*fractal and fractional*

Article

# An Existence Result for $\psi$-Hilfer Fractional Integro-Differential Hybrid Three-Point Boundary Value Problems

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**Abstract:** In this research work, we study a new class of $\psi$-Hilfer hybrid fractional integro-differential boundary value problems with three-point boundary conditions. An existence result is established by using a generalization of Krasnosel'skiĭ's fixed point theorem. An example illustrating the main result is also constructed.

**Keywords:** boundary value problem; $\psi$-Hilfer fractional derivative; hybrid fractional integro-differential equation; fixed point theorem



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## 1. Introduction

Differential equations of fractional order have recently received a lot of attention and now constitute a significant branch of nonlinear analysis, because some real world problems in physics, mechanics and other fields can be described better with the help of fractional differential equations. Numerous monographs have appeared devoted to fractional differential equations, for example, see [1–8]. Recently, differential equations and inclusions equipped with various boundary conditions have been widely investigated by many researchers (see [9–18] and the references cited therein).

Hybrid fractional differential equations have also been studied by several researchers. Hybrid fractional differential equations involve the fractional derivative of an unknown function hybrid with the nonlinearity depending on it. Hybrid systems play a key role in embedded control systems that interact with the physical situation. Time- and event-based behaviors are more accurately described by hybrid models as such models help to deal with challenging design requirements in the design of control systems. Examples include automotive control [19], mobile robotics [20], the process industry [21], real-time software verification [22], transportation systems [23], and manufacturing [24].

Some recent results on hybrid differential equations can be found in a series of papers [25–29].

In 2010, Dhage and Lakshmikantham [30] initiated the study of initial value problems for first order hybrid differential equation of the form:

$$\begin{cases} \dfrac{d}{dt}\left(\dfrac{x(t)}{f(t,x(t))}\right) = g(t,x(t)), \quad t \in [0,T], \\ x(0) = x_0 \in \mathbb{R}, \end{cases} \tag{1}$$

where $f \in C([0,T] \times \mathbb{R}, \mathbb{R} \setminus \{0\})$ and $g \in C([0,T] \times \mathbb{R}, \mathbb{R})$. They gave the existence, uniqueness results, and some theorems on differential inequalities.

In 2011, Zhao et al. [26] investigated the hybrid fractional initial value problem and Sun et al. [27] discussed fractional boundary value problems containing hybrid equations.

In [31], the authors proved the existence of solutions for a nonlocal boundary value problem of hybrid fractional integro-differential equations given by

$$\begin{cases} D^{\alpha}\left[\dfrac{x(t)-\sum_{i=1}^{n} I^{\beta_i}h_i(t,x(t))}{g(t,x(t))}\right] = f(t,x(t)), \quad t\in[0,1], \\ x(0)=\mu(x), \qquad x(1)=A\in\mathbb{R}, \end{cases} \tag{2}$$

where $D^{\alpha}$ is the Caputo fractional derivative of order $\alpha$ with $1<\alpha\le 2$, $I^{\beta_i}$ is the Riemann–Liouville fractional integral of order $\beta_i>0$ and functions $h_i\in C([0,1]\times\mathbb{R},\mathbb{R})$, for $i=1,2,\ldots,n$, $g\in C([0,1]\times\mathbb{R},\mathbb{R}\setminus\{0\})$, $f\in C([0,1]\times\mathbb{R},\mathbb{R})$, the functional $\mu: C([0,1],\mathbb{R})\to\mathbb{R}$. The main result was proved by using of a hybrid fixed point theorem for three operators in a Banach algebra from Dhage [32].

The existence of solutions of hybrid fractional integro-differential equations with initial conditions, given by

$$\begin{cases} D^{\alpha}\left[\dfrac{D^{\omega}x(t)-\sum_{i=1}^{n} I^{\beta_i}h_i(t,x(t))}{g(t,x(t))}\right] = f(t,x(t)), \quad t\in[0,1], \\ x(0)=0, \qquad D^{\omega}x(0)=0, \end{cases} \tag{3}$$

was studied in [33]. Here, $D^{\chi}$ is the Caputo fractional derivative of order $\chi\in\{\alpha,\omega\}$ with $0<\alpha,\omega\le 1$, $I^{\beta_i}$ is the Riemann–Liouville fractional integral of order $\beta_i>0$, $h_i\in C([0,1]\times\mathbb{R},\mathbb{R})$, for $i=1,2,\ldots,n$, $g\in C([0,1]\times\mathbb{R},\mathbb{R}\setminus\{0\})$, $f\in C([0,1]\times\mathbb{R},\mathbb{R})$. A generalization of Krasnosel'skiĭ's fixed point theorem ([32,34]) was applied to prove the existence result.

