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# Iterated Functions Systems Composed of Generalized $\theta$-Contractions

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**Abstract:** The theory of iterated function systems (IFSs) has been an active area of research on fractals and various types of self-similarity in nature. The basic theoretical work on IFSs has been proposed by Hutchinson. In this paper, we introduce a new generalization of Hutchinson IFS, namely generalized $\theta$-contraction IFS, which is a finite collection of generalized $\theta$-contraction functions $T_1, \ldots, T_N$ from finite Cartesian product space $X \times \cdots \times X$ into $X$, where $(X, d)$ is a complete metric space. We prove the existence of attractor for this generalized IFS. We show that the Hutchinson operators for countable and multivalued $\theta$-contraction IFSs are Picard. Finally, when the map $\theta$ is continuous, we show the relation between the code space and the attractor of $\theta$-contraction IFS.

**Keywords:** iterated function systems; fixed point; attractor; fractal; $\theta$-contraction; picard operator; code space

**MSC:** 37C25; 47H04; 47H09; 47H10; 28A80

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## 1. Introduction

In 1975, Mandelbrot [1] introduced the concept of fractal theory, which studies patterns in the highly complex and unpredictable structures that exist in nature. In 1981, Hutchinson [2] conceptualized a mathematical way to generate self-similar fractals from iterated function system (IFS). The IFS is a finite collection of continuous mappings on a complete metric space. It is known that a contraction map is also continuous. Banach [3] proved that every contraction map on a complete metric space has a unique fixed point. The Banach fixed point theorem is a very effective and popular tool to prove the existence and uniqueness of solutions of certain problems arising within and beyond mathematics. Using Banach fixed point theorem, we can get an attractor or a fractal by iteration of a finite collection of contraction maps of an IFS. An attractor is usually a non-empty self-similar set as it satisfies a self-referential equation, and it is a compact subset of a complete metric space. The IFS theory is used to construct fractal interpolation functions (FIFs) to model various complex scientific and natural phenomena. The fractal theory has found applications in diverse areas such as learning automata, modelling, image processing, signal processing, approximation theory, study of bio-electric recordings, etc. (see [4–12]).

The framework of IFS theory has been extended to generalized contractions, countable IFSs, multifunction systems and more general spaces by many authors in the last two decades, see for instance [13–19]. In particular, Mihail and Miculescu [20,21] considered mappings from a finite Cartesian product $X \times \cdots \times X$ into $X$ instead of self-mappings of a metric space $X$. Dumitru [22] enhanced the work of Miculescu and Mihail by taking a generalized IFS composed of Meir–Keeler type mappings. A similar extension performed by Strobin and Swaczyna [23] with a generalized IFS consisting of $\phi$-contractions.

Secelean [24] explored the IFSs composed of a countable family of Meir–Keeler contractive and $\phi$-contraction maps. Again, he [25] extended some fixed point results from the classical Hutchinson–Barnsley theory of IFS consisting of Banach contractions to IFS consisting of $F$-contractions. A multivalued approach of infinite iterated function systems accomplished by Leśniak [26]. Jeli and Samet [27] proposed a new type of contractive mappings known as $\theta$-contraction (or JS-contraction), and they proved a fixed point result in generalized metric spaces. In addition, they demonstrated that the Banach fixed point theorem remains as a particular case of $\theta$-contraction.

In the present paper, we propose an extension of IFS theory by including the left end-points of the domain and range of $\theta$-functions. The new system is called generalized $\theta$-contraction IFS, and it consists of a finite collection of $\theta^{k_n}$-contraction functions on a complete metric product space. Every $\theta$-contraction IFS is an IFS, but the converse is not generally true, hence the set of attractors for the $\theta$-contraction IFSs is a broader family than the set of attractors of the IFSs. This paper is organized as follows: We discuss the basics and elementary properties of IFS, $\theta$-contractions, multivalued map, and code space in Section 2. In Section 3, we construct generalized $\theta$-contraction IFSs and prove the existence and uniqueness of its attractor. Further, we present the results for attractors of IFSs consisting of countable and multivalued $\theta^{k_n}$-contraction maps in Section 4. Finally in Section 5, we demonstrate the relation between the codes space and the attractor of $\theta$-contraction IFS, when the map $\theta$ is continuous.

## 2. Preliminary Facts

We discuss some basics and elementary results on iterated function systems, $\theta$-contractions, multivalued map, and code spaces in this section. The details can be found in the references [3,25,27–29].

**Definition 1.** *A mapping $T : X \to X$ on a metric space $(X, d)$ is called a contraction mapping if there is a constant $0 \le k < 1$ such that*

$$d(Tx, Ty) \le k\, d(x, y),\ \forall x, y \in X,$$

*where k is called contractivity factor for $T$. In most of the text, this map is also called contractivity map.*

Banach [3] proved that if $T : X \to X$ is a contraction map on a complete metric space $(X, d)$, then $T$ has unique fixed point $\bar{x} \in X$. Moreover, $\lim_{n\to\infty} T^n(x) = \bar{x}$ for each $x \in X$.

Let $K(X)$ be the set of all non-empty compact subsets of a metric space $(X, d)$. It is a metric space with the Hausdorff metric $h$ defined by

$$h(A, B) := \max\{D(A, B), D(B, A)\},$$

where $D(A, B) := \sup_{x\in A} \inf_{y\in B} d(x, y)$. The space $(K(X), h)$ is called Hausdorff metric space. If $(X, d)$ is complete (compact) metric space, t hen $(K(X), h)$ is also complete (compact) metric space, respectively.

**Lemma 1** ([28])**.** *If $\{C_i\}_{i\in\Lambda}, \{D_i\}_{i\in\Lambda}$ are two arbitrary collections of sets in $(K(X), h)$, then*

$$h\left(\overline{\underset{i\in\Lambda}{\cup} C_i}\,,\, \overline{\underset{i\in\Lambda}{\cup} D_i}\right) = h\left(\underset{i\in\Lambda}{\cup} C_i\,,\, \underset{i\in\Lambda}{\cup} D_i\right) \le \sup_{i\in\Lambda} h(C_i, D_i).$$

**Lemma 2** ( [25])**.** *If $\{T_n\}_n$ is a sequence of contractive maps on a metric space $(X, d)$ and point-wise convergent to a map $T$ on $X$, then $T_n$ (defined on compacts) is point-wise convergent to $T$ with respect to the Hausdorff metric.*

**Lemma 3.** *Let $E, F \in K(X)$ for some metric space $(X, d)$. Then for any $x \in E$, there exists $y \in F$ such that $d(x, y) \le h(E, F)$. Also, there are $x^*$ in $E$ and $y^*$ in $F$ such that $d(x^*, y^*) = h(E, F)$.*

**Proof.** Let $x \in E$. By compactness of $F$, there exists $y \in F$ such that $d(x,y) = \inf_{\overline{y} \in F} d(x, \overline{y})$. Thus $d(x,y) \leq D(E,F) \leq h(E,F)$.

