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On Solvability of the Sonin–Abel Equation in the Weighted Lebesgue Space

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Abstract: In this paper we present a method of studying a convolution operator under the Sonin conditions imposed on the kernel. The particular case of the Sonin kernel is a kernel of the fractional integral Riemman–Liouville operator, other various types of the Sonin kernels are a Bessel-type function, functions with power-logarithmic singularities at the origin e.t.c. We pay special attention to study kernels close to power type functions. The main our aim is to study the Sonin–Abel equation in the weighted Lebesgue space, the used method allows us to formulate a criterion of existence and uniqueness of the solution and classify a solution, due to the asymptotics of the Jacobi series coefficients of the right-hand side.

Keywords: Riemann–Liouville operator; Abel equation; Jacobi polinomials; weighted Lebesgue spaces; convolution operators; Sonin conditions

MSC: 26A33; 45H05; 33C45; 12E10

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1. Introduction

The central point of fractional calculus is a concept of fractional differentiation. In this regard, we should admit that the Riemann–Liouville operator of fractional differentiation is at the origin of the concept and plays a special role in the science. Such operators as Caputo and Marchaud certainly are worth mentioning in this context, the first one is not interesting for us since it is more like a reduction of the Riemann–Liouville operator on smooth functions disappearing at the initial point (if we consider the matter from the point of view that is of functional analysis), but the second one does completely reflect a true mathematical nature of fractional derivative as a notion, since it has a representation in terms of infinitesimal generator of the corresponding semigroup [1]. It is clear that considering such an approach we are forced to deal with more general notions of the operator theory and in this way the understanding of the notion of fractional derivative as a fractional power of infinitesimal generator is harmoniously completed, on the one hand. On the other hand, we can make another generalization, if we interpret the fractional differential Riemann–Liouville operator as a particular case of the derivative of the convolution operator for which the so called Sonin condition holds [2]. Here we should note that initially our interest was inspired by lots of previously known results related to mapping theorems for fractional integral operators obtained by mathematicians such as Rubin B.S. [3–5], Vakulov B.G. [6], Samko S.G. [7,8], and Karapetyants N.K. [9,10]. Let us remember that the so called mapping theorem for the Riemann–Liouville operator (the particular case of the Sonin operator) were first studied by H. Hardy and Littlewoo [11] and nowadays is known as the Hardy–Littlewood theorem with limit index. However, there was an attempt to extend this theorem on some class of weighted Lebesgue spaces defined as functional spaces endowed with the following norm $\|f\|_{L_p(I,\beta,\gamma)} := \|f\|_{L_p(I,\mu)}$, $\mu(x) = \omega^{\beta,\gamma}(x) := (x-a)^{\beta}(b-x)^{\gamma}$, $\beta, \gamma \in \mathbb{R}$, $I := (a,b)$. In this direction the mathematicians such as Rubin B.S., Karapetyants N.K. [9] had success, the following problem was considered $I^{\alpha}_{a+} : L_p(I,\, \beta, \gamma) \longrightarrow ?$ However, the converse theorem was not considered. All these create the prerequisite to invent another approach for studying mapping properties of the Riemann–Liouville operator or, more generally, integral operators. Thus, trying to solve (at least in particular) the more general problem, in the paper [12] we deal with mapping theorems for operators acting on Banach spaces in order to obtain afterwards the desired results applicable to integral operators. In this regard the following papers are worth noticing [13,14], where in addition, a special technique based on the properties of the Jacobi polynomials was introduced. Based on this approach, in this paper we offer a method of studying the Sonin operator [2], which is defined as a convolution operator ${}_sI^{\varrho}_{a+}\varphi := \varrho *_a \varphi$ under some conditions (the so called Sonin conditions) imposed on the kernel ϱ, i.e., there exists the function ϑ such that $\varrho * \vartheta = 1$. The particular case of the Sonin kernel is a kernel of the fractional integral Riemann–Liouville operator, many other examples can be found in the papers [15,16], the first one gives us a survey considering various types of kernels such as the Bessel-type function, the power-exponential function, the incomplete gamma function e.t.c., the main concept of the second one is to construct a widest class of functions being a Sonin kernel. In our study, we pay a special attention to kernels presented as $x^{\alpha-1}a(x)$, $\alpha \in (0,1)$, where $a(x)$ is an analytic function. The main our aim is to study the Sonin–Abel equation ${}_sI^{\varrho}_{a+}\varphi = f$ in the weighted Lebesgue space, the used method allows us to formulate the criterion of existence and uniqueness of the solution and in addition (what we particularly want to highlight) to classify a solution, in accordance with belonging to a weighted L_p space, due to the asymptotics of the Jacobi series coefficients of the right-hand side. Note that an opportunity to consider the whole problem in the matrix form is worth noticing itself and such remarkable results as [17,18] give us a tool—a basis property of the Jacobi polynomials. This approach leads to a similar problem, we can consider more wide class of operators if we use the matrix form and a valuable fact is that the criterion of the solvability of the Sonin–Abel equation is naturally formulated in the very matrix form. We stress that the used method was not previously considered in the well-known monographs [19,20] devoted to the topic. The paper is organized as follows. In Section 1 a brief historical review as well as some facts that motivated the author to write the paper are presented. In Section 2 some denotations and notions that are used throughout the paper are presented. Section 3.1 is devoted to the central result of the paper; the criterion of solvability of the Sonin–Abel equation in terms of the Jacobi series coefficients is formulated. Sections 3.2–3.5 are devoted to study of various particular cases of the Sonin kernel.

2. Preliminaries

2.1. Jacobi Polynomials

Let C be a real positive constant, we assume that a value of C can be different in various formulas and parts of formulas. We use special notations $q := \max\{\beta,\gamma\}, \sigma := \beta - 1$, $\varsigma := \gamma - 1, \beta, \gamma \in \mathbb{R}$, $\nu := -1/2$ for a more convenient form of writing and understand the following symbols as

$$\sum_{i=-k}^{n} a_i := \sum_{i=0}^{n} a_i,\ n,k \in \mathbb{N}_0,\ \ \sum_{i=k}^{n} a_i := 0,\ n,k \in \mathbb{Z},\ n < k.$$

We use the following notations for Jacobi polynomials and related expressions

$$p_n^{\beta,\gamma}(x) = \frac{(-1)^n \delta'_n(\beta,\gamma)}{(b-a)^{n+(\beta+\gamma+1)/2}} \times (x-a)^{-\beta}(b-x)^{-\gamma}\varphi_n^{(n)}(x),\ \beta,\gamma > -1,\ n \in \mathbb{N}_0,$$

where

$$\varphi_n^{(n)}(x) := (x-a)^{\beta+n}(b-x)^{\gamma+n},\ \delta'_n(\beta,\gamma) := \sqrt{\frac{(\beta+\gamma+2n+1)\Gamma(\beta+\gamma+n+1)}{n!\Gamma(\beta+n+1)\Gamma(\gamma+n+1)}},$$

$$\delta_0'(\beta,\gamma) = \frac{1}{\sqrt{\Gamma(\beta+1)\Gamma(\gamma+1)}}, \; \beta+\gamma+1=0,$$

We also use short-hand notations

$$\delta_n(\tilde{\beta},\gamma) = \delta_n'(\beta,\gamma)\Gamma(n+\beta+1), \; \delta_n(\beta,\tilde{\gamma}) = \delta_n'(\beta,\gamma)\Gamma(n+\gamma+1).$$

In accordance with Formula (3) [21], p. 282, we have

$$p_n^{\beta,\gamma\,(1)}(x) = \sqrt{n(\beta+\gamma+n+1)}\, p_{n-1}^{\beta+1,\gamma+1}(x),$$

It gives us the formula

$$p_n^{\beta,\gamma\,(k)}(a) = C_{n,k}(\beta,\gamma)\, p_{n-k}^{\beta+k,\gamma+k}(a), \; k \le n, \tag{1}$$

where the following denotations are used

$$C_{n,k}(\beta,\gamma) = \frac{\delta_{n-k}'(\beta+k,\gamma+k)}{\delta_n'(\beta,\gamma)}. \tag{2}$$

If we take into account the normalized multiplier, then Formula (6) [21], p. 283, in terms of the used notations, can be rewritten as follows

$$p_n^{\beta,\gamma}(a) = (-1)^n \frac{\delta_n'(\beta,\gamma)\Gamma(\beta+n+1)}{(b-a)^{(\beta+\gamma+1)/2}\Gamma(\beta+1)},$$

combining the above formulas, we have

$$p_n^{\beta,\gamma\,(k)}(a) = (-1)^{n+k} \frac{\delta_n'(\beta,\gamma)C_{n,k}^2\Gamma(\beta+n+1)}{(b-a)^{k+(\beta+\gamma+1)/2}\Gamma(\beta+k+1)}. \tag{3}$$

Using the Taylor series expansion for the Jacobi polynomials, we get

$$p_n^{\beta,\gamma}(x) = \sum_{k=0}^{n} (-1)^{n+k} \frac{\delta_n'(\beta,\gamma)C_{n,k}^2\Gamma(\beta+n+1)}{(b-a)^{k+(\beta+\gamma+1)/2}k!\Gamma(\beta+k+1)}(x-a)^k. \tag{4}$$

Applying the Formulas (2.44) and (2.45) [20], p. 40, of the fractional integral and derivative (see definitions in Section 2.2) of a power function, we obtain

$$\left(I_{a+}^{\alpha} p_n^{\beta,\gamma}\right)(x) = \sum_{k=0}^{n} (-1)^{n+k} \frac{\delta_n'(\beta,\gamma)C_{n,k}^2\Gamma(\beta+n+1)}{(b-a)^{k+(\beta+\gamma+1)/2}\Gamma(k+1+\alpha)\Gamma(\beta+k+1)}(x-a)^{k+\alpha}, \; \alpha \in \mathbb{R},$$

Here we used the formal notation $I_{a+}^{-\alpha} := D_{a+}^{\alpha}$. Thus, using integration by parts (see [13]), we get

$$\int_a^b p_m^{\beta,\gamma}(x)(I_{a+}^{\alpha} p_n^{\beta,\gamma})(x)\omega^{\beta,\gamma}(x)dx = \delta_m'\delta_n' \sum_{k=0}^{n} (-1)^{n+k} \frac{C_{n,k}^2\Gamma(\beta+n+1)B(\alpha+\beta+k+1,\gamma+m+1)}{\Gamma(\beta+k+1)\Gamma(k+\alpha-m+1)},$$
$$\alpha \in \mathbb{R} \setminus \{0\}, \; \alpha+\beta > -1. \tag{5}$$

