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Article

On the Nonlinear Integro-Differential Equations

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Abstract: The goal of this paper is to study the uniqueness of solutions of several nonlinear Liouville–Caputo integro-differential equations with variable coefficients and initial conditions, as well as an associated coupled system in Banach spaces. The results derived are new and based on Banach's contractive principle, the multivariate Mittag–Leffler function and Babenko's approach. We also provide a few examples to demonstrate the use of our main theorems by convolutions and the gamma function.

Keywords: Riemann–Liouville fractional integral; Liouville–Caputo derivative; Babenko's approach; Banach fixed point theorem; multivariate Mittag–Leffler function; gamma function

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1. Introduction

Let $T > 0$. The space $C[0, T]$ of continuous functions on $[0, T]$ is given by

$$C[0,T] = \left\{ u(x) : \ \|u\|_C = \max_{x\in[0,T]} |u(x)| < \infty \right\}.$$

Clearly, $C[0, T]$ is a Banach space.

We further define the space $C^n[0, T]$, for $n \in N = \{1, 2, \cdots\}$, of those functions on $[0, T]$ with up to nth order continuous derivatives by

$$C^n[0,T] = \left\{ u(x) : \ \frac{d^n}{dx^n} u(x) \in C[0,T] \ \text{ and } \ \|u\|_n < \infty \right\},$$

where

$$\|u\|_n = \max\left\{ \|u(x)\|_C, \left\| \frac{d}{dx} u(x) \right\|_C, \cdots, \left\| \frac{d^n}{dx^n} u(x) \right\|_C \right\}.$$

Obviously,

$$C^n[0,T] \subset C[0,T]$$

for all $n = 1, 2, \cdots$. Furthermore, $C^n[0, T]$ is a Banach space using Theorem 7.17 in [1] stated as follows:

Theorem 1. *If $\{u_n\}$ is a sequence of differentiable functions on $[a, b]$ such that $\lim_{n\to\infty} u_n(x_0)$ exists (and is finite) for some $x_0 \in [a, b]$, and the sequence $\{u'_n\}$ converges uniformly on $[a, b]$, then u_n converges uniformly to a function u on $[a, b]$, and $u'(x) = \lim_{n\to\infty} u'_n(x)$ for $x \in [a, b]$.*

In addition, the product space $C^n[0, T] \times C^m[0, T]$ $(m \in N)$ is given by

$$C^n[0,T] \times C^m[0,T] = \{\, (u,v) \mid u \in C^n[0,T], \ v \in C^m[0,T] \text{ and } \|(u,v)\| < \infty \},$$

where

$$\|(u,v)\| = \|u\|_n + \|v\|_m.$$

Clearly, $C^n[0,T] \times C^m[0,T]$ is also a Banach space. The fractional integral (or the Riemann–Liouville) I^α of fractional order $\alpha \in R^+$ of function $u(x)$ [2,3] is defined by

$$(I^\alpha u)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} u(t) dt.$$

In particular,

$$(I^0 u)(x) = u(x),$$

from [2]. Indeed,

$$(I^0 u)(x) = \delta(x) * u(x) = u(x),$$

where $\delta(x)$ is the Dirac delta function (or distribution) given by

$$(\delta(x), \phi(x)) = \int_{-\infty}^{\infty} \delta(x)\phi(x)dx = \phi(0),$$

for any sufficiently smooth function $\phi(x)$ with a compact support.

Let $u(x)$ and $a(x)$ be in $C[0,T]$. Then,

$$\|I^\alpha u\|_C = \max_{x\in[0,T]} \left| \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} u(t) dt \right| \leq \frac{T^\alpha}{\Gamma(\alpha+1)} \|u\|_C,$$
$$\|aI^\alpha u\|_C = \max_{x\in[0,T]} \left| a(x) \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} u(t) dt \right| \leq \frac{T^\alpha}{\Gamma(\alpha+1)} \|a\|_C \|u\|_C.$$

The Liouville–Caputo derivative of fractional order $\alpha \in R^+$ of function $u(x)$ is defined as

$$({}_C D^\alpha u)(x) = I^{n-\alpha} \frac{d^n}{dx^n} u(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-t)^{n-\alpha-1} u^{(n)}(t) dt,$$

where $n-1 < \alpha \leq n \in N$.

Assume that

$$u(0) = u'(0) = \cdots = u^{(n-1)}(0) = 0,$$

then integration by parts infers that

$$\begin{aligned} I^n \frac{d^n}{dx^n} u(x) &= \frac{1}{(n-1)!} \int_0^x (x-t)^{n-1} \, u^{(n)}(t) dt \\ &= \frac{1}{(n-1)!} (x-t)^{n-1} \, u^{(n-1)}(t) \Big|_0^x + \frac{1}{(n-2)!} \int_0^x (x-t)^{n-2} \, u^{(n-1)}(t) dt \\ &= \cdots = u(x). \end{aligned}$$

Assume that $a_i(x)$ and $b_j(x)$ are in $C[0,T]$ for all $i = 1,2,\cdots,n_1 \in N$ and $j = 1,2,\cdots,n_2 \in N$. In this paper, we begin to establish a unique and global solution in the space $C^n[0,T]$ using Babenko's method and the multivariate Mittag–Leffler function for the following integro-differential equation for $f \in C[0,T]$,

$$\begin{cases} u^{(n)}(x) + a_{n_1}(x) \, {}_C D^{\beta_{n_1}} u(x) + \cdots + a_1(x) \, {}_C D^{\beta_1} u(x) + \\ b_1(x) I^{\alpha_1} u(x) + \cdots + b_{n_2}(x) I^{\alpha_{n_2}} u(x) = f(x), \\ u(0) = u'(0) = \cdots = u^{(n-1)}(0) = 0, \end{cases} \tag{1}$$

where $0 \leq \beta_1 < \beta_2 < \cdots < \beta_{n_1} < n$ and $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_{n_2}$. Then, we further study the uniqueness of solutions in $C^n[0,T]$ for

$$\begin{cases} u^{(n)}(x) + a_{n_1}(x)\, {}_C D^{\beta_{n_1}} u(x) + \cdots + a_1(x)\, {}_C D^{\beta_1} u(x) + \\ b_1(x) I^{\alpha_1} u(x) + \cdots + b_{n_2}(x) I^{\alpha_{n_2}} u(x) = g\Big(x, u(x), u'(x), \cdots, u^{(n-1)}(x)\Big), \\ u(0) = u'(0) = \cdots = u^{(n-1)}(0) = 0, \end{cases} \tag{2}$$

where $g(x, y_1, \cdots, y_n)$ is a mapping from $[0,T] \times R^n$ to R and satisfies certain conditions. Finally, the sufficient conditions are provided for the uniqueness of solutions in $C^n[0,T] \times C^m[0,T]$ to the coupled system

$$\begin{cases} u^{(n)}(x) + a_{n_1}(x)\, {}_C D^{\beta_{n_1}} u(x) + \cdots + a_1(x)\, {}_C D^{\beta_1} u(x) + b_1(x) I^{\alpha_1} u(x) + \\ \cdots + b_{n_2}(x) I^{\alpha_{n_2}} u(x) = g_1\Big(x, u(x), \cdots, u^{(n-1)}(x), v(x), \cdots, v^{(m-1)}(x)\Big), \\ v^{(m)}(x) + c_{m_1}(x)\, {}_C D^{\mu_{m_1}} v(x) + \cdots + c_1(x)\, {}_C D^{\mu_1} v(x) + d_1(x) I^{\gamma_1} v(x) + \\ \cdots + d_{m_2}(x) I^{\gamma_{m_2}} v(x) = g_2\Big(x, u(x), \cdots, u^{(n-1)}(x), v(x), \cdots, v^{(m-1)}(x)\Big), \\ u(0) = u'(0) = \cdots = u^{(n-1)}(0) = 0, \\ v(0) = u'(0) = \cdots = v^{(m-1)}(0) = 0, \end{cases} \tag{3}$$

where both $g_1(x, y_1, \cdots, y_{n+m})$ and $g_2(x, y_1, \cdots, y_{n+m})$ are mappings from $[0,T] \times R^{n+m}$ to R, and all coefficient functions $c_i(x)$ and $d_j(x)$ are in $C[0,T]$. As far as we know, Equations (1)–(3) are new and have never been investigated before.

There have been intensive studies on the existence and uniqueness of different types of integral and fractional differential equations based on fixed point theory [4–9]. Marasi et al. [6] studied the existence and multiplicity of positive solutions for the following initial value problem using the fixed point index theory:

$$({}_C D^{\alpha})u(x) = f\Big(x, u(x), ({}_C D^{\beta})u(x)\Big), \quad x \in (0,1],$$
$$u^{(k)}(0) = \eta_k, \quad k = 0, 1, \cdots, n-1,$$

where $n-1 < \beta < \alpha < n \in N$. J. Deng and Z. Deng [10] considered the existence of the following initial value problems based on the Schauder fixed point theorem:

$$({}_C D^{\alpha})u(x) = f\Big(x, ({}_C D^{\beta})u(x)\Big), \quad x \in (0,1],$$
$$u^{(k)}(0) = \eta_k, \quad k = 0, 1, \cdots, m-1,$$

where $m-1 < \alpha < m \in N$, $n-1 < \beta < n \in N$ $(m-1 \geq n)$ and $f \in C([0,1] \times R)$ satisfies certain conditions.