Suppose $h(E,F) = D(E,F)$, then by compactness of $E$, there exists $x^* \in E$ such that

$$D(E,F) = \inf_{y \in F} d(x^*, y),$$

and by compactness of $F$ , there exists $y^* \in F$ such that

$$\inf_{y \in F} d(x^*, y) = d(x^*, y^*).$$

Similarly, we can prove for the case $h(E,F) = D(F,E)$. □

**Definition 2.** *An iterated function system (IFS) $\{X; T_1, T_2, \ldots, T_N\}$ on a topological space $X$ is given by a finite set of continuous maps $T_n : X \to X, n \in \mathbb{N}_N$, where $\mathbb{N}_N$ is the set of the first $N$ natural numbers. If $X$ is a complete metric space and the maps $T_n$ are contraction mappings with contraction factors $k_n$, $n \in \mathbb{N}_N$ , then the IFS is said to be hyperbolic.*

Note that each map $T_n$ on a topological space $X$ induces a map $\mathbb{T}_n$ on its hyperspace $K(X)$ for $n = 1, 2, \ldots, N$, and we will use this notion throughout the paper. A hyperbolic IFS induces a map $\mathbb{T} : K(X) \to K(X)$ defined by $\mathbb{T}(A) = \bigcup_{n=1}^{N} \mathbb{T}_n(A)$. In fact, $\mathbb{T}$ is also contracting with contractivity factor $k = \max k_n$, and $k$ is called the contractivity of the IFS. Barnsley [28] proved every IFS on a complete metric space has a unique invariant set $A$(say) in $K(X)$ such that

$$A = \mathbb{T}(A) = \bigcup_{n=1}^{N} \mathbb{T}_n(A).$$

Moreover, $A = \lim_{m \to \infty} \mathbb{T}^m(B)$ for any $B \in K(X)$. This set $A$ is called the attractor. It is also called self-similar set or fractal. The above map $\mathbb{T}$ is called the Hutchinson operator for the corresponding IFS.

Jleli and Samet [27] proposed a novel type of contractive maps, and proved a new fixed point theorem for such maps in the framework of generalized metric spaces. Consistent with [27], we define a similar class of maps on a metric space by modifying the left end-points of domain and range:

**Definition 3.** *We take $\Theta$ be the set of functions $\theta : [0, \infty) \to [1, \infty)$ satisfying the following conditions:*
$(\Theta_1)$ *$\theta$ is non-decreasing,*
$(\Theta_2)$ *for each sequence $\{t_n\} \subset [0, \infty)$, $\lim_{n \to \infty} \theta(t_n) = 1$ if and only if $\lim_{n \to \infty} t_n = 0$,*
$(\Theta_3)$ *there exists $r \in (0,1)$ and $l \in (0, \infty]$ such that $\lim_{t \to \infty} \frac{\theta(t)-1}{t^r} = l$.*

Let $\theta_1(t) = t^{k_1} + 1$, and $\theta_2(t) = e^{(t \log(t+2))^{k_2}}$, for some $k_1, k_2 \in (0,1)$. Observe that $\theta_1, \theta_2 \in \Theta$.

**Definition 4.** *A map $T$ on a metric space $(X,d)$ into itself is called $\theta^k$-contraction, if $T$ satisfies the following condition:*

$$\theta(d(Tx, Ty)) \leq [\theta(d(x,y))]^k, \ \forall x, y \in X,$$

*where $\theta \in \Theta$ and $k \in [0,1)$.*

**Example 1.** *Let $\theta : [0, \infty) \to [1, \infty)$ defined by $\theta(t) := e^{\sqrt{te^t}}$ and $T : \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}\} \cup [\frac{3}{4}, 1] \to \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}\} \cup [\frac{3}{4}, 1]$ defined by*

$$T(x) = \begin{cases} 1/2 & \text{if } x \in [\frac{3}{4}, 1], \\ 1/5 & \text{if } x \in \{\frac{1}{3}, \frac{1}{4}, \frac{1}{5}\}, \\ 1/3 & \text{if } x = \frac{1}{2}. \end{cases}$$

*Clearly, $\theta \in \Theta$. Our claim is, $T$ is a $\theta^k$-contraction with the usual metric. It is enough to prove*

$$\frac{|Tx - Ty|}{|x - y|} e^{|Tx - Ty| - |x - y|} \leq k^2.$$

**Remark 1.** *(i) If $T$ is a contraction map with contractivity factor $r$, then $T$ is a $\theta^k$-contraction map, where $\theta(t) = e^{\sqrt{t}}$ and $k = r^{1/2}$.*

*(ii) Every $\theta^k$-contraction map on a metric space $(X, d)$ is continuous on $X$, and moreover, a $\theta^k$-contraction map $T$ is contractive in the sense that*

$$d(Tx, Ty) < d(x, y), \ x \neq y.$$

*(iii) It is easy to see the following implications:*

*Contraction $\Rightarrow$ $\theta^k$-contraction $\Rightarrow$ Contractive $\Rightarrow$ Continuous.*

**Definition 5** ([30]). *An operator $T : X \to X$ is a Picard operator if $T$ has a unique fixed point $\bar{x}$ and $T^n(x) \to \bar{x}$ as $n \to \infty$ for all $x \in X$.*

**Theorem 1** ([27]). *Let $(X, d)$ be a complete metric space and $T : X \to X$ be a given map. Suppose that there exist $\theta \in \Theta$ and $k \in (0, 1)$ such that*

$$x, y \in X, \qquad d(x, y) \neq 0 \Rightarrow \theta(d(Tx, Ty)) \leq [\theta(d(x, y))]^k.$$

*Then $T$ is a Picard operator.*

**Corollary 1.** *Let $T : X \to X$ be a $\theta^k$-contraction on a complete metric space $(X, d)$, then $T$ is a Picard operator.*

Let $X$ and $Y$ be two metric spaces. A map $T : X \to Y$ is called multivalued if for every $x \in X$, $T(x)$ is a non-empty closed subset of $Y$. The point-to-set mapping $T : X \to Y$ extends to a set-to-set mapping by taking $T(C) = \underset{x \in C}{\cup} T(x)$, $C \subseteq X$. For a multivalued map $T : X \to Y$, denote $T^{-1}(C) := \{x \in X \mid T(x) \subseteq C\}$ and $T_+^{-1}(C) = \{x \in X \mid T(x) \cap C \neq \emptyset\}$.