Note that the last formula appeared in [14] in terms $\tilde{C}_n^k(\beta,\gamma) := C_{n,k}^2\Gamma(\beta+n+1)/\Gamma(\beta+k+1)$. Having noticed (3), we conclude that the latter formal equality becomes natural if we note that in accordance with notations given in [14] we have

$$\tilde{C}_n^k(\beta,\gamma) = (-1)^{n+k} p_n^{\beta,\gamma\,(k)}(a)\, \frac{(b-a)^{k+(\beta+\gamma+1)/2}}{\delta_n'(\beta,\gamma)}, \; k \le n.$$

Further, we use notations

$$f_n(\beta,\gamma) = \int_a^b f(x) p_n^{\beta,\gamma}(x)\omega^{\beta,\gamma}(x)dx,\ S_n^{\beta,\gamma} f := \sum_{k=0}^{n} f_n p_n^{\beta,\gamma}.$$

Throughout this paper we use conditions that guarantee passing to the limit in integral constructions. Muckenhoupt's result (see [17]) is in the following, assume that

$$f \in L_p(I,\beta,\gamma),\ |(\beta+1)/p - 1/2 - \beta'/2| < \min\{1/4, 1/2 + \beta'/2\},$$

and the same holds for γ, γ', then

$$S_k^{\beta',\gamma'} f \overset{L_p(I,\beta,\gamma)}{\longrightarrow} f,\ k \to \infty.$$

The particular case corresponding to $\beta,\gamma \geq -1/2$, $\beta' = \beta$, $\gamma' = \gamma$ is Pollard's result. In this case the condition can be reformulated in the author's (initial) form

$$M(\beta,\gamma) < p < m(\beta,\gamma),$$

$$M(\beta,\gamma) := 4\max\left\{\frac{\beta+1}{2\beta+3}, \frac{\gamma+1}{2\gamma+3}\right\},\ m(\beta,\gamma) := 4\min\left\{\frac{\beta+1}{2\beta+1}, \frac{\gamma+1}{2\gamma+1}\right\},$$

but the conclusion is obviously the same.

2.2. *Simple Properties of Convolution Operators with the Sonin Type Kernel*

We use a denotation

$$(f *_a g)(x) := \int_a^x f(x-\tau)g(\tau)d\tau,$$

if $a = 0$, then we write $f * g$. Throughout the paper we assume that the functions ϱ, $\vartheta \in L_1(I_0)$, $I_0 = (0, b-a) \subset \mathbb{R}$ are such that the so called Sonin condition holds $\varrho * \vartheta = 1$. Consider the left-hand side and right-hand side integral operators

$$\left({}_sI^{\varrho}_{a+}\varphi\right)(x) := \int_a^x \varrho(x-t)\varphi(t)dt,\ \left({}_sI^{\varrho}_{b-}\varphi\right)(x) := \int_x^b \varrho(t-x)\varphi(t)dt,\ \varphi \in L_1(I,\beta,\gamma);$$

and the differential operators

$${}_sD^{\vartheta}_{a+}f(x) := \frac{d}{dx}\left({}_sI^{\vartheta}_{a+}\varphi\right)(x),\ {}_sD^{\vartheta}_{b-}f(x) := -\frac{d}{dx}\left({}_sI^{\vartheta}_{b-}\varphi\right)(x),$$

where f belongs to the class of functions representable by the corresponding fractional integral ${}_sI^{\varrho}_{a+}(L_1(I))$, ${}_sI^{\varrho}_{b-}(L_1(I))$. The latter assumption gives us an opportunity to conclude that (compare with [20], p. 29)

$${}_sD^{\vartheta}_{a+}\,{}_sI^{\varrho}_{a+}\varphi = \frac{d}{dx}\left({}_sI^{\vartheta}_{a+}\,{}_sI^{\varrho}_{a+}\varphi\right)(x) = \frac{d}{dx}\int_a^x \varphi(t)dt = \varphi(x),\ \varphi \in L_1(I).$$

Indeed, by virtue of the obvious relation $\|\,|\varrho| *_a |\varphi|\,\|_{L_1(I)} \leq \|\varrho\|_{L_1(I_0)}\|\varphi\|_{L_1(I)}$ (the proof is by direct application of the corollary of the Fubini theorem) we can deduce that $|\vartheta| *_a |\varrho| *_a |\varphi| \in L_1(I)$. Therefore the following integral is convergent for almost all $x \in I$

$$\int_a^x |\vartheta(x-t)|dt \int_a^t |\varrho(t-\tau)||\varphi(\tau)|d\tau < \infty\ a.e.$$

It is clear that the latter integral converges for $x \in I$, since the functions under the integral are non-negative. Hence, applying the Fubini theorem, we can change the order of integration

$$\int_a^x \vartheta(x-t)dt \int_a^t \varrho(t-\tau)\varphi(\tau)d\tau = \int_a^x \varphi(\tau)d\tau \int_\tau^x \vartheta(x-t)\varrho(t-\tau)dt,\ x \in I.$$

Thus, by virtue of the Sonin conditions $\varrho * \vartheta = 1$ we obtain the desired result in the case $\varphi \in L_1(I)$. Absolutely analogously we obtain ${}_sD^{\vartheta}_{b-}\, {}_sI^{\varrho}_{b-}\varphi = \varphi,\ \varphi \in L_1(I)$. Note that in the particular case $\varrho(x) = x^{\alpha-1}/\Gamma(\alpha),\ \vartheta(x) = x^{-\alpha}/\Gamma(1-\alpha),\ 0 < \alpha < 1$, we have reduction to the fractional integral and differential Riemann–Liouville operators i.e., ${}_sI^{\varrho}_{a+} = I^{\alpha}_{a+},\ {}_sD^{\vartheta}_{a+} = D^{\alpha}_{a+}$. We use the following notations

$$A^{\vartheta,\beta,\gamma}_{mn} := \int_a^b p^{\beta,\gamma}_m(x)\left({}_sI^{\vartheta}_{a+}p^{\beta,\gamma}_n\right)(x)\,\omega^{\beta,\gamma}(x)dx,\ \mathfrak{B}^{\beta,\gamma}_p(f,\xi) := \sum_{n=1}^{\infty}|f_n(\beta,\gamma)|^p n^{\xi},$$

$$\varphi^{\varrho}_n(\beta,\gamma) := \int_a^b ({}_sI^{\varrho}_{a+}\varphi)(x)p^{\beta,\gamma}_n(x)\omega^{\beta,\gamma}(x)dx$$

and short-hand notations $p_n := p^{\beta,\gamma}_n,\ f_n := f_n(\beta,\gamma)$, if their meaning is quite understandable.

3. Main Results

3.1. Criterion of Solvability of the Sonin–Abel Equation

Throughout this section we deal with the case $\beta,\gamma \in (0,1)$. Consider the Sonin–Abel equation under the previously made assumptions regarding the kernel (see preliminaries section) and most general assumptions regarding the right-hand side

$${}_sI^{\varrho}_{a+}\varphi = f \in L_1(I). \tag{6}$$

We have the following theorem

Theorem 1. *Assume that the following conditions hold*

$$\text{i) } \mathfrak{B}^{\sigma,\varsigma}_p({}_sI^{\vartheta}_{a+}f,\xi) < \infty,\ \text{ii) } \sum_{m=0}^{\infty} f^{\vartheta}_m(\sigma,\varsigma)p^{\sigma,\varsigma}_m(a) = 0, \tag{7}$$

where $\xi = (5/2 + \max\{\beta,\gamma\})(p-2) + 2$, then there exists a unique solution of the Sonin–Abel Equation (6), the solution is represented by the series convergent in the sense of norm $L_p(I,\beta,\gamma)$

$$\psi = \sum_{m=0}^{\infty} \psi_m(\beta,\gamma)p^{\beta,\gamma}_m.$$

Moreover, in the case $p = 2$ we claim that conditions (7) are necessary, so that we have a criterion.

Proof. The sufficient part of existence. Consider the following coefficients, let us denote them

$$g_m := C_{m+1,1}(\sigma,\varsigma)\left|\int_a^b p^{\sigma,\varsigma}_m(x)\left({}_sI^{\vartheta}_{a+}f\right)(x)\omega^{\sigma,\varsigma}(x)dx\right|,$$

where according to Formula (2), we can calculate $C_{m+1,1}(\sigma,\varsigma) = \sqrt{(m+1)(\beta+\gamma+m)}$. Thus, due to the theorem conditions we have

$$\sum_{m=1}^{\infty}|g_m|^p m^{\xi-p} \leq C\,\mathfrak{B}^{\sigma,\varsigma}_p({}_sI^{\varrho}_{a+}f,\xi) < \infty.$$

Let us calculate

$$\xi - p = (5/2 + \max\{\beta,\gamma\})(p-2) + 2 - p = (1/2 + \max\{\beta,\gamma\})(p-2) + p - 2.$$

It implies that

$$\sum_{m=1}^{\infty} |g_m|^p M_m^{p-2} m^{p-2} < \infty,\ M_m = m^{1/2+\max\{\beta,\gamma\}}.$$

Having applied the Zygmund–Marcinkiewicz theorem (see [22]), we get that there exists a function $\psi \in L_p(I,\beta,\gamma)$, such that

$$\psi_m(\beta,\gamma) = g_m,\ m \in \mathbb{N}_0,\ S_k^{\beta,\gamma}\psi \overset{L_p(I,\beta,\gamma)}{\longrightarrow} \psi,\ k \to \infty.$$

In other terms, we have

$$\int_a^b p_m^{\beta,\gamma}(x)\psi(x)\omega^{\beta,\gamma}(x)dx = C_{m+1,1}(\sigma,\varsigma)\int_a^b p_{m+1}^{\sigma,\varsigma}(x)\left({}_sI_{a+}^{\vartheta} f\right)(x)\,\omega^{\sigma,\varsigma}(x)dx. \tag{8}$$

Using the integration by parts formulae, then the Fubini theorem, it is not hard to calculate the following relation

$$\int_a^b p_m^{\beta,\gamma}(x)\left(S_k^{\beta,\gamma}\psi\right)(x)\,\omega^{\beta,\gamma}(x)dx = C_{m+1,1}(\sigma,\varsigma)\int_a^b p_{m+1}^{\sigma,\varsigma}(x)\omega^{\sigma,\varsigma}(x)dx\int_a^x \left(S_k^{\beta,\gamma}\psi\right)(t)dt, \quad m,k = 0,1,\ldots,. \tag{9}$$