The Hadamard derivative of fractional order α of a function $u : [1, \infty) \to R$ is defined as

$$D^{\alpha} u(x) = \frac{1}{\Gamma(n-\alpha)} \left(x \frac{d}{dx}\right)^n \int_1^x \left(\log \frac{x}{s}\right)^{n-\alpha-1} \frac{u(s)}{s} ds, \quad n-1 < \alpha < n, \quad n = [\alpha] + 1,$$

where $[\alpha]$ denotes the integer part of the real number α and $\log(\cdot) = \log_e(\cdot)$.

Ahmad and Ntouyas [11] studied the existence and uniqueness of solutions to the following initial value problem of Hadamard sequential fractional order neutral functional differential equations using fixed point theory

$$D^{\alpha}[D^{\beta} u(x) - g(x, u_x)] = f(x, u_x), \quad x \in J = [1, b],$$
$$u(x) = \phi(x), \quad x \in [1-r, 1],$$
$$D^{\beta} u(1) = \eta \in R,$$

where $0 < \alpha, \beta < 1$, $f, g : J \times C([-r, 0], R) \to R$ are given functions and $\phi \in C([1-r, 1], R)$. For any function u defined on $[1-r, b]$ and $x \in [1, b]$, we denote u_x as the element of $C([-r, 0], R)$ defined by

$$u_x(\theta) = u(x+\theta), \quad \theta \in [-r, 0].$$

Let $AC[a, b]$ denote the set of all absolutely continuous functions on $[a, b]$. The Banach space $AC_0[a, b]$ is defined as

$$AC_0[a,b] = \left\{ u : u(x) \in AC[a,b] \text{ with } u(a) = 0 \text{ and } \|u\|_0 = \int_a^b |u'(x)|dx < \infty \right\}.$$

The fractional version of the Hadamard-type integral and derivative is given by

$$(\mathcal{J}^{\alpha}_{a+,\,\mu}\, u)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \left(\frac{t}{x}\right)^{\mu} \left(\log \frac{x}{t}\right)^{\alpha-1} u(t) \frac{dt}{t} \quad \alpha > 0, \ \ 0 < a < x < b,$$

and

$$(\mathcal{D}^{\alpha}_{a+,\,\mu}\, u)(x) = x^{-\mu} \left(x \frac{d}{dx}\right)^{[\alpha]+1} x^{\mu} (\mathcal{J}^{n-\alpha}_{a+,\,\mu}\, u)(x).$$

Let $0 < \alpha_0 < \alpha_1 < \cdots < \alpha_n < 1$ and $0 \le \beta_{n+1} < \cdots < \beta_m \in R$ where $n = 0, 1, \cdots$ and $m > n$. Very recently, Li [4] considered the uniqueness of solutions for the following nonlinear Hadamard-type integro-differential equation with constant coefficients for all $\mu \in R$, in the space $AC_0[a, b]$:

$$\begin{aligned} &\mathcal{D}^{\alpha_n}_{a+,\,\mu} u + a_{n-1} \mathcal{D}^{\alpha_{n-1}}_{a+,\,\mu} u + \cdots + a_0 \mathcal{D}^{\alpha_0}_{a+,\,\mu} u + b_{n+1} \mathcal{J}^{\beta_{n+1}}_{a+,\,\mu} u + \cdots + b_m \mathcal{J}^{\beta_m}_{a+,\,\mu} u \\ &= \int_a^x f(\tau, u'(\tau)) d\tau, \end{aligned}$$

according to Banach's contractive principle and Babenko's approach [12]. Babenko's approach is a very useful method in solving differential and integral equations by treating integral operators as variables and derives convergent infinite series as solutions in spaces under consideration. Li also investigated Abel's integral equations of the first [13] and second kind with variable coefficients in distribution by Babenko's technique [14,15].

The following multivariate Mittag–Leffler function was introduced by Hadid and Luchko [16,17] for solving linear fractional differential equations with constant coefficients by the operational calculus:

$$E_{(\alpha_1, \cdots, \alpha_m), \beta}(z_1, \cdots, z_m) = \sum_{k=0}^{\infty} \sum_{k_1 + \cdots + k_m = k} \binom{k}{k_1, \cdots, k_m} \frac{z_1^{k_1} \cdots z_m^{k_m}}{\Gamma(\alpha_1 k_1 + \cdots + \alpha_m k_m + \beta)},$$

where $\alpha_i, \beta > 0$ for $i = 1, 2, \cdots, m$.

2. The Main Results

In this section, we begin to construct a unique and global solution to Equation (1) in space $C^n[0, T]$ using Babenko's method and the multivariate Mittag–Leffler function.

Theorem 2. *Assume that $a_i(x)$ and $b_j(x)$ are continuous functions on $[0, T]$ for all $i = 1, 2, \cdots, n_1 \in N$ and $j = 1, 2, \cdots, n_2 \in N$, and $0 \le \beta_1 < \beta_2 < \cdots < \beta_{n_1} < n$ and $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_{n_2}$. Then, Equation (1) has a unique, convergent, and global solution in $C^n[0, T]$ for $f \in C[0, T]$.*

$$\begin{aligned} u(x) &= I^n \sum_{k=0}^{\infty} (-1)^k \sum_{k_1 + k_2 + \cdots + k_{n_1+n_2} = k} \binom{k}{k_1, k_2, \cdots, k_{n_1+n_2}} (a_{n_1}(x)\, I^{n-\beta_{n_1}})^{k_1} \cdots \\ &\quad (a_1(x)\, I^{n-\beta_1})^{k_{n_1}} (b_1(x) I^{\alpha_1+n})^{k_{n_1+1}} \cdots (b_{n_2}(x) I^{\alpha_{n_2}+n})^{k_{n_1+n_2}} f(x). \end{aligned}$$

Proof. Equation (1) turns out to be

$$u(x) + I^n a_{n_1}(x)\ {}_C D^{\beta_{n_1}} u(x) + \cdots + I^n a_1(x)\ {}_C D^{\beta_1} u(x) +$$
$$I^n b_1(x) I^{\alpha_1} u(x) + \cdots + I^n b_{n_2}(x) I^{\alpha_{n_2}} u(x) = I^n f(x),$$

by applying the operator I^n to both sides of Equation (1) and using initial conditions

$$u(0) = u'(0) = \cdots = u^{(n-1)}(0) = 0.$$

Hence,

$$(1 + I^n a_{n_1}(x)\ {}_C D^{\beta_{n_1}} + \cdots + I^n a_1(x)\ {}_C D^{\beta_1} +$$
$$I^n b_1(x) I^{\alpha_1} + \cdots + I^n b_{n_2}(x) I^{\alpha_{n_2}}) u(x) = I^n f(x).$$

According to Babenko's approach,

$$\begin{aligned}
&u(x) = \\
&\Big(1 + I^n a_{n_1}(x)\ {}_C D^{\beta_{n_1}} + \cdots + I^n a_1(x)\ {}_C D^{\beta_1} + I^n b_1(x) I^{\alpha_1} + \cdots + I^n b_{n_2}(x) I^{\alpha_{n_2}}\Big)^{-1} \cdot \\
&I^n f(x) \\
&= \sum_{k=0}^{\infty} (-1)^k \Big(I^n a_{n_1}(x)\ {}_C D^{\beta_{n_1}} + \cdots + I^n a_1(x)\ {}_C D^{\beta_1} + \cdots + I^n b_{n_2}(x) I^{\alpha_{n_2}}\Big)^k I^n f(x) \\
&= \sum_{k=0}^{\infty} (-1)^k \sum_{k_1 + k_2 + \cdots + k_{n_1+n_2} = k} \binom{k}{k_1, k_2, \cdots, k_{n_1+n_2}} (I^n a_{n_1}(x)\ {}_C D^{\beta_{n_1}})^{k_1} \cdots \\
&\quad (I^n a_1(x)\ {}_C D^{\beta_1})^{k_{n_1}} (I^n b_1(x) I^{\alpha_1})^{k_{n_1+1}} \cdots (I^n b_{n_2}(x) I^{\alpha_{n_2}})^{k_{n_1+n_2}} I^n f(x).
\end{aligned}$$

Clearly,

$$\begin{aligned}
&(I^n b_{n_2}(x) I^{\alpha_{n_2}})^{k_{n_1+n_2}} I^n f(x) = (I^n b_{n_2}(x) I^{\alpha_{n_2}}) \cdots (I^n b_{n_2}(x) I^{\alpha_{n_2}}) I^n f(x) \\
&= I^n \big(b_{n_2}(x) I^{\alpha_{n_2}+n}\big)^{k_{n_1+n_2}} f(x).
\end{aligned}$$