**Definition 6.** *If $X \subset Y$ and $T : X \to Y$ is a multivalued map, then a point $x \in X$ is called a fixed point of $T$ provided $x \in T(x)$. Thus, the set of fixed point of $T$ is given by $Fix(T) = \{x \in X \mid x \in T(x)\}$.*

**Definition 7** ([29]). *A multivalued map $T : X \to Y$ is called*

*(i) upper semicontinuous (u.s.c) if $T^{-1}(C)$ is open in $X$ for all open sets $C \subseteq Y$,*

*(ii) lower semicontinuous (l.s.c) if $T_+^{-1}(C)$ is open in $X$ for all open sets $C \subseteq Y$.*

**Theorem 2** ([29]). *A mapping $T : X \to K(Y)$ is Hausdorff continuous if and only if it is both u.s.c. and l.s.c.*

**Lemma 4** ([29]). *Let $T : X \to K(Y)$ be an u.s.c and $A \in K(X)$. Then $T(A) \in K(Y)$.*

**Definition 8.** *Let $(\sum_N, d_c)$ be a code space on $N$ symbols $\{1, 2, \ldots, N\}$, with the metric $d_c$ defined by*

$$d_c(\alpha, \beta) = \sum_{n=1}^{\infty} \frac{|\alpha_n - \beta_n|}{(N+1)^n},$$

*where $\alpha = (\alpha_n)_{n=1}^{\infty}$, $\beta = (\beta_n)_{n=1}^{\infty} \in \sum_N$ and $\alpha_n, \beta_n \in \mathbb{N}_N$.*

## 3. Generalized $\theta$-Contraction Iterated Function Systems

**Definition 9.** *A $\theta$-contraction IFS is a finite collection of $\theta^{k_n}$-contraction maps $T_n : X \to X$, $n \in \mathbb{N}_N$ on a complete metric space $(X, d)$.*

**Lemma 5.** *If $f$ is a continuous map on a metric space $(X, d)$ into a metric space $(Y, d')$, $A$ is a compact subset of $X$ and $\theta$ is a non-decreasing self map on $\mathbb{R}$, then $\theta(\sup_{x\in A} f(x)) = \sup_{x\in A} \theta(f(x))$.*

**Proof.** Since $\theta$ is non-decreasing,

$$\theta(f(a)) \leq \theta(\sup_{x\in A} f(x)), \ \forall a \in A.$$

This implies,

$$\sup_{x\in A} \theta(f(x)) \leq \theta(\sup_{x\in A} f(x)).$$

By continuity of $f$ and compactness of $A$, there exists $x_0 \in A$ such that $\sup_{x\in A} f(x) = f(x_0)$. Therefore,

$$\theta(\sup_{x\in A} f(x)) = \theta(f(x_0)) \leq \sup_{x\in A} \theta(f(x)).$$

Combining the above two inequalities, we get the desired result. □

We introduce the following concepts for our results in this section:

(i) Let $(X, d)$ be a metric space, we define a metric $d_m$ on $X^m := X \times \cdots \times X$, ($m$-times) for some $m \in \mathbb{N}$ as follows

$$d_m((x_1, \ldots, x_m), (y_1, \ldots, x_m)) := \max_{1\leq j\leq m} d(x_j, y_j).$$

(ii) For any map $T : X^m \to X$, define a corresponding self-map $\widetilde{T}$ on $X$ is $\widetilde{T}(x) := T(x, \ldots, x)$.

(iii) Let $T : X^m \to X$ and $x = (x_1, \ldots, x_m) \in X^m$, define the iterative sequence $(y_n)_{n\geq 0}$ of the map $T$ at the point $x$ as $y_0 := T(x), y_n := T(\widetilde{T}^n(x_1), \ldots, \widetilde{T}^n(x_m))$.

**Definition 10.** *Let $T : X^m \to X$ be a map for some $m \in \mathbb{N}$. Then we say that $x \in X$ is a fixed point of $T$ if $T(x, \ldots, x) = x$.*

**Definition 11.** *A map $T : X^m \to X$ is called a generalized $\theta^k-$contraction on a metric space $(X, d)$, if $T$ satisfies the following condition:*

$$\theta(d(Tx, Ty)) \leq [\theta(d_m(x, y))]^k, \ \forall x, y \in X^m,$$

*where $\theta \in \Theta$ and $k \in [0, 1)$.*

Note that, if we take $m = 1$ in the above definition, we get the map $T$ as a $\theta^k$-contraction on $(X, d)$. Every generalized $\theta^k$-contraction is uniformly continuous because of that,

$$d(Tx, Ty) \leq d_m(x, y), \ \forall x, y \in X^m.$$

**Theorem 3.** *Let $T : X^m \to X$ be a generalized $\theta^k$-contraction on a complete metric space $(X, d)$ for some $m \in \mathbb{N}$. Then $T$ satisfies the following properties:*

*(i) $T$ has a unique fixed point $x^*$ and for any $x \in X$, $\lim_{n\to\infty} \widetilde{T}^n(x) = x^*$.*

*(ii) The iterative sequence $(y_n)_{n\geq 0}$ of $f$ at any point in $X^m$ converges to $x^*$.*

**Proof.** Observe that, $\widetilde{T}$ is a $\theta^k$-contraction on $(X, d)$. Therefore, $\lim_{n\to\infty} \widetilde{T}^n(x) = x^*$ for any $x \in X$, where $x^*$ is the unique fixed point of $\widetilde{T}$.

Let $(x_1, \ldots, x_m) \in X^m$ and $\epsilon > 0$. Then for all $j \in \mathbb{N}_m$, there exists $N_j \in \mathbb{N}$ such that $d(\widetilde{T}^n(x_j), x^*) < \epsilon, \forall n \geq N_j$. Thus, we obtain

$$\begin{aligned} d(y_n, x^*) &= d(T(\widetilde{T}^n(x_1), \ldots, \widetilde{T}^n(x_m)), T(x^*, \ldots, x^*)) \\ &\leq \max_{1\leq j\leq m} d(\widetilde{T}^n(x_j), x^*) < \epsilon, \ \ \forall n \geq N := \max\{N_j : j \in \mathbb{N}_m\}. \end{aligned}$$

From the above inequality, we conclude that $\lim_{n\to\infty} y_n = x^*$. □

**Theorem 4.** *If a map $T : X^m \to X$ is a generalized $\theta^k$-contraction on a metric space $(X, d)$, then the set-valued map $\mathbb{T} : K(X)^m \to K(X)$ is also a generalized $\theta^k$-contraction on $(K(X), h)$.*