Also, we have

$$\left(\int_a^b \left|\int_a^x \left(S_k^{\beta,\gamma}\psi\right)(t)dt\right|^p \omega^{\sigma,\varsigma}(x)dx\right)^{1/p} \le \int_a^b \left(S_k^{\beta,\gamma}\psi\right)(t)dt\left(\int_t^b \omega^{\sigma,\varsigma}(x)dx\right)^{1/p} \le$$

$$\le \|S_k^{\beta,\gamma}\psi\|_{L_p(I,\beta,\gamma)}\left(\int_a^b \omega^{-p'/p}(t)dt\int_t^b \omega^{\sigma,\varsigma}(x)dx\right)^{1/p} \le C\|S_k^{\beta,\gamma}\psi\|_{L_p(I,\beta,\gamma)},\ k = 0,1,\ldots,.$$

Taking into account this fact and passing to the limit in the left-hand and right hand sides of the inequality (9), we get

$$\int_a^b p_m^{\beta,\gamma}(x)\psi(x)\,\omega^{\beta,\gamma}(x)dx = C_{m+1,1}(\sigma,\varsigma)\int_a^b p_{m+1}^{\sigma,\varsigma}(x)\omega^{\sigma,\varsigma}(x)dx\int_a^x \psi(t)dt,\ m = 0,1,\ldots,.$$

Substituting the relation ${}_sI_{a+}^{\vartheta}\,{}_sI_{a+}^{\varrho}\psi = I_{a+}^{1}\psi$ a.e. to the obtained formula, we get

$$\int_a^b p_m^{\beta,\gamma}(x)\psi(x)\omega^{\beta,\gamma}(x)dx = C_{m+1,1}(\sigma,\varsigma)\int_a^b p_{m+1}^{\sigma,\varsigma}(x)\left({}_sI_{a+}^{\vartheta}\,{}_sI_{a+}^{\varrho}\psi\right)(x)\omega^{\sigma,\varsigma}(x)dx = = C_{m+1,1}(\sigma,\varsigma)\int_a^b p_{m+1}^{\sigma,\varsigma}(x)\left({}_sI_{a+}^{\varrho}\,{}_sI_{a+}^{\vartheta}\psi\right)(x)\omega^{\sigma,\varsigma}(x)dx. \tag{10}$$

Combining (8) with (10), we obtain

$$\int_a^b p_{m+1}^{\sigma,\varsigma}(x)\left({}_sI_{a+}^{\vartheta}\,{}_sI_{a+}^{\varrho}\psi\right)(x)\omega^{\sigma,\varsigma}(x)dx = \int_a^b p_{m+1}^{\sigma,\varsigma}(x)\left({}_sI_{a+}^{\vartheta} f\right)(x)\omega^{\sigma,\varsigma}(x)dx,\ m \in \mathbb{N}_0.$$

It implies that

$$_sI^{\vartheta}_{a+}\,{}_sI^{\varrho}_{a+}\psi = {}_sI^{\vartheta}_{a+}f + \delta \text{ a.e.}, \tag{11}$$

where

$$\delta = \int_a^b (p_0^{\sigma,\varsigma})^2 \Big\{ \Big({}_sI^{\vartheta}_{a+}\,{}_sI^{\varrho}_{a+}\psi\Big)(x) - \Big({}_sI^{\vartheta}_{a+}f\Big)(x) \Big\} \omega^{\sigma,\varsigma}(x)dx. \tag{12}$$

Using ordinary properties of the convolution operator (see preliminaries section), it is not hard to obtain the formula ${}_sI^{\varrho}_{a+}\,{}_sI^{\vartheta}_{a+}\,{}_sI^{\varrho}_{a+}\psi = I^1_{a+}\,{}_sI^{\varrho}_{a+}\psi$ *a.e.* Note that the proof of the formula ${}_sI^{\vartheta}_{a+}\,{}_sI^{\varrho}_{a+}f = I^1_{a+}f$ *a.e.* is given at the same place. Having applied the obtained results to (11), we get $I^1_{a+}\,{}_sI^{\varrho}_{a+}\psi = I^1_{a+}f + {}_sI^{\varrho}_{a+}\delta$ *a.e.* Since the given functions at the left-hand and right-hand sides of the previous inequality are absolutely continuous, then

$$I^1_{a+}\,{}_sI^{\varrho}_{a+}\psi = I^1_{a+}f + {}_sI^{\varrho}_{a+}\delta.$$

Differentiating the left-hand side and right-hand side, we obtain

$${}_sI^{\varrho}_{a+}\psi = f + \delta\cdot\varrho(x-a)\; a.e.$$

Now, let us express the constant δ in terms of the theorem conditions. By easy calculation, we have

$$\int_a^b \left| \int_a^x \psi(t)dt \right| \omega^{\sigma,\varsigma}(x)dx \le \int_a^b |\psi(t)|dt \left(\int_t^b \omega^{\sigma,\varsigma}(x)dx \right) \le C\|\psi\|_{L_2(I,\beta,\gamma)}.$$

By virtue of this relation (we need it to pass to the limit), using the formula ${}_sI^{\vartheta}_{a+}\,{}_sI^{\varrho}_{a+}\psi = I^1_{a+}\psi$ *a.e.* and the fact

$$S_k^{\beta,\gamma}\psi \overset{L_2(I,\beta,\gamma)}{\longrightarrow} \psi,\; k\to\infty,$$

we can easily obtain

$$\int_a^b \Big(I^1_{a+}S_k^{\beta,\gamma}\psi\Big)(x)\omega^{\sigma,\varsigma}(x)dx \to \int_a^b \Big({}_sI^{\vartheta}_{a+}\,{}_sI^{\varrho}_{a+}\psi\Big)(x)\omega^{\sigma,\varsigma}(x)dx,\; k\to\infty.$$

Therefore, combining this relation with (12), we get

$$\int_a^b (p_0^{\sigma,\varsigma})^2 \Big\{ \Big(I^1_{a+}S_k^{\beta,\gamma}\psi\Big)(x) - \Big({}_sI^{\vartheta}_{a+}f\Big)(x) \Big\} \omega^{\sigma,\varsigma}(x)dx \to \delta,\; k\to\infty. \tag{13}$$

Using the formula (see [21], p. 282)

$$(p^{\sigma,\varsigma}_{m+1})' = \sqrt{(m+1)(\beta+\gamma+m)}\; p_m^{\beta,\gamma},$$

we have

$$\int_a^x \Big(S_k^{\beta,\gamma}\psi\Big)(t)dt = \sum_{m=0}^k \psi_m(\beta,\gamma)\int_a^x p_m^{\beta,\gamma}(t)dt = \sum_{m=0}^k \frac{\psi_m(\beta,\gamma)\big(p^{\sigma,\varsigma}_{m+1}(x) - p^{\sigma,\varsigma}_{m+1}(a)\big)}{\sqrt{(m+1)(\beta+\gamma+m)}}.$$

It follows that

$$\int_a^b \Big(I^1_{a+}S_k^{\beta,\gamma}\psi\Big)(x)\,\omega^{\sigma,\varsigma}(x)dx = \int_a^b \sum_{m=0}^k \frac{\psi_m(\beta,\gamma)\big(p^{\sigma,\varsigma}_{m+1}(x) - p^{\sigma,\varsigma}_{m+1}(a)\big)}{\sqrt{(m+1)(\beta+\gamma+m)}}\,\omega^{\sigma,\varsigma}(x)dx =$$

$$= -\sum_{m=0}^k \frac{\psi_m(\beta,\gamma)p^{\sigma,\varsigma}_{m+1}(a)}{\sqrt{(m+1)(\beta+\gamma+m)}} \int_a^b \omega^{\sigma,\varsigma}(x)dx = -B(\beta,\gamma)(b-a)^{\beta+\varsigma}\sum_{m=0}^k \frac{\psi_m(\beta,\gamma)p^{\sigma,\varsigma}_{m+1}(a)}{\sqrt{(m+1)(\beta+\gamma+m)}}.$$

Taking into account the established above fact $\psi_m(\beta,\gamma) = C_{m+1,1} f^{\vartheta}_{m+1}(\sigma,\varsigma)$, we get

$$\int_a^b \left(I^1_{a+} S^{\beta,\gamma}_k \psi\right)(x)\, \omega^{\sigma,\varsigma}(x)dx = -B(\beta,\gamma)(b-a)^{\beta+\varsigma} \sum_{m=0}^{k} f^{\vartheta}_{m+1}(\sigma,\varsigma)\, p^{\sigma,\varsigma}_{m+1}(a).$$

Hence, a substitution in (13) gives us

$$\int_a^b (p_0^{\sigma,\varsigma})^2 \left\{ \left(I^1_{a+} S^{\beta,\gamma}_k \psi\right)(x) - \left({}_sI^{\vartheta}_{a+} f\right)(x) \right\} \omega^{\sigma,\varsigma}(x)dx =$$

$$= -f^{\vartheta}_0(\sigma,\varsigma) p_0^{\sigma,\varsigma} - \sum_{m=0}^{k} f^{\vartheta}_{m+1}(\sigma,\varsigma) p^{\sigma,\varsigma}_{m+1}(a) = -\sum_{m=0}^{k+1} f^{\vartheta}_{m}(\sigma,\varsigma) p^{\sigma,\varsigma}_{m}(a) \to \delta,\ k\to\infty.$$

Thus, the theorem conditions guarantee that $\delta = 0$ and we complete the proof of the sufficient part.

The proof of the necessary part of existence. Now assume that there exists a solution of the Sonin–Abel equation in $L_2(I,\beta,\gamma)$, $\beta,\gamma \in (0,1)$. Then, using the fact that Jacobi polynomials form a basis in $L_2(I,\beta,\gamma)$, applying Formula (10), we get

$$\int_a^b p^{\beta,\gamma}_m(x)\psi(x)\omega^{\beta,\gamma}(x)dx = C_{m+1,1}(\sigma,\varsigma) \int_a^b p^{\sigma,\varsigma}_{m+1}(x) \left({}_sI^{\vartheta}_{a+}\, {}_sI^{\varrho}_{a+} \psi\right)(x)\omega^{\sigma,\varsigma}(x)dx =$$

$$= C_{m+1,1}(\sigma,\varsigma) \int_a^b p^{\sigma,\varsigma}_{m+1}(x) \left({}_sI^{\vartheta}_{a+} f\right)(x)\omega^{\sigma,\varsigma}(x)dx,\ m\in\mathbb{N}_0.$$

It follows that

$$\mathfrak{B}^{\sigma,\varsigma}_2({}_sI^{\vartheta}_{a+} f, 2) < \infty.$$

Since ψ is a solution, then (11) is fulfilled, where $\delta = 0$. We can also establish (13), in the way that was used above. Further, having repeated the above reasonings, we come to the relation

$$\sum_{m=0}^{\infty} f^{\vartheta}_m(\sigma,\varsigma) p^{\sigma,\varsigma}_m(a) = 0.$$

The proof of the necessary part is complete.