Therefore,

$$\begin{aligned}
u(x) &= \sum_{k=0}^{\infty} (-1)^k \sum_{k_1 + k_2 + \cdots + k_{n_1+n_2} = k} \binom{k}{k_1, k_2, \cdots, k_{n_1+n_2}} I^n\ (a_{n_1}(x)\ I^{n-\beta_{n_1}})^{k_1} \cdots \\
&\quad (a_1(x)\ I^{n-\beta_1})^{k_{n_1}} \big(b_1(x) I^{\alpha_1+n}\big)^{k_{n_1+1}} \cdots \big(b_{n_2}(x) I^{\alpha_{n_2}+n}\big)^{k_{n_1+n_2}} f(x).
\end{aligned} \quad (4)$$

Since $a_i(x)$ and $b_j(x)$ are continuous functions on $[0, T]$ for all $i = 1, 2, \cdots, n_1 \in N$ and $j = 1, 2, \cdots, n_2 \in N$, there exist $A_i > 0$ and $B_j > 0$ such that

$$\max_{x \in [0,T]} |a_i(x)| \le A_i, \quad \max_{x \in [0,T]} |b_j(x)| \le B_j.$$

Thus,

$$\begin{aligned}
\|u(x)\|_C &\leq \sum_{k=0}^{\infty} \sum_{k_1+k_2+\cdots+k_{n_1+n_2}=k} \binom{k}{k_1, k_2, \cdots, k_{n_1+n_2}} A_{n_1}^{k_1} \cdots A_1^{k_{n_1}} B_1^{k_{n_1+1}} \cdots B_{n_2}^{k_{n_1+n_2}} \cdot \\
&\quad \left\| I^{n+k_1(n-\beta_{n_1})+\cdots+k_{n_1}(n-\beta_1)+k_{n_1+1}(n+\alpha_1)+\cdots+k_{n_1+n_2}(n+\alpha n_2)} \right\|_C \|f\|_C \\
&\leq \sum_{k=0}^{\infty} \sum_{k_1+k_2+\cdots+k_{n_1+n_2}=k} \binom{k}{k_1, k_2, \cdots, k_{n_1+n_2}} A_{n_1}^{k_1} \cdots A_1^{k_{n_1}} B_1^{k_{n_1+1}} \cdots B_{n_2}^{k_{n_1+n_2}} \cdot \\
&\quad \frac{T^{n+k_1(n-\beta_{n_1})+\cdots+k_{n_1}(n-\beta_1)+\cdots+k_{n_1+n_2}(n+\alpha_{n_2})}}{\Gamma(n+1+k_1(n-\beta_{n_1})+\cdots+k_{n_1}(n-\beta_1)+\cdots+k_{n_1+n_2}(n+\alpha_{n_2}))} \|f\|_C \\
&= T^n E_{(n-\beta_{n_1},\cdots,n+\alpha_{n_2}),\, n+1}\left(T^{n-\beta_{n_1}} A_{n_1}, \cdots, T^{n+\alpha_{n_2}} B_{n_2}\right) \|f\|_C < \infty.
\end{aligned}$$

This implies that the series on the right-hand side of Equation (4) is uniformly convergent on $[0, T]$ with respect to x and

$$\begin{aligned}
u(x) &= I^n \sum_{k=0}^{\infty} (-1)^k \sum_{k_1+k_2+\cdots+k_{n_1+n_2}=k} \binom{k}{k_1, k_2, \cdots, k_{n_1+n_2}} (a_{n_1}(x)\ I^{n-\beta_{n_1}})^{k_1} \cdots \\
&\quad (a_1(x)\ I^{n-\beta_1})^{k_{n_1}} \left(b_1(x) I^{\alpha_1+n}\right)^{k_{n_1+1}} \cdots \left(b_{n_2}(x) I^{\alpha_{n_2}+n}\right)^{k_{n_1+n_2}} f(x),
\end{aligned}$$

which is well defined over $[0, T]$. Clearly,

$$u(0) = u'(0) = \cdots = u^{(n-1)}(0) = 0,$$

due to the integral operator I^n.

It remains to show that $u(x) \in C^n[0, T]$ and is a unique solution. Obviously,

$$\begin{aligned}
u'(x) &= I^{n-1} \sum_{k=0}^{\infty} (-1)^k \sum_{k_1+k_2+\cdots+k_{n_1+n_2}=k} \binom{k}{k_1, k_2, \cdots, k_{n_1+n_2}} (a_{n_1}(x)\ I^{n-\beta_{n_1}})^{k_1} \cdots \\
&\quad (a_1(x)\ I^{n-\beta_1})^{k_{n_1}} \left(b_1(x) I^{\alpha_1+n}\right)^{k_{n_1+1}} \cdots \left(b_{n_2}(x) I^{\alpha_{n_2}+n}\right)^{k_{n_1+n_2}} f(x),
\end{aligned}$$

and

$$\left\|u'(x)\right\|_C \leq T^{n-1} E_{(n-\beta_{n_1},\cdots,n+\alpha_{n_2}),\, n}\left(T^{n-\beta_{n_1}} A_{n_1}, \cdots, T^{n+\alpha_{n_2}} B_{n_2}\right) \|f\|_C < \infty.$$

Similarly,

$$\left\|u^{(n)}(x)\right\|_C \leq E_{(n-\beta_{n_1},\cdots,n+\alpha_{n_2}),\, 1}\left(T^{n-\beta_{n_1}} A_{n_1}, \cdots, T^{n+\alpha_{n_2}} B_{n_2}\right) \|f\|_C < \infty.$$

So $u(x) \in C^n[0, T]$. To see $u(x)$ is a solution to Equation (1), we have

$$\begin{aligned}
u^{(n)}(x) &= \sum_{k=0}^{\infty} (-1)^k \sum_{k_1+k_2+\cdots+k_{n_1+n_2}=k} \binom{k}{k_1, k_2, \cdots, k_{n_1+n_2}} (a_{n_1}(x)\ I^{n-\beta_{n_1}})^{k_1} \cdots \\
&\quad (a_1(x)\ I^{n-\beta_1})^{k_{n_1}} \left(b_1(x) I^{\alpha_1+n}\right)^{k_{n_1+1}} \cdots \left(b_{n_2}(x) I^{\alpha_{n_2}+n}\right)^{k_{n_1+n_2}} f(x) \\
&= f(x) + \sum_{k=1}^{\infty} (-1)^k \sum_{k_1+\cdots+k_{n_1+n_2}=k} \binom{k}{k_1, \cdots, k_{n_1+n_2}} (a_{n_1}(x)\ I^{n-\beta_{n_1}})^{k_1} \cdots \\
&\quad (a_1(x)\ I^{n-\beta_1})^{k_{n_1}} \left(b_1(x) I^{\alpha_1+n}\right)^{k_{n_1+1}} \cdots \left(b_{n_2}(x) I^{\alpha_{n_2}+n}\right)^{k_{n_1+n_2}} f(x) \\
&\quad = f(x) + X,
\end{aligned}$$

where

$$X = \sum_{k=1}^{\infty}(-1)^k \sum_{k_1+\cdots+k_{n_1+n_2}=k} \binom{k}{k_1,\cdots,k_{n_1+n_2}} \left(a_{n_1}(x)\ I^{n-\beta_{n_1}}\right)^{k_1}\cdots \left(a_1(x)\ I^{n-\beta_1}\right)^{k_{n_1}} \left(b_1(x)I^{\alpha_1+n}\right)^{k_{n_1+1}}\cdots\left(b_{n_2}(x)I^{\alpha_{n_2}+n}\right)^{k_{n_1+n_2}} f(x).$$

On the other hand,

$$\begin{aligned} Y &= a_{n_1}(x)\ {}_C D^{\beta_{n_1}} u(x) + \cdots + a_1(x)\ {}_C D^{\beta_1} u(x) + b_1(x) I^{\alpha_1} u(x) + \cdots + b_{n_2}(x) I^{\alpha_{n_2}} u(x) \\ &= \sum_{k=0}^{\infty}(-1)^k \sum_{k_1+k_2+\cdots+k_{n_1+n_2}=k} \binom{k}{k_1,k_2,\cdots,k_{n_1+n_2}} \left(a_{n_1}(x)\ I^{n-\beta_{n_1}}\right)^{k_1+1}\cdots \\ &\quad \left(a_1(x)\ I^{n-\beta_1}\right)^{k_{n_1}} \left(b_1(x)I^{\alpha_1+n}\right)^{k_{n_1+1}}\cdots\left(b_{n_2}(x)I^{\alpha_{n_2}+n}\right)^{k_{n_1+n_2}} f(x) + \\ &\quad \sum_{k=0}^{\infty}(-1)^k \sum_{k_1+k_2+\cdots+k_{n_1+n_2}=k} \binom{k}{k_1,k_2,\cdots,k_{n_1+n_2}} \left(a_{n_1}(x)\ I^{n-\beta_{n_1}}\right)^{k_1}\cdots \\ &\quad \left(a_1(x)\ I^{n-\beta_1}\right)^{k_{n_1}+1} \left(b_1(x)I^{\alpha_1+n}\right)^{k_{n_1+1}}\cdots\left(b_{n_2}(x)I^{\alpha_{n_2}+n}\right)^{k_{n_1+n_2}} f(x) + \\ &\quad \sum_{k=0}^{\infty}(-1)^k \sum_{k_1+k_2+\cdots+k_{n_1+n_2}=k} \binom{k}{k_1,k_2,\cdots,k_{n_1+n_2}} \left(a_{n_1}(x)\ I^{n-\beta_{n_1}}\right)^{k_1}\cdots \\ &\quad \left(a_1(x)\ I^{n-\beta_1}\right)^{k_{n_1}} \left(b_1(x)I^{\alpha_1+n}\right)^{k_{n_1+1}+1}\cdots\left(b_{n_2}(x)I^{\alpha_{n_2}+n}\right)^{k_{n_1+n_2}} f(x) + \\ &\quad \sum_{k=0}^{\infty}(-1)^k \sum_{k_1+k_2+\cdots+k_{n_1+n_2}=k} \binom{k}{k_1,k_2,\cdots,k_{n_1+n_2}} \left(a_{n_1}(x)\ I^{n-\beta_{n_1}}\right)^{k_1}\cdots \\ &\quad \left(a_1(x)\ I^{n-\beta_1}\right)^{k_{n_1}} \left(b_1(x)I^{\alpha_1+n}\right)^{k_{n_1+1}}\cdots\left(b_{n_2}(x)I^{\alpha_{n_2}+n}\right)^{k_{n_1+n_2}+1} f(x). \end{aligned}$$