**Proof.** Let $A = (A_1, \ldots, A_m), B = (B_1, \ldots, B_m) \in K(X)^m$ and $x = (x_1, \ldots, x_m) \in A$. Then, there exists $\widetilde{y}_j \in B_j, \forall j \in \mathbb{N}_m$ such that $d(x_j, \widetilde{y}_j) = \inf_{y_j\in B_j} d(x_j, y_j)$. Consider,

$$\begin{aligned} \theta(d(T(x), \mathbb{T}(B))) &\leq \theta(d(T(x), T(\widetilde{y}))), \ \ \widetilde{y} = (\widetilde{y}_1, \ldots, \widetilde{y}_m) \\ &\leq [\theta(d_m(x, \widetilde{y}))]^k = [\theta(\max_{1\leq j\leq m} \inf_{y_j\in B_j} d(x_j, y_j))]^k \\ &\leq [\theta(\max_{1\leq j\leq m} D(A_j, B_j))]^k. \end{aligned}$$

Since $x$ is arbitrary, $\sup_{x\in A} \theta(d(T(x), \mathbb{T}(B))) \leq [\theta(h_m(A, B))]^k$.
Therefore, using Lemma 5, we have

$$\begin{aligned} \theta(D(\mathbb{T}(A), \mathbb{T}(B))) &= \theta(\sup_{x\in A} d(T(x), \mathbb{T}(B))) \\ &= \sup_{x\in A} \theta(d(T(x), \mathbb{T}(B))) \\ &\leq [\theta(h_m(A, B))]^k. \end{aligned}$$

Similarly, we can prove $\theta(D(\mathbb{T}(B), \mathbb{T}(A))) \leq [\theta(h_m(A, B))]^k$.
By the property $(\Theta_1)$, we conclude that,

$$\begin{aligned} \theta(h(\mathbb{T}(B), \mathbb{T}(A))) &= \max\{\theta(D(\mathbb{T}(A), \mathbb{T}(B))), \theta(D(\mathbb{T}(B), \mathbb{T}(A)))\} \\ &\leq [\theta(h_m(A, B))]^k. \end{aligned}$$

□

**Definition 12.** *A generalized $\theta$-contraction IFS is a finite collection of generalized $\theta^{k_n}$-contraction maps $T_n : X^m \to X, \ m \in \mathbb{N}, \ n \in \mathbb{N}_N$ on a complete metric space $(X, d)$.*

**Theorem 5.** *Let $T_n : X^m \to X, n \in \mathbb{N}_N$, be a finite collection of generalized $\theta^{k_n}$-contraction on a metric space $(X, d)$, then the Hutchinson map $\mathbb{T} : K(X)^m \to K(X)$ defined by $\mathbb{T}(A) = \bigcup_{n=1}^{N} \mathbb{T}_n(A)$ is also a generalized $\theta^k$-contraction on $(K(X), h)$ with the same $\theta$ and $k := \max\{k_n : n \in \mathbb{N}_N\}$.*

**Proof.** Let $A, B \in K(X)^m$. Our claim is

$$\theta(h(\mathbb{T}(A), \mathbb{T}(B))) \leq [\theta(h_m(A, B))]^k.$$

By using Lemma 1 and the property $(\Theta_1)$, we have

$$\begin{aligned}\theta(h(\mathbb{T}(A),\mathbb{T}(B))) &= \theta(h(\bigcup_{n=1}^{N}\mathbb{T}_n(A),\bigcup_{n=1}^{N}\mathbb{T}_n(B)))\\ &\leq \theta(\max_{1\leq n\leq N} h(\mathbb{T}_n(A),\mathbb{T}_n(B)))\\ &= \max_{1\leq n\leq N} \theta(h(\mathbb{T}_n(A),\mathbb{T}_n(B))).\end{aligned}$$

Using Theorem 4 in the above inequality, we obtain

$$\theta(h(\mathbb{T}(A),\mathbb{T}(B))) \leq \max_{1\leq n\leq N} [\theta(h_m(A,B))]^{k_n}. \tag{1}$$

By the non-decreasing property of log and $\theta$,

$$\begin{aligned}\log\,(\max_{1\leq n\leq N}(\theta(h_m(A,B)))^{k_n}) &= \max_{1\leq n\leq N}\{\log\,((\theta(h_m(A,B)))^{k_n})\}\\ &= \max_{1\leq n\leq N}\{k_n\log\,(\theta(h_m(A,B)))\}\\ &= k\log\,[(\theta(h_m(A,B)))]\\ &= \log\,[(\theta(h_m(A,B)))]^k.\end{aligned}$$

Since the logarithm is a one to one function,

$$\max_{1\leq n\leq N}[\theta(h_m(A,B))]^{k_n} = [(\theta(h_m(A,B)))]^k. \tag{2}$$

Finally, using (2) in (1), we get the desired claim of this proof. □

**Corollary 2.** *Every generalized $\theta$-contraction IFS has a unique attractor $A$ (say), and the iterative sequence at any point $B \in K(X)^m$ of the corresponding Hutchinson map $\mathbb{T}$ converges to $A$, that is*

$$\lim_{l\to\infty} A_l = A, \quad \text{where } A_0 := \mathbb{T}(B), A_l := \mathbb{T}(\widetilde{T}^l(B),\ldots,\widetilde{T}^l(B)).$$

**Proof.** Since, $(X,d)$ is complete, then $(K(X),h)$ is also complete. The proof follows from sequential use of Theorems 3–5. □

Note that the concept of $\theta^k$-contraction is a particular case of generalized $\theta^k$-contraction. The proofs of the following two theorems are straightforward by taking $m=1$ in Theorems 4 and 5, respectively, and hence omitted.

**Theorem 6.** *Let $T : X \to X$ be a $\theta^k$-contraction on a metric space $(X,d)$, then the set-valued map $\mathbb{T} : K(X) \to K(X)$ is also a $\theta^k$-contraction on $(K(X),h)$ with the same $\theta$ and $k$.*

**Theorem 7.** *Let $T_n : X \to X, n \in \mathbb{N}_N$, be a finite collection of $\theta^{k_n}$-contractions on a metric space $(X,d)$, then the Hutchinson map $\mathbb{T} : K(X) \to K(X)$ defined by $\mathbb{T}(A) = \bigcup_{n=1}^{N}\mathbb{T}_n(A)$ is also a $\theta^k$-contraction on $(K(X),h)$ with the same $\theta$, and $k := \max\{k_n : n \in \mathbb{N}_N\}$.*

**Corollary 3.** *Every $\theta$-contraction IFS has a unique attractor $A$ (say) and moreover, the corresponding Hutchinson operator $\mathbb{T} : K(X) \to K(X)$ is a Picard operator, that is*

$$\lim_{m\to\infty}\mathbb{T}^m(B) = A, \quad \text{for all } \; B \in K(X).$$

**Proof.** By Theorem 7, $\mathbb{T}$ is a $\theta$-contraction IFS on the complete metric space $(K(X),h)$. From Corollary 1, we conclude that $\mathbb{T}$ is a Picard operator. □