The problem (3) was extended to higher order fractional derivatives in [35] as a boundary value problem

$$\begin{cases} D^{\alpha}\left[\dfrac{D^{\omega}x(t)-\sum_{i=1}^{n} I^{\beta_i}h_i(t,x(t))}{g(t,x(t))}\right] = f(t,x(t)), \quad t\in[0,1], \\ x(0)=0, \qquad D^{\omega}x(0)=0, \quad x(1)=\delta x(\eta), \quad 0<\delta<1, 0<\eta<1, \end{cases} \tag{4}$$

where $D^{\chi}$ is the Caputo fractional derivative of order $\chi\in\{\alpha,\omega\}$ with $0<\alpha\le 1, 1<\omega\le 2$, $I^{\beta_i}$ is the Riemann–Liouville fractional integral of order $\beta_i>0$, $h_i\in C([0,1]\times\mathbb{R},\mathbb{R})$, for $i=1,2,\ldots,n$, $g\in C([0,1]\times\mathbb{R},\mathbb{R}\setminus\{0\})$, $f\in C([0,1]\times\mathbb{R},\mathbb{R})$. Dhage's fixed point theorem [32] was used to obtain an existence result.

For recent papers on hybrid boundary value problems of fractional differential equations and inclusions, we refer to [36–38] and references cited therein.

In [39], an initial value problem was studied for hybrid fractional differential equations containing a $\psi$-Hilfer fractional derivative of the form

$$\begin{cases} {}^{H}\mathfrak{D}^{\alpha,\beta;\psi}\left(\dfrac{x(t)}{f(t,x(t))}\right) = g(t,x(t)), \ \ t\in[a,b], \\ I^{1-\gamma,\psi}\left(\dfrac{x(0)}{f(0,x(0))}\right) = x_0\in\mathbb{R}, \end{cases} \tag{5}$$

where ${}^{H}\mathfrak{D}^{\alpha,\beta;\psi}$ is the $\psi$-Hilfer fractional derivative with $0 < \alpha < 1$, $0 \le \beta \le 1$, $\alpha \le \gamma = \alpha + (1-\alpha)\beta < 1$, $f \in C_{1-\gamma;\psi}([a,b] \times \mathbb{R}, \mathbb{R} \setminus \{0\})$, $g \in C_{1-\gamma;\psi}([a,b] \times \mathbb{R}, \mathbb{R})$, $I^{1-\gamma,\psi}$ is the $\psi$-Hilfer fractional integral of order $1-\gamma$. Here, $C_{1-\gamma;\psi}[a,b] = \{h : (a,b] \to \mathbb{R} : (\psi(t) - \psi(a))^{1-\gamma}h(t) \in C[a,b]\}, 0 < \gamma \le 1$. For some recent research papers on $\psi$-Hilfer fractional initial value problems, see [40–42] and references cited therein.

In the present work, we study a three-point $\psi$-Hilfer hybrid fractional integro-differential nonlocal boundary value problem of the form

$$\begin{cases} {}^{H}\mathfrak{D}_a^{\alpha,\rho;\psi}\left[\dfrac{{}^{H}\mathfrak{D}_a^{p,\rho;\psi}x(t)}{g(t,x(t))} - \displaystyle\sum_{i=1}^{n}\mathcal{I}_a^{\beta_i;\psi}h_i(t,x(t))\right] = f(t,x(t)), \quad t \in [a,b], \\ \\ x(a) = 0, \;\; {}^{H}\mathfrak{D}_a^{p,\rho;\psi}x(a) = 0, \;\; x(b) = \theta x(\xi), \end{cases} \tag{6}$$

where ${}^{H}\mathfrak{D}_a^{\omega,\rho;\psi}$ is the $\psi$-Hilfer fractional derivative operator of order $\omega \in \{\alpha, p\}$, with $0 < \alpha \le 1$, $1 < p \le 2$, $0 \le \rho < 1$, $\mathcal{I}_a^{\beta_i;\psi}$ is $\psi$-Riemann–Liouville fractional integral of order $\beta_i > 0$, for $i = 1,2,\ldots,n$, $g \in C([a,b] \times \mathbb{R}, \mathbb{R} \setminus \{0\})$, $f \in C([a,b] \times \mathbb{R}, \mathbb{R})$, $h_i \in C([a,b] \times \mathbb{R}, \mathbb{R})$ for $i = 1,2,\ldots,n$, $\xi \in [a,b]$ and $\theta \in \mathbb{R}$. An existence result is established via a generalization of the Krasnosel'skiĭ fixed point theorem ([32,34]).

The rest of the paper is organized as follows: In Section 2, we recall some notations, definitions, and lemmas from fractional calculus needed in our study. We also prove an auxiliary lemma helping us to transform the hybrid boundary value problem (6) into an equivalent integral equation. The main existence result for the $\psi$-Hilfer hybrid boundary value problem (6) is contained in Section 3. The obtained result is illustrated by a numerical example.