The proof of uniqueness: the proof can be obtained by direct calculation, we should apply the inverse operator to the left-hand and the right-hand side of the Sonin–Abel equation, then the desired result follows directly from the obtained formula for the solution. But here we want to stress more general nature of the solution existence considering the abstract scheme, presented in the paper [12], of the proof. Assume that there exists a solution ψ and another solution ϕ in $L_p(I,\beta,\gamma)$ of the Sonin–Abel equation, and let us denote $\xi := \psi - \phi$. Denote $I_n := (a + 1/n, b - 1/n)$, then the following assumptions are fulfilled

$$\bigcup_{n=1}^{\infty} I_n = I,\ I_n \subset I_{n+1},\ \mu(I\backslash I_n) \to 0,\ n\to\infty,\ L_p(I,\beta,\gamma) \subset L_p(I_n).$$

The verification is left to the reader. In terms of these notations, it is also clear that

$$\bigcup_{n=1}^{\infty} C_0^{\infty}(I_n) = C_0^{\infty}(I),\ C_0^{\infty}(I_n) \subset C_0^{\infty}(I_{n+1}).$$

Thus to apply the scheme of the proof presented in [12], we need to show that

$$\forall \eta \in C_0^{\infty}(I),\ \exists g \in L_{p'}(I,\beta,\gamma):\ (\xi,\eta)_{L_2(I)} = (\xi,\, {}_sI^{\varrho *}_{a+} g)_{L_2(I,\beta,\gamma)}.$$

For this purpose it is sufficient to show that

$$\exists h \in L_\infty(I) : \int_a^b \xi(x)\eta(x)dx = \int_a^b \xi(x)\Big({}_sI^{\varrho}_{b-}\omega^{\beta,\gamma}h\Big)(x)dx. \tag{14}$$

Let us prove that ${}_sD^{\vartheta}_{b-}\eta(x) \in L_\infty(I)$. Making substitution of the expression

$$\eta(x) = -\int_x^b \eta'(\tau)d\tau + \eta(b),$$

to the formula ${}_sD^{\vartheta}_{b-}\eta(x)$, we can calculate easily

$$({}_sD^{\vartheta}_{b+}\eta)(x) = \eta(b)\vartheta(b-x) - \int_x^b \vartheta(t-x)\eta'(t)dt = -\int_x^b \vartheta(t-x)\eta'(t)dt \ \text{ a.e.} \tag{15}$$

It is clear that

$$\left|\int_x^b \vartheta(t-x)\eta'(t)dt\right| \le C\int_x^b |\vartheta(t-x)|dt \le \int_0^{b-a} |\vartheta(t)|dt < \infty,$$

thus we have $D^{\vartheta}_{b-}\eta \in L_\infty(I)$. Let the desired function be $h := \omega^{-\beta,-\gamma} \cdot {}_sD^{\vartheta}_{b-}\eta$, then by direct calculation, we get $\omega^{\beta,\gamma}h \in L_{p'}(I,\beta,\gamma)$. Using Formula (15), applying the Fubini theorem, then using the condition $\varrho * \vartheta = 1$, we get

$$\Big({}_sI^{\varrho}_{b-}\, {}_sD^{\vartheta}_{b-}\eta\Big)(x) = -\int_x^b \varrho(\tau-x)d\tau \int_\tau^b \vartheta(t-\tau)\eta'(t)dt = -\int_x^b \eta'(t)dt\int_x^t \vartheta(t-\tau)\varrho(\tau-x)d\tau =$$
$$= -\int_x^b \eta'(t)dt \int_0^{x-t} \vartheta(y)\varrho(x-t-y)dy = -\int_x^b \eta'(t)dt = \eta(x).$$

Thus, combining the obtained facts, we have $\eta = {}_sI^{\varrho}_{b-}\omega^{\beta,\gamma}h$, $h \in L_\infty(I)$, from what follows Formula (14). Using the Hölder inequality, the generalized Cauchy–Schwarz inequality, we get

$$\int_a^b |\xi(x)|\Big({}_sI^{\varrho}_{b-}\omega^{\beta,\gamma}|h|\Big)(x)dx \le C\int_a^b |\xi(x)|\Big({}_sI^{\varrho}_{b-}\omega^{\beta,\gamma}\Big)(x)dx \le C\|\xi\|_{L_{p'}(I,\beta,\gamma)}.$$

Thus we can apply the Dirichlet formula to relation (14) and rewrite it as follows

$$\int_a^b \xi(x)\eta(x)dx = \int_a^b \xi(x)\Big({}_sI^{\varrho}_{b-}\omega^{\beta,\gamma}h\Big)(x)dx = \int_a^b \omega^{\beta,\gamma}(x)h(x)\Big({}_sI^{\varrho}_{a+}\xi\Big)(x)dx. \tag{16}$$

Hence $\omega^{-\beta,-\gamma} \cdot_s I^{\varrho}_{b-} \subset {}_sI^{\varrho *}_{a+}$ and we can complete the proof having applied Theorem 3 [12]. However, for readers' convenience, we present the rest part of the proof below. By virtue of (16), we have

$$\int_{I_n} \xi(x)\eta(x)dx = 0,\ \forall \eta \in C_0^\infty(I_n).$$

We claimed that $\xi \neq 0$. Note that in accordance with the corollary of the Hahn-Banach theorem there exists an element $\varpi \in L_{p'}(I_n)$, such that $(\varpi,\xi)_{L_2(I_n)} = \|\psi-\phi\|_{L_p(I_n)} > 0$. On the other hand, there exists the sequence $\{\eta_k\}_1^\infty \subset C_0^\infty(I_n)$, such that $\eta_k \to \varpi$ with respect to the norm $L_{p'}(I_n)$. Hence, using the Hölder inequality it is not hard to prove that $0 = (\eta_k, \xi)_{L_2(I_n)} \to (\varpi, \xi)_{L_2(I_n)}$. Therefore $\psi = \phi$ almost everywhere on the set I_n, $n = 1, 2, \ldots$. In its own turn, it implies that $\psi = \phi$ almost everywhere on the set I. The uniqueness has been proved. □

The next fact follows immediately from the proof of the previous theorem.

Corollary 1. *Consider a formal equation with a perturbed right-hand side*

$$ {}_sI^{\varrho}_{a+}\psi = f + \delta \cdot \varrho(x-a), \tag{17}$$

where

$$ \delta := -\sum_{m=0}^{\infty} f_m^{\vartheta}(\sigma,\varsigma) p_m^{\sigma,\varsigma}(a). \tag{18}$$

Assume that condition (i) of Theorem 1 holds, then there exists a unique solution of Equation (17), that is satisfied the conditions claimed in the conclusion part of Theorem 1.

Note that condition (i) gives us $|f_m^{\vartheta}(\sigma,\varsigma)| m^{(\xi-1)/p} \le C$. Combining this relation with the well-known fact $p_m^{\sigma,\varsigma}(a) \sim (-1)^m C m^{\sigma+1/2}$ (see [21], p. 288), calculating $(\xi-1)/p$ we can easily establish the convergence of series (18). Thus, under assumption (i), we get $\delta \in \mathbb{R}$; if $\delta = 0$, then we have a regular Sonin–Abel equation. If $\delta \neq 0$, then we call a solution of Equation (17) *quasi-solution of the Sonin–Abel equation*. In this case Equation (6) is unsolvable, but the question that may be relevant in applications is how large a perturbation of the right-hand side, in some sense, should be to obtain a regular solution? Thus, we come to the following definition in a natural way. We will call the following norm a *defect of the right-hand side*

$$ \| {}_sI^{\varrho}_{a+}\psi - f\|_{L_2(I,\beta,\gamma)} = |\delta| \cdot \|\varrho\|_{L_2(I_0,\beta,\gamma)}. $$

It is clear that the defect of the right-hand side can be completely described by a value of δ, the following corollary gives an example how we can estimate the latter in the concrete case. The following reasonings are based on the formula for a value of $p_m^{\sigma,\varsigma}(a)$, $m \in \mathbb{N}_0$ (see [21]) and the main property of the alternative series.

Example 1. *Assume additionally that*

$$ \exists k \in \mathbb{N}_0 : \operatorname{sign}\Big\{ f_m^{\vartheta}(\sigma,\varsigma) \Big\} = \text{const}, \; f_{m+1}^{\vartheta}(\sigma,\varsigma)/f_m^{\vartheta}(\sigma,\varsigma) < K_m(\beta,\gamma), \; m = k, k+1, \ldots, $$

$$ K_0(\beta,\gamma) = \sqrt{\frac{(\beta+1)(\gamma+1)}{(\beta+\gamma+3)}}, \; K_m(\beta,\gamma) = \sqrt{\frac{(m+1)(\beta+m)(\gamma+m)(\beta+\gamma+2m-1)}{(\beta+\gamma+m-1)(\beta+\gamma+2m+1)}}, $$

then

$$ |\delta| < \left| \sum_{m=0}^{k-1} f_m^{\vartheta}(\sigma,\varsigma) p_m^{\sigma,\varsigma}(a) \right| + |f_k^{\vartheta}(\sigma,\varsigma) p_k^{\sigma,\varsigma}(a)|. $$

The proof is simple and left to the reader. Thus, we can estimate δ by a finite sum.

