fractal and fractional

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Synchronization Analysis of Multiple Integral Inequalities Driven by Steklov Operator

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Abstract: We construct a subclass of Copson's integral inequality in this article. In order to achieve this goal, we attempt to use the Steklov operator for generalizing different inequalities of the Copson type relevant to the situations $\rho > 1$ as well as $\rho < 1$. We demonstrate the inequalities with the guidance of basic comparison, Holder's inequality, and the integration by parts approach. Moreover, some new variations of Hardy's integral inequality are also presented with the utilization of Steklov operator. We also formulate many remarks and two examples to show the novelty and authenticity of our results.

Keywords: Hardy's inequality; Steklov operator; Copson's inequality

Citation: Albalawi, W.; Khan, Z.A. Synchronization Analysis of Multiple Integral Inequalities Driven by Steklov Operator. *Fractal Fract.* **2021**, *5*, 97. https://doi.org/10.3390/fractalfract5030097

Academic Editor: António M. Lopes

Received: 8 July 2021
Accepted: 12 August 2021
Published: 14 August 2021

1. Introduction

Historically, the main and essential branch of mathematics throughout the last three decades seems to have been mathematical research. Evidently, inequalities have been the core of mathematical studies. Numerous brilliant mathematicians have contributed immensely to many recent discoveries in this subject, referring to the exploration of several other new inequalities in certain areas of mathematical physics and theoretical and applied mathematics, including evidence and useful applications. In reality, numerical inequalities became a significant part of current science in the 20th century through the spearheading work entitled Inequalities by Hardy, Littlewood, and P'olya, which is the first composition to be released in 1934 [1]. This distinctive publication symbolizes a framework of precise logic filled with inequalities for the betterment pf relevant technologies in mathematics.

In the twentieth century, integral inequalities have played a central position in science and have a broad range of applications in most other fields of pure and applied mathematics. In [2], Hardy produced the inequality of the following kind:

$$\mathcal{G}(z_1) = \int_0^{z_1} \mathfrak{g}(z_2)dz_2,$$

then

$$\int_0^{\infty} \left(\frac{\mathcal{G}}{z_1}\right)^{\psi_1} dz_1 < \psi_2^{\psi_1} \int_0^{\infty} \mathfrak{g}^{\psi_1}(z_1)dz_1 \tag{1}$$

unless $\mathfrak{g} \equiv 0$. The best possible constant is $\psi_2 = \psi_1(\psi_1 - 1)^{-1}$. The above inequality plays a crucial role in the assessment and its implementation. It is evident that, for parameters a and b where $0 < a < b < \infty$, the corresponding inequality is also true:

$$\int_a^{b} \left(\frac{\mathcal{G}}{z_1}\right)^{\psi_1} dz_1 < \psi_2^{\psi_1} \int_a^{b} \mathfrak{g}^{\psi_1}(z_1)dz_1, \tag{2}$$

where $0 < \int\limits_0^\infty \mathfrak{g}^{\psi_1}(z_1)dz_1 < \infty$. The classical Hardy inequality declares that if $\psi_1 > 1$ and $\mathfrak{g}$ is a non-negative measurable function on (a, b), then (2) is valid except $\mathfrak{g} \equiv 0$, i.e., in (a, b), where constant here is the best possible contant. This inequality stays relevant providing that $0 < a < b < 1$.

Then, the inequality (2) has now grown tremendously to what is now termed Hardy-type inequalities. One interesting explanation is that this principle is very valuable across both mathematics and technological sciences.

Particularly, in 1928, the following integral inequalities were established by Hardy in [3]. Let $\mathfrak{g}$ be a non-negative measurable function on $(0, \infty)$ and the following is the case:

$$(\mathcal{G}\mathfrak{g})(z_1) \leq \begin{cases} \int\limits_0^{z_1} \mathfrak{g}(z_2)dz_2 & \text{for} \quad a < \psi_1 - 1, \\ \int\limits_{z_1}^{\infty} \mathfrak{g}(z_2)dz_2 & \text{for} \quad a > \psi_1 - 1, \end{cases}$$

then

$$\int_0^\infty z_1^{a-\psi_1}(\mathcal{G}\mathfrak{g})^{\psi_1}(z_1)dz_1 \leq \left(\frac{\psi_1}{|\psi_1 - a - 1|}\right)^{\psi_1} \int_0^\infty z_1^a \mathfrak{g}^{\psi_1}(z_1)dz_1, \quad \text{for } \psi_1 > 1. \tag{3}$$

Later in 1976, integral inequalities ([4], Theorems 1 and 3) studied by Copson are as follows.

Let $\mathfrak{g}$, ς be non-negative measurable functions on $(0, \infty)$

$$\Omega(z_1) = \int_0^{z_1} \varsigma(z_2)dz_2 \text{ and } \quad (\mathcal{P}\mathfrak{g})(z_1) \leq \begin{cases} \int\limits_0^{z_1} \mathfrak{g}(z_2)\varsigma(z_2)dz_2 & \text{for} \quad c > 1, \\ \int\limits_{z_1}^{\infty} \mathfrak{g}(z_2)\varsigma(z_2)dz_2 & \text{for} \quad c < 1, \end{cases}$$

then

$$\int_0^\infty \Omega^{-c}(z_1)\varsigma(z_1)(\mathcal{P}\mathfrak{g})^{\psi_1}(z_1)dz_1 \leq \left(\frac{\psi_1}{|c-1|}\right)^{\psi_1} \int_0^\infty \Omega^{\psi_1 - c}(z_1)\varsigma(z_1)\mathfrak{g}^{\psi_1}(z_1)dz_1, \quad \text{for } \psi_1 \geq 1.$$

In view of their major significance and control throughout the long term, much exertion and time have been committed to the improvement and speculation of Hardy's and Copson's inequalities. However, these are not limited to the works in [5–15].

The aim of this research is to further generalize the basic integral inequalities of Hardy and Copson with the Steklov operator. The results can be achieved by using certain elementary techniques of analysis as well as integration by parts. We define the Hardy–Steklov operator in several formulas in order to prove a certain class of inequalities by inducing conditions related to Hardy–Steklov and Copson–Steklov operators in two cases, $\rho > 1$ and $\rho < 1$. Throughout the whole document, inequalities of the left-hand side exist if the ones on the right-hand side exist.

The paper is structured in the following manner: After the introduction, some basic concepts are mentioned in Section 2 that are essential for our main results. We construct a new class of generalized inequalities of the Copson form pertaining to Steklov operator in Section 3 and special cases are mentioned which result in some known results in the literature. Extensions of Hardy's integral inequality with the same operator are devoted to Section 4. Lastly, the conclusion of our results is presented.

2. Basic Concepts of Hardy–Steklov Operator

Some new inequalities are demonstrated through the Hardy–Steklov operator (see [16–20]). For the sake of completeness, before asserting the key results, we recall the following definitions:

Definition 1. *Hardy–Steklov operator is characterized as follows:*

$$(\mathcal{T}\mathfrak{g})(z_1) = \mathfrak{m}(z_1) \int_{y(z_1)}^{s(z_1)} \mathfrak{g}(z_2)dz_2, \quad \mathfrak{g} \geq 0, \tag{4}$$

where $\mathfrak{m}$ *is a non-negative measurable function and* $y(z_1)$ *and* $s(z_1)$ *are functions defined on an interval* (a, b) *provided with* $y(z_1) < s(z_1)$ *for all* $z_1 \in (a, b)$.

One such integral operator is connected towards other function transitions [21–24] and certain [25] embedding hypotheses. Furthermore, this operator chooses a few extraordinary cases as follows.

Definition 2. *The Hardy operator can be described as follows.*

$$(\mathcal{G}\mathfrak{g})(z_1) = \int_{0}^{z_1} \mathfrak{g}(z_2)dz_2.$$

Definition 3. *A hardy averaging operator is defined by the following.*

$$(\mathcal{G}_\mu\mathfrak{g})(z_1) = z_1^{\mu} \int_{0}^{z_1} \mathfrak{g}(z_2)dz_2.$$

Definition 4. *The operator related to Steklov is specified by the following:*

$$(\mathcal{S}\mathfrak{g})(z_1) = \int_{z_1-1}^{z_1+1} \mathfrak{g}(z_2)dz_2,$$

which has been concentrated on seriously (see [26] for instance).