**Theorem 8.** *Let $\{T_n^i\}_{n=1}^N, \forall i \in \mathbb{N}$, be a sequence of $\theta$-contraction IFSs. Assume that the following conditions are satisfied:*

*(i)* *For all $i \in \mathbb{N}, n \in \mathbb{N}_N$, $T_n^i$ is a $\theta^{k_{n,i}}$-contraction map on a complete metric space $(X,d)$ with $\theta$-continuous and for each $n \in \mathbb{N}_N, \sup_{i\in\mathbb{N}} k_{n,i} < 1$.*

*(ii)* *The sequence $\{T_n^i\}_i$ converges point-wise to a map $T_n, \forall n \in \mathbb{N}_N$.*

*(iii)* *For all $i \in \mathbb{N}$, $A_i$ is the attractor of the IFS $\{T_n^i\}_{n=1}^N$ and the sequence $\{A_i\}_i$ converges to a non-empty compact set $A$ with respect to the Hausdorff metric.*

*(iv)* *For all $i \in \mathbb{N}$, $\mathbb{T}_i$ is the Hutchinson operator of the IFS $\{T_n^i\}_{n=1}^N$, i.e. $\mathbb{T}_i := \bigcup_{n=1}^{N} \mathbb{T}_n^i(A)$.*

*Then $\{T_n\}_{n=1}^N$ is also a $\theta$-contraction IFS and $\{\mathbb{T}_i\}_i$ converges point-wise to the map $\mathbb{T}$, where $\mathbb{T}$ is the Hutchinson operator of the IFS $\{T_n\}_{n=1}^N$. In addition, $A$ is the attractor of the IFS $\{T_n\}_{n=1}^N$.*

**Proof.** (i) Let $x, y \in X$. By the given assumptions,

$$\theta(d(T_n^i x, T_n^i y)) \leq [\theta(d(x,y))]^{k_{n,i}}, \quad \forall n \in \mathbb{N}_N, \ \forall i \in \mathbb{N}.$$

Taking logarithms on both sides we get,

$$\begin{aligned}\log\theta(d(T_n^i x, T_n^i y)) &\leq k_{n,i}\log\theta(d(x,y)) \\ &\leq \sup_{i\in\mathbb{N}} k_{n,i}\log\theta(d(x,y)), \quad \forall n \in \mathbb{N}_N, \ \forall i \in \mathbb{N}.\end{aligned}$$

Let $k_n = \sup_{i\in\mathbb{N}} k_{n,i}$. Taking limit as $i \to \infty$ on both sides and by continuity of $\theta$, we conclude

$$\log\theta(d(T_n x, T_n y)) \leq k_n \log\theta(d(x,y)), \quad \forall n \in \mathbb{N}_N.$$

The above inequality proves that $T_n$'s are $\theta^{k_n}$-contractions, and by using Lemma 2, we obtain that $\{\mathbb{T}_i\}_i$ convergent point-wise to the map $\mathbb{T}$.

(ii) Since $\mathbb{T}_i$'s are $\theta^{k_i}$-contractions, where $k_i = \max_{n\in\mathbb{N}_N} k_{n,i}$, for each $i \in \mathbb{N}$, thus $\mathbb{T}_i$'s are contractive maps. Therefore,

$$\begin{aligned}h(A, \mathbb{T}(A)) &\leq h(A, \mathbb{T}_i(A_i)) + h(\mathbb{T}_i(A_i), \mathbb{T}_i(A)) + h(\mathbb{T}_i(A), \mathbb{T}(A)) \\ &\leq h(A, A_i) + h(A_i, A) + h(\mathbb{T}_i(A), \mathbb{T}(A)).\end{aligned}$$

Taking limit as $i \to \infty$ in the above inequality, we conclude $\mathbb{T}(A) = A$. □

## 4. Countable and Multivalued $\theta$-Contraction Iterated Function Systems

In this section, motivated by the work of Secelean [24] and Leśniak [26], we utilize our results to show the existence and uniqueness of attractors of countable and multivalued $\theta$-contraction IFSs, respectively, by proving the corresponding the set valued map is a Picard operator.

**Theorem 9.** *Let $\{T_n\}_{n=1}^\infty$ be a sequence of $\theta^{k_n}$-contraction functions on a compact metric space $(X,d)$, where $\theta$ is left continuous and $\sup_{n\in\mathbb{N}} k_n < 1$. Then the map $\mathbb{T} : K(X) \to K(X)$ defined by $\mathbb{T}(A) := \overline{\bigcup_{n\geqslant 1} \mathbb{T}_n(A)}$ is a $\theta^k$-contraction.*

**Proof.** Let $A, B \in K(X)$. By Lemma 1 and the property $(\Theta_1)$,

$$\begin{aligned}\theta(h(\mathbb{T}(A), \mathbb{T}(B))) &= \theta(h(\overline{\bigcup_{n\geqslant 1} \mathbb{T}_n(A)}, \overline{\bigcup_{n\geqslant 1} \mathbb{T}_n(B)})) \\ &\leq \theta(\sup_{n\in\mathbb{N}} h(\mathbb{T}_n(A), \mathbb{T}_n(B))) \\ &= \theta(\lim_{N\to\infty} \max_{1\leq n\leq N} h(\mathbb{T}_n(A), \mathbb{T}_n(B))).\end{aligned}$$

Since $\theta$ is non-decreasing and left continuous,

$$\theta(h(\mathbb{T}(A),\mathbb{T}(B))) \leq \lim_{N\to\infty} \max_{1\leq n\leq N} \theta(h(\mathbb{T}_n(A),\mathbb{T}_n(B))).$$

By Theorem 6,

$$\theta(h(\mathbb{T}(A),\mathbb{T}(B))) \leq \sup_{n\in\mathbb{N}} [\theta(h(A,B))]^{k_n}. \tag{3}$$

Consider,

$$\begin{aligned}
\log\left(\sup_{n\in\mathbb{N}} (\theta(h(A,B)))^{k_n}\right) &= \sup_{n\in\mathbb{N}} \{\log\left((\theta(h(A,B)))^{k_n}\right)\} \\
&= \sup_{n\in\mathbb{N}} \{k_n \log\left(\theta(h(A,B))\right)\} \\
&= k \log\left[(\theta(h(A,B)))\right] \\
&= \log\left[(\theta(h(A,B)))\right]^k.
\end{aligned}$$

