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A Note on Existence of Mild Solutions for Second-Order Neutral Integro-Differential Evolution Equations with State-Dependent Delay

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Abstract: This article is mainly devoted to the study of the existence of solutions for second-order abstract non-autonomous integro-differential evolution equations with infinite state-dependent delay. In the first part, we are concerned with second-order abstract non-autonomous integro-differential retarded functional differential equations with infinite state-dependent delay. In the second part, we extend our results to study the second-order abstract neutral integro-differential evolution equations with state-dependent delay. Our results are established using properties of the resolvent operator corresponding to the second-order abstract non-autonomous integro-differential equation and fixed point theorems. Finally, an application is presented to illustrate the theory obtained.

Keywords: second order differential system; state-dependent delay; integro-differential equations; neutral system; cosine function of operators; resolvent operator

1. Introduction

Differential systems that exhibit state-dependent delay frequently emerge when modeling physical phenomena. For this reason, research on different properties related to this class of equations has gained great interest in recent years. The literature related to this research topic is mainly dedicated to the study of functional differential systems with states in spaces of finite dimension or first-order autonomous systems (systems determined by a constant unbounded operator) with states in Banach spaces (the reader can review [1–17] for recent advances in this matter). Nevertheless, the study of non-autonomous abstract integro-differential systems of second-order with state-dependent delay using the properties of the resolvent operator associated with the homogeneous equation, as we will carry out in this work, does not seem to have been addressed yet. The study of the non-autonomous abstract Cauchy problems of first, second or fractional order via evolution families are discussed by many authors. We only mention here [18–27].

On the other hand, neutral functional equations arise in various areas of applied mathematics. For this reason, these equations have attracted the attention of numerous researchers in recent times. In particular, neutral functional equations defined in Banach spaces are used to model heat flow problems in materials, in the study of the behavior of visco-elastic materials, analysis of wave propagation in different media, and in the modeling of various other natural systems. For very useful discussions about first and second-order abstract integro-differential systems related to the current study, we can refer to [28–32].

Motivated by the theory developed in the works mentioned previously, our objective in this article is to study the existence of mild solutions for non-autonomous second-order abstract integro-differential evolution equations with infinite state-dependent delay of the following form:

$$z''(t) = A(t)z(t) + \int_0^t B(t,s)z(s)ds + E_1(t, z_{\rho(t,z_t)}), \quad t \in V = [0,c], \tag{1}$$

$$z_0 = \phi \in \mathcal{P}, \quad z'(0) = x^1 \in H, \tag{2}$$

and the second-order non-autonomous neutral integro-differential evolution systems with state-dependent delay of the following form:

$$\frac{d^2}{dt^2}N(t,z_t) = A(t)N(t,z_t) + \int_0^t B(t,s)N(s,z_s)ds + E_1(t, z_{\rho(t,z_t)}), \quad t \in V = [0,c], \tag{3}$$

with initial condition (2). In the above expressions, $A(t) : D(A(t)) \subseteq H \to H$ and $B(t,s) : D(B) \subseteq H \to H$ are closed linear operators in a Banach space H. Let us consider that $D(B)$ is independent of (t,s). The function $z_t : (-\infty, 0] \to H$, $z_t(s) = z(t+s)$ belongs to some abstract phase space $\mathcal{P}$ described axiomatically and $E_1, E_2, N : [0,c] \times \mathcal{P} \to H$, with $N(t,\psi) = \psi(0) + E_2(t,\psi)$, $\rho : V \times \mathcal{P} \to (-\infty, c]$ are appropriate functions.

The fundamental tool that we will use to study these problems is the existence of a resolvent operator connected with the homogeneous system

$$z''(t) = A(t)z(t) + \int_0^t B(t,s)z(s)ds, \quad t \in V = [0,c].$$

We subdivide this article into five sections. In Section 2, we include a few concepts and results related to this work, which will be used throughout the text. In Section 3, we present our results on the existence of mild solutions for the abstract second-order non-autonomous integro-differential evolution equations with state-dependent delay (1) and (2). In Section 4, we extend the theory developed in Section 3 to study the abstract second-order neutral integro-differential evolution equation with state-dependent delay (2) and (3). Finally, in Section 5, we apply our previous results to some specific models described by partial differential equations with state-dependent delay.

2. Preliminaries

This section is dedicated to introduce a few concepts and essential results needed to present our results. In the rest of this text, $(H, \|\cdot\|)$ is a Banach space and $A(t)$, $B(t,s)$, for $0 \le s \le t$ are closed linear operators defined on $D(A)$ and $D(B)$, respectively. We assume that $D(A)$ is dense in H. The space $D(A)$ provided with the graph norm induced by $A(t)$ is a Banach space. We will assume that all of these norms are equivalent. A simple condition for obtaining this property is that there exists $\lambda \in \rho(A(t))$, the resolvent set of $A(t)$, so that $(\lambda I - A(t))^{-1} : H \to D(A)$ is a bounded linear operator. In what follows, by $[D(A)]$ we represent the vector space $D(A)$ provided with any of these equivalent norms, and we denote

$$\|z\|_{[D(A)]} = \|z\| + \|A(t)z\|, \; z \in D(A).$$

For Banach spaces $(Z, \|\cdot\|_Z)$ and $(Y, \|\cdot\|_Y)$, we denote by $\mathcal{L}(Z,Y)$ the Banach space consisting of the bounded linear operators from Z into Y endowed with the uniform operator topology. When $Y = Z$, we abbreviate the notation writing $\mathcal{L}(Z)$. In addition, we use $B_p[z, Y]$ to symbolize the closed ball with center at z and radius $p > 0$ in Y. For a bounded function $z : [0, c] \to Y$ and $b \in [0, c]$ by $\|z\|_{Y,b}$, we denote

$$\|z\|_{Y,b} = \sup\{\|z(s)\|_Y : s \in [0, b]\}$$

or abbreviated $\|z\|_b$, when the space Y is clear from the context.

In recent times, there has been an increasing interest in studying the abstract non-autonomous second-order initial value problem

$$z''(t) = A(t)z(t) + f(t), \quad 0 \le s, t \le c, \tag{4}$$

$$z(s) = z^0, \quad z'(s) = z^1, \tag{5}$$

where $A(t) : D(A) \subseteq H \to H$, $t \in [0, c]$, is a closed linear operator densely defined and $f : [0, c] \to H$ is an appropriate function. Equations of this type have been considered in several papers. The reader is referred to [33–40] and the references mentioned in these works. In most of the works, the existence of solutions to the system (4) and (5) is related to the existence of an evolution operator $S(t, s)$ for the homogeneous system

$$z''(t) = A(t)z(t), \quad 0 \le t \le c.$$

Let us assume that for every $z \in D(A)$, $t \mapsto A(t)z$ is continuous. Hereafter, we consider that $A(\cdot)$ generates $(S(t, s))_{0 \le s \le t \le c}$, which was discussed by Kozak [41], Definition 2.1 (refer also Henríquez [42], Definition 1.1). We refer to these works for a careful study of this issue. We only regard here that $S(\cdot)z$ is continuously differentiable for all $z \in H$ with derivative uniformly bounded on bounded intervals, which in particular implies that there exists $M_1 > 0$ such that

$$\|S(t + h, s) - S(t, s)\| \le M_1 |h|,$$

for all $s, t, t + h \in [0, c]$. We define the operator $C(t, s) = -\dfrac{\partial S(t, s)}{\partial s}$. Consider that $f : [0, c] \to H$ is an integrable function.