3. Formulation of Copson Type Integral Inequalities via Steklov Operator

Throughout this paper, we set $(\mathcal{G}_s\mathfrak{g})$ as the Hardy–Steklov type operator, whereas $(\mathcal{P}_s\mathfrak{g})$ is the Copson–Steklov type operator and $0 < b \leq \infty$ by considering that the integrals exist and are finite. In addition, $y(z_1)$ and $s(z_1)$ are increasing differentiable functions on $[0, \infty)$, such that the following is the case.

$$\begin{cases} 0 < y(z_1) < s(z_1) < \infty & \text{for all } \ z_1 \in (0, \infty), \\ y(0) = s(0) = 0 \ \text{ and } \ y(\infty) = s(\infty) = \infty. \end{cases} \tag{5}$$

Theorem 1. *Let* $\mathfrak{g}$ *and* φ *be non-negative measurable functions on* $(0, \infty)$ *and* $\mathfrak{j}$ *and* σ*, be positive and absolutely continuous function on* $(0, \infty)$*. Moreover,* $1 < \psi_1 \leq \psi_2 < \infty$, $\rho > 1$, y, s *is satisfied* (5)*. Let the following be the case:*

$$(\mathcal{P}_s\mathfrak{g})(z_1) = \mathfrak{j}(z_1) \int_{y(z_1)}^{s(z_1)} \frac{1}{\mathfrak{j}(z_2)} \frac{\varphi(z_2)}{\Upsilon(z_2)} \mathfrak{g}(z_2)dz_2, \tag{6}$$

and

$$1 - \frac{1}{\rho - 1} \frac{\sigma'(z_1)}{\sigma(z_1)} \frac{\Upsilon(z_1)}{\varphi(z_1)} + \frac{\psi_1}{\rho - 1} \frac{\mathfrak{j}'(z_1)}{\mathfrak{j}(z_1)} \frac{\Upsilon(z_1)}{\varphi(z_1)} \geq \frac{1}{\lambda}, \tag{7}$$

for some $\lambda > 0$. *Then, we have the following:*

$$\int_0^b \sigma(z_1)\frac{\varphi(z_1)}{\Upsilon^{\rho}(z_1)}(\mathcal{P}_s\mathfrak{g})^{\psi_1}(z_1)dz_1 \le \Big(\frac{\lambda\psi_1}{\rho-1}\Big)^{\psi_1}\Big(\int_0^b \sigma(z_1)\varphi(z_1)dz_1\Big)^{1-\frac{\psi_1}{\psi_2}} \times \Big(\int_0^b \sigma(z_1)\frac{\varphi(z_1)}{\Upsilon^{\rho\frac{\psi_2}{\psi_1}}(z_1)}\mathcal{M}^{\psi_2}(z_1)dz_1\Big)^{\frac{\psi_1}{\psi_2}}, \tag{8}$$

where

$$\Upsilon(z_1) = \int_0^{z_1} \varphi(z_2)dz_2, \quad \textit{for} \quad z_1 \in (0,\infty), \tag{9}$$

$$\mathcal{M}(z_1) = \frac{\Upsilon(z_1)}{\varphi(z_1)}\Big\{\mathfrak{j}(z_1)\Big[\frac{[\Upsilon(z_2)]'}{\Upsilon(z_2)}\frac{\mathfrak{g}(z_2)}{\mathfrak{j}(z_2)}\Big]_{y(z_1)}^{s(z_1)}\Big\}. \tag{10}$$

Proof. Differentiating both sides of (6) with respect to z_1, we have the following.

$$\frac{d(\mathcal{P}_s f)(z_1)}{dz_1} = \mathfrak{j}'(z_1)\int_{y(z_1)}^{s(z_1)} \frac{1}{\mathfrak{j}(z_2)}\frac{\varphi(z_2)}{\Upsilon(z_2)}\mathfrak{g}(z_2)dz_2 + \mathfrak{j}(z_1)\Big[\frac{1}{\mathfrak{j}(z_2)}\frac{\varphi(z_2)}{\Upsilon(z_2)}\mathfrak{g}(z_2)z_2'\Big]_{y(z_1)}^{s(z_1)},$$

Obviously $\Upsilon'(z_1) = \varphi(z_1)$ from (9), and by using (10), we attain the following.

$$\begin{aligned}(\mathcal{P}_s\mathfrak{g})'(z_1) &= \frac{\mathfrak{j}'(z_1)}{\mathfrak{j}(z_1)}(\mathcal{P}_s\mathfrak{g})(z_1) \\ &+ \mathfrak{j}(z_1)\Big[\frac{1}{\mathfrak{j}(s(z_1))}\frac{[\Upsilon(s(z_1))]'}{\Upsilon(s(z_1))}\mathfrak{g}(s(z_1)) - \frac{1}{\mathfrak{j}(y(z_1))}\frac{[\Upsilon(y(z_1))]'}{\Upsilon(s(z_1))}\mathfrak{g}(y(z_1))\Big] \\ &= \frac{\mathfrak{j}'(z_1)}{\mathfrak{j}(z_1)}(\mathcal{P}_s\mathfrak{g})(z_1) + \mathcal{M}(z_1)\frac{\varphi(z_1)}{\Upsilon(z_1)}.\end{aligned}$$

By integrating the left hand side of (8) by parts from 0 to b and by using (9), we obtain the following:

$$\begin{aligned}\int_0^b \sigma(z_1)\frac{\varphi(z_1)}{\Upsilon^{\rho}(z_1)}(\mathcal{P}_s\mathfrak{g})^{\psi_1}(z_1)dz_1 &= \Big[\frac{\Upsilon^{-\rho+1}(z_1)}{-\rho+1}\sigma(z_1)(\mathcal{P}_s\mathfrak{g})^{\psi_1}(z_1)\Big]_0^b \\ &- \frac{1}{-\rho+1}\int_0^b \Upsilon^{-\rho+1}(z_1)\Big\{\sigma'(z_1)(\mathcal{P}_s\mathfrak{g})^{\psi_1}(z_1) \\ &- \psi_1\frac{\mathfrak{j}'(z_1)}{\mathfrak{j}(z_1)}\sigma(z_1)(\mathcal{P}_s\mathfrak{g})^{\psi_1}(z_1) \\ &+ \psi_1\frac{\varphi(z_1)}{\Upsilon(z_1)}\sigma(z_1)K(z_1)(\mathcal{P}_s\mathfrak{g})^{\psi_1-1}(z_1)\Big\}dz_1,\end{aligned}$$

which results in the following.

$$\int_0^b \sigma(z_1)\frac{\varphi(z_1)}{\Upsilon^{\rho}(z_1)}(\mathcal{P}_s\mathfrak{g})^{\psi_1}(z_1)\Big[1 - \frac{1}{\rho-1}\frac{\sigma'(z_1)}{\sigma(z_1)}\frac{\Upsilon(z_1)}{\varphi(z_1)} + \frac{\psi_1}{\rho-1}\frac{\mathfrak{j}'(z_1)}{\mathfrak{j}(z_1)}\frac{\Upsilon(z_1)}{\varphi(z_1)}\Big]dz_1$$

$$= \frac{\Upsilon^{-\rho+1}(b)}{-\rho+1}\sigma(b)(\mathcal{P}_s\mathfrak{g})^{\psi_1}(b) - \frac{\psi_1}{-\rho+1}\int_0^b \sigma(z_1)\frac{\varphi(z_1)}{\Upsilon^{\rho}(z_1)}(\mathcal{P}_s\mathfrak{g})^{\psi_1-1}(z_1)\mathcal{M}(z_1)dz_1.$$

Since $\mathcal{P}_s\mathfrak{g}$ is positive, from (7) and $\rho > 1$, we obtain the following.