This gives,

$$\sup_{n\in\mathbb{N}} (\theta(h(A,B)))^{k_n} \leq [(\theta(h(A,B)))]^k. \tag{4}$$

Using (4) in (3), we conclude the proof. □

**Corollary 4.** *If $\{T_n\}_{n=1}^{\infty}$ is a sequence of $\theta^{k_n}$-contraction functions on a compact metric space, where $\theta$ is left continuous and $\sup\limits_{n\in\mathbb{N}} k_n < 1$ , then the map $\mathbb{T}$ defined as in the above statement is a Picard operator.*

**Definition 13.** *A countable collection of $\theta^{k_n}$-contraction maps $\{T_n\}_{n\geq 1}, n \in \mathbb{N}$ on a compact metric space, where $\theta$ is left continuous and $\sup\limits_{n\in\mathbb{N}} k_n < 1$ is called a countable $\theta$-contraction IFS.*

**Definition 14.** *A multivalued map $T : X \to K(X)$ is said to be a multivalued $\theta^k$-contraction on a metric space $(X,d)$, if there exists $\theta \in \Theta$ and $k \in [0,1)$ such that*

$$\theta(h(Tx,Ty)) \leq [\theta(d(x,y))]^k, \ \forall x,y \in X.$$

**Definition 15.** *A finite collection of multivalued $\theta^{k_n}$-contraction maps on a complete metric space is called a multivalued $\theta$-contraction IFS.*

If a map $T$ is a multivalued $\theta$-contraction on a metric space $(X,d)$, then $T$ is continuous on $X$, and it satisfies

$$h(Tx,Ty) \leq d(x,y)$$

**Theorem 10.** *Let $T_n, n \in \mathbb{N}_N$, be a finite collection of multivalued $\theta^{k_n}$-contraction on a metric space $(X,d)$, then the map $\mathbb{T} : K(X) \to K(X)$ defined by $\mathbb{T}(A) := \bigcup\limits_{n\in\mathbb{N}_N, x\in A} T_n(x)$ is a $\theta^k$-contraction on the metric space $(K(X),h)$, where $k := \max\{k_n : n \in \mathbb{N}_N\}$.*

**Proof.** By Theorem 2 and Lemma 4, $\mathbb{T}$ is well defined. Let $A,B \in K(X)$ and choose $u \in \mathbb{T}(A)$ such that $\sup\limits_{x\in\mathbb{T}(A)} d(x,\mathbb{T}(B)) = d(u,\mathbb{T}(B))$. Then there exists $n \in \mathbb{N}_N$ and $a \in A$ such that $u \in T_n(a)$ and there exists $b \in B$ such that $d(a,b) \leq h(A,B)$. Then we have,

$$\theta(D(\mathbb{T}(A),\mathbb{T}(B))) = \theta(d(u,\mathbb{T}(B))) \leq \theta(h(T_n(a),T_n(b))) \leq [\theta(d(a,b))]^{k_n} \leq [\theta(h(A,B))]^k.$$

Similarly,

$$\theta(D(\mathbb{T}(B),\mathbb{T}(A))) \leq [\theta(h(A,B))]^k.$$

Combining the above two inequalities, we obtain

$$\theta(h(\mathbb{T}(A),\mathbb{T}(B))) \leq [\theta(d(A,B))]^k,\ \forall A,B \in K(X),$$

and hence the proof. □

**Corollary 5.** *If $T_n, n \in \mathbb{N}_N$, is a finite collection of multivalued $\theta^{k_n}$-contractions on a complete metric space, then the map $\mathbb{T}$ defined as in the above statement is a Picard operator.*

## 5. Code Space and Attractor of $\theta$-Contraction IFS

Our goal is to construct a continuous transformation $\psi$ from the code space onto the attractor of a restrictive class $\Omega$ of $\theta$-contraction IFS so that it generalizes the classical result proved in Barnsley [28] for usual contractions.

**Definition 16.** *Let $\Omega$ be the set of functions $\theta : [0,\infty) \to [1,\infty)$ satisfying the following conditions:*
$(\Omega_1)$ *$\theta$ is nondecreasing,*
$(\Omega_2)$ *for each sequence $\{t_n\} \subset [0,\infty)$, $\lim_{n\to\infty} \theta(t_n) = 1$ if and only if $\lim_{n\to\infty} t_n = 0$,*
$(\Omega_3)$ *there exists $r \in (0,1)$ and $l \in (0,\infty]$ such that $\lim_{t\to\infty} \frac{\theta(t)-1}{t^r} = l$,*
$(\Omega_4)$ *$\theta$ is continuous.*

Note that $\Omega$ is a subset of the collection $\Theta$.

**Lemma 6.** *Let $\{T_i\}_{i=1}^N, N \in \mathbb{N}$ be a family of $\theta^{k_i}$-contraction maps on a complete metric space $(X,d)$, where $\theta \in \Theta$. Let $M \in K(X)$. Then there exists $M_0 \in K(X)$ such that $M \subset M_0$, and the restriction maps $\{T_i\}_{i=1}^N$ on $M_0$ forms a $\theta$-contraction IFS. In other words, $T_i(M_0) \subset M_0$, $\forall i \in \mathbb{N}_N$.*

**Proof.** Take $T_0(B) = M$ for all $B \in K(X)$ as a condensation set. Denote as $\mathbb{T}$ and $\widetilde{\mathbb{T}}$ the Hutchinson operators for the $\theta$-contraction IFSs $\{X : T_1,\dots,T_N\}$ and $\{X : T_0, T_1,\dots,T_N\}$, respectively. By Theorem 7, both $\mathbb{T}$ and $\widetilde{\mathbb{T}}$ are $\theta^k$-contractions with $k = \max\{k_i : i \in \mathbb{N}\}$.