For each fixed $0 \le s \le c$, we define the mild solution $z : [s, c] \to H$ by

$$z(t) = C(t, s)z^0 + S(t, s)z^1 + \int_s^t S(t, \tau) f(\tau) d\tau$$

for $s \le t \le c$. To avoid notations that make the text difficult to read, in this case we prefer to implicitly leave $z(\cdot)$ depends on the initial value s.

Next, we consider the second-order integro-differential systems

$$z''(t) = A(t)z(t) + \int_s^t B(t, \tau) z(\tau) d\tau, \quad s \le t \le c, \tag{6}$$

$$z(s) = 0, \quad z'(s) = x \in H, \tag{7}$$

for $0 \le s \le c$. This problem was discussed in [43]. We denote $\Delta = \{(t, s) : 0 \le s \le t \le c\}$.

We now introduce some conditions fulfilling the operator $B(\cdot)$:

(B1) For each $0 \le s \le t \le c$, $B(t, s) : [D(A)] \to H$ is a bounded linear operator, for every $z \in D(A)$, $B(\cdot, \cdot)z$ is continuous and

$$\|B(t, s)z\| \le b\|z\|_{[D(A)]},$$

for $b > 0$ which is a constant independent of $s, t \in \Delta$.

(B2) There exists $L_B > 0$ such that

$$\|B(t_2, s)z - B(t_1, s)z\| \le L_P |t_2 - t_1| \|z\|_{[D(A)]},$$

for all $z \in D(A)$, $0 \le s \le t_1 \le t_2 \le c$.

(B3) There exists $b_1 > 0$ such that

$$\left\| \int_{\sigma}^{t} S(t,s)B(s,\sigma)zds \right\| \leq b_1 \|z\|,$$

for all $z \in D(A)$.

Under these conditions, it has been established that there exists a resolvent operator $(\mathcal{V}(t,s))_{t \geq s}$ associated with the systems (6) and (7). From now on, we are going to consider that such a resolvent operator exists, and we adopt its properties as a definition.

Definition 1 ([43]). *A family of bounded linear operators $(\mathcal{V}(t,s))_{t \geq s}$ on H is said to be a resolvent operator for the systems (6) and (7) if it satisfies:*

(a) *The map $\mathcal{V} : \Delta \to \mathcal{L}(H)$ is strongly continuous, $\mathcal{V}(t,\cdot)z$ is continuously differentiable for all $z \in H$, $\mathcal{V}(s,s) = 0$, $\frac{\partial}{\partial t}\mathcal{V}(t,s)|_{t=s} = I$ and $\frac{\partial}{\partial s}\mathcal{V}(t,s)|_{s=t} = -I$.*

(b) *Assume $x \in D(A)$. The function $\mathcal{V}(\cdot,s)x$ is a solution for the systems (6) and (7). This means that*

$$\frac{\partial^2}{\partial t^2}\mathcal{V}(t,v)x = A(t)\mathcal{V}(t,s)x + \int_{s}^{t} B(t,\tau)\mathcal{V}(\tau,s)xd\tau,$$

for all $0 \leq s \leq t \leq c$.

It follows from condition (a) that there are constants $P > 0$ and $\widetilde{P} > 0$ such that

$$\|\mathcal{V}(t,s)\| \leq P, \quad \|\frac{\partial}{\partial s}\mathcal{V}(t,s)\| \leq \widetilde{P}, \quad (t,s) \in \Delta.$$

Moreover, the linear operator

$$G(t,\sigma)x = \int_{\sigma}^{t} B(t,s)\mathcal{V}(s,\sigma)xds, \ x \in D(A), \ 0 \leq \sigma \leq t \leq c,$$

can be extended to H. Portraying this expansion by the similar notation $G(t,\sigma)$, $G : \Delta \to \mathcal{L}(H)$ is strongly continuous, and it is verified that

$$\mathcal{V}(t,\sigma)x = S(t,\sigma) + \int_{\sigma}^{t} S(t,s)G(s,\sigma)xds, \text{ for all } x \in H. \tag{8}$$

It follows from this property that $\mathcal{V}(\cdot)$ is uniformly Lipschitz continuous, that is, there exists a constant $L_{\mathcal{V}} > 0$ such that

$$\|\mathcal{V}(t+h,\sigma) - \mathcal{V}(t,\sigma)\| \leq L_{\mathcal{V}}|h|, \text{ for all } t, t+h, \sigma \in [0,c]. \tag{9}$$

Let $g : V \to H$ be an integrable function. The non-homogeneous system

$$z''(t) = A(t)z(t) + \int_{0}^{t} B(t,s)z(s)ds + g(t), \quad t \in V = [0,c], \tag{10}$$

$$z(0) = x^0, \quad z'(0) = x^1, \tag{11}$$

was discussed in [43]. We now introduce the concept of mild solution for the systems (10) and (11).

Definition 2 ([43]). *Assume $x^0, x^1 \in H$. The function $z : [0,c] \to H$ given by*

$$z(t) = -\frac{\partial \mathcal{V}(t,s)x^0}{\partial s}\Big|_{s=0} + \mathcal{V}(t,0)x^1 + \int_{0}^{t} \mathcal{V}(t,s)g(s)ds,$$

is called the mild solution for the systems (10) *and* (11).

It is clear that $z(\cdot)$ in Definition 2 is a continuous function.

On the other hand, to study a retarded functional differential equation with infinite delay, we need to describe the system with states in an appropriately defined phase space. In order to develop a general theory, and using the theory established in [44], in this article, we will use an axiomatic definition of phase space. The phase space $\mathcal{P}$ is a vector space of functions from $(-\infty, 0]$ into H endowed with a seminorm $\|\cdot\|_{\mathcal{P}}$ that satisfies the following conditions:

(A) If $z : (-\infty, \eta + c) \to H$, $c > 0$, $\eta \in \mathbb{R}$ is continuous on $[\eta, \eta + c)$ and $z_\eta \in \mathcal{P}$, then for every $t \in [\eta, \eta + c)$, the following conditions hold:

- **(i)** z_t is in $\mathcal{P}$.
- **(ii)** $\|z(t)\| \le K_1 \|z_t\|_{\mathcal{P}}$.
- **(iii)** $\|z_t\|_{\mathcal{P}} \le K_2(t - \eta) \sup\{\|z(s)\| : \eta \le s \le t\} + K_3(t - \eta)\|z_\eta\|_{\mathcal{P}}$, where $K_1 > 0$ is a constant; $K_2, K_3 : [0, \infty) \to [1, \infty)$, $K_2(\cdot)$ is continuous, $K_3(\cdot)$ is locally bounded, and K_1, K_2, K_3 are independent of $z(\cdot)$.

(A1) For $z(\cdot)$ in $(\mathbf{A})$, $t \to z_t$ is continuous from $[\eta, \eta + c)$ into $\mathcal{P}$.

(B) $\mathcal{P}$ is complete.

A detailed theory about the properties of axiomatically defined phase spaces as above is found in [44].

Finally, we complete these preliminary observations by recalling two theorems about the existence of fixed points of applications that will be essential to establish the existence results. The following Theorem 1 is known as the Leray–Schauder's alternative theorem, and Theorem 2 has been established by Krasnoselskii (see [45], II.6.9(C.4)).