$$\int_0^b \sigma(z_1)\frac{\varphi(z_1)}{\Upsilon^\rho(z_1)}(\mathcal{P}_s\mathfrak{g})^{\psi_1}(z_1)dz_1 \leq \frac{\lambda\psi_1}{\rho-1}\int_0^b \sigma(z_1)\frac{\varphi(z_1)}{\Upsilon^\rho(z_1)}(\mathcal{P}_s\mathfrak{g})^{\psi_1-1}(z_1)\mathcal{M}(z_1)dz_1. \quad (11)$$

The right hand side of (11) by applying Holder's inequality with indices ψ_1, $\frac{\psi_1}{\psi_1-1}$, takes the following form.

$$\begin{aligned}\int_0^b \sigma(z_1)\frac{\varphi(z_1)}{\Upsilon^\rho(z_1)}(\mathcal{P}_s\mathfrak{g})^{\psi_1}(z_1)dz_1 &\leq \frac{\lambda\psi_1}{\rho-1}\int_0^b \Big(\sigma(z_1)\frac{\varphi(z_1)}{\Upsilon^\rho(z_1)}\Big)^{\frac{\psi_1-1}{\psi_1}} \\ &\quad \times (\mathcal{P}_s\mathfrak{g})^{\psi_1-1}(z_1)\Big(\sigma(z_1)\frac{\varphi(z_1)}{\Upsilon^\rho(z_1)}\Big)^{\frac{1}{\psi_1}}\mathcal{M}(z_1)dz_1 \\ &\leq \frac{\lambda\psi_1}{\rho-1}\Big(\int_0^b (\sigma(z_1)\frac{\varphi(z_1)}{\Upsilon^\rho(z_1)}) \\ &\quad \times (\mathcal{P}_s\mathfrak{g})^{\psi_1}(z_1)dz_1\Big)^{\frac{\psi_1-1}{\psi_1}}\Big(\int_0^b (\sigma(z_1)\frac{\varphi(z_1)}{\Upsilon^\rho(z_1)})\mathcal{M}^{\psi_1}(z_1)dz_1\Big)^{\frac{1}{\psi_1}},\end{aligned}$$

Equivalently, we have the following:

$$\Big(\int_0^b \sigma(z_1)\frac{\varphi(z_1)}{\Upsilon^\rho(z_1)}(\mathcal{P}_s\mathfrak{g})^{\psi_1}(z_1)dz_1\Big)^{\frac{1}{\psi_1}} \leq \frac{\lambda\psi_1}{\rho-1}\Big(\int_0^b \sigma(z_1)\frac{\varphi(z_1)}{\Upsilon^\rho(z_1)}\mathcal{M}^{\psi_1}(z_1)dz_1\Big)^{\frac{1}{\psi_1}},$$

Hence, the following is the case.

$$\int_0^b \sigma(z_1)\frac{\varphi(z_1)}{\Upsilon^\rho(z_1)}(\mathcal{P}_s\mathfrak{g})^{\psi_1}(z_1)dz_1 \leq \Big(\frac{\lambda\psi_1}{\rho-1}\Big)^{\psi_1}\int_0^b \sigma(z_1)\frac{\varphi(z_1)}{\Upsilon^\rho(z_1)}\mathcal{M}^{\psi_1}(z_1)dz_1.$$

The last inequality can be restated as follows:

$$\begin{aligned}\int_0^b \sigma(z_1)\frac{\varphi(z_1)}{\Upsilon^\rho(z_1)}(\mathcal{P}_s\mathfrak{g})^{\psi_1}(z_1)dz_1 &\leq \Big(\frac{\lambda\psi_1}{\rho-1}\Big)^{\psi_1}\Big(\int_0^b \sigma(z_1)\varphi(z_1)dz_1\Big)^{1-\frac{\psi_1}{\psi_2}} \\ &\quad \times \Big(\int_0^b \sigma(z_1)\frac{\varphi(z_1)}{\Upsilon^{\rho\frac{\psi_2}{\psi_1}}(z_1)}\mathcal{M}^{\psi_2}(z_1)dz_1\Big)^{\frac{\psi_1}{\psi_2}},\end{aligned}$$

which is the desired inequality (8). □

Some of the important remarks are listed below.

Remark 1. *Theorem 1 of the Copson inequality can be transformed into Hardy inequality (3) by substituting* $\sigma(z_1) = 1$, $\psi_1 = p$, $\psi_2 = 1$, $\varphi(z_2) = \Upsilon(z_2) = 1$, $\Upsilon^\rho(z_1) = x^{p-\alpha}$, $\rho = \alpha - p$, $\mathfrak{g}(z_2) = f(t)$, $y(z_1) = 0$, $s(z_1) = x$, $\mathfrak{j}(z_1) = \mathfrak{j}(z_2) = 1$, *and* $\mathcal{P}_s\mathfrak{g} = Ff$.

Remark 2. *By taking* $\mathfrak{j}(z_1) = \frac{1}{x}$, $\mathfrak{j}(z_2) = 1$, $\psi_1 = p$, $\psi_2 = 1$, $\varphi(z_2) = \Upsilon(z_2) = \Upsilon^\rho(z_1) = 1$, $\sigma(z_1) = 1$, $y(z_1) = 0$, $s(z_1) = x$, *and* $\mathcal{P}_s\mathfrak{g} = F$, *then Theorem 1 changes into [2].*

Remark 3. *It is quite impressive to know that Theorem 1 reduces to [27] by substituting* $y(z_1) = 1$, $\psi_1 = p$, $\psi_2 = 1$, $s(z_1) = x$, $\mathfrak{j}(z_1) = \mathfrak{j}(z_2) = 1$, $\varphi(z_2) = \Upsilon(z_2) = \Upsilon^\rho(z_1) = 1$, $\sigma(z_1) = x^{-1}$, $(\mathcal{P}_s\mathfrak{g})^{\psi_1} = (\frac{F}{logx})^{\psi_1}$ *where* $x \in (1, \infty)$.

Corollary 1. *Assume that* $\mathfrak{g}$, φ, $\mathfrak{j}$, σ, ψ_1, ψ_2, *and* ρ *are similar as of Theorem 1. Moreover,* $y(z_1) = 0$ *in (6) such that the following is the case.*

$$\begin{cases} 0 < s(z_1) < \infty & \text{for all } z_1 \in (0, \infty), \\ s(0) = 0 \text{ and } s(\infty) = \infty. \end{cases} \tag{12}$$

Then, we obtain the following:

$$\int_0^b \sigma(z_1) \frac{\varphi(z_1)}{\Upsilon^\rho(z_1)} (\mathcal{P}_s\mathfrak{g})^{\psi_1}(z_1) dz_1 \leq (\frac{\lambda\psi_1}{\rho - 1})^{\psi_1} \Big(\int_0^b \sigma(z_1)\varphi(z_1) dz_1 \Big)^{1 - \frac{\psi_1}{\psi_2}} \times \Big(\int_0^b \sigma(z_1) \frac{\varphi(z_1)}{\Upsilon^{\rho\frac{\psi_2}{\psi_1}}(z_1)} (\mathcal{M}^*(z_1))^{\psi_2} dz_1 \Big)^{\frac{\psi_1}{\psi_2}}, \tag{13}$$

where

$$\mathcal{M}^*(z_1) = \mathfrak{j}(z_1) \frac{\Upsilon(z_1)}{\varphi(z_1)} \Big[\frac{[\Upsilon(z_2)]'}{\Upsilon(z_2)} \frac{\mathfrak{g}(z_2)}{\mathfrak{j}(z_2)} \Big]_0^{s(z_1)}.$$

Corollary 2. *Under the same supposition of* $\mathfrak{g}$, σ, φ, $\mathfrak{j}$, *and* ρ *of Theorem 1 and* $y(z_1)$ *and* $s(z_1)$ *of Corollary 1 if* $\psi_2 = \psi_1$, *we have the following.*

$$\int_0^b \sigma(z_1) \frac{\varphi(z_1)}{\Upsilon^\rho(z_1)} (\mathcal{P}_s\mathfrak{g})^{\psi_1}(z_1) dz_1 \leq (\frac{\lambda\psi_1}{\rho - 1})^{\psi_1} \int_0^b \sigma(z_1) \frac{\varphi(z_1)}{\Upsilon^\rho(z_1)} (\mathcal{M}^*(z_1))^{\psi_1} dz_1. \tag{14}$$