By Corollary 3, $\{\widetilde{\mathbb{T}}^n(M)\}_{n\geq 1}$ converges to an attractor $M_0$ (say). Observe that $\{\widetilde{\mathbb{T}}^n(M)\}_{n\geq 1}$ is an increasing sequence, i.e.,

$$\widetilde{\mathbb{T}}(M) \subset \widetilde{\mathbb{T}}^2(M) \subset \cdots \subset \widetilde{\mathbb{T}}^n(M) \subset \dots$$

and

$$\widetilde{\mathbb{T}}^n(M) = M \cup \mathbb{T}(M) \cup \mathbb{T}^2(M) \cdots \cup \mathbb{T}^n(M), \quad \forall n \in \mathbb{N}.$$

Therefore, we have

$$M_0 = \lim_{n\to\infty} \widetilde{\mathbb{T}}^n(M) = \overline{M \cup \mathbb{T}(M) \cup \mathbb{T}^2(M) \cdots \cup \mathbb{T}^n(M) \dots},$$

where $\overline{A}$ means the closure of $A$. Observe that $M \subset M_0$ and $\mathbb{T}(M_0) \subset \widetilde{\mathbb{T}}(M_0) = M_0$. Therefore, the set $M_0$ satisfies the desired conclusion. □

**Lemma 7.** *Let $\{T_i\}_{i=1}^N, N \in \mathbb{N}$ be a family of $\theta^{k_i}$-contraction maps on a complete metric space $(X,d)$, where $\theta \in \Omega$. Denote*

$$\varphi(\alpha,n,x) = T_{\alpha_1} \circ \cdots \circ T_{\alpha_n}(x), \quad \textit{for all} \ \ \alpha \in \sum_N, n \in \mathbb{N}, x \in X.$$

*Let $M \in K(X)$. Then there exists a finite constant $\lambda$ such that*

$$\theta(d(\varphi(\alpha,m,x),\varphi(\alpha,n,y))) \leq \lambda^{k^m}, \ \textit{for all}\ \alpha \in \sum_N, m \leq n \in \mathbb{N}, x,y \in M,$$

*where $k = \max\{k_i : i \in \mathbb{N}_N\}$.*

**Proof.** Let $\alpha \in \sum_N, m \leq n \in \mathbb{N}, M \in K(X)$ and $x, y \in M$. By Lemma 6, there exists $M_0 \in K(X)$ such that $M \subset M_0$. Consider

$$\varphi(\alpha, n, y) = \varphi(\alpha, m, z),$$

where $z = \varphi(\gamma, n-m, y) \in M_0$ and $\gamma = (\alpha_i)_{i=m+1}^{\infty}$. Therefore, we have

$$\begin{aligned}\theta(d(\varphi(\alpha, m, x), \varphi(\alpha, n, y)) &= \theta(d(T_{\alpha_1} \circ \cdots \circ T_{\alpha_m}(x), T_{\alpha_1} \circ \cdots \circ T_{\alpha_m}(z))) \\ &\leq [\theta(d(x, y))]^{k^m} \leq \lambda^{k^m},\end{aligned}$$

where $\lambda = \max\{\theta(d(x, y)) : x, y \in M_0\}$. By continuity of $\theta$ and compactness of $M_0$, $\lambda$ is finite. □

**Theorem 11.** *Let $\{T_i\}_{i=1}^N, N \in \mathbb{N}$ be a family of $\theta^{k_i}$-contraction maps on a complete metric space $(X, d)$, where $\theta \in \Omega$. Let $A$ denote the attractor of a $\theta$-contraction IFS $\{(X, d); T_1, \ldots, T_N\}$. Define a map $\psi : \sum_N \to A$ by*

$$\psi(\alpha) = \lim_{n\to\infty} \varphi(\alpha, n, x), \text{ for all } \alpha \in \sum_N, x \in X$$

*is well-defined (i.e, the limit exists, belongs to $A$ and is independent of $x \in X$), continuous and onto, where $\varphi$ is defined as in Lemma 7.*

**Proof.** Our first claim is that $\psi$ is well defined. It's enough to prove the existence and independence of $x$ of

$$\lim_{n\to\infty} \varphi(\alpha, n, x) \in A, \quad \forall \alpha \in \sum_N, x \in A.$$

Let $x \in X, M \in K(X)$ such that $x \in M$ and $\alpha \in \sum_N$. According to Lemma 7,

$$1 \leq \theta(d(\varphi(\alpha, m, y), \varphi(\alpha, n, z))) \leq \lambda^{k^m}, \ \forall \ m \leq n \in \mathbb{N}, y, z \in M,$$

$$\Rightarrow \quad \theta(d(\varphi(\alpha, m, y), \varphi(\alpha, n, z))) \to 1 \text{ as } m, n \to \infty \ \forall \ y, z \in M,$$

$$\Rightarrow \quad d(\varphi(\alpha, m, y), \varphi(\alpha, n, z)) \to 0 \text{ as } m, n \to \infty \ \forall \ y, z \in M, \tag{5}$$

$$\Rightarrow \quad d(\varphi(\alpha, m, x), \varphi(\alpha, n, x)) \to 0 \text{ as } m, n \to \infty.$$

Therefore, $\lim_{n\to\infty} \varphi(\alpha, n, x)$ exists. It is easy to observe that $\varphi(\alpha, n, x) \in \mathbb{T}^n(M), \forall n \in \mathbb{N}$, where $\mathbb{T}$ is the Hutchinson operator. From Corollary 3, $\mathbb{T}$ is a Picard operator, and consequently, $\lim_{n\to\infty} \varphi(\alpha, n, x) \in A$. Suppose $\lim_{n\to\infty} \varphi(\alpha, n, x) = a$ and $\lim_{n\to\infty} \varphi(\alpha, n, y) = b$ for some $a \neq b$ and $x, y \in M$. Let $\epsilon = d(a, b)$. Then there exists $N \in \mathbb{N}$ such that for all $n \geq N$,

$$d(\varphi(\alpha, n, x), a) < \epsilon/4 \text{ and } d(\varphi(\alpha, n, y), b) < \epsilon/4.$$

Consider

$$d(a, b) \leq d(a, \varphi(\alpha, n, x)) + d(\varphi(\alpha, n, x), b).$$

$$\Rightarrow \quad d(\varphi(\alpha, n, x), b) \geq d(a, b) - d(a, \varphi(\alpha, n, x)) > \epsilon - \epsilon/4 = 3\epsilon/4.$$

and

$$d(\varphi(\alpha, n, x), b) \leq d(\varphi(\alpha, n, x), \varphi(\alpha, n, y)) + d(\varphi(\alpha, n, y), b).$$

$$\Rightarrow \quad d(\varphi(\alpha, n, x), \varphi(\alpha, n, y)) \geq d(\varphi(\alpha, n, x), b) - d(\varphi(\alpha, n, y), b) > 3\epsilon/4 - \epsilon/4 = \epsilon/2,$$

which is a contradiction to (5). Therefore, the limit of the sequence $(\varphi(\alpha, n, x))$ is independent of $x$.

Our next claim is $\psi : \sum_N \to A$ is continuous. Let $\epsilon > 0$. Then, there exists $n \in \mathbb{N}$ such that

$$d(\varphi(\alpha, n, x), \varphi(\alpha, n, y)) < \epsilon, \ \forall\, \alpha \in \sum_N, x, y \in M_0,$$

where $M_0$ is defined from $M$ as in Lemma 6. The above inequality is true because $\lambda^{k^n}$ is not depending on $\alpha$ in $\sum_N$.