This implies that

$$X + Y = 0,$$

due to the sign change and the fact that all series above are absolutely convergent.

To illustrate more in detail, we can easily deduce that

$$\begin{aligned} &- \sum_{k_1+\cdots+k_{n_1+n_2}=1} \binom{1}{k_1,\cdots,k_{n_1+n_2}} \left(a_{n_1}(x)\ I^{n-\beta_{n_1}}\right)^{k_1}\cdots \\ &\left(a_1(x)\ I^{n-\beta_1}\right)^{k_{n_1}} \left(b_1(x)I^{\alpha_1+n}\right)^{k_{n_1+1}}\cdots\left(b_{n_2}(x)I^{\alpha_{n_2}+n}\right)^{k_{n_1+n_2}} f(x) + \\ &\sum_{k_1+k_2+\cdots+k_{n_1+n_2}=0} \binom{0}{k_1,k_2,\cdots,k_{n_1+n_2}} \left(a_{n_1}(x)\ I^{n-\beta_{n_1}}\right)^{k_1+1}\cdots \\ &\left(a_1(x)\ I^{n-\beta_1}\right)^{k_{n_1}} \left(b_1(x)I^{\alpha_1+n}\right)^{k_{n_1+1}}\cdots\left(b_{n_2}(x)I^{\alpha_{n_2}+n}\right)^{k_{n_1+n_2}} f(x) + \\ &\sum_{k_1+k_2+\cdots+k_{n_1+n_2}=0} \binom{0}{k_1,k_2,\cdots,k_{n_1+n_2}} \left(a_{n_1}(x)\ I^{n-\beta_{n_1}}\right)^{k_1}\cdots \\ &\left(a_1(x)\ I^{n-\beta_1}\right)^{k_{n_1}+1} \left(b_1(x)I^{\alpha_1+n}\right)^{k_{n_1+1}}\cdots\left(b_{n_2}(x)I^{\alpha_{n_2}+n}\right)^{k_{n_1+n_2}} f(x) + \\ &\sum_{k_1+k_2+\cdots+k_{n_1+n_2}=0} \binom{0}{k_1,k_2,\cdots,k_{n_1+n_2}} \left(a_{n_1}(x)\ I^{n-\beta_{n_1}}\right)^{k_1}\cdots \end{aligned}$$

$$(a_1(x)\ I^{n-\beta_1})^{k_{n_1}} \left(b_1(x) I^{\alpha_1+n}\right)^{k_{n_1+1}+1} \cdots \left(b_{n_2}(x) I^{\alpha_{n_2}+n}\right)^{k_{n_1+n_2}} f(x) +$$
$$\sum_{k_1+k_2+\cdots+k_{n_1+n_2}=0} \binom{0}{k_1, k_2, \cdots, k_{n_1+n_2}} (a_{n_1}(x)\ I^{n-\beta_{n_1}})^{k_1} \cdots$$
$$(a_1(x)\ I^{n-\beta_1})^{k_{n_1}} \left(b_1(x) I^{\alpha_1+n}\right)^{k_{n_1+1}} \cdots \left(b_{n_2}(x) I^{\alpha_{n_2}+n}\right)^{k_{n_1+n_2}+1} f(x)$$
$$= 0.$$

The pattern follows similarly for other values of k. The uniqueness can be derived from the fact that the integro-differential equation

$$u^{(n)}(x) + a_{n_1}(x)\ {}_C D^{\beta_{n_1}} u(x) + \cdots + a_1(x)\ {}_C D^{\beta_1} u(x) +$$
$$b_1(x) I^{\alpha_1} u(x) + \cdots + b_{n_2}(x) I^{\alpha_{n_2}} u(x) = 0,$$

has only the zero solution by Babenko's method. This completes the proof of Theorem 2. □

Remark 1. *(i) If all coefficient functions* $a_i(x) = b_j(x) = 0$ *over* $[0, T]$*, then Equation (1) becomes*

$$u^{(n)}(x) = f(x),$$
$$u(0) = \cdots = u^{(n-1)}(0) = 0,$$

with the solution

$$u(x) = I^n f(x) = \frac{1}{(n-1)!} \int_0^x (x-t)^{n-1} f(t) dt.$$

(ii) If all coefficient functions $a_i(x) = b_j(x) = 1$ *over* $[0, T]$*, then Equation (1) has the solution for* $f \in C[0, T]$*:*

$$u(x) = I^n \sum_{k=0}^{\infty} (-1)^k \sum_{k_1+k_2+\cdots+k_{n_1+n_2}=k} \binom{k}{k_1, k_2, \cdots, k_{n_1+n_2}} \cdot I^{k_1(n-\beta_{n_1})+\cdots+k_{n_1+n_2}(n+\alpha_{n_2})} f(x)$$

in $C^n[0, T]$*.*

(iii) If all coefficient functions $b_j(x) = 0$ *on* $[0, T]$*, then Equation (1) turns out to be the fractional differential equation with the following initial conditions:*

$$u^{(n)}(x) + a_{n_1}(x)\ {}_C D^{\beta_{n_1}} u(x) + \cdots + a_1(x)\ {}_C D^{\beta_1} u(x) = f(x),$$
$$u(0) = u'(0) = \cdots = u^{(n-1)}(0) = 0.$$

The solution is given as

$$u(x) = I^n \sum_{k=0}^{\infty} (-1)^k \sum_{k_1+k_2+\cdots+k_{n_1}=k} \binom{k}{k_1, k_2, \cdots, k_{n_1}} (a_{n_1}(x)\ I^{n-\beta_{n_1}})^{k_1} \cdots (a_1(x)\ I^{n-\beta_1})^{k_{n_1}} f(x).$$

(iv) If all coefficient functions $a_i(x) = 0$ *on* $[0, T]$*, then Equation (1) turns out to be the integro-differential equation with the following initial conditions:*

$$u^{(n)}(x) + b_1(x) I^{\alpha_1} u(x) + \cdots + b_{n_2}(x) I^{\alpha_{n_2}} u(x) = f(x),$$
$$u(0) = u'(0) = \cdots = u^{(n-1)}(0) = 0.$$

The solution is given as

$$u(x) = I^n \sum_{k=0}^{\infty} (-1)^k \sum_{k_1+k_2+\cdots+k_{n_2}=k} \binom{k}{k_1, k_2, \cdots, k_{n_2}} \left(b_1(x) I^{\alpha_1+n}\right)^{k_1} \cdots \left(b_{n_2}(x) I^{\alpha_{n_2}+n}\right)^{k_{n_2}} f(x).$$

Let $\alpha \in R$. *Define*

$$x_+^{\alpha} = \begin{cases} x^{\alpha} & \text{if } x > 0, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\Phi_{\alpha}(x) = \frac{x_+^{\alpha-1}}{\Gamma(\alpha)}.$$

Then, it follows that $\Phi_0(x) = \delta(x)$ *(the Dirac delta function) and*

$$\Phi_{\alpha} * \Phi_{\beta} = \Phi_{\alpha+\beta}, \tag{5}$$

for $\alpha,\ \beta \in R$ *[18].*

Example 1. *The following integro-differential equation*

$$u'(x) + x^{0.5}\ {}_C D^{0.5} u(x) + I^{0.5} u(x) = x,$$
$$u(0) = 0$$

has a unique and global solution in $C^1[0, T]$

$$u(x) = \sum_{k=0}^{\infty} (-1)^k \sum_{j=0}^{k} \binom{k}{j} C_{j,k} \Phi_{2+j+1.5(k-j)},$$

where the coefficient $C_{j,k}$ *is given below.*

Proof. Based on Theorem 2,

$$u(x) = I \sum_{k=0}^{\infty} (-1)^k \sum_{j=0}^{k} \binom{k}{j} \left(x^{0.5} I^{0.5}\right)^j I^{1.5(k-j)}\ x.$$