Corollary 3. *Suppose* $\mathfrak{g}$, ψ_2, ψ_1, σ, φ, y, $\mathfrak{j}$, *and* ρ *are mentioned of Corollary 2 and take* $s(z_1) = z_1$, *we obtain the following.*

$$\int_0^b \sigma(z_1) \frac{\varphi(z_1)}{\Upsilon^\rho(z_1)} (\mathcal{P}_s\mathfrak{g})^{\psi_1}(z_1) dz_1 \leq (\frac{\lambda\psi_1}{\rho - 1})^{\psi_1} \int_0^b \sigma(z_1) \frac{\varphi(z_1)}{\Upsilon^\rho(z_1)} \mathfrak{g}^{\psi_1}(z_1) dz_1.$$

Example 1. *By choosing* $\rho = \frac{\psi_2}{\psi_1}$, $\psi_2 = 5/3$, *and* $\psi_1 = 3/2$ *in Theorem 1, we have the following:*

$$\int_0^b \sigma(z_1) \frac{\varphi(z_1)}{\Upsilon^{\frac{10}{9}}(z_1)} (\mathcal{P}_s\mathfrak{g})^{3/2}(z_1) dz_1 \leq \frac{81\sqrt{3}\lambda^{3/2}}{2\sqrt{2}} \Big(\int_0^b \sigma(z_1)\varphi(z_1) dz_1 \Big)^{\frac{1}{10}} \times \Big(\int_0^b \sigma(z_1) \frac{\varphi(z_1)}{\Upsilon^{\frac{100}{81}}(z_1)} \mathcal{M}^{5/3}(z_1) dz_1 \Big)^{\frac{9}{10}},$$

where

$$1 - 9\frac{\sigma'(z_1)}{\sigma(z_1)} \frac{\Upsilon(z_1)}{\varphi(z_1)} + \frac{27}{2} \frac{\mathfrak{j}'(z_1)}{\mathfrak{j}(z_1)} \frac{\Upsilon(z_1)}{\varphi(z_1)} \geq \frac{1}{\lambda},$$

for some $\lambda > 0$. *Therefore, the example satisfies (8) for any* $\rho > 1$.

Theorem 2. *If* $\mathfrak{g}$, φ, σ, $\mathfrak{j}$, y, s, ψ_1, *and* ψ_2 *are the same as of Theorem 1 and satisfies the following:*

$$(\mathcal{P}_s\mathfrak{g})(z_1) = \frac{1}{\mathfrak{j}(z_1)} \int_{y(z_1)}^{s(z_1)} \mathfrak{j}(z_2) \frac{\varphi(z_2)}{\Upsilon(z_2)} \mathfrak{g}(z_2) dy, \tag{15}$$

with

$$\Upsilon(z_1) = \int_{z_1}^{\infty} \varphi(z_2) dz_2, \quad for \quad z_1 \in (0, \infty), \tag{16}$$

for $\rho < 1$. *Then the following is the case:*

$$\int_0^b \sigma(z_1) \frac{\varphi(z_1)}{\Upsilon^{\rho}(z_1)} (\mathcal{P}_s \mathfrak{g})^{\psi_1}(z_1) dz_1 \leq \left(\frac{\beta \psi_1}{1-\rho}\right)^{\psi_1} \Big(\int_0^b \sigma(z_1) \varphi(z_1) dz_1 \Big)^{1-\frac{\psi_1}{\psi_2}} \times \Big(\int_0^b \sigma(z_1) \frac{\varphi(z_1)}{\Upsilon^{\rho \frac{\psi_2}{\psi_1}}(z_1)} \mathcal{M}_2^{\psi_2}(z_1) dz_1 \Big)^{\frac{\psi_1}{\psi_2}}, \tag{17}$$

where

$$\mathcal{M}_1(z_1) = \frac{\Upsilon(z_1)}{\varphi(z_1)} \Big\{ \frac{1}{\mathfrak{j}(z_1)} \Big[\frac{[\Upsilon(z_2)]'}{\Upsilon(z_2)} \mathfrak{j}(z_2) \mathfrak{g}(z_2) \Big]_{s(z_1)}^{y(z_1)} \Big\}, \tag{18}$$

$$1 - \frac{1}{1-\rho} \frac{\sigma'(z_1)}{\sigma(z_1)} \frac{\Upsilon(z_1)}{\varphi(z_1)} + \frac{\psi_1}{1-\rho} \frac{\mathfrak{j}'(z_1)}{\mathfrak{j}(z_1)} \frac{\Upsilon(z_1)}{\varphi(z_1)} \geq \frac{1}{\beta}.$$

Proof. Differentiate (15), we obtain the following:

$$\frac{d(\mathcal{P}_s \mathfrak{g})(z_1)}{dz_1} = \frac{-\mathfrak{j}'(z_1)}{\mathfrak{k}^2(z_1)} \int_{y(z_1)}^{s(z_1)} \mathfrak{j}(z_2) \frac{\varphi(z_2)}{\Upsilon(z_2)} \mathfrak{g}(z_2) dz_2 + \frac{1}{\mathfrak{j}(z_1)} \Big[\mathfrak{j}(z_2) \frac{\varphi(z_2)}{\Upsilon(z_2)} \mathfrak{g}(z_2) z_2' \Big]_{y(z_1)}^{s(z_1)},$$

since $\Upsilon'(z_1) = -\varphi(z_1)$ from (16) and by using (18), one obtains the following.

$$\begin{aligned} (\mathcal{P}_s \mathfrak{g})'(z_1) &= \frac{-\mathfrak{j}'(z_1)}{\mathfrak{j}(z_1)} (\mathcal{P}_s \mathfrak{g})(z_1) \\ &+ \frac{1}{\mathfrak{j}(z_1)} \Big[-\mathfrak{j}(s(z_1)) \frac{[\Upsilon(s(z_1))]'}{\Upsilon(h(x))} \mathfrak{g}(h(x)) + \mathfrak{j}(y(z_1)) \frac{[\Upsilon(y(z_1))]'}{\Upsilon(s(z_1))} \mathfrak{g}(y(z_1)) \Big] \\ &= \frac{-\mathfrak{j}'(z_1)}{\mathfrak{j}(z_1)} (\mathcal{P}_s \mathfrak{g})(z_1) + \mathcal{M}_1(z_1) \frac{\varphi(z_1)}{\Upsilon(z_1)}. \end{aligned}$$

Integrate the left side of (17) by parts from 0 to b and use (16), we deduce the following:

$$\begin{aligned} \int_0^b \sigma(z_1) \frac{\varphi(z_1)}{\Upsilon^{\rho}(z_1)} (\mathcal{P}_s \mathfrak{g})^{\psi_1}(z_1) dz_1 &= \Big[\frac{-\Upsilon^{-\rho+1}(z_1)}{1-\rho} \sigma(z_1) (\mathcal{P}_s \mathfrak{g})^{\psi_1}(z_1) \Big]_0^b \\ &+ \frac{1}{1-\rho} \int_0^b \Upsilon^{-\rho+1}(z_1) \Big\{ \sigma'(z_1) (\mathcal{P}_s \mathfrak{g})^{\psi_1}(z_1) \\ &- \psi_1 \frac{\mathfrak{j}'(z_1)}{\mathfrak{j}(z_1)} \sigma(z_1) (\mathcal{P}_s \mathfrak{g})^{\psi_1}(z_1) \\ &+ \psi_1 \frac{\varphi(x)}{\Upsilon(z_1)} \sigma(z_1) \mathcal{M}_1(z_1) (\mathcal{P}_s \mathfrak{g})^{\psi_1 - 1}(z_1) \Big\} dz_1, \end{aligned}$$

or

$$\int_0^b \sigma(z_1) \frac{\varphi(z_1)}{\Upsilon^{\rho}(z_1)} (\mathcal{P}_s \mathfrak{g})^{\psi_1}(z_1) \Big[1 - \frac{1}{1-\rho} \frac{\sigma'(z_1)}{\sigma(z_1)} \frac{\Upsilon(z_1)}{\varphi(z_1)} + \frac{\psi_1}{1-\rho} \frac{\mathfrak{j}'(z_1)}{\mathfrak{j}(z_1)} \frac{\Upsilon(z_1)}{\varphi(z_1)} \Big] dz_1$$

$$= \frac{-\Upsilon^{-\rho+1}(b)}{1-\rho}\sigma(b)(\mathcal{P}_s\mathfrak{g})^{\psi_1}(b) + \frac{\psi_1}{1-\rho}\int_0^b \sigma(z_1)\frac{\varphi(z_1)}{\Upsilon^\rho(z_1)}(\mathcal{P}_s\mathfrak{g})^{\psi_1-1}(z_1)\mathcal{M}_1(z_1)dz_1.$$