Let $\delta = \frac{1}{(N+1)^{n+1}}$. Since $\sum_{i=n+2}^{\infty} \frac{N}{(N+1)^i} = \frac{1}{(N+1)^{n+1}}$, we have

$$\text{if } d_c(\alpha, \beta) < \delta \Rightarrow \alpha_i = \beta_i, \forall i \in \mathbb{N}_n.$$

This implies

$$d(\varphi(\alpha, m, x), \varphi(\beta, m, x)) = d(\varphi(\alpha, n, y), \varphi(\alpha, n, z)) < \epsilon, \forall m \geq n.$$

where $y = T_{\alpha_{n+1}} \circ \cdots \circ T_{\alpha_m}(x)$, $z = T_{\beta_{n+1}} \circ \cdots \circ T_{\beta_m}(x) \in M_0$.

Taking limits as $m \to \infty$, we have

$$d(\psi(\alpha), \psi(\beta)) = d(\lim_{m\to\infty} \varphi(\alpha, m, x), \lim_{m\to\infty} \varphi(\beta, m, x)) < \epsilon.$$

Finally, we need to prove $\psi$ is onto. Let $a \in A$. Since $A = \lim_{n\to\infty} T^n(\{x\})$, there exists a sequence $\alpha^{(n)} \in \sum_N, n \in \mathbb{N}$ such that

$$\lim_{n\to\infty} \varphi(\alpha^{(n)}, n, x) = a.$$

By the compactness of $(\sum_N, d_c)$, there exists a convergent subsequence $\{\alpha^{(n_k)}\}$, whose limit is $\alpha$. For all $n \in \mathbb{N}$, define $\gamma(n)$ as the number of elements in $\{m \in \mathbb{N} : \alpha_j^{(n)} = \alpha_j, j \in \mathbb{N}_m\}$. Consider

$$\begin{aligned}\theta(d(\varphi(\alpha^{(n_k)}, n_k, x), \varphi(\alpha, n_k, x))) &= \theta(d(\varphi(\alpha, \gamma(n_k), y_{n_k}), \varphi(\alpha, \gamma(n_k), z_{n_k}))) \\ &\leq \lambda^{k^{\gamma(n_k)}}.\end{aligned}$$

for some $y_{n_k}, z_{n_k} \in M_0$. Observe that $\gamma(n) \to \infty$ as $n \to \infty$. Therefore,

$$\lim_{k\to\infty} d(\varphi(\alpha^{(n_k)}, n_k, x), \varphi(\alpha, n_k, x)) = 0.$$

$$\Rightarrow d(a, \psi(\alpha)) = 0.$$

Hence the proof. □

**Definition 17.** *Suppose $A$ is the attractor of a $\theta$-contraction IFS $\{(X, d); T_1, \ldots, T_N\}$, where $T_i$ is $\theta^{k_i}$-contraction on a complete metric space $(X, d)$ and $\theta \in \Omega$. Let $\psi : \sum_N \to A$ defined as in Theorem 11. For any $a \in A$,*

$$\psi^{-1}(a) := \{\alpha \in \sum_N : \psi(\alpha) = a\}$$

*is called the set of addresses of $a \in A$.*

When we assume the map $\theta$ is continuous, then it is possible to compute the addresses for each point on the attractor of $\theta$-contraction IFS as per the description given in Definition 17.

## 6. Conclusions

In the present work, we have investigated a generalization of the Banach-contraction principle through the novel generalized $\theta$-contraction. For construction of new type self-similar sets, we have developed a new IFS consisting of finite collection of generalized $\theta^{k_n}$-contractions $T_n : X^m \to X$, $n \in \mathbb{N}_N$, and named it as generalized $\theta$-contraction IFS. We have proved the existence and uniqueness of attractor for the generalized $\theta$-contraction IFS. Further, the Hutchinson operators for countable and multivalued $\theta$-contraction IFSs are proven as a Picard operator. Finally, we have demonstrated the relation between code space and the attractor of $\theta$-contraction IFS, when the map $\theta$ is continuous.

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## Short Biography of Authors

**Pasupathi Rajan** received a Bachelor of Science with specialization in Mathematics from Arignar Anna College, Krishnagiri, Tamilnadu in 2014, and a Master of Science in Mathematics from Bharathidasan University, Thiruchirapalli, Tamilnadu in 2016. He has been pursuing doctoral program in Mathematics at IIT Madras, Chennai since 2017 under the joint-guidance of Prof. Arya Kumar Bedabrata Chand and Prof. María A. Navascués, University of Zaragoza, Spain. His areas of research are iterated function systems, fixed point theory, fractal interpolation functions, approximation by fractal functions.

**María A. Navascués** is a researcher of the Instituto Universitario de Matemáticas y Aplicaciones (IUMA) of the University of Zaragoza, Spain. She has been Visitor Professor at several foreign Universities. She is Member of the Editorial Board of various international publications and is Topics Editor of the journal Fractal & Fractional. Professor Navascués has dozens of articles indexed in the Journal of Citation Reports data base. Her article "Fractal Bases of $L_p$ Spaces" was in the list of Most Read Papers of the journal Fractals during the whole year 2013. She has been occasional referee of more than hundred publications in Mathematics and Physics. She has participated in several International Conferences taking part in the Organizer as well as the Scientific Committee. Professor Navascués has also collaborated in research international projects, some of them sponsored by the National Science Foundation. She has been an international external examiner of several Ph.D. candidates of Australia and India. For two years she was the Secretary of the Real Sociedad Matemática Española (Royal Spanish Mathematical Society), and now belongs to the Editorial Board of the Bulletin of the same Society.

**Arya Kumar Bedabrata Chand** is a native of Bhubaneswar, Odisha, India. He received a Bachelor of Science (Mathematics-Honours) from Samanta Chandra Sekhar College, Puri in 1994, then Master of Science and Master of Philosophy in Mathematics from Utkal University, Bhubaneswar, Odisha in 1996 and 1997, respectively. He investigated spline and coalescence fractal interpolation functions during his doctoral research (1997–2004) at IIT Kanpur, Uttar Pradesh, India. He worked as Assistant Professor in BITS-Pilani Goa campus prior to his post-doctoral position at University of Zaragoza, Spain in 2007. Since 2008, he has been working as a faculty at IIT Madras and currently, he is Professor. He has been visiting professor at several foreign universities. His research group developed the theory of shape preserving spline fractal interpolation functions/surfaces. Prof. Chand guided six Ph.D. students in theory and geometric modelling of fractal functions and surfaces, and published around 90 research articles. Notably, his research papers were in the top 20 articles in Fractals journal, and Journal of Approximation Theory. His research interests include shape preserving fractals, approximation by fractal functions, fixed point theory, computer-aided geometric design, wavelets, and fractal signal/image processing.