Theorem 1 ([45], Theorem II.6.5.4)**.** *Assume E is a closed convex subset of a Banach space $(X, \|\cdot\|)$ and $0 \in E$. Assume the map $F : E \to E$ is completely continuous. Then the map F has a fixed point in E or $\{z \in E : z = \lambda F(z),\ 0 < \lambda < 1\}$ is an unbounded set.*

Incidentally, condition $0 \in E$ is not necessary. It is sufficient that the set E is not empty. Specifically, if $x_0 \in E$, by translation in x_0, one can determine the subsequent result.

Corollary 1. *Let E be a closed convex subset of a Banach space $(X, \|\cdot\|)$ and $x_0 \in E$. Assume the map $F : E \to E$ is completely continuous. Then, F has a fixed point in E or $\{z \in E : z = (1 - \lambda)x_0 + \lambda F(z)$, for some $0 < \lambda < 1\}$ is an unbounded set.*

Theorem 2. *Assume M is a closed, convex, and bounded subset of a Banach space $(X, \|\cdot\|)$. Assume $F : M \to M$ is a continuous map of the form $F = A + B$, where $A : M \to X$ is compact and $B : M \to X$ is a contraction. Then F has a fixed point.*

3. Integro-Differential Systems

We are primarily focusing on the discussion about the existence of mild solutions for the systems (1) and (2). The general framework for studying the mentioned topic is as follows. We consider $(\mathcal{V}(t,s))_{t \ge s}$ is a resolvent operator for the systems (6) and (7). Moreover, $\phi \in \mathcal{P}$ and $\rho : V \times \mathcal{P} \to (-\infty, c]$ is a continuous function such that $\rho(t, \psi) \le t$, for all $t \in V$ and $\psi \in \mathcal{P}$. For a function $z \in C([0, c], H)$ such that $z(0) = \phi(0)$, we identify z with its extending to $(-\infty, c]$, and it is determined by $z(\theta) = \phi(\theta)$ for $\theta \le 0$. We also introduce the set

$$\mathcal{N}(\rho^-) = \{\rho(s, \psi) : (s, \psi) \in V \times \mathcal{P},\ \rho(s, \psi) \le 0\}.$$

Comparing with the developments in [43], we establish the following concept of mild solution for (1) and (2).

Definition 3. *A continuous function $z : (-\infty, c] \to H$ is said to be a mild solution for* (1) *and* (2) *if $z_0 = \phi \in \mathcal{P}$ and*

$$z(t) = -\frac{\partial \mathcal{V}(t,s)\phi(0)}{\partial s}\Big|_{s=0} + \mathcal{V}(t,0)x^1 + \int_0^t \mathcal{V}(t,s)E(s, z_{\rho(s,z_s)})ds,\ t \in V,$$

is satisfied.

Remark 1. In general, for a mild solution $z(\cdot)$ of (1) and (2), the derivative $z'(0)$ does not exist. However, it follows from [43] that if $z(0) = \phi(0) \in D(A)$, then $z'(0)$ exists and $z'(0) = x^1$.

To reach our aim, we will introduce the required assumptions as follows:

($\mathbf{H_1}$) The function $\mathcal{N}(\rho^-) \to \mathcal{P}$, $r \to \phi_r$ is well defined and continuous. Moreover, there exists a bounded continuous function $J^\phi : \mathcal{N}(\rho^-) \to [0, \infty)$ such that $\|\phi_r\|_{\mathcal{P}} \leq J^\phi(r)\|\phi\|_{\mathcal{P}}$ for all $r \in \mathcal{N}(\rho^-)$.

($\mathbf{H_2}$) The function $E_1 : V \times \mathcal{P} \to H$ fulfills the following conditions:

(i) For each $\psi \in \mathcal{P}$, $E_1(\cdot, \psi) : V \to H$ is strongly measurable.
(ii) For every $t \in V$, $E_1(t, \cdot) : \mathcal{P} \to H$ is continuous.
(iii) There exists $k : V \to [0, \infty)$, which is integrable, and $U : [0, \infty) \to [0, \infty)$, which is a continuous and non-decreasing function such that

$$\|E_1(t, \psi)\| \leq k(t)U(\|\psi\|_{\mathcal{P}}), \quad (t, \psi) \in V \times \mathcal{P}.$$

(iv) For each $t \in [0, c]$ and $r > 0$, $W(t, r) = \{\mathcal{V}(t,s)E_1(s, \psi) : s \in [0, t],\ \psi \in \mathcal{P}, \|\psi\|_{\mathcal{P}} \leq r\}$ is relatively compact in H.

Lemma 1 ([8], Lemma 3.1)**.** *Assume that $z : (-\infty, c] \to H$ is continuous on $[0, c]$ and $z_0 = \phi$. Then,*

$$\|z_s\|_{\mathcal{P}} \leq (K_{(3,c)} + J_0^\phi)\|\phi\|_{\mathcal{P}} + K_{(2,c)} \sup\{\|z(s)\|; s \in [0, \max\{0, s\}]\}, \quad s \in \mathcal{N}(\rho^-) \cup V,$$

where $J_0^\phi = \sup_{r \in \mathcal{R}(\rho^-)} J^\phi(r)$, $K_{(2,c)} = \sup_{t \in [0,c]} K_2(t)$ and $K_{(3,c)} = \sup_{t \in [0,c]} K_3(t)$.

Theorem 3. *If* ($\mathbf{H_1}$) *and* ($\mathbf{H_2}$) *are fulfilled, and*

$$PK_{(2,c)} \int_0^c k(s)ds < \int_m^\infty \frac{d\tau}{U(\tau)}, \tag{12}$$

where $m = (K_{(2,c)}K_1(\widetilde{P} + 1) + K_{(3,c)} + J_0^\phi)\|\phi\|_{\mathcal{P}} + PK_{(2,c)}\|x^1\|$, then there exists a mild solution for (1) *and* (2) *on V.*

Proof. We introduce the set $\mathcal{T}(c) = \{z \in C(V, H) : z(0) = \phi(0)\}$. Clearly, $\mathcal{T}(c)$ is a convex closed subset of $C(V, H)$ for the uniform convergence topology. Define the map $\Upsilon : \mathcal{T}(c) \to \mathcal{T}(c)$ as follows:

$$\Upsilon z(t) = -\frac{\partial \mathcal{V}(t,s)\phi(0)}{\partial s}\Big|_{s=0} + \mathcal{V}(t,0)x^1 + \int_0^t \mathcal{V}(t,s)E_1(s, z_{\rho(s,z_s)})ds,$$

for $t \in [0, c]$. Clearly $\Upsilon : \mathcal{T}(c) \to \mathcal{T}(c)$. In addition, as a consequence of our hypotheses and applying the Lebesgue dominated convergence theorem, one can argue that Υ is continuous.

We now establish that Υ is completely continuous. Assume $p > 0$ and set $B_p = \{z \in \mathcal{T}(c) : \|z(t)\| \leq p\}$.