The remaining proof follows the same proof of Theorem 1 with some changes. We omit the details here. □

Remark 4. *If* $\mathfrak{j}(z_1) = \mathfrak{j}(z_2) = 1$, $\psi_1 = p$, $\psi_2 = 1$, $\varphi(z_2) = \Upsilon(z_2) = \Upsilon^\rho(z_1) = 1$, $y(z_1) = x$, $s(z_1) = \infty$, $\sigma(z_1) = x^{-m}$, *and* $\mathcal{P}_s\mathfrak{g} = F$, *then Theorem 2 can be converted into the inequality proved by Hardy [3].*

Remark 5. *If* $\psi_1 = p$, $\psi_2 = 1$, $y(z_1) = x$, $s(z_1) = \infty$, $\mathfrak{j}(z_1) = q_k(x)$, $\mathfrak{j}(z_2) = q_k(t)$, $\varphi(z_2) = 1$, $\sigma(z_1) = x^{-m}$, $\Upsilon(z_2) = \Upsilon^\rho(z_1) = 1$, $\mathfrak{g}(z_2) = \frac{f_j(t)}{t}$, *and* $(\mathcal{P}_s\mathfrak{g})^{\psi_1} = F_1^{p_1}F_2^{p_2}[F_2^{p_2}F_3^{p_3} + F_3^{p_3}F_1^{p_1} + F_1^{p_1}F_2^{p_2}]$, $k = 1,2,3$, *then we can easily obtain the result in [28] from Theorem 2.*

Corollary 4. *Assuming* $\mathfrak{g}, \sigma, \varphi, \mathfrak{j}, \rho, \psi_1, y$, *and* ψ_2 *satisfies Theorem 1 and* $s(z_1) = \infty$ *in (15). Then, we have the following:*

$$\int_0^b \sigma(z_1)\frac{\varphi(z_1)}{\Upsilon^\rho(z_1)}(\mathcal{P}_s\mathfrak{g})^{\psi_1}(z_1)dz_1 \le \left(\frac{\beta\psi_1}{1-\rho}\right)^{\psi_1}\left(\int_0^b \sigma(z_1)\varphi(z_1)dz_1\right)^{1-\frac{\psi_1}{\psi_2}}$$
$$\times\left(\int_0^b \sigma(z_1)\frac{\varphi(z_1)}{\Upsilon^{\rho\frac{\psi_2}{\psi_1}}(z_1)}(\mathcal{M}_1^*(z_1))^{\psi_2}dz_1\right)^{\frac{\psi_1}{\psi_2}}, \quad (19)$$

where

$$\mathcal{M}_1^*(z_1) = \frac{\Upsilon(z_1)}{\varphi(z_1)}\left\{\frac{1}{\mathfrak{j}(z_1)}\left[\frac{[\Upsilon(z_2)]'}{\Upsilon(z_2)}\mathfrak{j}(z_2)\mathfrak{g}(z_2)\right]_{y(z_1)}^{\infty}\right\}.$$

Corollary 5. *If* $\psi_2 = \psi_1$ *and with the same assumptions of* $\mathfrak{g}, s, \sigma, \varphi, \mathfrak{j}, \rho$, *and* y *of Corollary 4, we have the following:*

$$\int_0^b \sigma(z_1)\frac{\varphi(z_1)}{\Upsilon^\rho(z_1)}(\mathcal{P}_s\mathfrak{g})^{\psi_1}(z_1)dz_1 \le \left(\frac{\beta\psi_1}{1-\rho}\right)^{\psi_1}\int_0^b \sigma(z_1)\frac{\varphi(z_1)}{\Upsilon^\rho(z_1)}(\mathcal{M}_1^*(z_1))^{\psi_1}dz_1. \quad (20)$$

Corollary 6. *Let* $\mathfrak{g}, s, \sigma, \varphi, \mathfrak{j}, \rho, \psi_2$, *and* ψ_1 *be described as of Corollary 5 and take* $y(z_1) = z_1$, *we obtain the following.*

$$\int_0^b \sigma(z_1)\frac{\varphi(z_1)}{\Upsilon^\rho(z_1)}(\mathcal{P}_s\mathfrak{g})^{\psi_1}(z_1)dz_1 \le \left(\frac{\beta\psi_1}{1-\rho}\right)^{\psi_1}\int_0^b \sigma(z_1)\frac{\varphi(z_1)}{\Upsilon^\rho(z_1)}\mathfrak{g}^{\psi_1}(z_1)dz_1.$$

4. Induction of Steklov Operator on Hardy Integral Inequalities

This section is related to the generalizations of Hardy style integral inequalities with the application of the Steklov operator.

Theorem 3. *Let* $\mathfrak{g}, \sigma, \mathfrak{j}, y, s, \psi_1$, *and* ψ_2 *be defined as in Theorem 1 and* $\mathfrak{u}$ *be a non-negative measurable function on* $(0,\infty)$. *For* $\rho > 1$, *if the following is the case:*

$$(\mathcal{G}_s\mathfrak{g})(z_1) = \frac{1}{\mathfrak{j}(z_1)}\int_{y(z_1)}^{s(z_1)} \mathfrak{j}(z_2)\mathfrak{u}(z_2)\mathfrak{g}(z_2)dy, \quad (21)$$

and

$$\Theta(z_1) = \int_0^{z_1} \mathfrak{u}(z_2)dz_2, \tag{22}$$

then

$$\int_0^b \sigma(z_1)\frac{\mathfrak{u}(z_1)}{\Theta^\rho(z_1)}(\mathcal{G}_s\mathfrak{g})^{\psi_1}(z_1)dz_1 \leq (\frac{\lambda_1\psi_1}{\rho-1})^{\psi_1}\Big(\int_0^b \sigma(z_1)\mathfrak{u}(z_1)dz_1\Big)^{1-\frac{\psi_1}{\psi_2}}$$
$$\times\Big(\int_0^b \sigma(z_1)\frac{\mathfrak{u}(z_1)}{\Theta^{\rho\frac{\psi_2}{\psi_1}}(z_1)}\mathcal{M}_2^{\psi_2}(z_1)dz_1\Big)^{\frac{\psi_1}{\psi_2}}, \tag{23}$$

where

$$\mathcal{M}_2(z_1) = \frac{\Theta(z_1)}{\mathfrak{u}(z_1)}\Big\{\frac{1}{\mathfrak{j}(z_1)}\Big[\mathfrak{j}(z_2))[\Theta(z_2)]'\mathfrak{g}(z_2)\Big]_{y(z_1)}^{s(z_1)}\Big\}, \tag{24}$$

and for some λ_1, we have the following.

$$1 - \frac{1}{\rho-1}\frac{\sigma'(z_1)}{\sigma(z_1)}\frac{\Theta(z_1)}{\mathfrak{u}(z_1)} + \frac{\psi_1}{\rho-1}\frac{\mathfrak{j}'(z_1)}{\mathfrak{j}(z_1)}\frac{\Theta(z_1)}{\mathfrak{u}(z_1)} \geq \frac{1}{\lambda_1}. \tag{25}$$