## 2. Preliminaries

This section defines some notation in relation to fractional calculus.

**Definition 1** ([2])**.** *Let $(a,b)$, $a \ge 0$, $(-\infty \le a < b \le \infty)$, be a finite or infinite interval of the half-axis $\mathbb{R}^+$ and $\alpha > 0$. Let $\psi(x)$ be an increasing and positive monotone function on $(a,b]$, having a continuous derivative $\psi'(x)$ on $(a,b)$. The $\psi$-Riemann–Liouville fractional integral of a function $f$ with respect to another function $\psi$ on $[a,b]$ is defined by*

$$\mathcal{I}_{a^+}^{\alpha;\psi}f(t) = \frac{1}{\Gamma(\alpha)}\int_a^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}f(s)ds, \quad t > a > 0, \tag{7}$$

*where $\Gamma(\cdot)$ is the Euler gamma function.*

**Definition 2** ([2])**.** *Let $\psi \in C^n([a,b],\mathbb{R})$ with $\psi'(t) \ne 0$ and $\alpha > 0$, $n \in \mathbb{N}$. The Riemann–Liouville derivative of a function $f$ with respect to another function $\psi$ of order $\alpha$ is defined by*

$$\mathfrak{D}_{a^+}^{\alpha;\psi}f(t) = \left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n \mathcal{I}_{a^+}^{n-\alpha;\psi}f(t) \tag{8}$$

$$= \frac{1}{\Gamma(n-\alpha)}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n \int_a^t \psi'(s)(\psi(t)-\psi(s))^{n-\alpha-1}f(s)ds, \tag{9}$$

*where $n = [\alpha] + 1$, $[\alpha]$ represents the integer part of the real number $\alpha$.*

**Definition 3** ([40])**.** *Let $n-1 < \alpha < n$ with $n \in \mathbb{N}$, $[a,b]$ is the interval such that $-\infty \le a < b \le \infty$ and $f, \psi \in C^n([a,b],\mathbb{R})$ two functions such that $\psi$ is increasing and $\psi'(t) \ne 0$, for all*

*$t \in [a,b]$. The $\psi$-Hilfer fractional derivative of a function $f$ of order $\alpha$ and type $0 \le \rho \le 1$ is defined by*

$$ {}^{H}\mathfrak{D}_{a^+}^{\alpha,\rho;\psi} f(t) = \mathcal{I}_{a^+}^{\rho(n-\alpha);\psi} \left( \frac{1}{\psi'(t)} \frac{d}{dt} \right)^n \mathcal{I}_{a^+}^{(1-\rho)(n-\alpha);\psi} f(t) = \mathcal{I}_{a^+}^{\gamma-\alpha;\psi} \mathfrak{D}_{a^+}^{\gamma;\psi} f(t), \tag{10} $$

*where $n = [\alpha] + 1$, $[\alpha]$ represents the integer part of the real number $\alpha$ with $\gamma = \alpha + \rho(n-\alpha)$.*

**Lemma 1** ([2]). *Let $\alpha, \beta > 0$. Then, we have the following semigroup property given by*

$$ \mathcal{I}_{a^+}^{\alpha;\psi} \mathcal{I}_{a^+}^{\beta;\psi} f(t) = \mathcal{I}_{a^+}^{\alpha+\beta;\psi} f(t), \quad t > a. \tag{11} $$

Next, we present the $\psi$-fractional integral and derivatives of a power function.

**Proposition 1** ([2,40]). *Let $\alpha > 0$, $v > 0$ and $t > a$. Then, we have*

*(i)* $\mathcal{I}_{a^+}^{\alpha;\psi} (\psi(s) - \psi(a))^{v-1}(t) = \dfrac{\Gamma(v)}{\Gamma(v+\alpha)} (\psi(t) - \psi(a))^{v+\alpha-1}$.

*(ii)* ${}^{H}\mathfrak{D}_{a^+}^{\alpha,\rho;\psi} (\psi(s) - \psi(a))^{v-1}(t) = \dfrac{\Gamma(v)}{\Gamma(v-\alpha)} (\psi(t) - \psi(a))^{v-\alpha-1}$, $n-1 < \alpha < n$, $v > n$.