First, we show that $\Upsilon(B_p)$ is equicontinuous on $[0,c]$. Let $z \in B_p$. From Lemma 1, we get

$$\begin{aligned}\|E_1(s, z_{\rho(s,z_s)})\| &\leq k(s)U(\|z_{\rho(s,z_s)}\|_{\mathcal{P}}) \\ &\leq k(s)U\Big((K_{(3,c)} + J_0^{\phi})\|\phi\|_{\mathcal{P}} + K_{(2,c)}p\Big) \\ &\leq C_1,\end{aligned}$$

for all $s \in [0,c]$ and for some constant $C_1 > 0$, which is independent of z. Therefore, using (9), we obtain

$$\begin{aligned}\|\Upsilon z(t+h) - \Upsilon z(t)\| \leq& \int_0^t \|[\mathcal{V}(t+h,s) - \mathcal{V}(t,s)]E_1(s, z_{\rho(s,z_s)})\|ds \\ &+ \int_t^{t+h} \|\mathcal{V}(t+h,s)E_1(s, z_{\rho(s,z_s)})\|ds \\ \leq& (tL_{\mathcal{V}} + P)C_1 h,\end{aligned}$$

for all $t \in [0,c]$ and $h \geq 0$ such that $t+h \leq c$. From this, we conclude that $\Upsilon(B_p)$ is right equicontinuous. By using the same process, one can easily prove the left equicontinuity. Hence, $\Upsilon(B_p)$ is equicontinuous.

We now show that $\Upsilon(B_p)(t)$ is relatively compact for each $t \in [0,c]$. Consider $z \in B_p$. From the mean value theorem, we infer that

$$\begin{aligned}\Upsilon z(t) &\in -\frac{\partial \mathcal{V}(t,s)\phi(0)}{\partial s}\Big|_{s=0} + \mathcal{V}(t,0)x^1 + \overline{tco\Big(\mathcal{V}(t,s)E_1(s, z_{\rho(s,z_s)})\Big)} \\ &\subseteq -\frac{\partial \mathcal{V}(t,s)\phi(0)}{\partial s}\Big|_{s=0} + \mathcal{V}(t,0)x^1 + \overline{tco(W(t,C_1))},\end{aligned}$$

where by $co(R)$ we denote the convex hull of a set R. Using now condition $(\mathbf{H_2})$(iv) and Mazur's theorem, we derive that $\Upsilon(B_p)(t)$ is relatively compact.

Consequently, collecting the preceding properties and by the Ascoli-Arzelà theorem, we conclude that Υ is completely continuous.

Assume that $z^{\lambda} \in \mathcal{T}(c)$ and $z^{\lambda} = (1-\lambda)\phi(0) + \lambda \Upsilon z^{\lambda}$ for $\lambda \in (0,1)$. Applying Lemma 1, we can estimate

$$\begin{aligned}\|z^{\lambda}(t)\| \leq& K_1(\widetilde{P}+1)\|\phi\|_{\mathcal{P}} + P\|z^1\| + P\int_0^t \Big\|E_1(s, z^{\lambda}_{\rho(s,z_s^{\lambda})})\Big\|ds \\ \leq& K_1(\widetilde{P}+1)\|\phi\|_{\mathcal{P}} + P\|x^1\| + P\int_0^t k(s)U\Big((K_{(3,c)} + J_0^{\phi})\|\phi\|_{\mathcal{P}} + K_{(2,c)}\|z^{\lambda}\|_{\max\{0,\rho(s,z_s^{\lambda})\}}\Big)ds \\ \leq& K_1(\widetilde{P}+1)\|\phi\|_{\mathcal{P}} + P\|x^1\| + P\int_0^t k(s)U\Big((K_{(3,c)} + J_0^{\phi})\|\phi\|_{\mathcal{P}} + K_{(2,c)}\|z^{\lambda}\|_s\Big)ds,\end{aligned} \tag{13}$$

where we have used $\rho(s, z_s^{\lambda}) \leq s$ for all $s \in [0,c]$. Let

$$\beta^{\lambda}(t) = (K_{(3,c)} + J_0^{\phi})\|\phi\|_{\mathcal{P}} + K_{(2,c)}\|z^{\lambda}\|_t.$$

From the preceding estimate, we find that

$$\begin{aligned}\beta^{\lambda}(t) \leq& \Big(K_{(2,c)}K_1(\widetilde{P}+1) + K_{(3,c)} + J_0^{\phi}\Big)\|\phi\|_{\mathcal{P}} + PK_{(2,c)}\|x^1\| \\ &+ PK_{(2,c)}\int_0^t k(s)U\Big(\beta^{\lambda}(s)\Big)ds.\end{aligned} \tag{14}$$

By denoting $\gamma_{\lambda}(t)$, the right-hand side of (14), one can get

$$\gamma'_{\lambda}(t) \leq PK_{(2,c)}k(t)U(\gamma_{\lambda}(t)).$$

Therefore,

$$\int_m^{\gamma_\lambda(t)} \frac{d\tau}{U(\tau)} \leq PK_{(2,c)} \int_0^c k(s)ds.$$

This inequality combined with the condition (12) allows us to conclude that $\{\gamma_\lambda : \lambda \in (0,1)\}$ is a bounded set and which means $\{z^\lambda : \lambda \in (0,1)\}$ is a bounded set.

The existence of a fixed point $z(\cdot)$ of Υ is a consequence of Corollary 1. From the definition of Υ, we find that $z(\cdot)$ is a mild solution for (1) and (2), and completes the proof. □

We need to introduce some additional conditions for continuing our study.

$(\mathbf{H_3})$ The function $F_1 : V \times \mathcal{P} \to H$ fulfills:

(i) For every $\psi \in \mathcal{P}$, $F_1(\cdot, \psi) : V \to H$ is strongly measurable and $F_1(\cdot, 0)$ is integrable.

(ii) There exists a continuous function $L_{F,1} : V \to [0,\infty)$ such that

$$\|F_1(t,\psi^2) - F_1(t,\psi^1)\| \leq L_{F,1}(t)\|\psi^2 - \psi^1\|_{\mathcal{P}},$$

for all $\psi^1, \psi^2 \in \mathcal{P}$ and $t \in V$.

(iii) There exists a positive continuous function $L_{F,2} : V \to [0,\infty)$ such that

$$\|F_1(t,z_{t_2}) - F_1(t,z_{t_1})\| \leq L_{F,2}(t)|t_2 - t_1|,\ t \in [0,c],$$

for all function $z : (-\infty, c] \to H$ such that $z_0 = \phi$ and $z : [0,c] \to H$ is continuous.

$(\mathbf{H_4})$ There exists a function $L_\rho : V \to [0,\infty)$ such that

$$\|\rho(t,\psi^2) - \rho(t,\psi^1)\| \leq L_\rho(t)\|\psi^2 - \psi^1\|_{\mathcal{P}},$$

for all $\psi^1, \psi^2 \in \mathcal{P}$ and $t \in V$.

We mention that, in general, the function $L_{F,2}(\cdot)$ depends on ϕ. Because in the most part of this work we treat with a fixed function $\phi \in \mathcal{P}$, for the sake of brevity we omit the dependence on ϕ. Furthermore, condition $(\mathbf{H_3})$(iii) is very demanding. However, it is easy to find examples of functions F_1 that satisfy it.

Example 1. *Assume $Q : H \to H$ is a bounded continuous function. We assume that $\|Q(x)\| \leq \eta$ for all $x \in H$ and that Q satisfies additional conditions, which depends on $\mathcal{P}$, so that the map $F_1 : \mathcal{P} \to H$ given by*

$$F_1(\psi) = \int_{-\infty}^0 Q(\psi(\theta))d\theta,$$

is well defined and continuous. It is clear that

$$\|F_1(t,z_{t_2}) - F_1(t,z_{t_1})\| \leq \int_{t_1}^{t_2} \|Q(\psi(\tau))\|d\tau \leq \eta(t_2 - t_1),\ t \in [0,c],$$

for all $z : (-\infty, c] \to H$ such that $z_0 = \phi$, $z : [0,c] \to H$ is continuous and $0 \leq t_1 \leq t_2 \leq c$.