Proof. We calculate the derivative of $\mathfrak{G}_s\mathfrak{g}$ as follows:

$$\frac{d(\mathcal{G}_s\mathfrak{g})(z_1)}{dz_1} = \frac{-1}{\mathfrak{j}^2(z_1)}\mathfrak{j}'(z_1)\int_{y(z_1)}^{s(z_1)} \mathfrak{j}(z_2)\mathfrak{u}(z_2)\mathfrak{g}(z_2)dz_2 + \frac{1}{\mathfrak{j}(z_1)}\Big[\mathfrak{j}(z_2)\mathfrak{u}(z_2)\mathfrak{g}(z_2)z_2'\Big]_{y(z_1)}^{s(z_1)},$$

clearly $\Theta'(z_1) = \mathfrak{u}(z_1)$ from (22) and with (24), hence the following is the case.

$$\begin{aligned}(\mathcal{G}_s\mathfrak{g})'(z_1) &= \frac{-\mathfrak{j}'(z_1)}{\mathfrak{j}^2(z_1)}\int_{y(z_1)}^{s(z_1)} \mathfrak{j}(z_2)\mathfrak{u}(z_2)\mathfrak{g}(z_2)dz_2 \\ &+ \frac{1}{\mathfrak{j}(z_1)}\Big[\mathfrak{j}(s(z_1))[\Theta(s(z_1))]'\mathfrak{g}(s(z_1)) - \mathfrak{j}(y(z_1))[\Theta(y(z_1))]'\mathfrak{g}(y(z_1))\Big] \\ &= \frac{-\mathfrak{j}'(z_1)}{\mathfrak{j}(z_1)}(\mathcal{G}_s\mathfrak{g})(z_1) + \frac{\mathfrak{u}(z_1)}{\Theta(z_1)}\mathcal{M}_2(z_1).\end{aligned}$$

Integrating (23) by parts and utilizing (22), we obtain the following:

$$\begin{aligned}\int_0^b \sigma(z_1)\frac{\mathfrak{u}(z_1)}{\Theta^\rho(z_1)}(\mathcal{G}_s\mathfrak{g})^{\psi_1}(z_1)dz_1 &= \Big[\frac{\Theta^{-\rho+1}(z_1)}{-\rho+1}\sigma(z_1)(\mathcal{G}_s\mathfrak{g})^{\psi_1}(z_1)\Big]_0^b \\ &- \frac{1}{-\rho+1}\int_0^b \sigma^{-\rho+1}(z_1)\Big\{\sigma'(z_1)(\mathcal{G}_s\mathfrak{g})^{\psi_1}(z_1) \\ &- \psi_1\frac{\mathfrak{j}'(z_1)}{\mathfrak{j}(z_1)}\sigma(z_1)(\mathcal{G}_s\mathfrak{g})^{\psi_1}(z_1) \\ &+ \psi_1\frac{\mathfrak{u}(z_1)}{\Theta(z_1)}\sigma(z_1)\mathcal{M}_2(z_1)(\mathcal{G}_s\mathfrak{g})^{\psi_1-1}(z_1)\Big\}dz_1,\end{aligned}$$

which implies the following.

$$\int_0^b \sigma(z_1)\frac{\mathfrak{u}(z_1)}{\Theta^\rho(z_1)}(\mathcal{G}_s\mathfrak{g})^{\psi_1}(z_1)\Big[1-\frac{1}{\rho-1}\frac{w\sigma'(z_1)}{\sigma(z_1)}\frac{\Theta(z_1)}{\mathfrak{u}(z_1)}+\frac{\psi_1}{\rho-1}\frac{\mathfrak{j}'(z_1)}{\mathfrak{j}(z_1)}\frac{\Theta(z_1)}{\mathfrak{u}(z_1)}\Big]dz_1$$

$$=\frac{\Theta^{-\rho+1}(b)}{-\rho+1}\sigma(b)(\mathcal{G}_s\mathfrak{g})^{\psi_1}(b)-\frac{\psi_1}{-\rho+1}\int_0^b \sigma(z_1)\frac{\mathfrak{u}(z_1)}{\Theta^\rho(z_1)}(\mathcal{G}_s\mathfrak{g})^{\psi_1-1}(z_1)\mathcal{M}_2(z_1)dz_1.$$

As $\rho>1$, $\mathcal{G}_s\mathfrak{g}$ is positive, the above equation with (25) becomes the following.

$$\int_0^b \sigma(z_1)\frac{\mathfrak{u}(z_1)}{\Theta^\rho(z_1)}(\mathcal{G}_s\mathfrak{g})^{\psi_1}(z_1)dz_1\le\frac{\lambda_1\psi_1}{\rho-1}\int_0^b \sigma(z_1)\frac{\mathfrak{u}(z_1)}{\Theta^\rho(z_1)}(\mathcal{G}_s\mathfrak{g})^{\psi_1-1}(z_1)\mathcal{M}_2(z_1)dz_1. \quad (26)$$

Employing Holder's inequality with indices ψ_1, $\frac{\psi_1}{\psi_1-1}$ on (26) yields the following:

$$\begin{aligned}\int_0^b \sigma(z_1)\frac{\mathfrak{u}(z_1)}{\Theta^\rho(z_1)}(\mathcal{G}_s\mathfrak{g})^{\psi_1}(z_1)dz_1 &\le \frac{\lambda_1\psi_1}{\rho-1}\int_0^b\Big(\sigma(z_1)\frac{\mathfrak{u}(z_1)}{\Theta^\rho(z_1)}\Big)^{\frac{\psi_1-1}{\psi_1}}\\ &\times(\mathcal{G}_s\mathfrak{g})^{\psi_1-1}(z_1)\Big(\sigma(z_1)\frac{\mathfrak{u}(z_1)}{\Theta^\rho(z_1)}\Big)^{\frac{1}{\psi_1}}\mathcal{M}_2(z_1)dz_1\\ &\le\frac{\lambda_1\psi_1}{\rho-1}\Big(\int_0^b(\sigma(z_1)\frac{\mathfrak{u}(z_1)}{\Theta^\rho(z_1)})\\ &\times(\mathcal{G}_s\mathfrak{g})^{\psi_1}(z_1)dz_1\Big)^{\frac{p-1}{\psi_1}}\Big(\int_0^b(\sigma(z_1)\frac{\mathfrak{u}(z_1)}{\Theta^\rho(z_1)})\mathcal{M}_2^{\psi_1}(z_1)dz_1\Big)^{\frac{1}{\psi_1}},\end{aligned}$$

from which one obtains

$$\Big(\int_0^b \sigma(z_1)\frac{\mathfrak{u}(z_1)}{\Theta^\rho(z_1)}(\mathcal{G}_s\mathfrak{g})^{\psi_1}(z_1)dz_1\Big)^{\frac{1}{\psi_1}}\le\frac{\lambda_1\psi_1}{\rho-1}\Big(\int_0^b \sigma(z_1)\frac{\mathfrak{u}(z_1)}{\Theta^\rho(z_1)}\mathcal{M}_2^{\psi_1}(z_1)dz_1\Big)^{\frac{1}{\psi_1}},$$

and, therefore, the following obtains.

$$\int_0^b \sigma(z_1)\frac{\mathfrak{u}(z_1)}{\Theta^\rho(z_1)}(\mathcal{G}_s\mathfrak{g})^{\psi_1}(z_1)dz_1\le\Big(\frac{\lambda_1\psi_1}{\rho-1}\Big)^{\psi_1}\int_0^b \sigma(z_1)\frac{\mathfrak{u}(z_1)}{\Theta^\rho(z_1)}\mathcal{M}_2^{\psi_1}(z_1)dz_1.$$

The previous inequality provides the following estimation.

$$\begin{aligned}\int_0^b \sigma(z_1)\frac{\mathfrak{u}(z_1)}{\Theta^\rho(z_1)}(\mathcal{G}_s\mathfrak{g})^{\psi_1}(z_1)dz_1 &\le\Big(\frac{\lambda_1\psi_1}{\rho-1}\Big)^{\psi_1}\Big(\int_0^b\sigma(z_1)\mathfrak{u}(z_1)dz_1\Big)^{1-\frac{\psi_1}{\psi_2}}\\ &\times\Big(\int_0^b\sigma(z_1)\frac{\mathfrak{u}(z_1)}{\Theta^{\frac{\rho\psi_2}{\psi_1}}(z_1)}\mathcal{M}_2^{\psi_2}(z_1)dz_1\Big)^{\frac{\psi_1}{\psi_2}}.\end{aligned}$$