**Lemma 2** ([41]). *Let $m-1 < \alpha < m$, $n-1 < \beta < n$, $n,m \in \mathbb{N}$, $n \le m$, $0 \le \rho \le 1$ and $\alpha \ge \beta + \rho(n-\beta)$. If $f \in C^n([a,b],\mathbb{R})$, then*

$$ {}^{H}\mathfrak{D}_{a^+}^{\beta,\rho;\psi} \mathcal{I}_{a^+}^{\alpha;\psi} f(t) = \mathcal{I}_{a^+}^{\alpha-\beta;\psi} f(t). \tag{12} $$

**Lemma 3** ([40]). *If $f \in C^n([a,b],\mathbb{R})$, $n-1 < \alpha < n$, $0 \le \rho \le 1$ and $\gamma = \alpha + \rho(n-\alpha)$, then*

$$ \mathcal{I}_{a^+}^{\alpha;\psi} {}^{H}\mathfrak{D}_{a^+}^{\alpha,\rho;\psi} f(t) = f(t) - \sum_{k=1}^{n} \frac{(\psi(t) - \psi(a))^{\gamma-k}}{\Gamma(\gamma-k+1)} \nabla_{\psi}^{[n-k]} \mathcal{I}_{a^+}^{(1-\rho)(n-\alpha);\psi} f(a), \tag{13} $$

*for all $t \in J$, where $\nabla_{\psi}^{[n]} f(t) := \left( \frac{1}{\psi'(t)} \frac{d}{dt} \right)^n f(t)$.*

**Definition 4.** *A function $x \in C([a,b],\mathbb{R})$ is said to be a solution of the hybrid boundary value problem (6) if $x \to \dfrac{{}^{H}\mathfrak{D}_{a}^{p,\rho;\psi} x}{g(t,x)} - \sum_{i=1}^{n} \mathcal{I}_{a}^{\beta_i;\psi} h_i(t,x)$ is continuous for each $x \in \mathbb{R}$ and $x$ satisfies the fractional differential equation and the boundary conditions in (6).*

**Lemma 4.** *Let $0 < \alpha < 1$, $1 < p \le 2$, $0 \le \rho < 1$, $\gamma = p + \rho(2-p)$, $\gamma_1 = \alpha + \rho(1-\alpha) < 1$, $\Lambda \ne 0$, $g, h_i, i = 1,2,\ldots,n$ satisfy boundary value problem (6) and $z \in C([a,b],\mathbb{R})$. Then, $x$ is a solution of the $\psi$-Hilfer hybrid fractional integro-differential boundary value problem of the form:*

$$ \begin{cases} {}^{H}\mathfrak{D}_{a}^{\alpha,\rho;\psi} \left[ \dfrac{{}^{H}\mathfrak{D}_{a}^{p,\rho;\psi} x(t)}{g(t,x(t))} - \displaystyle\sum_{i=1}^{n} \mathcal{I}_{a}^{\beta_i;\psi} h_i(t,x(t)) \right] = z(t), \quad t \in [a,b], \\ x(a) = 0, \quad {}^{H}\mathfrak{D}_{a}^{p,\rho;\psi} x(a) = 0, \quad x(b) = \theta x(\xi), \end{cases} \tag{14} $$

*if and only if $x$ satisfies the equation*

$$ \begin{aligned} x(t) \quad = \quad & \int_a^t \frac{\psi'(s)(\psi(t) - \psi(s))^{p-1}}{\Gamma(p)} g(s,x(s)) \left\{ \sum_{i=1}^{n} \mathcal{I}_{a}^{\beta_i;\psi} h_i(s,x(s)) + \mathcal{I}_{a}^{\alpha;\psi} z(s) \right\} ds \\ & + \frac{(\psi(t) - \psi(a))^{\gamma-1}}{\Lambda} \left[ \theta \int_a^{\xi} \frac{\psi'(s)(\psi(\xi) - \psi(s))^{p-1}}{\Gamma(p)} g(s,x(s)) \right. \end{aligned} $$

$$\times\left\{\sum_{i=1}^{n}\mathcal{I}_{a}^{\beta_i;\psi}h_i(s,x(s))+\mathcal{I}_{a}^{\alpha;\psi}z(s)\right\}ds \tag{15}$$
$$-\int_a^b\frac{\psi'(s)(\psi(b)-\psi(s))^{p-1}}{\Gamma(p)}g(s,x(s))\left\{\sum_{i=1}^{n}\mathcal{I}_{a}^{\beta_i;\psi}h_i(s,x(s))+\mathcal{I}_{a}^{\alpha;\psi}z(s)\right\}ds\Bigg],$$

*where*

$$\Lambda:=(\psi(b)-\psi(a))^{\gamma-1}-\theta(\psi(\xi)-\psi(a))^{\gamma-1}.$$

**Proof.** Let $x\in C([a,b],\mathbb{R})$ be a solution of the problem (14). Applying the $\psi$-Riemann–Liouville fractional integral operator $\mathcal{I}_a^{\alpha;\psi}$ to both sides of (14) and using Lemma 3, we obtain

$$\frac{{}^{H}\mathfrak{D}_{a}^{p,\rho;\psi}x(t)}{g(t,x(t))}-\sum_{i=1}^{n}\mathcal{I}_{a}^{\beta_i;\psi}h_i(t,x(t))=\mathcal{I}_{a}^{\alpha;\psi}z(t)+\frac{(\psi(t)-\psi(a))^{\gamma_1-1}}{\Gamma(\gamma_1)}c_0, \tag{16}$$

where $c_0\in\mathbb{R}$. By using the boundary condition, ${}^{H}\mathfrak{D}_{a}^{p,\rho;\psi}x(a)=0$, we obtain the constant $c_0=0$. Thus, we have

$${}^{H}\mathfrak{D}_{a^+}^{p,\rho;\psi}x(t)=g(t,x(t))\left\{\sum_{i=1}^{n}\mathcal{I}_{a}^{\beta_i;\psi}h_i(t,x(t))+\mathcal{I}_{a}^{\alpha;\psi}z(t)\right\}. \tag{17}$$