In what follows in this section, we will be concerned with the system

$$z''(t) = A(t)z(t) + \int_0^t B(t,s)z(s)ds + F_1(t, z_{\rho(t,z_t)}) + E_1(t, z_{\rho(t,z_t)}), \quad t \in V, \tag{15}$$

with initial condition (2). It is important to point out that the following Theorem 4 is not an extension of Theorem 3 due to condition $(\mathbf{H_4})$. Fundamentally, this is because equations with state-dependent delay are intrinsically nonlinear even though the operators involved in the equation can be linear. In the following statement, we maintain the general conditions under which we are developing this section.

Theorem 4. *If* $(\mathbf{H_1})$–$(\mathbf{H_4})$ *are fulfilled,*

$$PK_{(2,c)}\int_0^c L_{F,1}(s)ds + PK_{(2,c)}\liminf_{\zeta\to\infty}\frac{U(\zeta)}{\zeta}\int_0^c k(s)ds < 1, \tag{16}$$

and

$$PK_{(2,c)}\int_0^c (L_{F,1}(s) + L_{F,2}(s)L_\rho(s))ds < 1, \tag{17}$$

hold. Then, there exists a mild solution for (15) *with initial conditions* (2) *on* V.

Proof. Let $\Upsilon_1 : \mathcal{T}(c) \to C(V,H)$ the map given by

$$\Upsilon_1 z(t) = \int_0^t \mathcal{V}(t,s)F_1(s, z_{\rho(s,z_s)})ds,$$

for $t \in [0,c]$ and $\widetilde{\Upsilon} = \Upsilon_1 + \Upsilon$.

As our first step, we will show that there is $p > 0$ sufficiently large such that $\|\phi(0)\| \le p$ and $\widetilde{\Upsilon}(B_p) \subseteq B_p$, where $B_p = \{z \in \mathcal{T}(c) : \|z(t)\| \le p,\ 0 \le t \le c\}$. Indeed, assuming the contrary, we infer that for every $p > 0$ there exist $z^p \in B_p$ and $t^p \in V$ such that $p < \|\widetilde{\Upsilon}z^p(t^p)\|$. Then, using Lemma 1, one can conclude that

$$\begin{aligned}
p &< \|\Upsilon_1(z^p)(t^p)\| + \|\Upsilon(z^p)(t^p)\| \\
&\le \left\| -\frac{\partial\mathcal{V}(t,s)\phi(0)}{\partial s}\Big|_{s=0} \right\| + \|\mathcal{V}(t^p,0)x^1\| + \int_0^{t^p} \left\|\mathcal{V}(t^p,s)F_1(s, z^p_{\rho(s,z^p_s)})\right\| ds \\
&\quad + \int_0^{t^p} \left\|\mathcal{V}(t^p,s)E_1(s, z^p_{\rho(s,z^p_s)})\right\| ds \\
&\le K_1(\widetilde{P}+1)\|\phi\|_{\mathcal{P}} + P\|x^1\| + P\int_0^c \|F_1(s,0)\|ds + P\int_0^{t^p} L_{F,1}(s)\left\|z^p_{\rho(s,z^p_s)}\right\|_{\mathcal{P}} ds \\
&\quad + P\int_0^{t^p} k(s)U\left(\left\|z^p_{\rho(s,z^p_s)}\right\|_{\mathcal{P}}\right)ds \\
&\le K_1(\widetilde{P}+1)\|\phi\|_{\mathcal{P}} + P\|x^1\| + P\int_0^c \|F_1(s,0)\|ds \\
&\quad + P\int_0^{t^p} L_{F,1}(s)\Big((K_{(3,c)} + J_0^\phi)\|\phi\|_{\mathcal{P}} + K_{(2,c)}\|z^p\|_c\Big)ds \\
&\quad + P\int_0^{t^p} k(s)U\Big((K_{(3,c)} + J_0^\phi)\|\phi\|_{\mathcal{P}} + K_{(2,c)}\|z^p\|_c\Big)ds \\
&\le K_1(\widetilde{P}+1)\|\phi\|_{\mathcal{P}} + P\|x^1\| + P\int_0^c \|F_1(s,0)\|ds \\
&\quad + P\Big((K_{(3,c)} + J_0^\phi)\|\phi\|_{\mathcal{P}} + K_{(2,c)}(p + \|\phi(0)\|)\Big)\int_0^c L_{F,1}(s)ds \\
&\quad + PU\Big((K_{(3,c)} + J_0^\phi)\|\phi\|_{\mathcal{P}} + K_{(2,c)}(p + \|\phi(0)\|)\Big)\int_0^c k(s)ds,
\end{aligned} \tag{18}$$

and hence

$$1 \le PK_{(2,c)}\int_0^c L_{F,1}(s)ds + PK_{(2,c)}\liminf_{\zeta\to\infty}\frac{U(\zeta)}{\zeta}\int_0^c k(s)ds.$$

which contradicts the condition (16).

As our next step, we prove that Υ_1 is a contraction. Assume $w, z \in \mathcal{T}(c)$ and let $t_1 = \rho(s, w_s)$ and $t_2 = \rho(s, z_s)$. We have to consider three possibilities essentially different $0 \leq t_1 \leq t_2$; $t_1 < 0 \leq t_2$ and $t_1 \leq t_2 \leq 0$. Let us consider the first alternative.

$$\begin{aligned}
&\|F_1(s, z_{\rho(s,z_s)}) - F_1(s, w_{\rho(s,w_s)})\| \leq \|F_1(s, z_{t_1}) - F_1(s, w_{t_1})\| + \|F_1(s, z_{t_1}) - F_1(s, z_{t_2})\| \\
&\quad \leq \quad L_{F,1}(s)\|z_{t_1} - w_{t_1}\|_{\mathcal{P}} + L_{F,2}(s)|t_1 - t_2| \\
&\quad \leq \quad L_{F,1}(s)K_2(t_1) \max_{0\leq\tau\leq t_1} \|z(\tau) - w(\tau)\| + L_{F,2}(s)L_\rho K_2(s) \max_{0\leq\tau\leq s} \|z(\tau) - w(\tau)\| \\
&\quad \leq \quad K_{(2,c)}(L_{F,1}(s) + L_{F,2}(s)L_\rho) \max_{0\leq\tau\leq c} \|z(\tau) - w(\tau)\|.
\end{aligned}$$

For the second and third alternative, we can proceed similarly as above to obtain

$$\|F_1(s, z_{\rho(s,z_s)}) - F_1(s, w_{\rho(s,w_s)})\| \leq K_{(2,c)} L_{F,2}(s) L_\rho \max_{0\leq\tau\leq c} \|z(\tau) - w(\tau)\|.$$

Thus, in the general case, one can affirm that

$$\|F_1(s, z_{\rho(s,z_s)}) - F_1(s, w_{\rho(s,w_s)})\| \leq K_{(2,c)}(L_{F,1}(s) + L_{F,2}(s)L_\rho) \max_{0\leq\tau\leq c} \|z(\tau) - w(\tau)\|.$$