Obviously from Equation (5),

$$I^{1.5(k-j)}\ x = \Phi_{1.5(k-j)} * \frac{x}{\Gamma(2)} = \Phi_{1.5(k-j)} * \Phi_2 = \Phi_{2+1.5(k-j)},$$

$$\left(x^{0.5} I^{0.5}\right)^0 I^{1.5(k-j)}\ x = I^{1.5(k-j)}\ x = \Phi_{2+1.5(k-j)},$$

$$\left(x^{0.5} I^{0.5}\right)^1 I^{1.5(k-j)}\ x = x^{0.5} \Phi_{2.5+1.5(k-j)} = \frac{x^{2+1.5(k-j)}}{\Gamma(2.5+1.5(k-j))}$$
$$= \frac{\Gamma(3+1.5(k-j))}{\Gamma(2.5+1.5(k-j))} \Phi_{3+1.5(k-j)},$$

$$\left(x^{0.5} I^{0.5}\right)^2 I^{1.5(k-j)}\ x = \frac{\Gamma(3+1.5(k-j))}{\Gamma(2.5+1.5(k-j))} \left(x^{0.5} I^{0.5}\right) \Phi_{3+1.5(k-j)}$$
$$= \frac{\Gamma(3+1.5(k-j))}{\Gamma(2.5+1.5(k-j))}\ x^{0.5} \Phi_{3.5+1.5(k-j)}$$

$$
\begin{aligned}
&= \frac{\Gamma(3+1.5(k-j))\Gamma(4+1.5(k-j))}{\Gamma(2.5+1.5(k-j))\Gamma(3.5+1.5(k-j))}\ \Phi_{4+1.5(k-j)},\\
&\cdots,\\
&\left(x^{0.5}I^{0.5}\right)^{j} I^{1.5(k-j)}\ x\\
&= \frac{\Gamma(2+1+1.5(k-j))\cdots\Gamma(2+j+1.5(k-j))}{\Gamma(1.5+1+1.5(k-j))\cdots\Gamma(1.5+j+1.5(k-j))}\Phi_{2+j+1.5(k-j)}\\
&= C_{j,k}\Phi_{2+j+1.5(k-j)},\\
&I\left(x^{0.5}I^{0.5}\right)^{j} I^{1.5(k-j)}\ x = C_{j,k}\Phi_{3+j+1.5(k-j)},
\end{aligned}
$$

where

$$
C_{j,k} = \begin{cases} 1 & \text{if } j=0,\\ \dfrac{\Gamma(2+1+1.5(k-j))\cdots\Gamma(2+j+1.5(k-j))}{\Gamma(1.5+1+1.5(k-j))\cdots\Gamma(1.5+j+1.5(k-j))} & \text{if } 1\le j\le k. \end{cases}
$$

This completes the proof of Example 1. □

Using Banach's contractive principle, we are now ready to show the uniqueness of solutions to Equation (2) in space $C^n[0,T]$.

Theorem 3. *Assume that $a_i(x)$ and $b_j(x)$ are continuous functions on $[0,T]$ for all $i = 1,2,\cdots,n_1 \in N$ and $j = 1,2,\cdots,n_2 \in N$, and $0 \le \beta_1 < \beta_2 < \cdots < \beta_{n_1} < n$ and $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_{n_2}$. Furthermore, suppose $g(x,y_1,\cdots,y_n) \in C([0,T]\times R^n)$ and that there exist constants $C_1,\cdots,C_n$ such that*

$$
|g(x,y_1,\cdots,y_n) - g(x,z_1,\cdots,z_n)| \le C_1|y_1-z_1| + \cdots + C_n|y_n-z_n|,
$$

and

$$
\begin{aligned}
q &= n\max\{C_1,\cdots,C_n\}\cdot\\
&\quad \max\{T^n E_{(n-\beta_{n_1},\cdots,n+\alpha_{n_2}),\,n+1}\left(T^{n-\beta_{n_1}}A_{n_1},\cdots,T^{n+\alpha_{n_2}}B_{n_2}\right),\cdots,\\
&\quad E_{(n-\beta_{n_1},\cdots,n+\alpha_{n_2}),\,1}\left(T^{n-\beta_{n_1}}A_{n_1},\cdots,T^{n+\alpha_{n_2}}B_{n_2}\right)\} < 1,
\end{aligned}
$$

where

$$
\max_{x\in[0,T]} |a_i(x)| \le A_i, \quad \max_{x\in[0,T]} |b_j(x)| \le B_j.
$$

Then Equation (2) has a unique solution in $C^n[0,T]$.

Proof. Clearly, $g(x,u,\cdots,u^{(n-1)}) \in C[0,T]$ for $u \in C^n[0,T]$. Define a mapping on $C^n[0,T]$ as

$$
\begin{aligned}
T(u) &= I^n \sum_{k=0}^{\infty}(-1)^k \sum_{k_1+k_2+\cdots+k_{n_1+n_2}=k} \binom{k}{k_1,k_2,\cdots,k_{n_1+n_2}} (a_{n_1}(x)\ I^{n-\beta_{n_1}})^{k_1}\cdots\\
&\quad (a_1(x)\ I^{n-\beta_1})^{k_{n_1}} \left(b_1(x)I^{\alpha_1+n}\right)^{k_{n_1+1}}\cdots\left(b_{n_2}(x)I^{\alpha_{n_2}+n}\right)^{k_{n_1+n_2}} g(x,u,\cdots,u^{(n-1)}).
\end{aligned}
$$

It follows from the proof of Theorem 2 that

$$\|T(u)\|_C \le T^n E_{(n-\beta_{n_1},\cdots,n+\alpha_{n_2}),\, n+1}\left(T^{n-\beta_{n_1}}A_{n_1},\cdots,T^{n+\alpha_{n_2}}B_{n_2}\right)\cdot \left\|g(x,u,\cdots,u^{(n-1)})\right\|_C < \infty,$$

$$\cdots,$$

$$\left\|\frac{d^n}{dx^n}T(u)\right\|_C \le E_{(n-\beta_{n_1},\cdots,n+\alpha_{n_2}),\, 1}\left(T^{n-\beta_{n_1}}A_{n_1},\cdots,T^{n+\alpha_{n_2}}B_{n_2}\right)\cdot \left\|g(x,u,\cdots,u^{(n-1)})\right\|_C < \infty.$$

This infers that T is a mapping from $C^n[0,T]$ to itself. It remains to prove that T is contractive. Indeed,

$$\begin{aligned}&\left\|g(x,u,\cdots,u^{(n-1)})-g(x,v,\cdots,v^{(n-1)})\right\|_C\\&=\max_{x\in[0,T]}|g(x,u(x),\cdots,u^{(n-1)}(x))-g(x,v(x),\cdots,v^{(n-1)}(x))|\\&\le C_1\|u-v\|_C+\cdots+C_n\left\|u^{(n-1)}(x)-v^{(n-1)}(x)\right\|_C\\&\le n\max\{C_1,C_2,\cdots,C_n\}\|u-v\|_n,\end{aligned}$$

and

$$\begin{aligned}&\|Tu-Tv\|_C\le n\max\{C_1,C_2,\cdots,C_n\}\\&T^nE_{(n-\beta_{n_1},\cdots,n+\alpha_{n_2}),\,n+1}\left(T^{n-\beta_{n_1}}A_{n_1},\cdots,T^{n+\alpha_{n_2}}B_{n_2}\right)\|u-v\|_n,\\&\cdots,\\&\left\|T^{(n)}u-T^{(n)}v\right\|_C\le n\max\{C_1,C_2,\cdots,C_n\}\\&E_{(n-\beta_{n_1},\cdots,n+\alpha_{n_2}),\,1}\left(T^{n-\beta_{n_1}}A_{n_1},\cdots,T^{n+\alpha_{n_2}}B_{n_2}\right)\|u-v\|_n.\end{aligned}$$

Hence,

$$\begin{aligned}&\|Tu-Tv\|_n=\max\left\{\|Tu-Tv\|_C,\cdots,\left\|T^{(n)}u-T^{(n)}v\right\|_C\right\}\\&\le n\max\{C_1,\cdots,C_n\}\max\{T^nE_{(n-\beta_{n_1},\cdots,n+\alpha_{n_2}),\,n+1}\left(T^{n-\beta_{n_1}}A_{n_1},\cdots,T^{n+\alpha_{n_2}}B_{n_2}\right),\\&\cdots,E_{(n-\beta_{n_1},\cdots,n+\alpha_{n_2}),\,1}\left(T^{n-\beta_{n_1}}A_{n_1},\cdots,T^{n+\alpha_{n_2}}B_{n_2}\right)\}\|u-v\|_n\\&=q\|u-v\|_n,\end{aligned}$$

where $q < 1$. This completes the proof of Theorem 3. □

Remark 2. *If all coefficient functions $a_i(x) = b_j(x) = 0$ over $[0,T]$, then Equation (2) becomes the integral equation*

$$u(x)=\frac{1}{(n-1)!}\int_0^x (x-t)^{n-1}g(x,u(x),\cdots,u^{(n-1)}(x))dx.$$

Example 2. *The following integro-differential equation with initial conditions:*

$$\begin{aligned}&u^{(2)}(x)+\sin x^2\ {}_CD^{0.8}u(x)-\sqrt{x}\ I^{0.5}u(x)+\frac{Iu(x)}{x^2+1}=\frac{1}{42((u')^2(x)+1)}+\frac{1}{43}\cos u(x),\\&u(0)=u'(0)=0,\end{aligned}$$

has a unique solution in $C^2[0,1]$.