Inserting the $\psi$-Riemann–Liouville fractional integral operator $\mathcal{I}_a^{p;\psi}$ into both sides of (17) and using Lemma 3, we obtain

$$\begin{aligned}x(t)&=\int_a^t\frac{\psi'(s)(\psi(t)-\psi(s))^{p-1}}{\Gamma(p)}g(s,x(s))\left\{\sum_{i=1}^{n}\mathcal{I}_{a}^{\beta_i;\psi}h_i(s,x(s))+\mathcal{I}_{a}^{\alpha;\psi}z(s)\right\}ds\\&\quad+\frac{(\psi(t)-\psi(a))^{\gamma-1}}{\Gamma(\gamma)}c_1+\frac{(\psi(t)-\psi(a))^{\gamma-2}}{\Gamma(\gamma-1)}c_2,\end{aligned} \tag{18}$$

where $c_1,c_2\in\mathbb{R}$. From the boundary condition $x(a)=0$, we obtain $c_2=0$, while from the boundary condition $x(b)=\theta x(\xi)$, we find that

$$\begin{aligned}c_1&=\frac{\Gamma(\gamma)}{\Lambda}\Bigg[\theta\int_a^\xi\frac{\psi'(s)(\psi(\xi)-\psi(s))^{p-1}}{\Gamma(p)}g(s,x(s))\left\{\sum_{i=1}^{n}\mathcal{I}_{a}^{\beta_i;\psi}h_i(s,x(s))+\mathcal{I}_{a}^{\alpha;\psi}z(s)\right\}ds\\&\quad-\int_a^b\frac{\psi'(s)(\psi(b)-\psi(s))^{p-1}}{\Gamma(p)}g(s,x(s))\left\{\sum_{i=1}^{n}\mathcal{I}_{a}^{\beta_i;\psi}h_i(s,x(s))+\mathcal{I}_{a}^{\alpha;\psi}z(s)\right\}ds\Bigg].\end{aligned}$$

Substituting the constants $c_1$ and $c_2$ into (18), we obtain the integral equation in (15) as desired.

Conversely, by a direct computation, it is easy to show that the solution $x$ given by (15) satisfies the problem (14). The proof of Lemma 4 is completed. □

Let $\mathbb{X}=C([a,b],\mathbb{R})$ be the Banach space of continuous real-valued functions defined on $[a,b]$, equipped with the norm $\|x\|=\sup_{t\in[a,b]}|x(t)|$ and a multiplication $(xy)(t)=x(t)y(t),\forall t\in[a,b]$. Then, clearly, $\mathbb{X}$ is a Banach algebra with the above-defined supremum norm and multiplication in it.

**Lemma 5** ([32,34])**.** *Let $S$ be a nonempty, convex, closed, and bounded set such that $S\subset\mathbb{X}$, and let $\mathcal{A}:\mathbb{X}\to\mathbb{X}$ and $\mathcal{B}:S\to\mathbb{X}$ be two operators which satisfy the following:*

*(i)* $\mathcal{A}$ *is contraction,*
*(ii)* $\mathcal{B}$ *is completely continuous, and*
*(iii)* $x=\mathcal{A}x+\mathcal{B}y,\forall y\in S\Rightarrow x\in S.$

*Then, there exists a solution of the operator equation $x = \mathcal{A}x + \mathcal{B}x$.*

## 3. Existence Result

In view of Lemma 4, we define an operator $\mathcal{Q} : \mathbb{X} \to \mathbb{X}$ by

$$
\begin{aligned}
(\mathcal{Q}x)(t) \quad = \quad & \int_a^t \frac{\psi'(s)(\psi(t)-\psi(s))^{p-1}}{\Gamma(p)} g(s,x(s)) \left\{ \sum_{i=1}^{n} \mathcal{I}_a^{\beta_i;\psi} h_i(s,x(s)) + \mathcal{I}_a^{\alpha;\psi} f(s,x(s)) \right\} ds \\
& + \frac{(\psi(t)-\psi(a))^{\gamma-1}}{\Lambda} \left[ \theta \int_a^{\xi} \frac{\psi'(s)(\psi(\xi)-\psi(s))^{p-1}}{\Gamma(p)} g(s,x(s)) \right. \\
& \times \left\{ \sum_{i=1}^{n} \mathcal{I}_a^{\beta_i;\psi} h_i(s,x(s)) + \mathcal{I}_a^{\alpha;\psi} f(s,x(s)) \right\} ds \qquad (19) \\
& \left. - \int_a^b \frac{\psi'(s)(\psi(b)-\psi(s))^{p-1}}{\Gamma(p)} g(s,x(s)) \left\{ \sum_{i=1}^{n} \mathcal{I}_a^{\beta_i;\psi} h_i(s,x(s)) + \mathcal{I}_a^{\alpha;\psi} f(s,x(s)) \right\} ds \right].
\end{aligned}
$$

