}) &= {}^C\partial^{\frac{1+\alpha_1}{2}}_{x_1, a_1^+} \left(I^{\frac{1+\alpha_1}{2}}_{a_1^+}[f_1](x_1, a_2, a_3) + I^{\frac{1+\alpha_2}{2}}_{a_2^+}[f_2](x_1, x_2, a_3) + I^{\frac{1+\alpha_3}{2}}_{a_3^+}[f_3](x_1, x_2, x_3) \right) \\
&= f_1(x_1, a_2, a_3) + I^{\frac{1+\alpha_2}{2}}_{a_2^+} {}^C\partial^{\frac{1+\alpha_1}{2}}_{x_1, a_1^+}[f_2](x_1, x_2, a_3) + I^{\frac{1+\alpha_3}{2}}_{a_3^+} {}^C\partial^{\frac{1+\alpha_1}{2}}_{x_1, a_1^+}[f_3](x_1, x_2, x_3).
\end{aligned}
\tag{78}
$$

Now, by hypothesis ${}^C\operatorname{curl}^\alpha_{a+} \vec{f} = 0$ or, equivalently, the following identities are satisfied:

$$
{}^C\partial^{\frac{1+\alpha_2}{2}}_{x_2, a_2}[f_3] - {}^C\partial^{\frac{1+\alpha_3}{2}}_{x_3, a_3}[f_2] = {}^C\partial^{\frac{1+\alpha_3}{2}}_{x_3, a_3}[f_1] - {}^C\partial^{\frac{1+\alpha_1}{2}}_{x_1, a_1}[f_3] = {}^C\partial^{\frac{1+\alpha_1}{2}}_{x_1, a_1}[f_2] - {}^C\partial^{\frac{1+\alpha_2}{2}}_{x_2, a_2}[f_1] = 0.
\tag{79}
$$

Substituting (79) into (78) and using the composition rule (10), it follows that

$$^C\partial_{x_1,a_1^+}^{\frac{1+\alpha_1}{2}} \mathcal{A}_{a^+}^\alpha [\vec{f}](\vec{x}) = f_1(x_1,a_2,a_3) + f_1(x_1,x_2,a_3) - f_1(x_1,a_2,a_3) + f_1(x_1,x_2,x_3) - f_1(x_1,x_2,a_3) = f_1(\vec{x}). \quad (80)$$

Analogously, one can establish that $^C\partial_{x_i,a_i^+}^{\frac{1+\alpha_i}{2}} \mathcal{A}_{a^+}^\alpha [\vec{f}](\vec{x}) = f_i(\vec{x})$, for $i = 2,3$. The conclusion readily follows now. $\square$

The next proposition shows that the fractional anti-gradient operator $\mathcal{A}_{a^+}^\alpha$ allows us to construct a Caputo fractional hyper-conjugate pair w_0 when $\vec{w} \in \text{Ker}(^C\Delta_{a^+}^\alpha)$ is known beforehand. The proposition is clearly a generalization of [20], Proposition 2.1.

Proposition 9. *Let $\vec{w} \in \text{Ker}(^C\Delta_{a^+}^\alpha) \cap \mathcal{Z}_{a^+}^\alpha(\Omega)$. A necessary and sufficient condition for the existence of a Caputo fractional hyper-conjugate pair of $\vec{w}$ is that $^C\text{div}_{a^+}^\alpha \vec{w} = 0$. In that case, there is w_0 such that $w = w_0 + \vec{w} \in \text{Ker}(^C\mathcal{D}_{a^+}^\alpha)$.*

Proof. The necessity is clear due to the characterization of Caputo fractional hyper-conjugate pairs provided by (76). Suppose now that $^C\text{div}_{a^+}^\alpha \vec{w} = 0$. By Proposition 1(iv), it follows that $^C\text{curl}_{a^+}^\alpha {}^C\text{curl}_{a^+}^\alpha \vec{w} = 0$. On the other hand, Proposition 8 ensures that

$$^C\text{grad}_{a^+}^\alpha \mathcal{A}_{a^+}^\alpha [^C\text{curl}_{a^+}^\alpha \vec{w}] = {}^C\text{curl}_{a^+}^\alpha \vec{w}. \quad (81)$$

The conclusion of this result follows from (76) if we let $w_0 = -\mathcal{A}_{a^+}^\alpha [^C\text{curl}_{a^+}^\alpha \vec{w}]$. $\square$

6. Conclusions

In this work, we extend some results from vector calculus to the fractional case—for instance, the space fractional Helmholtz Decomposition Theorem provided by Propositions 6 and 7. The key tools used are the decompositions of the fractional Teodorescu transform in the Riemann–Liouville case (52) and in the Caputo case (69) as well as various properties associated with these fractional operators, which are thoroughly established in this manuscript. To this end, we consider fractional derivatives in the senses of Riemann–Liouville and Caputo, and we analyze fractional forms of various integer-order differential operators, including the divergence, the rotational, the gradient, the Dirac and the Laplace operators. As the most important result, we prove an existence theorem for the solutions of a div-curl system, considering fractional differential operators of the Riemann–Liouville and Caputo types (see Theorems 2 and 3). Other important generalizations of well-known theorems from vector calculus are proven in this way. More precisely, we present fractional versions of the classical Divergence and Stokes Theorems for vector fields (see Proposition 2). Furthermore, we focus on the construction of fractional hyper-conjugate pairs, which represent a fractional generalization of the well-known Moisil–Teodorescu system in quaternionic analysis. Finally, we note that we are also able to provide an explicit expression for an inverse of the fractional gradient operator when we restrict ourselves to vector fields whose fractional rotational is zero, when we consider fractional derivatives of the Caputo type (see Proposition 8).

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