Using these estimates, we get

$$\begin{aligned}
\|\Upsilon_1 z(t) - \Upsilon_1 w(t)\| \quad &\leq \quad \int_0^t \|\mathcal{V}(t,s)F_1(s, z_{\rho(s,z_s)}) - \mathcal{V}(t,s)F_1(s, z_{\rho(s,z_s)})\|ds \\
&\leq \quad PK_{(2,c)} \int_0^c (L_{F,1}(s) + L_{F,2}(s)L_\rho)ds \max_{0\leq\tau\leq c} \|z(\tau) - w(\tau)\|.
\end{aligned}$$

It follows from (17) that Υ_1 is a contraction. In particular, Υ_1 is a continuous map. Moreover, using the proof of Theorem 3, we know that Υ is continuous, which in turn implies that $\widetilde{\Upsilon}$ also is continuous. Arguing in a similar way, we can affirm that the map Υ is compact. Collecting these properties, we infer that hypotheses of Theorem 2 are fulfilled. Hence, we conclude that $\widetilde{\Upsilon}$ has a fixed point $z(\cdot)$, which is a mild solution for (15) with initial conditions (2). □

4. Existence for Neutral Systems

This section is devoted to prove the existence of (2) and (3). In this section, we remain within the general framework of hypotheses discussed in Section 3.

Remark 2. As explained in Remark 1, we should not expect that a mild solution to (2) and (3) is differentiable. Therefore, to determine the concept of mild solution $z(\cdot)$ of (2) and (3), we have to decide in which sense we are going to interpret $z'(0)$. Comparing with what is stated in Remark 1, in this case $z'(0)$ does not depend only on $\phi(0)$, but also on $E_2(0, \phi)$ and $\frac{d}{dt}E_2(t, z_t)|_{t=0}$, whose existence will depend in turn on E_2, ϕ and the properties of phase space $\mathcal{P}$. If we assume, formally, that $\frac{d}{dt}E_2(t, z_t)|_{t=0} = \nu^1$, this motivates us to specify the initial conditions of (2) and (3) as $z_0 = \phi$, $z'(0) = x^1$ and $\frac{d}{dt}N(t, z_t)|_{t=0} = x^1 + \nu^1$. In the case that $\phi(0) + E_2(0, \phi) \in D(A)$ and the derivative $\frac{d}{dt}E_2(t, z_t)|_{t=0} = \nu^1$, this way of interpreting the mild solution of (2) and (3) determines exactly that $z'(0) = x^1$. Consequently, we define the concept of mild solution in terms of an arbitrary $\nu^1 \in H$.

Definition 4. *A function $z : (-\infty, c] \to H$ is called a mild solution for* (2) *and* (3) *if z is continuous on $[0, c]$, $z_0 = \phi \in \mathcal{P}$, $\frac{d}{dt}N(t, z_t)|_{t=0} = x^1 + v^1$ and the integral equation*

$$\begin{aligned} z(t) = &- \frac{\partial \mathcal{V}(t,s)[\phi(0) + E_2(0,\phi)]}{\partial s}\Big|_{s=0} + \mathcal{V}(t,0)[x^1 + y^1] - E_2(t, z_t) \\ &+ \int_0^t \mathcal{V}(t,s)E_1(s, z_{\rho(s,z_s)})ds, \end{aligned}$$

is satisfied for $t \in V$.

To reach our aim, we consider appropriated conditions for the function E_2.

$(\mathbf{H_5})$ The function $E_2 : [0, c] \times \mathcal{P} \to H$ is continuous and fulfills:

(i) For each $r > 0$, the set $\{E_2(\cdot, \psi) : \|\psi\|_{\mathcal{P}} \le r\}$ is equicontinuous on $[0, c]$ and, for every $t \in [0, c]$, the set $\{E_2(t, \psi) : \|\psi\|_{\mathcal{P}} \le r\}$ is relatively compact in H.

(ii) There exist $e_1, e_2 > 0$ such that $e_2 K_{(2,c)} < 1$ and

$$\|E_2(t, \psi)\| \le e_1 + e_2 \|\psi\|_{\mathcal{P}},$$

for all $t \in [0, c]$ and $\psi \in \mathcal{P}$.

(iii) There exists a positive continuous function $L_{E,2} : [0, c] \times [0, \infty) \to [0, \infty)$ such that

$$\|E_2(t, z_{t_2}) - E_2(t, z_{t_1})\| \le L_{E,2}(t, r)|t_2 - t_1|, \ t \in [0, c],$$

for all function $z : (-\infty, c] \to H$ such that $z_0 = \phi$ and $z : [0, c] \to H$ is continuous with $\|z\|_c \le r$.

$(\mathbf{H_6})$ The function $E_2 : [0, c] \times \mathcal{P} \to H$ is continuous and there exists $L_2 > 0$ such that

$$\|E_2(t, \psi^1) - E_2(t, \psi^2)\| \le L_2 \|\psi^1 - \psi^2\|_{\mathcal{P}},$$

for all $t \in [0, c]$ and $\psi^1, \psi^2 \in \mathcal{P}$.

Theorem 5. *If* $(\mathbf{H_1})$, $(\mathbf{H_2})$, $(\mathbf{H_5})$ *are fulfilled, and*

$$M_2 K_{(2,c)} \int_0^c k(s)ds < \int_m^\infty \frac{d\tau}{U(\tau)},$$

where

$$\begin{aligned} M_1 =& \frac{1}{1 - e_2 K_{(2,c)}} \Big[(\widetilde{P} + 1)e_1 + \Big(\widetilde{P}e_2 + K_1(\widetilde{P} + 1) + e_2 K_{(3,c)} \Big) \|\phi\|_{\mathcal{P}} + P(\|x^1\| + \|v^1\|) \Big], \\ M_2 =& \frac{1}{1 - e_2 K_{(2,c)}} P, \\ m =& K_{(2,c)} M_1 + (K_{(3,c)} + J_0^\phi)\|\phi\|_{\mathcal{P}}, \end{aligned}$$

then there exists a mild solution for (2) *and* (3) *on V.*

Proof. We define $\Upsilon_1 : \mathcal{T}(c) \to C([0, c], H)$ by

$$\Upsilon_1 z(t) = -\frac{\partial \mathcal{V}(t,s)E_2(0,\phi)}{\partial s}\Big|_{s=0} + \mathcal{V}(t,0)v^1 - E_2(t, z_t),$$

for $t \in [0, c]$ and $\widetilde{\Upsilon} = \Upsilon_1 + \Upsilon$.

Initially, we verify that Υ_1 is completely continuous. Clearly, Υ_1 is continuous. Moreover, it follows from $(\mathbf{H_5})$ that for each $r > 0$ and $t \in [0,c]$, the set $\{E_2(t, z_t) : \|z\|_c \leq r\}$ is relatively compact in H. In addition,

$$\begin{aligned}
&\|E_2(t+h, z_{t+h}) - E_2(t, z_t)\| \\
&\leq \quad \|E_2(t+h, z_t) - E_2(t, z_t)\| + \|E_2(t+h, z_{t+h}) - E_2(t+h, z_t)\| \\
&\leq \quad \|E_2(t+h, z_t) - E_2(t, z_t)\| + L_{E,2}(r)|h| \to 0, \ h \to 0,
\end{aligned}$$

uniformly for $z \in C([0,c], H)$ with $\|z\|_c \leq r$. Due to Ascoli-Arzelà theorem, the set $\Upsilon_1(B_r)$ is relatively compact in $C([0,c], H)$.