Notice that the problem (6) has solutions if and only if the operator $\mathcal{Q}$ has fixed points.

**Theorem 1.** *Assume that:*

($A_1$) *The functions $g : [a,b] \times \mathbb{R} \to \mathbb{R} \setminus \{0\}$, $f : [a,b] \times \mathbb{R} \to \mathbb{R}$, and $h_i : [a,b] \times \mathbb{R} \to \mathbb{R}$ for $i = 1,2,\ldots,n$, are continuous and there exist positive functions $\phi$, $\omega$, $\chi_i$, $i = 1,2,\ldots,n$, with bounds $\|\phi\|$, $\|\omega\|$, and $\|\chi_i\|$, $i = 1,2,\ldots,n$, respectively, such that*

$$|g(t,x) - g(t,y)| \le \phi(t)|x-y|, \qquad (20)$$

$$|f(t,x) - f(t,y)| \le \omega(t)|x-y|, \qquad (21)$$

*and*

$$|h_i(t,x) - h_i(t,y)| \le \chi_i(t)|x-y|, \quad i = 1,2,\ldots,n, \qquad (22)$$

*for $t \in [a,b]$ and $x, y \in \mathbb{R}$;*

($A_2$) $|h_i(t,x)| \le \lambda_i(t)$, $\lambda_i \in C([a,b],\mathbb{R})$, $\forall (t,x) \in [a,b] \times \mathbb{R}, i = 1,2,\ldots,m$, $|f(t,x)| \le \mu(t)$, $|g(t,x)| \le \nu(t)$, $\forall (t,x) \in [a,b] \times \mathbb{R}$, $\mu, \nu \in C([a,b],\mathbb{R})$;

($A_3$) *Assume that*

$$
\begin{aligned}
K: \quad = \quad & \left[ 1 + (|\theta|+1) \frac{(\psi(b)-\psi(a))^{\gamma-1}}{|\Lambda|} \right] \frac{(\psi(b)-\psi(a))^{p}}{\Gamma(p+1)} \left( \|\nu\| \sum_{i=1}^{n} \frac{(\psi(b)-\psi(a))^{\beta_i}}{\Gamma(\beta_i+1)} \|\chi_i\| \right. \\
& \left. + \|\phi\| \sum_{i=1}^{n} \frac{(\psi(b)-\psi(a))^{\beta_i}}{\Gamma(\beta_i+1)} \|\lambda_i\| \right) < 1. \qquad (23)
\end{aligned}
$$

*Then, the $\psi$-Hilfer hybrid fractional integro-differential three-point boundary value problem (6) has at least one solution on $[a,b]$.*

**Proof.** Firstly, we consider a subset $S$ of $\mathbb{X}$ defined by $S = \{x \in \mathbb{X} : \|x\| \le r\}$, where $r$ is given by

$$
\begin{aligned}
r = & \left[ 1 + (|\theta|+1) \frac{(\psi(b)-\psi(a))^{\gamma-1}}{|\Lambda|} \right] \frac{(\psi(b)-\psi(a))^{p}}{\Gamma(p+1)} \|\nu\| \\
& \times \left[ \sum_{i=1}^{n} \frac{(\psi(b)-\psi(a))^{\beta_i}}{\Gamma(\beta_i+1)} \|\lambda_i\| + \frac{(\psi(b)-\psi(a))^{\alpha}}{\Gamma(\alpha+1)} \|\mu\| \right]. \qquad (24)
\end{aligned}
$$

Observe that $S$ is a closed, convex, and bounded subset of the Banach space $\mathbb{X}$. Now, we set $\sup_{t\in[a,b]}|\lambda_i(t)| = \|\lambda_i\|$, $i = 1,2,\ldots,n$, $\sup_{t\in[a,b]}|\mu(t)| = \|\mu\|$, $\sup_{t\in[a,b]}|\nu(t)| = \|\nu\|$.