□

Remark 6. *If* $\mathfrak{j}(z_1)=q_k(x)$, $\mathfrak{j}(z_2)=q_k(t)$, $\mathfrak{u}(z_2)=1$, $\mathfrak{g}(z_2)=\frac{f_j(t)}{t}$, $y(z_1)=x$, $s(z_1)=\infty$, $\sigma(z_1)=x^{-m}$, $\psi_1=p$, $\psi_2=1$, $\mathfrak{u}(z_1)=\Theta^\rho(z_1)=1$, *and* $\mathcal{G}_s\mathfrak{g}=F_1F_2$, $k=1,2$ *in Theorem 3, then we obtain the inequality obtained by Pachpatte in [29].*

Remark 7. *Inequality (23) of Theorem 3 converts into the inequality given in [27], if we take* $\mathfrak{j}(z_1) = \mathfrak{j}(z_2) = 1$, $\psi_1 = p$, $\psi_2 = 1$, $y(z_1) = x$, $s(z_1) = 1$, $\mathfrak{u}(z_2) = 1$, $\Theta^\rho(z_1) = 1$, $(\mathcal{G}_s\mathfrak{g})^{\psi_1} = (\frac{F(x)}{logx})^{\psi_1}$, $\sigma(z_1) = x^{-1}$, *and* $x \in (0,1)$.

Corollary 7. *Suppose* $\mathfrak{g}$, ρ, σ, $\mathfrak{j}$, *and* Θ *be fulfilled as in Theorem 3. Furthermore, by setting* $y(z_1) = 0$ *and* $\psi_2 = \psi_1$ *and satisfying (12), we have the following:*

$$\int_0^b \sigma(z_1)\frac{\mathfrak{u}(z_1)}{\Theta^\rho(z_1)}(\mathcal{G}_s\mathfrak{g})^{\psi_1}(z_1)dz_1 \leq (\frac{\lambda_1\psi_1}{\rho-1})^{\psi_1}\int_0^b \sigma(z_1)\frac{\mathfrak{u}(z_1)}{\Theta^\rho(z_1)}(\mathcal{M}_2^*(z_1))^{\psi_1}dz_1,$$

where the following obtains.

$$\mathcal{M}_2^*(z_1) = \frac{\Theta(z_1)}{\mathfrak{u}(z_1)}\left\{\frac{1}{\mathfrak{j}(z_1)}\Big[\mathfrak{j}(z_2)[\Theta(z_2)]'\mathfrak{g}(z_2)\Big]_0^{s(z_1)}\right\}.$$

Corollary 8. *Assume that* $\mathfrak{g}$, y, ρ, σ, $\mathfrak{j}$, Θ, ψ_2, *and* ψ_1 *are given as in Corollary 7 and substitute* $s(z_1) = z_1$, *we obtain the following.*

$$\int_0^b \sigma(z_1)\frac{\mathfrak{u}(z_1)}{\Theta^\rho(z_1)}(\mathcal{G}_s\mathfrak{g})^{\psi_1}(z_1)dz_1 \leq (\frac{\lambda_1\psi_1}{\rho-1})^{\psi_1}\int_0^b \sigma(z_1)\frac{\mathfrak{u}(z_1)}{\Theta^\rho(z_1)}\mathfrak{g}^{\psi_1}(z_1)dz_1.$$

Theorem 4. *Let* $\mathfrak{g}$, $\mathfrak{u}$, σ, $\mathfrak{j}$, y, s, ψ_1, *and* ψ_2 *be satisfied under the same hypotheses of Theorem 2 and the following:*

$$(\mathcal{G}_s\mathfrak{g})(z_1) = \mathfrak{j}(z_1)\int_{y(z_1)}^{s(z_1)}\frac{1}{\mathfrak{j}(z_2)}\mathfrak{u}(z_2)\mathfrak{g}(z_2)dz_2, \tag{27}$$

with

$$\Theta(z_1) = \int_{z_1}^{\infty}\mathfrak{u}(z_2)dz_2, \quad \textit{for} \quad z_1 \in (0,\infty), \tag{28}$$

for $\rho < 1$. *Then the following obtains:*

$$\int_0^b \sigma(z_1)\frac{\mathfrak{u}(z_1)}{\Theta^\rho(z_1)}(\mathcal{G}_s\mathfrak{g})^{\psi_1}(z_1)dz_1 \leq (\frac{\beta_1\psi_1}{1-\rho})^{\psi_1}\Big(\int_0^b \sigma(z_1)\mathfrak{u}(z_1)dz_1\Big)^{1-\frac{\psi_1}{\psi_2}}$$
$$\times\Big(\int_0^b \sigma(z_1)\frac{\mathfrak{u}(z_1)}{\Theta^{\rho\frac{\psi_2}{\psi_1}}(z_1)}\mathcal{M}_3^{\psi_2}(z_1)dz_1\Big)^{\frac{\psi_1}{\psi_2}}, \tag{29}$$

where the following is the case.

$$\mathcal{M}_3(z_1) = \frac{\Theta(z_1)}{\mathfrak{u}(z_1)}\left\{\mathfrak{j}(z_1)\Big[\frac{\mathfrak{g}(z_2)}{\mathfrak{j}(z_2)}[\Theta(z_2)]'\Big]_{s(z_1)}^{y(z_1)}\right\}, \tag{30}$$

$$1 - \frac{1}{1-\rho}\frac{\sigma'(z_1)}{\sigma(z_1)}\frac{\Theta(z_1)}{\mathfrak{u}(z_1)} - \frac{\psi_1}{1-\rho}\frac{\mathfrak{j}'(z_1)}{\mathfrak{j}(z_1)}\frac{\Theta(z_1)}{\mathfrak{u}(z_1)} \geq \frac{1}{\beta_1}.$$

Proof. The proof can be carried out with suitable alterations of Theorem 3. Details are left to the readers. □

Remark 8. *We can achieve, from Theorem 4, the discrete type of inequality studied by Hardy [1], if* $\mathcal{G}_s\mathfrak{g} = A_n$, $\mathfrak{j}(z_1) = \frac{1}{n}$, $\mathfrak{u}(z_2) = 1$, $\mathfrak{g}(z_2) = a_r$, $\psi_1 = p$, $\psi_2 = 1$, $\sigma(z_1) = 1$, $\mathfrak{u}(z_1) = \Theta^\rho(z_1) = 1$, $y(z_1) = 0$, *and* $s(z_1) = x$.

Remark 9. *Place* $y(z_1) = 0$, $s(z_1) = x$, $\mathfrak{j}(z_1) = \mathfrak{j}(z_2) = 1$, $\sigma(z_1) = x^{-m}$, $\psi_1 = p$, $\psi_2 = 1$, $\mathcal{G}_s\mathfrak{g} = F$ *and* $\mathfrak{u}(z_1) = \mathfrak{u}(z_2) = \Theta^{\rho}(z_1) = 1$ *in Theorem 4, then it reduces to an inequality due to Hardy [3].*

Corollary 9. *Consider* $\mathfrak{g}$, $\mathfrak{u}$, Θ, σ, $\mathfrak{j}$, y, *and* ρ *as defined as of Theorem 4. In addition, place* $s(z_1) = \infty$ *and* $\psi_2 = \psi_1$ *in (27). Therefore, we have the following:*

$$\int_0^b \sigma(z_1)\frac{\mathfrak{u}(z_1)}{\Theta^{\rho}(z_1)}(\mathcal{G}_s\mathfrak{g})^{\psi_1}(z_1)dz_1 \le (\frac{\beta_1\psi_1}{1-\rho})^{\psi_1}\Big(\int_0^b \sigma(z_1)\mathfrak{u}(z_1)dz_1\Big)^{1-\frac{\psi_1}{\psi_2}} \times \Big(\int_0^b \sigma(z_1)\frac{\mathfrak{u}(z_1)}{\Theta^{\rho\frac{\psi_2}{\psi_1}}(z_1)}(\mathcal{M}_3^*(z_1))^{\psi_2}dz_1\Big)^{\frac{\psi_1}{\psi_2}},$$

where the following results obtain.