Now, let $z^\lambda \in \mathcal{T}(c)$ such that $z^\lambda = (1-\lambda)\phi(0) + \lambda\widetilde{\Upsilon} z^\lambda$ for $\lambda \in (0,1)$. Using the estimate (13), we get

$$\begin{aligned}
&\|z^\lambda(t)\| \leq \|\Upsilon_1 z^\lambda(t)\| + K_1(\widetilde{P}+1)\|\phi\|_{\mathcal{P}} + P\|x^1\| \\
&\quad + P\int_0^t k(s)U\Big((K_{(3,c)} + J_0^\phi)\|\phi\|_{\mathcal{P}} + K_{(2,c)}\|z^\lambda\|_s\Big)ds \\
&\leq \widetilde{P}\|E_2(0,\phi)\| + P\|v^1\| + \|E_2(t, z_t^\lambda)\| + K_1(\widetilde{P}+1)\|\phi\|_{\mathcal{P}} + P\|x^1\| \\
&\quad + P\int_0^t k(s)U\Big((K_{(3,c)} + J_0^\phi)\|\phi\|_{\mathcal{P}} + K_{(2,c)}\|z^\lambda\|_s\Big)ds \\
&\leq \widetilde{P}(e_1 + e_2\|\phi\|_{\mathcal{P}}) + P\|v^1\| + e_1 + e_2\|z_t^\lambda\|_{\mathcal{P}} + K_1(\widetilde{P}+1)\|\phi\|_{\mathcal{P}} + P\|x^1\| \\
&\quad + P\int_0^t k(s)U\Big((K_{(3,c)} + J_0^\phi)\|\phi\|_{\mathcal{P}} + K_{(2,c)}\|z^\lambda\|_s\Big)ds \\
&\leq \widetilde{P}(e_1 + e_2\|\phi\|_{\mathcal{P}}) + P\|v^1\| + e_1 + e_2K_{(2,c)}\max_{0\leq\tau\leq t}\|z^\lambda(\tau)\| + e_2K_{(3,c)}\|\phi\|_{\mathcal{P}} + K_1(\widetilde{P}+1)\|\phi\|_{\mathcal{P}} \\
&\quad + P\|x^1\| + P\int_0^t k(s)U\Big((K_{(3,c)} + J_0^\phi)\|\phi\|_{\mathcal{P}} + K_{(2,c)}\|z^\lambda\|_s\Big)ds,
\end{aligned}$$

which implies that

$$\begin{aligned}
\max_{0\leq s\leq t}\|z^\lambda(s)\| \leq& \frac{1}{1 - e_2K_{(2,c)}}\Big[\widetilde{P}(e_1 + e_2\|\phi\|_{\mathcal{P}}) + P\|v^1\| \\
&+ e_1 + \Big(e_2K_{(3,c)} + K_1(\widetilde{P}+1)\Big)\|\phi\|_{\mathcal{P}}\Big] \\
&+ \frac{1}{1 - e_2K_{(2,c)}}\Big[P\|x^1\| + P\int_0^t k(s)U\Big((K_{(3,c)} + J_0^\phi)\|\phi\|_{\mathcal{P}} + K_{(2,c)}\|z^\lambda\|_s\Big)ds\Big] \\
\leq& M_1 + M_2\int_0^t k(s)U\Big((K_{(3,c)} + J_0^\phi)\|\phi\|_{\mathcal{P}} + K_{(2,c)}\|z^\lambda\|_s\Big)ds.
\end{aligned}$$

Let

$$\beta^\lambda(t) = (K_{(3,c)} + J_0^\phi)\|\phi\|_{\mathcal{P}} + K_{(2,c)}\|z^\lambda\|_t.$$

From the preceding estimate, we obtain that

$$\beta^\lambda(t) \leq K_{(2,c)}M_1 + (K_{(3,c)} + J_0^\phi)\|\phi\|_{\mathcal{P}} + K_{(2,c)}M_2\int_0^t k(s)U\Big(\beta^\lambda(s)\Big)ds. \tag{19}$$

By denoting $\gamma_\lambda(t)$, the right-hand side of (19), one can get

$$\gamma'_\lambda(t) \leq M_2K_{(2,c)}k(t)U(\gamma_\lambda(t)).$$

Therefore,

$$\int_m^{\gamma_\lambda(t)} \frac{d\tau}{U(\tau)} \leq M_2K_{(2,c)}\int_0^c k(s)ds.$$

As this inequality contradicts hypothesis (5), we infer that $\{\gamma_\lambda : \lambda \in (0,1)\}$ is a bounded set, which provides $\{z^\lambda : \lambda \in (0,1)\}$ is a bounded set in $C([0,c],H)$. Applying Corollary 1, we conclude that $\widetilde{\Upsilon}$ has a fixed point $z(\cdot)$, which is a mild solution for (2) and (3) on V. □

Theorem 6. *If* $(\mathbf{H_1})$, $(\mathbf{H_2})$, $(\mathbf{H_6})$ *are fulfilled, and*

$$L_2 K_{(2,c)} + P K_{(2,c)} \liminf_{\zeta \to \infty} \frac{U(\zeta)}{\zeta} \int_0^c k(s)ds < 1, \tag{20}$$

then there exists a mild solution for (2) *and* (3) *on* V.

Proof. We keep the notations introduced in Theorem 5.

Initially, we will show that there is $p > 0$ sufficiently large such that $\|\phi(0)\| \leq p$ and $\widetilde{\Upsilon}(B_p) \subseteq B_p$, where $B_p = \{z \in \mathcal{T}(c) : \|z(t)\| \leq p,\ 0 \leq t \leq c\}$. In fact, assuming the contrary, we infer that for every $p > 0$ there exist $z^p \in B_p$ and $t^p \in V$ such that

$$p < \|\widetilde{\Upsilon} z^p(t^p)\| \leq \|\Upsilon_1(z^p)(t^p)\| + \|\Upsilon(z^p)(t^p)\|,$$

and using the estimate (18), we find that

$$\begin{aligned} p <& \widetilde{P}\|E_2(0,\phi)\| + P\|\nu^1\| + \|E_2(t^p, z^p_{t^p})\| + K_1(\widetilde{P}+1)\|\phi\|_{\mathcal{P}} + P\|x^1\| \\ &+ PU\Big((K_{(3,c)} + J_0^\phi)\|\phi\|_{\mathcal{P}} + K_{(2,c)}(p + \|\phi(0)\|)\Big)\int_0^c k(s)ds \\ \leq& K_1(\widetilde{P}+1)\|\phi\|_{\mathcal{P}} + \widetilde{P}\|E_2(0,\phi)\| + P(\|x^1\| + \|\nu^1\|) + \max_{0\leq t\leq c}\|E_2(t,0)\| + K_{(3,c)}\|\phi\|_{\mathcal{P}} \\ &+ L_2 K_{(2,c)} p + PU\Big((K_{(3,c)} + J_0^\phi)\|\phi\|_{\mathcal{P}} + K_{(2,c)}(p + \|\phi(0)\|)\Big)\int_0^c k(s)ds. \end{aligned}$$

From this estimate, we find that

$$1 \leq L_2 K_{(2,c)} + P K_{(2,c)} \liminf_{\zeta\to\infty} \frac{U(\zeta)}{\zeta}\int_0^c k(s)ds,$$

which contradicts the condition (20).