Proof. Clearly,

$$g(x, y_1, y_2) = \frac{1}{42(y_2^2+1)} + \frac{1}{43}\cos y_1,$$

and

$$|g(x, y_1, y_2) - g(x, z_1, z_2)| \leq \frac{1}{43}|y_1 - z_1| + \frac{1}{42}|y_2 - z_2|.$$

Thus,

$$\max\{C_1, C_2\} = \max\left\{\frac{1}{43}, \frac{1}{42}\right\} = \frac{1}{42}.$$

Obviously,

$$\max_{x\in[0,1]} |\sin x^2| \leq 1, \quad \max_{x\in[0,1]} |-\sqrt{x}| \leq 1, \quad \max_{x\in[0,1]} \left|\frac{1}{x^2+1}\right| \leq 1.$$

Therefore,

$$\begin{aligned} q &= 2 \cdot \frac{1}{42}\max\left\{E_{(1.2,2.5,3),3}(1,1,1), E_{(1.2,2.5,3),2}(1,1,1), E_{(1.2,2.5,3),1}(1,1,1)\right\} \\ &= \frac{1}{21}E_{(1.2,2.5,3),1}(1,1,1), \end{aligned}$$

since

$$E_{(1.2,2.5,3),3}(1,1,1) \leq E_{(1.2,2.5,3),2}(1,1,1) \leq E_{(1.2,2.5,3),1}(1,1,1).$$

Furthermore,

$$E_{(1.2,2.5,3),1}(1,1,1) = \sum_{k=0}^{\infty} \sum_{k_1+k_2+k_3=k} \binom{k}{k_1, k_2, k_3} \frac{1}{\Gamma(1.2k_1 + 2.5k_2 + 3k_3 + 1)},$$

$$\sum_{k_1+k_2+k_3=k} \binom{k}{k_1, k_2, k_3} = 3^k,$$

$$\frac{1}{\Gamma(1.2k_1 + 2.5k_2 + 3k_3 + 1)} \leq \frac{1}{\Gamma(k+1)}, \quad k = 0, 1, \cdots,$$

by noting that $\Gamma(x+1)$ is an increasing function if $x \geq 1$ [19].

It follows that

$$\begin{aligned} &E_{(1.2,2.5,3),1}(1,1,1) \leq \sum_{k=0}^{\infty} \frac{3^k}{k!} = 1 + \sum_{k=1}^{\infty} \frac{3^k}{k!} \\ &= 1 + \frac{3}{1} + \frac{3\cdot 3}{1\cdot 2} + \frac{3\cdot 3\cdot 3}{1\cdot 2\cdot 3} + \frac{3\cdot 3\cdot 3\cdot 3}{1\cdot 2\cdot 3\cdot 4} + \frac{3\cdot 3\cdot 3\cdot 3\cdot 3}{1\cdot 2\cdot 3\cdot 4\cdot 5} + \frac{3\cdot 3\cdot 3\cdot 3\cdot 3\cdot 3}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6} + \\ &\quad \frac{3\cdot 3\cdot 3\cdot 3\cdot 3\cdot 3\cdot 3}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7} + \frac{3\cdot 3\cdot 3\cdot 3\cdot 3\cdot 3\cdot 3\cdot 3}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8} + \cdots \\ &< 1 + 3 + 4.5 + 4.5 + 3.375 + 2.025 + \left(\frac{1}{80} + 1\right) + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots \\ &= 20.4125. \end{aligned}$$

This implies that

$$q < \frac{20.4125}{21} < 1.$$

This completes the proof of Example 2. □

Finally, we provide sufficient conditions for the uniqueness of system (3) in the product space $C^n[0,T] \times C^m[0,T]$.

Theorem 4. *Assume that all coefficient functions $a_i(x)$, $b_j(x)$, $c_{i_1}(x)$ and $d_{j_1}(x)$ in system (3) are continuous functions on $[0,T]$ for all $i = 1,2,\cdots,n_1 \in N$, $j = 1,2,\cdots,n_2 \in N$, $i_1 = 1,2,\cdots,m_1 \in N$ and $j_1 = 1,2,\cdots,m_2 \in N$. Furthermore, $0 \leq \beta_1 < \beta_2 < \cdots < \beta_{n_1} < n$, $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_{n_2}$, $0 \leq \mu_1 < \mu_2 < \cdots < \mu_{m_1} < m$ and $0 < \gamma_1 < \gamma_2 < \cdots < \gamma_{m_2}$. In addition, suppose $g_1(x,y_1,\cdots,y_{n+m})$, $g_2(x,y_1,\cdots,y_{n+m}) \in C([0,T] \times R^{n+m})$ and that there exist constants $K_1,\cdots,K_{n+m}$ such that*

$$
\begin{array}{l}
|g_1(x,y_1,\cdots,y_{n+m}) - g_1(x,z_1,\cdots,z_{n+m})| \leq K_1|y_1 - z_1| + \cdots + \\
K_{n+m}|y_{n+m} - z_{n+m}|, \\
|g_2(x,y_1,\cdots,y_{n+m}) - g_2(x,z_1,\cdots,z_{n+m})| \leq K_1|y_1 - z_1| + \cdots + \\
K_{n+m}|y_{n+m} - z_{n+m}|,
\end{array}
$$

and

$$
\begin{aligned}
Q \;=\; & (n+m)\max\{K_1,\cdots,K_{n+m}\}\cdot \\
& \max\{T^n E_{(n-\beta_{n_1},\cdots,n+\alpha_{n_2}),\,n+1}\Big(T^{n-\beta_{n_1}}A_{n_1},\cdots,T^{n+\alpha_{n_2}}B_{n_2}\Big),\cdots, \\
& E_{(n-\beta_{n_1},\cdots,n+\alpha_{n_2}),\,1}\Big(T^{n-\beta_{n_1}}A_{n_1},\cdots,T^{n+\alpha_{n_2}}B_{n_2}\Big), \\
& T^m E_{(m-\mu_{m_1},\cdots,m+\gamma_{m_2}),\,m+1}\big(T^{m-\mu_{m_1}}C_{m_1},\cdots,T^{m+\gamma_{m_2}}D_{m_2}\big),\cdots, \\
& E_{(m-\mu_{m_1},\cdots,m+\gamma_{m_2}),\,1}\big(T^{m-\mu_{m_1}}C_{m_1},\cdots,T^{m+\gamma_{m_2}}D_{m_2}\big)\} < 1,
\end{aligned}
$$

where

$$
\max_{x\in[0,T]}|a_i(x)| \leq A_i, \quad \max_{x\in[0,T]}|b_j(x)| \leq B_j,
$$
$$
\max_{x\in[0,T]}|c_{i_1}(x)| \leq C_{i_1}, \quad \max_{x\in[0,T]}|d_{j_1}(x)| \leq D_{j_1}.
$$

Then, system (3) has a unique solution in $C^n[0,T] \times C^m[0,T]$.

Proof. Clearly,

$$
g_1\Big(x,u(x),\cdots,u^{(n-1)}(x),v(x),\cdots,v^{(m-1)}(x)\Big) \in C[0,T],
$$
$$
g_2\Big(x,u(x),\cdots,u^{(n-1)}(x),v(x),\cdots,v^{(m-1)}(x)\Big) \in C[0,T],
$$

for $(u,v) \in C^n[0,T] \times C^m[0,T]$. Define a mapping T on $C^n[0,T] \times C^m[0,T]$ as

$$
T(u,v) = (T_1(u,v), T_2(u,v)),
$$

where

$$
\begin{aligned}
T_1(u,v) \;=\; & I^n \sum_{k=0}^{\infty}(-1)^k \sum_{k_1+k_2+\cdots+k_{n_1+n_2}=k} \binom{k}{k_1,k_2,\cdots,k_{n_1+n_2}} \left(a_{n_1}(x)\ I^{n-\beta_{n_1}}\right)^{k_1}\cdots \\
& \left(a_1(x)\ I^{n-\beta_1}\right)^{k_{n_1}}\left(b_1(x)I^{\alpha_1+n}\right)^{k_{n_1+1}}\cdots\left(b_{n_2}(x)I^{\alpha_{n_2}+n}\right)^{k_{n_1+n_2}} \\
& g_1(x,u,\cdots,u^{(n-1)},v,\cdots,v^{(m-1)}).
\end{aligned}
$$