$$K_3^*(z_1) = \frac{\Theta(z_1)}{\mathfrak{u}(z_1)}\Big\{\mathfrak{j}(z_1)\Big[\frac{[\Theta(z_2)]'}{\Theta(z_2)}\frac{1}{\mathfrak{j}(z_2)}\mathfrak{g}(z_2)\Big]_{y(z_1)}^{\infty}\Big\}.$$

Corollary 10. *Under the same assumptions* $\mathfrak{g}$, $\mathfrak{u}$, ρ, σ, Θ, $\mathfrak{j}$, y, *and* s *of Corollary 9 and by substituting* $\psi_2 = \psi_1$, *we obtain the following.*

$$\int_0^b \sigma(z_1)\frac{\mathfrak{u}(z_1)}{\Theta^{\rho}(z_1)}(\mathcal{G}_s\mathfrak{g})^{\psi_1}(z_1)dx \le (\frac{\beta_1\psi_1}{1-\rho})^{\psi_1}\int_0^b \sigma(z_1)\frac{\mathfrak{u}(z_1)}{\Theta^{\rho}(z_1)}\big(\mathcal{M}_3^*(z_1)\big)^{\psi_1}dz_1.$$

Corollary 11. *If* $y(z_1) = z_1$ *in Corollary 10, we have the following.*

$$\int_0^b \sigma(z_1)\frac{\mathfrak{u}(z_1)}{\Theta^{\rho}(z_1)}(\mathcal{G}_s\mathfrak{g})^{\psi_1}(z_1)dz_1 \le (\frac{\beta_1\psi_1}{1-\rho})^{\psi_1}\int_0^b \sigma(z_1)\frac{\mathfrak{u}(z_1)}{\Theta^{\rho}(z_1)}\mathfrak{g}^{\psi_1}(z_1)dz_1.$$

Theorem 5. *Let* $\mathfrak{g}$, σ, $\mathfrak{j}$, y, s, ψ_1, ψ_2, $\mathfrak{u}$, *and* Θ *be as they are in Theorem 3 and* $\rho > 1$. *If the following is the case:*

$$(\mathcal{G}_s\mathfrak{g})(z_1) = \mathfrak{j}(z_1)\int_0^{z_1}\frac{1}{\mathfrak{j}(z_2)}\frac{\mathfrak{u}(z_2)}{\Theta(z_2)}\mathfrak{g}(z_2)dz_2,$$

then

$$\int_0^b \sigma(z_1)\frac{\mathfrak{u}(z_1)}{\Theta^{\rho}(x)}(\mathcal{G}_s\mathfrak{g})^{\psi_1}(z_1)dz_1 \le (\frac{\beta_2\psi_1}{\rho-1})^{\psi_1}\Big(\int_0^b \sigma(z_1)\mathfrak{u}(z_1)dz_1\Big)^{1-\frac{\psi_1}{\psi_2}} \times \Big(\int_0^b \sigma(z_1)\frac{\mathfrak{u}(z_1)}{\Theta^{\rho\frac{\psi_2}{\psi_1}}(z_1)}\mathfrak{g}^{\psi_2}(z_1)dz_1\Big)^{\frac{\psi_1}{\psi_2}},$$

where for some β_2, *we obtain the following.*

$$1 - \frac{1}{\rho-1}\frac{\sigma'(z_1)}{\sigma(z_1)}\frac{\Theta(z_1)}{\mathfrak{u}(z_1)} - \frac{\psi_1}{\rho-1}\frac{\mathfrak{j}'(z_1)}{\mathfrak{j}(z_1)}\frac{\Theta(z_1)}{\mathfrak{u}(z_1)} \ge \frac{1}{\beta_2}.$$

Proof. The application is the same as of Theorem 3 with appropriate modifications. □

The counter example of Theorem 3 is as follows.

Example 2. *Let $\rho = 1/2$ in Theorem 3. Then, we achieve the following:*

$$\int_0^b \sigma(z_1)\frac{\mathfrak{u}(z_1)}{\Theta^{1/2}(z_1)}(\mathcal{G}_s\mathfrak{g})^{\psi_1}(z_1)\Big[1+2\frac{\sigma'(z_1)}{\sigma(z_1)}\frac{\Theta(z_1)}{\mathfrak{u}(z_1)} - 2\psi_1\frac{\mathfrak{j}'(z_1)}{\mathfrak{j}(z_1)}\frac{\Theta(z_1)}{\mathfrak{u}(z_1)}\Big]dz_1$$

$$= 2\Theta^{1/2}(b)\sigma(b)(\mathcal{G}_s\mathfrak{g})^{\psi_1}(b) - 2\psi_1\int_0^b \sigma(z_1)\frac{\mathfrak{u}(z_1)}{\Theta^{\rho}(z_1)}(\mathcal{G}_s\mathfrak{g})^{\psi_1-1}(z_1)\mathcal{M}_2(z_1)dz_1.$$

implying the following results.

$$\int_0^b \sigma(z_1)\frac{\mathfrak{u}(z_1)}{\Theta^{\rho}(z_1)}(\mathcal{G}_s\mathfrak{g})^{\psi_1}(z_1)\Big[1+2\frac{\sigma'(z_1)}{\sigma(z_1)}\frac{\Theta(z_1)}{\mathfrak{u}(z_1)} - 2\psi_1\frac{\mathfrak{j}'(z_1)}{\mathfrak{j}(z_1)}\frac{\Theta(z_1)}{\mathfrak{u}(z_1)}\Big]dz_1$$

$$\leq 2\Theta^{1/2}(b)\sigma(b)(\mathcal{G}_s\mathfrak{g})^{\psi_1}(b).$$

If we set the following:

$$1+2\frac{\sigma'(z_1)}{\sigma(z_1)}\frac{\Theta(z_1)}{\mathfrak{u}(z_1)} - 2\psi_1\frac{\mathfrak{j}'(z_1)}{\mathfrak{j}(z_1)}\frac{\Theta(z_1)}{\mathfrak{u}(z_1)} \geq \frac{1}{\lambda_1}.$$

for some $\lambda_1 > 0$, we obtain the following.

$$\int_0^b \sigma(z_1)\frac{\mathfrak{u}(z_1)}{\Theta^{1/2}(z_1)}(\mathcal{G}_s\mathfrak{g})^{\psi_1}(z_1)dz_1 \leq 2\lambda_1\Theta^{1/2}(b)\sigma(b)(\mathcal{G}_s\mathfrak{g})^{\psi_1}(b).$$

Since $\rho < 1$, hence the prescribed example is a contradiction of (23).

5. Conclusions

Inequalities are considered in rather general forms and contain various special integral and discrete inequalities. In the present article, several new integral inequalities have been developed and demonstrated through the operators of Hardy–Steklov and Copson–Steklov form and by the inclusion of a second q integrability parameter. These existing inequalities, such as those of Hardy, Copson and Pachpatte, are generalized by the fundamental inequalities and the technique is based on the applications of well-known literature. In consideration of this, we recommend the implementation of these results to $\mathbb{R}^n$ or for $n \geq 2$ on subsets of $\mathbb{R}^n$. Likewise, it would also be of great research interest to pursue any of these integral inequalities in the analysis of various mathematical areas such as partial differential equations, mathematical simulation, functional spaces, and so on. Our work formulates the results of the Hardy and Copson type inequalities and the generalization of the Hardy and Copson integral inequalities in one variable. In future research, we will continue to generalize some of the more established inequalities by using the Steklov operator based on two variables.

Author Contributions: W.A.; Conceptualization with writing—review and editing. Z.A.K.; Conceptualization and writing—original draft. Both authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: The authors would like to express their sincere thanks to the editor and anonymous reviewers for their helpful comments and suggestions. This research was funded by the Deanship of Scientific Research at Princess Nourah bint Abdulrahman University through the Fast-track Research Funding Program.

Conflicts of Interest: The authors declare that there are no competing interests.

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