Furthermore, for $w, z \in \mathcal{T}(c)$, we find that

$$\|E_2(s,z_s) - E_2(s,w_s)\| \leq L_2\|z_s - w_s\|_{\mathcal{P}} \leq L_2 K_{(2,c)}\|z - w\|_c.$$

Since condition (20) implies that $L_2 K_{(2,c)} < 1$, we conclude that Υ_1 is a contraction.

Therefore, all the requirements of Theorem 2 are fulfilled with $A = \Upsilon$ and $B = \Upsilon_1$. This allows us to conclude that $\widetilde{\Upsilon}$ has a fixed point $z(\cdot)$, and which is a mild solution for (2) and (3) on V. □

5. Applications

This section aims to illustrate the application of the theory developed in the previous sections to the study of a wave motion of a bar located in $[0,\pi]$ and with fixed ends. Specifically, we will consider the following problem described by a second-order partial integro-differential equation with the state-dependent delay of the following form

$$\begin{aligned} \frac{\partial^2}{\partial t^2} z(t,y) =& \frac{\partial^2}{\partial y^2} z(t,y) + a(t)z(t,y) + \int_0^t b(t-s)\frac{\partial^2 z(s,y)}{\partial y^2}ds \\ &+ \int_{-\infty}^t a_0(\tau - t) z(\tau - \rho_1(t)\rho_2(\|z(t)\|), y)d\tau,\ (t,y) \in [0,c]\times[0,\pi], \end{aligned} \tag{21}$$

$$z(t,0) = z(t,\pi) = 0, \quad t \in [0,c], \tag{22}$$

$$z(\theta, y) = \phi(\theta, y), \quad \frac{\partial}{\partial t} z(t, y)|_{t=0} = \hbar(y), \quad \theta \in (-\infty, 0], \ y \in [0, \pi], \tag{23}$$

where $a, b : [0, c] \to \mathbb{R}$, $a_0 : (-\infty, 0] \to \mathbb{R}$, $\rho_1 : [0, c] \to [0, \infty)$ and $\rho_2 : [0, \infty) \to [0, \infty)$ are continuous functions, $\phi : (-\infty, 0] \times [0, \pi] \to \mathbb{R}$ and $\hbar : [0, \pi] \to \mathbb{R}$ satisfy appropriate conditions described below.

We model the systems (21) and (22) in the space $H = L^2([0, \pi])$ endowed with its classical inner product $\langle \cdot \rangle$. Accordingly, we consider $\phi(\theta, \cdot), \hbar(\cdot) \in H$. We assume that $\mathcal{P}$ is a phase space for functions with values in H. This implies that $\rho : [0, c] \times \mathcal{P} \to [0, \infty))$ is given by $\rho(t, \psi) = t - \rho_1(t)\rho(\|\psi(0)\|)$ is continuous such that $\rho(t, \psi) \leq t$ for all $0 \leq t \leq c$. Moreover, identifying $\phi(\theta)(y) = \phi(\theta, y)$ for $\theta \in (-\infty, 0]$ and $y \in [0, \pi]$, consider $\phi \in \mathcal{P}$ has the following property: for every $\tau \in (-\infty, 0]$, $\phi_\tau \in \mathcal{P}$ and the function $(-\infty, 0] \to \mathcal{P}$, $s \mapsto \phi_\tau$, is continuous and there exists $J^\phi(\tau) \geq 0$ such that $\|\phi_\tau\|_{\mathcal{P}} \leq J^\phi(\tau)\|\phi\|_{\mathcal{P}}$. This implies that condition (H_1) is fulfilled. We define

$$E_1(t, \psi) = \int_{-\infty}^{0} a_0(\theta)\psi(\theta) d\theta, \ \psi \in \mathcal{P}.$$

We will assume that the map $E_1(t, \psi)$ is well defined and E_1 is bounded linear. Hence, there exists $k > 0$ such that

$$\|E_1(t, \psi)\| \leq k\|\psi\|_{\mathcal{P}}.$$

We denote by A_0 the operator given by $A_0 z(\tau) = z''(x)$ with domain

$$D(A) = \{z \in H^2([0, \pi]) : z(0) = z(\pi) = 0\}.$$

Then, A_0 is the infinitesimal generator of a cosine function of operators $(C_0(t))_{t \in \mathbb{R}}$ on H associated with sine function $(S_0(t))_{t \in \mathbb{R}}$. Additionally, A_0 has discrete spectrum which consists of eigenvalues $-n^2$ for $n \in \mathbb{N}$, with corresponding eigenvectors

$$w_n(x) = \frac{1}{\sqrt{2\pi}} e^{inx}, \quad n \in \mathbb{N}.$$

The set $\{w_n : n \in \mathbb{N}\}$ is an orthonormal basis of H. Applying this basis, we can write that

$$A_0 z = \sum_{n=1}^{\infty} -n^2 \langle z, w_n \rangle w_n,$$

for $z \in D(A_0)$, $(C_0(t))_{t \in \mathbb{R}}$ is given by

$$C_0(t)z = \sum_{n=1}^{\infty} \cos(nt) \langle z, w_n \rangle w_n, \quad t \in \mathbb{R},$$

and the sine function is

$$S_0(t)z = \sum_{n=1}^{\infty} \frac{\sin(nt)}{n} \langle z, w_n \rangle w_n, \quad t \in \mathbb{R}.$$

It is immediate from these representations that $\|C_0(t)\| \leq 1$ and that $S_0(t)$ is compact for all $t \in \mathbb{R}$.

We define $A(t)z = A_0 z + a(t)z$ on $D(A)$. Clearly, $A(t)$ is a closed linear operator. Therefore, $A(t)$ generates $(S(t, s))_{0 \leq s \leq t \leq c}$ such that $S(t, s)$ is compact, for all $0 \leq s \leq t \leq c$ [43].

We complete the terminology by defining $B(t, s) = b(t - s)A_0$ for $0 \leq s \leq t \leq c$ on $D(A)$. Collecting these definitions, it is clear that we can represent the system (21) and (23) in the abstract form (1) and (2). Furthermore, it is not difficult to see that conditions **(B1)–(B3)** from Section 2 are fulfilled, which in turn implies that there exists a resolvent operator $(\mathcal{V}(t, s))_{0 \leq s \leq t \leq c}$ associated to (21)–(23). In addition, it follows from (8) that each operator $\mathcal{V}(t,s)$ is compact. This allow us to conclude $(\mathbf{H_2})$ is fulfilled with $U(\tau) = \tau$. Since $\displaystyle\int_m^\infty \frac{d\tau}{U(\tau)} = \infty$ for all $m < \infty$, we conclude that the condition (12) is also satisfied.

Therefore, the next proposition is a simple outcome of Theorem 3.

Proposition 1. *Under the above conditions, there exists a mild solution for* (21)–(23) *with values in H and defined on* $[0,c]$.

6. Conclusions

The main focus of this paper is on finding mild solutions for second-order abstract non-autonomous integro-differential evolution systems with infinite state-dependent delay. We initially studied the existence of solutions to second-order abstract evolution systems to later expand the scope of our study to include the second-order abstract neutral evolution systems in our analysis. The features of the resolvent operator analogous to second-order integro-differential systems were used to arrive at our conclusions. Finally, we gave an application to back up the discussion's authenticity. We will focus on the existence of mild solutions for second-order abstract non-autonomous stochastic integro-differential evolution systems with infinite state-dependent delay as our future work.

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