Clearly,

$$\begin{aligned}
&\|T_1(u,v)\|_C \le T^n E_{(n-\beta_{n_1},\cdots,n+\alpha_{n_2}),\, n+1}\Big(T^{n-\beta_{n_1}}A_{n_1},\cdots,T^{n+\alpha_{n_2}}B_{n_2}\Big)\cdot \\
&\qquad\qquad \Big\|g_1(x,u,\cdots,u^{(n-1)},v,\cdots,v^{(m-1)})\Big\|_C < \infty, \\
&\cdots, \\
&\left\|\frac{d^n}{dx^n}T_1(u,v)\right\|_C \le E_{(n-\beta_{n_1},\cdots,n+\alpha_{n_2}),\,1}\Big(T^{n-\beta_{n_1}}A_{n_1},\cdots,T^{n+\alpha_{n_2}}B_{n_2}\Big)\cdot \\
&\qquad\qquad \Big\|g_1(x,u,\cdots,u^{(n-1)},v,\cdots,v^{(m-1)})\Big\|_C < \infty.
\end{aligned}$$

Hence,

$$T_1(u,v) \in C^n[0,T].$$

Similarly,

$$\begin{aligned}
T_2(u,v) \;&=\; I^m\sum_{k=0}^{\infty}(-1)^k \sum_{k_1+k_2+\cdots+k_{m_1+m_2}=k}\binom{k}{k_1,k_2,\cdots,k_{m_1+m_2}}\,(c_{m_1}(x)\;I^{m-\mu_{m_1}})^{k_1}\cdots \\
&\quad (c_1(x)\;I^{m-\mu_1})^{k_{m_1}}\big(d_1(x)I^{\gamma_1+m}\big)^{k_{m_1+1}}\cdots\big(d_{m_2}(x)I^{\alpha_{m_2}+m}\big)^{k_{m_1+m_2}} \\
&\quad g_2(x,u,\cdots,u^{(n-1)},v,\cdots,v^{(m-1)}).
\end{aligned}$$

Clearly,

$$\begin{aligned}
&\|T_2(u,v)\|_C \le T^m E_{(m-\mu_{m_1},\cdots,m+\gamma_{m_2}),\, m+1}\big(T^{m-\mu_{m_1}}C_{m_1},\cdots,T^{m+\gamma_{m_2}}D_{m_2}\big)\cdot \\
&\qquad\qquad \Big\|g_2(x,u,\cdots,u^{(n-1)},v,\cdots,v^{(m-1)})\Big\|_C < \infty, \\
&\cdots, \\
&\left\|\frac{d^m}{dx^m}T_2(u,v)\right\|_C \le E_{(m-\mu_{m_1},\cdots,m+\gamma_{m_2}),\,1}\big(T^{m-\mu_{m_1}}C_{m_1},\cdots,T^{m+\gamma_{m_2}}D_{m_2}\big)\cdot \\
&\qquad\qquad \Big\|g_2(x,u,\cdots,u^{(n-1)},v,\cdots,v^{(m-1)})\Big\|_C < \infty.
\end{aligned}$$

Hence,

$$T_2(u,v) \in C^m[0,T],$$

and T is a mapping from $C^n[0,T]\times C^m[0,T]$ to itself. We need to show that T is contractive. In fact,

$$\begin{aligned}
&\Big\|g_1(x,u_1,\cdots,u_1^{(n-1)},v_1,\cdots,v_1^{(m-1)}) - g_1(x,u_2,\cdots,u_2^{(n-1)},v_2,\cdots,v_2^{(m-1)})\Big\|_C \\
&= \max_{x\in[0,T]}|g_1(x,u_1(x),\cdots,u_1^{(n-1)}(x),v_1(x),\cdots,v_1^{(m-1)}(x)) \\
&\qquad\quad -g_1(x,u_2(x),\cdots,u_2^{(n-1)}(x),v_2(x),\cdots,v_2^{(m-1)}(x))| \\
&\le K_1\|u_1-u_2\|_C + \cdots + K_{n+m}\Big\|v_1^{(m-1)}(x)-v_2^{(m-1)}(x)\Big\|_C \\
&\le (n+m)\max\{K_1,K_2,\cdots,K_{n+m}\}\|(u_1,v_1)-(u_2,v_2)\|,
\end{aligned}$$

by noting that

$$\begin{aligned}
&\|(u_1,v_1)-(u_2,v_2)\| = \|(u_1-u_2,v_1-v_2)\| = \|u_1-u_2\|_n + \|v_1-v_2\|_m, \\
&\|u_1-u_2\|_C \le \|u_1-u_2\|_n \le \|(u_1,v_1)-(u_2,v_2)\|, \\
&\cdots, \\
&\Big\|v_1^{(m-1)}(x)-v_2^{(m-1)}(x)\Big\|_C \le \|v_1-v_2\|_m \le \|(u_1,v_1)-(u_2,v_2)\|.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
&\|T_1(u_1,v_1)-T_1(u_2,v_2)\|_C \le (n+m)\max\{K_1,K_2,\cdots,K_{n+m}\}\\
&\cdot T^n E_{(n-\beta_{n_1},\cdots,n+\alpha_{n_2}),\,n+1}\left(T^{n-\beta_{n_1}}A_{n_1},\cdots,T^{n+\alpha_{n_2}}B_{n_2}\right)\\
&\cdot\|(u_1,v_1)-(u_2,v_2)\|,\\
&\cdots,\\
&\left\|T_1^{(n)}(u_1,v_1)-T_1^{(n)}(u_2,v_2)\right\|_C \le (n+m)\max\{K_1,K_2,\cdots,K_{n+m}\}\\
&\cdot E_{(n-\beta_{n_1},\cdots,n+\alpha_{n_2}),\,1}\left(T^{n-\beta_{n_1}}A_{n_1},\cdots,T^{n+\alpha_{n_2}}B_{n_2}\right)\\
&\cdot\|(u_1,v_1)-(u_2,v_2)\|.
\end{aligned}$$

This deduces that

$$\begin{aligned}
\|T_1(u_1,v_1)-T_1(u_2,v_2)\|_n \le\; & (n+m)\max\{K_1,K_2,\cdots,K_{n+m}\}\cdot\\
&\max\{T^n E_{(n-\beta_{n_1},\cdots,n+\alpha_{n_2}),\,n+1}\left(T^{n-\beta_{n_1}}A_{n_1},\cdots,T^{n+\alpha_{n_2}}B_{n_2}\right),\cdots,\\
&E_{(n-\beta_{n_1},\cdots,n+\alpha_{n_2}),\,1}\left(T^{n-\beta_{n_1}}A_{n_1},\cdots,T^{n+\alpha_{n_2}}B_{n_2}\right)\}\\
&\cdot\|(u_1,v_1)-(u_2,v_2)\|.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\|T_2(u_1,v_1)-T_2(u_2,v_2)\|_m \le\; & (n+m)\max\{K_1,K_2,\cdots,K_{n+m}\}\cdot\\
&\max\{T^m E_{(m-\mu_{m_1},\cdots,m+\gamma_{m_2}),\,m+1}(T^{m-\mu_{m_1}}C_{m_1},\cdots,T^{m+\gamma_{m_2}}D_{m_2}),\\
&\cdots,E_{(m-\mu_{m_1},\cdots,m+\gamma_{m_2}),\,1}(T^{m-\mu_{m_1}}C_{m_1},\cdots,T^{m+\gamma_{m_2}}D_{m_2})\}\cdot\\
&\|(u_1,v_1)-(u_2,v_2)\|.
\end{aligned}$$

In summary,

$$\begin{aligned}
&\|T(u_1,v_1)-T(u_2,v_2)\| = \|(T_1u_1,T_2v_1)-(T_1u_2,T_2v_2\|\\
&= \|T_1u_1-T_1u_2\|_n+\|T_2v_1-T_2v_2\|_m\\
&\le (n+m)\max\{K_1,K_2,\cdots,K_{n+m}\}\cdot\\
&\max\{T^n E_{(n-\beta_{n_1},\cdots,n+\alpha_{n_2}),\,n+1}\left(T^{n-\beta_{n_1}}A_{n_1},\cdots,T^{n+\alpha_{n_2}}B_{n_2}\right),\cdots,\\
&E_{(n-\beta_{n_1},\cdots,n+\alpha_{n_2}),\,1}\left(T^{n-\beta_{n_1}}A_{n_1},\cdots,T^{n+\alpha_{n_2}}B_{n_2}\right),\\
&T^m E_{(m-\mu_{m_1},\cdots,m+\gamma_{m_2}),\,m+1}(T^{m-\mu_{m_1}}C_{m_1},\cdots,T^{m+\gamma_{m_2}}D_{m_2}),\\
&\cdots,E_{(m-\mu_{m_1},\cdots,m+\gamma_{m_2}),\,1}(T^{m-\mu_{m_1}}C_{m_1},\cdots,T^{m+\gamma_{m_2}}D_{m_2})\}\\
&\cdot\|(u_1,v_1)-(u_2,v_2)\| = Q\|(u_1,v_1)-(u_2,v_2)\|,
\end{aligned}$$

where

$$\begin{aligned}
Q =\; & (n+m)\max\{K_1,K_2,\cdots,K_{n+m}\}\cdot\\
&\max\{T^n E_{(n-\beta_{n_1},\cdots,n+\alpha_{n_2}),\,n+1}\left(T^{n-\beta_{n_1}}A_{n_1},\cdots,T^{n+\alpha_{n_2}}B_{n_2}\right),\cdots,\\
&E_{(n-\beta_{n_1},\cdots,n+\alpha_{n_2}),\,1}\left(T^{n-\beta_{n_1}}A_{n_1},\cdots,T^{n+\alpha_{n_2}}B_{n_2}\right),\\
&T^m E_{(m-\mu_{m_1},\cdots,m+\gamma_{m_2}),\,m+1}(T^{m-\mu_{m_1}}C_{m_1},\cdots,T^{m+\gamma_{m_2}}D_{m_2}),\\
&\cdots,E_{(m-\mu_{m_1},\cdots,m+\gamma_{m_2}),\,1}(T^{m-\mu_{m_1}}C_{m_1},\cdots,T^{m+\gamma_{m_2}}D_{m_2})\} < 1.
\end{aligned}$$