Corollary 2. *Assume that $\mathfrak{B}_2^{\sigma,\varsigma}({}_sI^{\vartheta}_{a+}f, 2) < \infty$, then ${}_sI^{\vartheta}_{a+}f \in AC(\bar{I})$.*

Proof. In accordance with Corollary 1, we have that there exists a quasi-solution $\psi \in L_2(I,\beta,\gamma)$, such that

$$ \left({}_sI^{\varrho}_{a+}\psi \right)(x) = f(x) - \varrho(x-a) \sum_{m=0}^{\infty} f_m^{\vartheta}(\sigma,\varsigma) p_m^{\sigma,\varsigma}(a). $$

Using the Cauchy–Schwarz inequality, it is not hard to prove that $\psi \in L_1(I)$. Applying the operator ${}_sI^{\vartheta}_{a+}$ to both sides of the previous relation, we get

$$\int_a^x \psi(t)dt = \left({}_sI^{\vartheta}_{a+}f\right)(x) - \sum_{m=0}^{\infty} f^{\vartheta}_m(\sigma,\varsigma)p^{\sigma,\varsigma}_m(a)\int_a^x \vartheta(x-t)\varrho(t-a)dt =$$

$$= \left({}_sI^{\vartheta}_{a+}f\right)(x) - \sum_{m=0}^{\infty} f^{\vartheta}_m(\sigma,\varsigma)p^{\sigma,\varsigma}_m(a).$$

This relation proves the desired result. □

3.2. *Kernels Close to Power-Type Functions*

In this section we consider concrete Sonin kernels such that:

(i) Bessel-type functions present Sonin kernels in two variants, the first one

$$\varrho(x) = x^{-\nu/2}J_\nu(2\sqrt{x}),\ \vartheta(x) = x^{(\nu-1)/2}I_{\nu-1}(2\sqrt{x}),$$

and the second one

$$\varrho(x) = x^{-\nu/2}I_\nu(2\sqrt{x}),\ \vartheta(x) = x^{(\nu-1)/2}J_{\nu-1}(2\sqrt{x}),$$

where

$$J_\nu = \sum_{n=0}^{\infty} \frac{(-1)^n(x/2)^{2n+\nu}}{n!\Gamma(n+\nu+1)},\ I_\nu = \sum_{n=0}^{\infty} \frac{(x/2)^{2n+\nu}}{n!\Gamma(n+\nu+1)}.$$

(ii) Incomplete gamma function and the power-exponential function:

$$\varrho(x) = \lambda^{1-\alpha} + \frac{1-\alpha}{\Gamma(\alpha)}\int_x^{\infty} e^{-\lambda t}t^{\alpha-2}dt,\ \vartheta(x) = \frac{e^{-\lambda x}}{\Gamma(1-\alpha)x^\alpha},\ \alpha \in (0,1),\ \lambda \geq 0.$$

(iii) Product of the power and Kummer functions

$$\varrho(x) = x^{\alpha-1}\Phi(\beta,\alpha,-\lambda x),\ \vartheta(x) = \frac{x^{-\alpha}\Phi(-\beta,1-\alpha,-\lambda x)}{\Gamma(\alpha)\Gamma(1-\alpha)},\ \alpha \in (0,1),\ \beta,\lambda > 0.$$

where the following function is the Kummer function

$$\Phi(\beta,\alpha,z) = \sum_{n=0}^{\infty} \frac{(\beta)_k z^k}{(\alpha)_k k!}.$$

Now, consider more general construction, assume that

$$\varrho(x) := \frac{x^{\alpha-1}a(x)}{\Gamma(\alpha)},\ a(x) = \sum_{n=0}^{\infty} a_n x^n,\ a_0 \neq 0,\ \alpha \in (0,1), \tag{19}$$

then in accordance with Theorem 7.1 [23], there exists a unique analytic function $b(x)$ such that the function

$$\vartheta(x) = \frac{x^{-\alpha}b(x)}{\Gamma(1-\alpha)} \tag{20}$$

is an associated kernel with respect to ϑ; the series for $b(x)$ converges where the series for $a(x)$ converges. To apply naturally Theorem 1 in this concrete case, let us establish the asymptotics of the coefficients of ${}_sI^{\vartheta}_{a+}f$ having known the asymptotics of the coefficients of f. Note that this problem can be principally solved due to the results of [14], here we also offer a slight improvement of them. Let us describe the coefficients $f^{\vartheta}_n(\sigma,\varsigma)$ via the coefficients $f_n(\sigma,\varsigma)$.

3.3. Convolutions Operators in the Matrix Form

It is a well-known fact that we have a representation of a bounded operator in a matrix form if we have a basis in the corresponding space. Below we present an approach that allows us to reformulate Theorem 1 in the matrix form. The following lemma is devoted to a power-type kernel.

Lemma 1. *Suppose $\vartheta(x) = x^{-\alpha}$, $\alpha \in (-\infty, 0) \cap (0,1)$, $f \in L_s(I,\sigma,\varsigma)$, where we assume that $s > 1$, if both numbers β, γ belong to the set $(0, 1/2]$, and $s > 4/3$, if at least one of the numbers β, γ belongs to the set $(1/2, 1)$. Then the following representation holds*

$$f_m^{\vartheta}(\sigma,\varsigma) = \sum_{n=0}^{\infty} f_n^{\sigma,\varsigma} A_{mn}^{\vartheta,\sigma,\varsigma},\ m \in \mathbb{N}_0,$$

where

$$A_{mn}^{\vartheta,\sigma,\varsigma} := (-1)^n \delta_m(\sigma,\varsigma)\delta_n(\sigma,\varsigma) \sum_{k=0}^{n} (-1)^k \frac{C_{n,k}^2 \Gamma(\sigma+n+1) B(2-\alpha+\sigma+k, \varsigma+m+1)}{\Gamma(\sigma+k+1)\Gamma(k-\alpha-m+2)}. \tag{21}$$

In the matrix form this Lemma can be formulated as follows

$$\begin{pmatrix} A_{00}^{\vartheta,\sigma,\varsigma} & A_{01}^{\vartheta,\sigma,\varsigma} & \dots \\ A_{10}^{\vartheta,\sigma,\varsigma} & A_{11}^{\vartheta,\sigma,\varsigma} & \dots \\ . & & \\ . & & \\ . & & \dots \end{pmatrix} \times \begin{pmatrix} f_0(\sigma,\varsigma) \\ f_1(\sigma,\varsigma) \\ . \\ . \\ . \end{pmatrix} = \begin{pmatrix} f_0^{\vartheta}(\sigma,\varsigma) \\ f_1^{\vartheta}(\sigma,\varsigma) \\ . \\ . \\ . \end{pmatrix}.$$

Proof. Consider the following reasonings

$$\int_a^b |\left({}_sI_{a+}^{\vartheta} f\right)(x)| \omega^{\sigma,\varsigma}(x)dx = \int_a^b |f(t)| \int_t^b \omega^{\sigma,\varsigma}(x)(x-t)^{-\alpha}dx \le C \int_a^b |f(t)| \int_t^b \omega^{\sigma,\varsigma}(x)(x-t)^{-\alpha}dx \le$$

$$\le C \int_a^b |f(t)|(t-a)^{\beta-1} \int_t^b (b-x)^{\gamma-1}(x-t)^{-\alpha}dx \le$$

$$\le C \left\{ \int_a^b (t-a)^{p(\beta-1)+d_1}(b-t)^{d_2} \left| \int_t^b (b-x)^{\gamma-1}(x-t)^{-\alpha}dx \right|^p dt \right\}^{1/p} \|f\|_{L_{p'}(I,\beta',\gamma')} =$$

$$= C \cdot B(\gamma, 1-\alpha) \left\{ \int_a^b (t-a)^{p(\beta-1)+d_1}(b-t)^{p(\gamma-\alpha)+d_2}dt \right\}^{1/p} \|f\|_{L_{p'}(I,\beta',\gamma')} \le C\|f\|_{L_{s_1}(I,\beta',\gamma')},$$

where $\beta' = -d_1p'/p$, $\gamma' = -d_2p'/p$, numbers p', s_1 are chosen so that $0 < p' \le s_1 < s$, numbers d_1, d_2 are chosen so that $p(\beta-1)+d_1 > -1$, $p(\gamma-1)+d_2 > -1$. It follows that $\beta' < p'\beta - 1$, $\gamma' < p'\gamma - 1$. Thus, we should assume that $\beta' = p'\beta - 1 - \varepsilon$, $\gamma' = p'\gamma - 1 - \varepsilon$ to guarantee the finiteness of the integral. Note, that we can rewrite the previously obtained relation in the following form, we have omitted the reasonings since they are too obvious

$$\left| \int_a^b p_m^{\sigma,\varsigma}(x) \left({}_sI_{a+}^{\vartheta} f\right)(x) \omega^{\sigma,\varsigma}(x)dx \right| \le C\|f\|_{L_{s_1}(I,\beta',\gamma')}. \tag{22}$$

In this case consider the question whether the Muckenhoupt conditions are satisfied or not (see Theorem 1 [17]), i.e., in the used terms, we should verify the fulfilment of the conditions

$$\left| \frac{p'\beta - \varepsilon}{s_1} - \frac{\beta}{2} \right| < \min\left\{ \frac{1}{4}, \frac{\beta}{2} \right\},\ \left| \frac{p'\gamma - \varepsilon}{s_1} - \frac{\gamma}{2} \right| < \min\left\{ \frac{1}{4}, \frac{\gamma}{2} \right\}.$$

It is clear that the last relation holds for $\beta \in (0,1/2]$, $p' \leq s_1$ and for $\beta \in (1/2,1)$, $4/3 \leq s_1/p' < 4$ (compare the last condition with the Pollard condition of the basis property of the Legendre polynomials). The same reasonings are held for γ. Therefore by virtue of the proved basis property, we have

$$S_k^{\sigma,\varsigma} f \stackrel{L_{s_1}(I,\beta',\gamma')}{\longrightarrow} f,\ k \to \infty.$$

Now, combining (22) and this relation, we can easily establish the fact

$$\sum_{n=0}^{k} f_n^{\sigma,\varsigma} A_{mn}^{\vartheta,\sigma,\varsigma} = \int_a^b p_m^{\sigma,\varsigma}(x)\left({}_sI_{a+}^{\vartheta} S_k^{\sigma,\varsigma} f\right)(x)\omega^{\sigma,\varsigma}(x)dx \to \int_a^b p_m^{\sigma,\varsigma}(x)\left({}_sI_{a+}^{\vartheta} f\right)(x)\omega^{\sigma,\varsigma}(x)dx,$$

$$k \to \infty,\ m \in \mathbb{N}_0.$$

We should note in the reminder that we can chose ε such that $\|f\|_{L_{s_1}(I,\beta',\gamma')} \leq C\|f\|_{L_s(I,\sigma,\varsigma)}$. Indeed, having applied the Hölder inequality, we have

$$\|f\|_{L_{s_1}(\beta',\gamma',I)} = \left\{\int_a^b |f(t)|^{s_1}(t-a)^{p'\beta-1-\varepsilon}(b-t)^{p'\gamma-1-\varepsilon}dt\right\}^{1/s_1} \leq$$

$$\leq C\left\{\int_a^b |f(t)|^{s_1}(t-a)^{\beta-1-\varepsilon}(b-t)^{\gamma-1-\varepsilon}dt\right\}^{1/s_1} \leq C\left\{\int_a^b |f(t)|^{s}(t-a)^{\beta-1}(b-t)^{\gamma-1}dt\right\}^{1/s}.$$

To complete the proof, we should note that representation (21) follows directly from Formula (5). □

The following lemma establishes a similar result, in comparison with the previous one, under more strong conditions imposed on the kernel. It can be justified by the opportunity to consider a kernel in an abstract form in the contrary to Lemma 1. Such an approach completely suits us since we deal with the kernels close to a power-type function which admit a decomposition on a sum where the summands are described in the lemmas.

Lemma 2. *Suppose $\vartheta \in L_p(I_0,\nu,\nu)$, $\nu = -1/2$, $p > 1$, $f \in L_{p'}(I,\sigma,\varsigma)$, where we assume that $p' > 1$, if both numbers β,γ belong to the set $(0,1/2]$ and $p' > 4/3$, if at least one of the numbers β,γ belongs to the set $(1/2,1)$. Then*

$$f_m^{\vartheta}(\sigma,\varsigma) = \sum_{n=0}^{\infty} f_n^{\sigma,\varsigma} A_{mn}^{\vartheta,\sigma,\varsigma},\ m \in \mathbb{N}_0,$$

where

$$A_{mn}^{\vartheta,\sigma,\varsigma} = \delta_m(\sigma,\tilde{\varsigma})\delta_n(\tilde{\sigma},\varsigma)\sum_{j=0}^{n}\frac{(-1)^{n+j}}{\Gamma(\sigma+j+1)}\sum_{i=m-j-1}^{\infty}(-1)^i\vartheta_i(\nu,\nu)\Psi_{m-j-1}^{i}(j,m), \tag{23}$$

$$\Psi_{m-j-1}^{i}(j,m) := \sum_{l=m-j-1}^{i}\frac{(-1)^l\sqrt{2}(b-a)C_{i,l}^2\Gamma(l+j+\sigma+2)}{\Gamma(l+1/2)(l+j+1-m)!\Gamma(l+j+\sigma+m+\varsigma+3)},\ i>0,$$

$$\Psi_{m-j-1}^{0}(j,m) := \frac{(b-a)\Gamma(j+\sigma+2)}{\sqrt{\pi}(j+1-m)!\Gamma(j+\sigma+m+\varsigma+3)}.$$

Proof. Applying the Hölder inequality and then the generalized Minkovskii inequality, we get

$$I := \int_a^b |f(t)|dt \int_t^b |\vartheta(x-t)|\omega^{\sigma,\varsigma}(x)dx \leq C\|f\|_{L_{p'}(I,\sigma,\varsigma)} \left(\int_a^b \left| \int_t^b \vartheta(x-t)\omega^{\sigma,\varsigma}(x)dx \right|^p dt \right)^{1/p} \leq$$

$$\leq C\|f\|_{L_{p'}(I,\sigma,\varsigma)} \int_a^b \omega^{\sigma,\varsigma}(x) \left(\int_a^x |\vartheta(x-t)|^p dt \right)^{1/p} dx =$$

$$= C\|f\|_{L_{p'}(I,\sigma,\varsigma)} \int_a^b \omega^{\sigma,\varsigma}(x) \left(\int_0^{x-a} |\vartheta(t)|^p dt \right)^{1/p} dx \leq C\|f\|_{L_{p'}(I,\sigma,\varsigma)}.$$

Thus, using the Fubini theorem, we get

$$\left| \int_a^b p_m^{\sigma,\varsigma}(x)\, ({}_sI^{\vartheta}_{a+}f)(x)\omega^{\sigma,\varsigma}(x)dx \right| = \left| \int_a^b f(t)dt \int_t^b \vartheta(x-t)p_m(x)\omega^{\sigma,\varsigma}(x)dx \right| \leq C \cdot I.$$

Hence

$$\left| \int_a^b p_m^{\sigma,\varsigma}(x)\, ({}_sI^{\vartheta}_{a+}f)(x)\omega^{\sigma,\varsigma}(x)dx \right| \leq C\|f\|_{L_{p'}(I,\sigma,\varsigma)}. \tag{24}$$

Let us verify the fulfilment of the Muckenhoupt conditions (see Theorem 1 [17]). By virtue of the lemma conditions, in the used terms, we have

$$\left| \frac{\beta}{p'} - \frac{\beta}{2} \right| < \min\left\{ \frac{1}{4}, \frac{\beta}{2} \right\}, \left| \frac{\gamma}{p'} - \frac{\gamma}{2} \right| < \min\left\{ \frac{1}{4}, \frac{\gamma}{2} \right\},$$

here we assume without loss of generality, that $p' < 4$. It is clear that it is possible due to the inequality $\|f\|_{L_{p_1}(I,\sigma,\varsigma)} \leq C\|f\|_{L_{p_2}(I,\sigma,\varsigma)}$, $p_1 < p_2$, $1 \leq p_1 < \infty$, which can be obtained by direct application of the Hölder inequality. Thus, using the basis property, we have

$$S_k^{\sigma,\varsigma} f \overset{L_{p'}(I,\sigma,\varsigma)}{\longrightarrow} f,\ k \to \infty.$$

Therefore, by virtue of (24), we can easily get

$$\sum_{n=0}^{k} f_n^{\sigma,\varsigma} A_{mn}^{\vartheta,\sigma,\varsigma} = \int_a^b p_m^{\sigma,\varsigma}(x)\left({}_sI^{\vartheta}_{a+} S_k^{\sigma,\varsigma} f\right)(x)\omega^{\sigma,\varsigma}(x)dx \to \int_a^b p_m^{\sigma,\varsigma}(x)\left({}_sI^{\vartheta}_{a+} f\right)(x)\omega^{\sigma,\varsigma}(x)dx,$$

$$k \to \infty,\ m \in \mathbb{N}_0.$$

Let us establish (23), note that due to such a choice of the power $\nu = -1/2$, we have

$$1 + \varepsilon > M\left(-\frac{1}{2}, -\frac{1}{2} \right),\ \forall \varepsilon > 0.$$

Hence the Pollard conditions are fulfilled, thus by virtue of the basis property, we get

$$\left\| \vartheta - S_k^{\nu,\nu} \vartheta \right\|_{L_{1+\varepsilon}(I_0,\nu,\nu)} \to 0,\ k \to \infty,$$

It is clear that

$$\left| \int_0^{x-a} p_n^{\sigma,\varsigma}(x-t)\left[\vartheta(t) - (S_k^{\nu,\nu}\vartheta)(t) \right] dt \right| \leq C \int_0^{x-a} \frac{|p_n^{\sigma,\varsigma}(x-t)|\left| \vartheta(t) - (S_k^{\nu,\nu}\vartheta)(t) \right|}{\sqrt{t(b-a-t)}} dt \leq$$

$$\leq C\left\|\vartheta - S_k^{\nu,\nu}\vartheta\right\|_{L_{1+\varepsilon}(I_0,\nu,\nu)}.$$

Taking into account these reasonings, we obtain

$$\int_0^{x-a} p_n^{\sigma,\varsigma}(x-t)(S_k^{\nu,\nu}\vartheta)(t)dt \to \int_a^x p_n^{\sigma,\varsigma}(t)\vartheta(x-t)dt,\ k\to\infty.$$

Denote

$$f_k(x) := p_m^{\sigma,\varsigma}(x)\omega^{\sigma,\varsigma}(x)\int_0^{x-a} p_n^{\sigma,\varsigma}(x-t)(S_k^{\nu,\nu}\vartheta)(t)dt,\ f(x) := p_m^{\sigma,\varsigma}(x)\omega^{\sigma,\varsigma}(x)\int_a^x p_n^{\sigma,\varsigma}(t)\vartheta(x-t)dt.$$

Thus, it is clear that in the given terms $f_k \to f$ pointwise. Let us show that there exists a Lebesgue summable function ξ such that $|f_k| \leq \xi$, $\forall k \in \mathbb{N}$. Using simple reasonings, we get

$$|f_k(x)| \leq C\omega^{\sigma,\varsigma}(x)\int_0^{x-a}\frac{|p_n^{\sigma,\varsigma}(x-t)|\left|(S_k^{\nu,\nu}\vartheta)(t)\right|}{\sqrt{t(b-a-t)}}dt \leq$$

$$\leq C\omega^{\sigma,\varsigma}(x)\left\|S_k^{\nu,\nu}\vartheta\right\|_{L_{1+\varepsilon}(I_0,\nu,\nu)} \leq C\omega^{\sigma,\varsigma}(x)\|\vartheta\|_{L_{1+\varepsilon}(I_0,\nu,\nu)}.$$

Thus, applying the majoring Lebesgue theorem (assuming that $\xi := C\omega^{\sigma,\varsigma}$), we obtain

$$\int_a^b p_m^{\sigma,\varsigma}(x)\omega^{\sigma,\varsigma}(x)dx\int_a^x p_n^{\sigma,\varsigma}(x)(S_k^{\nu,\nu}\vartheta)(x-t)dt \to A_{mn}^{\vartheta,\sigma,\varsigma}.$$

Using Formula (4), we obtain

$$I_k := \int_a^b p_m^{\sigma,\varsigma}(x)\omega^{\sigma,\varsigma}(x)dx\int_a^x p_n^{\sigma,\varsigma}(t)(S_k^{\nu,\nu}\vartheta)(x-t)dt =$$

$$= \int_a^b (x-a)^{m+\sigma}(b-x)^{m+\varsigma}\left(\int_a^x p_n^{\sigma,\varsigma}(t)(S_k^{\nu,\nu}\vartheta)(x-t)dt\right)^{(m)}dx =$$

$$= \frac{\delta_m'(\sigma,\varsigma)\delta_n'(\sigma,\varsigma)}{(b-a)^{m+\sigma+\varsigma+1}}\sum_{i=0}^{k}\vartheta_i(\nu,\nu)\delta_i'(\nu,\nu)\sum_{l=0}^{i}\frac{C_{i,l}^2\Gamma(i+1/2)}{\Gamma(l+1/2)}\sum_{j=0}^{n}\frac{(-1)^{n+j+i+l}\Gamma(n+\sigma+1)}{(b-a)^{j+l}j!l!\Gamma(j+\sigma+1)}\times$$

$$\times\int_a^b (x-a)^{m+\sigma}(b-x)^{m+\varsigma}\left(\int_a^x (x-t)^l(t-a)^j dt\right)^{(m)}dx.$$

Calculating the fractional integral of a power function, we have

$$\int_a^x (x-t)^l(t-a)^j dt = l!I_{a+}^{l+1}(t-a)^j = \frac{l!j!}{(l+j+1)!}(x-a)^{l+j+1}.$$

Hence

$$\left(\int_a^x (x-t)^l(t-a)^j dt\right)^{(m)} = \begin{cases}\dfrac{l!j!(x-a)^{l+j+1-m}}{(l+j+1-m)!}, & m \leq l+j+1,\\ 0 & , m > l+j+1\end{cases}.$$