This completes the proof of Theorem 4. □

Remark 3. *If all coefficient functions $a_i(x) = b_j(x) = c_{i_1}(x) = d_{j_1}(x) = 0$ in system (3), then it is equivalent to*

$$u(x) = \frac{1}{(n-1)!}\int_0^x (x-t)^{n-1} g_1(x, u(x), \cdots, u^{(n-1)}(x), v(x), \cdots, v^{(m-1)}(x))dx,$$
$$v(x) = \frac{1}{(m-1)!}\int_0^x (x-t)^{m-1} g_2(x, u(x), \cdots, u^{(n-1)}(x), v(x), \cdots, v^{(m-1)}(x))dx.$$

Example 3. *The following integro-differential system with initial conditions*

$$u^{(2)}(x) + x\,{}_C D^{0.5} u(x) - \frac{1}{x+1} I^{0.5} u(x) = \frac{1}{51}\sin v(x),$$
$$u(0) = u'(0) = 0,$$
$$v^{(3)}(x) = \frac{1}{42}\cos u(x),$$
$$v(0) = v'(0) = v''(0) = 0$$

has a unique nonzero solution in $C^2[0,1] \times C^3[0,1]$.

Proof. Clearly,

$$g_1(x, y_1, \cdots, y_5) = \frac{1}{51}\sin y_3, \quad g_2(x, y_1, \cdots, y_5) = \frac{1}{42}\cos y_1$$

and

$$|g_1(x, y_1, \cdots, y_5) - g_1(x, z_1, \cdots, z_5)| \le \frac{1}{51}|y_3 - z_3|,$$
$$|g_2(x, y_1, \cdots, y_5) - g_1(x, z_1, \cdots, z_5)| \le \frac{1}{42}|y_1 - z_1|.$$

Thus,

$$(n+m)\max\{K_1, K_2, \cdots, K_{n+m}\} = (2+3)\max\left\{\frac{1}{42}, \frac{1}{51}\right\} = \frac{5}{42}.$$

Obviously,

$$\max_{x\in[0,1]} |x| \le 1, \quad \max_{x\in[0,1]} \left|-\frac{1}{x+1}\right| \le 1.$$

Using Theorem 4, we derive that

$$Q = \frac{5}{42}\max\{E_{(1.5,2.5),3}(1,1), E_{(1.5,2.5),2}(1,1), E_{(1.5,2.5),1}(1,1), \frac{1}{\Gamma(3+1)}, \frac{1}{\Gamma(2+1)}, \frac{1}{\Gamma(1+1)}, \frac{1}{\Gamma(1)}\} = \frac{5}{42}E_{(1.5,2.5),1}(1,1),$$

by noting that

$$c_{i_1}(x) = d_{j_1}(x) = 0$$

over $[0,1]$. Furthermore,

$$\begin{aligned} E_{(1.5,2.5),1}(1,1) &= \sum_{k=0}^{\infty}\sum_{k_1+k_2=k}\binom{k}{k_1,k_2}\frac{1}{\Gamma(1.5k_1+2.5k_2+1)} \\ &\leq \sum_{k=0}^{\infty}\frac{2^k}{\Gamma(k+1)} = \sum_{k=0}^{\infty}\frac{2^k}{k!} = 1+\frac{2}{1!}+\frac{2\cdot 2}{1\cdot 2}+\frac{2\cdot 2\cdot 2}{1\cdot 2\cdot 3}+ \\ &\quad \frac{2\cdot 2\cdot 2\cdot 2}{1\cdot 2\cdot 3\cdot 4}+\frac{2\cdot 2\cdot 2\cdot 2\cdot 2}{1\cdot 2\cdot 3\cdot 4\cdot 5}+\cdots \\ &\leq 1+2+2+\left(\frac{1}{3}+\left(\frac{2}{3}\right)^0\right)+\left(\frac{2}{3}\right)^1+\left(\frac{2}{3}\right)^2+\cdots \\ &= \frac{25}{3}. \end{aligned}$$

Therefore,

$$Q \leq \frac{5}{42}\cdot\frac{25}{3} = \frac{125}{126} < 1,$$

and $(u,v)=0$ is clearly not a solution. This completes the proof of Example 3. □

3. Conclusions

Using Banach's contractive principle, the multivariate Mittag–Leffler function and Babenko's approach, we studied the uniqueness of solutions of several nonlinear Liouville–Caputo integro-differential equations with variable coefficients and initial conditions, as well as the associated coupled system in Banach spaces. The results obtained are new and original. We also presented three examples to demonstrate the use of our main theorems.

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References

1. Rudin, W. *Principle of Mathematical Analysis*, 3rd ed.; McGraw Hill: New York, NY, USA, 1976.
2. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*; Elsevier: Amsterdam, The Netherlands, 2006.
3. Samko, S.G.; Kilbas, A.A.; Marichev, O.I. *Fractional Integrals and Derivatives: Theory and Applications*; Gordon and Breach: New York, NY, USA, 1993.
4. Li, C. On the nonlinear Hadamard-type integro-differential equation. *Fixed Point Theory Algorithms Sci. Eng.* **2021**, *2021*, 7. [CrossRef] [PubMed]
5. Zhang, P.; Hao, X.; Liu, L. Existence and uniqueness of the global solution for a class of nonlinear fractional integro-differential equations in a Banach space. *Adv. Differ. Equ.* **2019**, *2019*, 135. [CrossRef]
6. Marasi, H.; Piri, H.; Aydi, H. Existence and multiplicity of solutions for nonlinear fractional differential equations. *J. Nonlinear Sci. Appl.* **2016**, *9*, 4639–4646. [CrossRef]

7. Ahmad, B.; Ntouyas, S.K. A fully Hadamard type integral boundary value problem of a coupled system of fractional differential equations. *Fract. Calc. Appl. Anal.* **2014**, *17*, 348–360. [CrossRef]
8. Li, C.P.; Sarwar, S. Existence and continuation of solutions for Caputo type fractional differential equations. *Electron. J. Differ. Equ.* **2016**, *2016*, 207. [CrossRef]
9. Ahmad, B.; Nieto, J.J. Existence results for nonlinear boundary value problems of fractional integro-differential equations with integral boundary conditions. *Bound. Value Probl.* **2009**, *2009*, 36.
10. Deng, J.; Deng, Z. Existence of solutions of initial value problems for nonlinear fractional differential equations. *Appl. Math. Lett.* **2014**, *32*, 6–12. [CrossRef]
11. Ahmad, B.; Ntouyas, S.K. Existence and uniqueness of solutions for Caputo-Hadamard sequential fractional order neutral functional differential equations. *Electron. J. Differ. Equ.* **2017**, *2017*, 36.
12. Babenkos, Y.I. *Heat and Mass Transfer;* Khimiya: Leningrad, Russia, 1986. (In Russian)
13. Li, C.; Li, C.P.; Clarkson, K. Several results of fractional differential and integral equations in distribution. *Mathematics* **2018**, *6*, 97. [CrossRef]
14. Li, C.; Huang, J. Remarks on the linear fractional integro-differential equation with variable coefficients in distribution. *Fract. Differ. Calc.* **2020**, *10*, 57–77.
15. Li, C.; Beaudin, J. Uniqueness of Abel's integral equations of the second kind with variable coefficients. *Symmetry* **2021**, *13*, 1064. [CrossRef]
16. Hadid, S.B.; Luchko, Y. An operational method for solving fractional differential equations of an arbitrary real order. *Panam. Math. J.* **1996**, *6*, 57–73.
17. Podlubny, I. *Fractional Differential Equations*; Academic Press: New York, NY, USA, 1999.
18. Li, C. Several results of fractional derivatives in $\mathcal{D}'(R_+)$. *Fract. Calc. Appl. Anal.* **2015**, *18*, 192–207. [CrossRef]
19. Gradshteyn, I.S.; Ryzhik, I.M. *Tables of Integrals, Series, and Products*; Academic Press: New York, NY, USA, 1980.