Using this relation, we obtain

$$\int_a^b (x-a)^{m+\sigma}(b-x)^{m+\varsigma}\left(\int_a^x (x-t)^l (t-a)^j dt\right)^{(m)} dx =$$

$$= \frac{l!j!}{(l+j+1-m)!}\int_a^b (x-a)^{l+j+1+\sigma}(b-x)^{m+\varsigma}dx =$$

$$= \frac{l!j!}{(l+j+1-m)!}(b-a)^{l+j+m+\sigma+\varsigma+2}B(l+j+\sigma+2, m+\varsigma+1),\ m \leq l+j+1.$$

Combining the above formulas, we get

$$I_k = (b-a)\delta'_m(\sigma,\varsigma)\delta'_n(\sigma,\varsigma)\sum_{j=0}^{n}(-1)^{n+j}\frac{\Gamma(n+\beta+1)}{\Gamma(j+\beta+1)}\sum_{i=m-j-1}^{k}\vartheta_i(\nu,\nu)\delta'_i(\nu,\nu)\times$$

$$\times\sum_{l=m-j-1}^{i}(-1)^{i+l}\frac{C_{i,l}^2\Gamma(i+1/2)B(l+j+\sigma+2,m+\varsigma+1)}{\Gamma(l+1/2)(l+j+1-m)!}.$$

Taking into account the proved above fact $I_k \to A_{mn}^{\vartheta,\sigma,\varsigma},\ k\to\infty$ we obtain the desired result. To show the form of writing used in the claim of the theorem, we should note that $\delta'_0(\nu,\nu) = 1/\sqrt{\pi},\ \delta'_n(\nu,\nu) = \sqrt{2}/\Gamma(n+1/2),\ n\in\mathbb{N}$, and use the formula connecting the beta and gamma functions. The proof is complete. □

3.4. Application to Existence and Uniqueness Theorems

Consider Equation (17), assuming that ϱ, ϑ are defined by (19), (20) respectively. The central aim of this paragraph is to provide a technique that guarantees proofs of existence and uniqueness theorems. It is preferable to formulate conditions in terms of the Jacobi series coefficients of the right-hand side of the Sonin–Abel equation instead of the coefficients of the image of the corresponding convolution operator (it has been done in Theorem 1), thus we are motivated to formulate mapping theorems in the matrix form for the convolution operator. The latter approach is rather reasonable if we take into account a tool that is given by Theorem 1 in accordance with which we can claim the existence an uniqueness of the quasi-solution of the Sonin–Abel equation just having a proper asymptotics of the Jacobi series coefficients. Generally, if we have an estimate $A_{mn}^{\vartheta,\sigma,\varsigma} \leq C\phi_n m^{-\theta},\ \phi_n \in \mathbb{R}$, where $\theta > 0$ is such that condition (i) of Theorem 1 is satisfied, then the conclusion part of the theorem holds for Equation (17) with the corresponding right-hand side f. Thus, it becomes possible to reformulate Corollary 1 in terms of the above estimate. Here we have faced with difficulties—it does not seem to be easy to obtain the estimate in the general case; however, the attempt was made in the paper [14]. Moreover, even in the case when the required estimate has been found then the following problem appears. It is not clear how we can obtain such an estimate where ϕ_n is a power type function. The latter assumption (under the corresponding assumptions regarding the power) gives us an opportunity to formulate conditions in terms of L_p classes by virtue of the Zygmund–Marcinkiewicz theorem. Below, we present the adopted version of the reasonings (see Theorem 2), made in the paper [14], that can be applicable to kernels close to the power type functions (19), (20). The subsequent reasonings jointly with Lemmas 1, 2, Theorem 1 can be treated as a technique creating prerequisite for more subtle proofs of existence and uniqueness theorems.

Theorem 2. *Assume that the right-hand side of Equation (17), where the kernel is the power type function (19), is such that the following condition holds*

$$\sum_{n=0}^{\infty} \phi_n |f_n(\sigma,\varsigma)| < \infty,$$

where $\phi_n = (n!)^2 4^{-n} n^{\beta+\gamma-3/2}$. *Then, there exists a unique solution of Equation (17). Moreover, the expansion on the series of the Jacobi polynomials holds for the solution, where convergence is understood in the sense of* $L_p(\beta,\gamma)$ *norm, where* $p<\infty$, *if* $0<\alpha\leq(\beta-\gamma)/2$ *and* $p<2(q+1)/(2\alpha-\beta+q)$, *if* $(\beta-\gamma)/2<\alpha<(\beta+1)/2$.

Proof. In accordance with Theorem 7.1 in [23] consider a kernel (20) such that $\varrho * \vartheta = 1$, and let us use the following denotation

$$\vartheta(x) := \frac{x^{-\alpha}}{\Gamma(1-\alpha)} \sum_{s=0}^{\infty} b_s x^s = \sum_{s=0}^{\infty} \vartheta_s(x). \tag{25}$$

Using Formula (5) consider the following relation for $s = 0,1,2,\ldots$

$$C\int_a^b p_m^{\sigma,\varsigma}(x) \int_a^x (x-t)^{s-\alpha} p_n^{\sigma,\varsigma}(t)\,dt\,\omega^{\sigma,\varsigma}(x)dx =$$

$$= \delta'_m(\sigma,\xi)\delta'_n(\tilde{\sigma},\varsigma) \sum_{k=0}^{\min\{c,n\}} \frac{(-1)^{n+k} C^2_{n,k}(\sigma,\varsigma)\Gamma^{-1}(\sigma+k+1)\Gamma(k+s-\alpha+\sigma+2)}{\Gamma(k+s-\alpha-m+2)\Gamma(k+s-\alpha+\sigma+\varsigma+m+3)} +$$

$$+\delta'_m(\sigma,\xi)\delta'_n(\tilde{\sigma},\varsigma) \sum_{k=c+1}^{n} \frac{(-1)^{n+k} C^2_{n,k}(\sigma,\varsigma)\Gamma^{-1}(\sigma+k+1)\Gamma(k+s-\alpha+\sigma+2)}{\Gamma(k+s-\alpha-m+2)\Gamma(k+s-\alpha+\sigma+\varsigma+m+3)} = I_1 + I_2,$$

where $c = m-s-1$. Consider I_1, note that under assumption $0\leq k\leq c$, we have

$$\frac{1}{\Gamma(k+s-\alpha-m+2)} = \frac{(k+s-\alpha-m+2)}{\Gamma(k+s-\alpha-m+3)} = \ldots = \frac{(-1)^{m+k+s}\prod_{i=1}^{m-k-s}(m-k-s-1+\alpha-i)}{\Gamma(2-\alpha)} = \tag{26}$$
$$= \frac{(-1)^{m+k+s}\Gamma(m-k-s-1+\alpha)}{\Gamma(\alpha-1)\Gamma(2-\alpha)}.$$

Using the facts obtained in Lemma 3 [14], we have $A_{mn}^{\vartheta_0,\sigma,\varsigma} \leq C\phi_n m^{2\alpha-\beta-5/2}$, $m,n\in\mathbb{N}$ (we should just substitute the parameters). Moreover, we claim that the scheme of the proof presented in the paper [14] remains true for ϑ_s, $s\in\mathbb{N}$, more precisely using Formula (26) and considering the order $s+1-\alpha$ of the integral, we can reformulate Lemma 1 [14] (the proof is the same) as follows

$$I_{mk} \leq Cm^{2\alpha-\beta-5/2-2s},\ I_{mk} \leq Ce^{2k} m^{2\xi+2\alpha-\beta-7/2-2s-2k},\ m>s,\ s\in\mathbb{N}_0,$$
$$k = 0,1,\ldots,m-s-1,$$

where

$$I_{mk} := \delta'_m(\sigma,\xi) \frac{\prod_{i=1}^{m-k-s}(m-k-s-1+\alpha-i)}{\Gamma(k+s+1-\alpha+\sigma+\varsigma+m+2)},$$

$\xi = 0.577215\ldots$ is the Mascheroni constant. Taking into account (26), we can write

$$I_1 = (-1)^{n+m+s}\delta'_n(\sigma,\varsigma) \sum_{k=0}^{\min\{c,n\}} \frac{\tilde{C}^k_n(\sigma,\varsigma)\Gamma(k+s-\alpha+\sigma+2) I_{mk}}{\Gamma(2-\alpha)}.$$

Using the estimate for $\tilde{C}_n^k(\sigma,\varsigma)$ obtained in Lemma 3 [14] and conducting its reasonings step by step, we get $|I_1| \leq C\phi_n m^{2\alpha-\beta-5/2-2s}$, $m,n \in \mathbb{N}$. The estimate $|I_2| \leq C\phi_n m^{\gamma-1/2}2^{-m-s}$, $m,n \in \mathbb{N}$ can be verified without any difficulties either. Combining these estimates, we obtain

$$A_{mn}^{\vartheta_s,\sigma,\varsigma} \leq 2^{-s}C\phi_n m^{2\alpha-\beta-5/2},\ m,n \in \mathbb{N},\ s \in \mathbb{N}_0. \tag{27}$$

Let us prove that

$$A_{mn}^{\vartheta,\sigma,\varsigma} = \sum_{s=0}^{\infty} A_{mn}^{\vartheta_s,\sigma,\varsigma}.$$

For this purpose it is sufficient to consider the following estimate and use the absolute convergence of series (25), we have

$$\left| A_{mn}^{\vartheta,\sigma,\varsigma} - \sum_{s=0}^{k} A_{mn}^{\vartheta_s,\sigma,\varsigma} \right| \leq C \int_a^b \omega^{\sigma,\varsigma}(x)dx \int_a^x \left|\tilde{S}_k\vartheta(x-t) - \vartheta(x-t)\right|dt \leq$$

$$\leq C\|\tilde{S}_k\vartheta - \vartheta\|_{C(\bar{I}_0)} \to 0,\ k \to \infty,$$

here, we put $\tilde{S}_k\vartheta := \sum_{s=0}^{k} \vartheta_s$. Therefore using (27), we get

$$|A_{mn}^{\vartheta,\sigma,\varsigma}| \leq \sum_{s=0}^{\infty} |A_{mn}^{\vartheta_s,\sigma,\varsigma}| \leq C\sum_{s=0}^{\infty} 2^{-s}\phi_n m^{2\alpha-\beta-5/2} \leq C\phi_n m^{2\alpha-\beta-5/2},\ m,n \in \mathbb{N}.$$

Let us show that

$$\sum_{n=0}^{k} f_n A_{mn}^{\vartheta,\sigma,\varsigma} \to f_m^{\vartheta}(\sigma,\varsigma),\ k \to \infty,\ m \in \mathbb{N}.$$

For this purpose we should consider a representation $\tilde{\vartheta} = \vartheta_0 + \vartheta_1,\ \hat{\vartheta} = \sum_{s=2}^{\infty} \vartheta_s$ and notice that conditions of Lemma 1 and Lemma 2 hold for $\tilde{\vartheta}, \hat{\vartheta}$ respectively. Therefore, we get

$$\sum_{n=0}^{k} f_n A_{mn}^{\tilde{\vartheta},\sigma,\varsigma} \to f_m^{\tilde{\vartheta}}(\sigma,\varsigma),\ \sum_{n=0}^{k} f_n A_{mn}^{\hat{\vartheta},\sigma,\varsigma} \to f_m^{\hat{\vartheta}}(\sigma,\varsigma),\ k \to \infty,\ m \in \mathbb{N}.$$

Having noticed the fact

$$\sum_{n=0}^{k} f_n A_{mn}^{\vartheta,\sigma,\varsigma} = \sum_{n=0}^{k} f_n A_{mn}^{\tilde{\vartheta},\sigma,\varsigma} + \sum_{n=0}^{k} f_n A_{mn}^{\hat{\vartheta},\sigma,\varsigma},\ f_m^{\vartheta}(\sigma,\varsigma) = f_m^{\tilde{\vartheta}}(\sigma,\varsigma) + f_m^{\hat{\vartheta}}(\sigma,\varsigma),\ m \in \mathbb{N},$$

we obtain the desired result. Using the above reasonings, we conclude

$$|f_m^{\vartheta}(\sigma,\varsigma)| = \left|\sum_{n=0}^{\infty} f_n A_{mn}^{\vartheta,\sigma,\varsigma}\right| \leq Cm^{2\alpha-\beta-5/2},\ m \in \mathbb{N}.$$

Consider the condition under which being imposed upon the parameters α, β it guaranties that condition (i) of Theorem 1 is fulfilled. Using the above it is clear that to prove the fact that the quasi-solution exists and lies in $L_p(\beta,\gamma)$ we need to establish the estimate $(5/2+q)(p-2)+2+p(2\alpha-\beta-5/2) < -1$. Note that the last relation holds, if $2\alpha-\beta+q \leq 0$. In the contrary case, we have $p < 2(1+q)/(2\alpha-\beta+q)$. Taking into account that p must be more or equals two, we come to the conclusion that the necessary condition for existence such an index p is $(1+q)/(2\alpha-\beta+q) > 1$. The final implication of the proof follows immediately from the latter relation. The proof is complete. □

It is remarkable that analogous approach can be implemented in the general case if we impose additional conditions on the kernel. In this case, by virtue of the peculiarity of the construction we can weaken conditions imposed on the right-hand side by means of strengthening conditions imposed on the kernel. Below, we present a theorem which reflects this idea.

Theorem 3. *Assume that the kernel in the left-hand side and the function in the right-hand side of Equation (17) are such that*

$$|\vartheta_n(\nu,\nu)| \le C(n!)^{-2}n^{-3/2-\varepsilon},\ \varepsilon > 0,\ \ |f_n(\sigma,\varsigma)| \le Cn^{-\upsilon},\ \upsilon > 1/2+\beta,\ n \in \mathbb{N}.$$

Then there exists a unique solution of Equation (17). Moreover the expansion on the series of the Jacobi polynomials holds for the solution, where convergence is understood in the sense of $L_p(\beta,\gamma)$ norm, where $p < \infty$.

Proof. Consider the estimate that can be obtained by direct calculations

$$C_{n,k}(\nu,\nu) = \sqrt{\frac{(n+k-1)!i!}{(n-1)!(n-k)!}} \le C\sqrt{\frac{(2n-1)!n}{4^{n-k}}} \le Cn!n^{-1/4}2^k,\ k \le n,\ n \in \mathbb{N}.$$

Using this estimate, we get

$$|\Psi^i_{m-j-1}(j,m)| \le C(i!)^2 i^{-1/2} \sum_{l=m-j-1}^{i} \frac{4^l\Gamma(l+j+\sigma+2)}{\Gamma(l+1/2)(l+j+1-m)!\Gamma(l+j+\sigma+m+\varsigma+3)}.$$

By virtue of the monotonous property of the expression, for values $m \ge j+2$, $i > 0$, we have

$$|\Psi^i_{m-j-1}(j,m)| \le \frac{C(i!)^2 i^{1/2}4^{m-j-1}\Gamma(m+\sigma+1)}{\Gamma(m-j-1/2)\Gamma(2m+\sigma+\varsigma+2)} \le C\frac{(i!)^2 i^{1/2}m^{1/2-\gamma}}{4^m m!}.$$

In the case $m < j+2$, we obtain analogously

$$|\Psi^i_{m-j-1}(j,m)| \le \frac{C(i!)^2 i^{1/2}\Gamma(j+\sigma+2)}{\Gamma(1/2)(j+1-m)!\Gamma(j+\sigma+m+\varsigma+3)} \le \frac{C(i!)^2 i^{1/2}\Gamma(\sigma+2)}{\Gamma(1/2)\Gamma(\sigma+m+\varsigma+3)}.$$

Combining these estimates with Formula (23), we get $A^{\vartheta,\sigma,\varsigma}_{mn} \le Cn^{\beta-1/2}/m^{\beta+\gamma}m!$, m, $n \in \mathbb{N}$. In accordance with Lemma 2, we have

$$|f^{\vartheta}_m(\sigma,\varsigma)| = \left|\sum_{n=0}^{\infty} f_n A^{\vartheta,\sigma,\varsigma}_{mn}\right| \le \sum_{n=0}^{\infty} |f_n A^{\vartheta,\sigma,\varsigma}_{mn}| \le \frac{C}{m^{\beta+\gamma}m!} \le Cm^{-\zeta},\ \zeta < \infty,\ m \in \mathbb{N}.$$

Applying Corollary 1 we complete the proof. □

3.5. Prospective Results

Note that the reasonings of the previous subsection indicate that a crucial point of existence and uniqueness questions is a problem how to obtain a relevant estimate for $A^{\vartheta,\sigma,\varsigma}_{mn}$. Below we present an approach under most general assumptions regarding the kernel and the right-hand side. Assume that ϑ satisfies conditions of Lemma 2, where $p = 2$, then we can easily establish the fact

$$\sum_{i=m-j-1}^{\infty} |\vartheta_i(\nu,\nu)\Psi^i_{m-j-1}(j,m)| < \infty. \tag{28}$$

For this purpose, note that

$$\sum_{i=m-j-1}^{\infty} |\vartheta_i(\nu,\nu)\Psi^i_{m-j-1}(j,m)| = \sum_{i=m-j-1}^{\infty} (-1)^i \tilde{\vartheta}_i(\nu,\nu)\Psi^i_{m-j-1}(j,m) < \infty,$$

where $|\tilde{\vartheta}_i(\nu,\nu)| = |\vartheta_i(\nu,\nu)|$, sign $\tilde{\vartheta}_i(\nu,\nu) = (-1)^i \text{sign}\, \Psi^i_{m-j-1}(j,m)$. It is clear that in consequence of the Riesz–Fischer theorem there exists a function $\tilde{\vartheta} \in L_2(I_0,\nu,\nu)$, this fact provides convergence of series (28). Thus, we can claim that if $\vartheta \in L_2(I_0,\nu,\nu)$, then series (23) is absolutely convergent. This fact can be treated as a prerequisite for the following hypothesis.

$$\exists \lambda \in (1/2,\infty) : \sum_{i=m-j-1}^{\infty} |\Psi^i_{m-j-1}(j,m)| i^{-\lambda} \leq C \cdot \Gamma(j+\sigma) j^{-\varepsilon} m^{-\theta},\ \varepsilon > 0,\ \theta > \gamma + 1.$$

Suppose the hypothesis is true and assume that the kernel in the left hand-side and the function in the right-hand side of equation (17) are such that $|\vartheta_n(\nu,\nu)| \leq Cn^{-\lambda}$, $|f_n(\sigma,\varsigma)| \leq Cn^{-\upsilon}$, $\upsilon > 1/2 + \beta$, $n \in \mathbb{N}$. Taking into account the made assumptions we can easily prove, for this purpose we should use the reasonings of the previous subsection, that conditions of Corollary 1 are satisfied.

4. Conclusions

In this paper we offer a method of studying the class of the convolution operators named the Sonin operators, we assume that the Sonin conditions hold regarding the kernel. The most well-known particular case of the Sonin kernel is a kernel of the fractional integral Riemann–Liouville operator as well as others presented in the section "Kernels close to power-type functions" the Bessel-type function, the power-exponential function, the incomplete gamma function e.t.c. In our study, we pay a special attention to kernels presented as a multiplication of the power function and analytic function, this case covers lots of known Sonin kernels useful in various mathematical and physical applications. The crucial point of the research is the study of the Sonin–Abel equation in the weighted Lebesgue space, the used method allows us to formulate the criterion of existence and uniqueness of the solution and classify the solution in accordance with belonging to a weighted L_p space due to the asymptotic of the Jacobi series coefficients of the right-hand side. Note that an opportunity to consider the whole problem in the matrix form is worth noticing itself for it leads to an abstract problem, we can consider a wider class of operators if we use the matrix form and a valuable fact is that the criterion of the solvability of the Sonin–Abel equation is naturally formulated in the very matrix form.

Below, we present questions which may be considered as continuation of this paper results. First of all, the lemmas of Section 3.3 are formulated with the minimal assumptions regarding the kernels, it has a rather significant disadvantage for we should compensate such lack of restrictions by necessity to consider the right-hand side from the class $L_p(I,\sigma,\varsigma)$, where the powers of the weighted function are negative. However, we may consider the matter from quite another point of view, i.e., we may impose conditions on index p corresponding to the kernel class, and in this way to loosen conditions regarding powers of the weighted function. Having taken this concept we can obtain results, similar to the results of Section 3.3 corresponding to the positive powers (see in the context). The second question is how to adapt the obtained results in the way to study an abstract class of Sonin kernels considered in [16] (the concrete representatives of the class are the functions with power-logarithmic singularities at the origin). The third, and most relevant question, is devoted to constructing the theory of convolution operators in the matrix form. The technique used in Section 3.4 allows us to define and consider a convolution operator in terms of its coefficients in a basis formed by Jacobi polynomials. In this regard the most interesting issue is to consider kernels represented by alternating series, we should stress that not all of them can be represented by functions, at the same time the corresponding convolution operator may be defined in the matrix form, it undoubtedly creates relevance of such an approach.

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