Let us define three operators $\mathcal{H},\mathcal{F},\mathcal{G} : \mathbb{X} \to \mathbb{X}$ such that

$$\mathcal{H}x(t) = \sum_{i=1}^{n} \mathcal{I}_a^{\beta_i;\psi} h_i(t,x(t)),\ \ t \in [a,b],$$

$$\mathcal{F}x(t) = \mathcal{I}_a^{\alpha;\psi} f(t,x(t)),\ \ t \in [a,b],$$

and

$$\mathcal{G}x(t) = g(t,x(t)),\ \ t \in [a,b].$$

Then, we have

$$\begin{aligned}|\mathcal{H}x(t) - \mathcal{H}y(t)| &\le \sum_{i=1}^{n}\int_a^t \frac{\psi'(s)(\psi(t)-\psi(s))^{\beta_i-1}}{\Gamma(\beta_i)}|h_i(s,x(s)) - h_i(s,y(s))|ds \\ &\le \sum_{i=1}^{n}\frac{(\psi(b)-\psi(a))^{\beta_i}}{\Gamma(\beta_i+1)}\|\chi_i\|\|x-y\|,\end{aligned}$$

and

$$|\mathcal{H}x(t)| \le \sum_{i=1}^{n}\frac{(\psi(b)-\psi(a))^{\beta_i}}{\Gamma(\beta_i+1)}\|\lambda_i\|.$$

In addition, we obtain

$$\begin{aligned}|\mathcal{F}x(t) - \mathcal{F}y(t)| &\le \int_a^t \frac{\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)}|f(s,x(s)) - f(s,y(s))|ds \\ &\le \frac{(\psi(b)-\psi(a))^{\alpha}}{\Gamma(\alpha+1)}\|\omega\|\|x-y\|,\end{aligned}$$

and

$$|\mathcal{F}x(t)| \le \frac{(\psi(b)-\psi(a))^{\alpha}}{\Gamma(\alpha+1)}\|\mu\|.$$

Moreover, we obtain

$$|\mathcal{G}x(t) - \mathcal{G}y(t)| = |g(t,x(t)) - g(t,y(t))| \le \|\phi\|\|x-y\|,$$

and

$$|\mathcal{G}x(t)| \le \|\nu\|.$$

Now we define two operators $\mathcal{A} : \mathbb{X} \to \mathbb{X}$ and $\mathcal{B} : S \to \mathbb{X}$ as follows:

$$\begin{aligned}\mathcal{A}x(t) &= \int_a^t \frac{\psi'(s)(\psi(t)-\psi(s))^{p-1}}{\Gamma(p)}\mathcal{G}x(s)\mathcal{H}x(s)ds \\ &\quad + \frac{(\psi(t)-\psi(a))^{\gamma-1}}{\Lambda}\left[\theta\int_a^{\xi}\frac{\psi'(s)(\psi(\xi)-\psi(s))^{p-1}}{\Gamma(p)}\mathcal{G}x(s)\mathcal{H}x(s)ds\right. \\ &\quad \left. - \int_a^{b}\frac{\psi'(s)(\psi(b)-\psi(s))^{p-1}}{\Gamma(p)}\mathcal{G}x(s)\mathcal{H}x(s)ds\right],\end{aligned}$$

and

$$\begin{aligned}\mathcal{B}x(t) &= \int_a^t \frac{\psi'(s)(\psi(t)-\psi(s))^{p-1}}{\Gamma(p)}\mathcal{G}x(s)\mathcal{F}x(s)ds \\ &\quad + \frac{(\psi(t)-\psi(a))^{\gamma-1}}{\Lambda}\left[\theta\int_a^{\xi}\frac{\psi'(s)(\psi(\xi)-\psi(s))^{p-1}}{\Gamma(p)}\mathcal{G}x(s)\mathcal{F}x(s)ds\right.\end{aligned}$$

$$-\int_a^b \frac{\psi'(s)(\psi(b)-\psi(s))^{p-1}}{\Gamma(p)}\mathcal{G}x(s)\mathcal{F}x(s)ds\Bigg].$$

Clearly, $\mathcal{Q}x = \mathcal{A}x + \mathcal{B}x$. In the next steps, we show that the operators $\mathcal{A}$ and $\mathcal{B}$ fulfill all the assumptions of Lemma 5. The proof is divided into three steps:

**Step 1**. The operator $\mathcal{A}$ is a contraction mapping. For any $x, y \in S$, we have

$$\begin{aligned}
&|\mathcal{A}x(t)-\mathcal{A}y(t)| \\
\leq\ & \int_a^t \frac{\psi'(s)(\psi(t)-\psi(s))^{p-1}}{\Gamma(p)}|\mathcal{G}x(s)\mathcal{H}x(s)-\mathcal{G}y(s)\mathcal{H}y(s)|ds \\
&+\frac{(\psi(t)-\psi(a))^{\gamma-1}}{|\Lambda|}\Bigg[|\theta|\int_a^\xi \frac{\psi'(s)(\psi(\xi)-\psi(s))^{p-1}}{\Gamma(p)}|\mathcal{G}x(s)\mathcal{H}x(s)-\mathcal{G}y(s)\mathcal{H}y(s)|ds \\
&+\int_a^b \frac{\psi'(s)(\psi(b)-\psi(s))^{p-1}}{\Gamma(p)}|\mathcal{G}x(s)\mathcal{H}x(s)-\mathcal{G}y(s)\mathcal{H}y(s)|ds\Bigg] \\
\leq\ & \int_a^b \frac{\psi'(s)(\psi(b)-\psi(s))^{p-1}}{\Gamma(p)}|\mathcal{G}x(s)\mathcal{H}x(s)-\mathcal{G}y(s)\mathcal{H}y(s)|ds \\
&+\frac{(\psi(b)-\psi(a))^{\gamma-1}}{|\Lambda|}\Bigg[|\theta|\int_a^\xi \frac{\psi'(s)(\psi(\xi)-\psi(s))^{p-1}}{\Gamma(p)}|\mathcal{G}x(s)\mathcal{H}x(s)-\mathcal{G}y(s)\mathcal{H}y(s)|ds \\
&+\int_a^b \frac{\psi'(s)(\psi(b)-\psi(s))^{p-1}}{\Gamma(p)}|\mathcal{G}x(s)\mathcal{H}x(s)-\mathcal{G}y(s)\mathcal{H}y(s)|ds\Bigg] \\
\leq\ & \left[1+(|\theta|+1)\frac{(\psi(b)-\psi(a))^{\gamma-1}}{|\Lambda|}\right]\int_a^b \frac{\psi'(s)(\psi(b)-\psi(s))^{p-1}}{\Gamma(p)} \\
&\times|\mathcal{G}x(s)\mathcal{H}x(s)-\mathcal{G}y(s)\mathcal{H}y(s)|ds \\
\leq\ & \left[1+(|\theta|+1)\frac{(\psi(b)-\psi(a))^{\gamma-1}}{|\Lambda|}\right]\int_a^b \frac{\psi'(s)(\psi(b)-\psi(s))^{p-1}}{\Gamma(p)}\Big(|\mathcal{G}x(s)||\mathcal{H}x(s)-\mathcal{H}y(s)| \\
&+|\mathcal{G}x(s)-\mathcal{G}y(s)||\mathcal{H}y(s)|\Big)ds \\
\leq\ & \left[1+(|\theta|+1)\frac{(\psi(b)-\psi(a))^{\gamma-1}}{|\Lambda|}\right]\frac{(\psi(b)-\psi(a))^{p}}{\Gamma(p+1)}\left(\|\nu\|\sum_{i=1}^n\frac{(\psi(b)-\psi(a))^{\beta_i}}{\Gamma(\beta_i+1)}\|\chi_i\|\|x-y\|\right. \\
&\left.+\|\phi\|\|x-y\|\sum_{i=1}^n\frac{(\psi(b)-\psi(a))^{\beta_i}}{\Gamma(\beta_i+1)}\|\lambda_i\|\right). \\
=\ & \left[1+(|\theta|+1)\frac{(\psi(b)-\psi(a))^{\gamma-1}}{|\Lambda|}\right]\frac{(\psi(b)-\psi(a))^{p}}{\Gamma(p+1)}\left(\|\nu\|\sum_{i=1}^n\frac{(\psi(b)-\psi(a))^{\beta_i}}{\Gamma(\beta_i+1)}\|\chi_i\|\right. \\
&\left.+\|\phi\|\sum_{i=1}^n\frac{(\psi(b)-\psi(a))^{\beta_i}}{\Gamma(\beta_i+1)}\|\lambda_i\|\right)\|x-y\|.
\end{aligned}$$

Consequently

$$\|\mathcal{A}x-\mathcal{A}y\| \leq K\|x-y\|,$$

which, by (23), the operator $\mathcal{A}$ is a contraction mapping. Thus, condition (a) of Lemma 5 is satisfied.

**Step 2**. The operator $\mathcal{B}$ is completely continuous on $S$. First, we will prove that $\mathcal{B}$ is continuous. Let $\{x_n\}$ be a sequence of functions in $S$ converging to a function $x \in S$. By Lebesgue domination theorem, for each $t \in [a, b]$,, we have

$$\begin{aligned}
\lim_{n\to\infty}\mathcal{B}x_n(t) &= \lim_{n\to\infty}\int_a^t \frac{\psi'(s)(\psi(t)-\psi(s))^{p-1}}{\Gamma(p)}\mathcal{G}x_n(s)\mathcal{F}x_n(s)ds \\
&\quad + \lim_{n\to\infty}\frac{(\psi(t)-\psi(a))^{\gamma-1}}{\Lambda}\left[\theta\int_a^\xi \frac{\psi'(s)(\psi(\xi)-\psi(s))^{p-1}}{\Gamma(p)}\mathcal{G}x_n(s)\mathcal{F}x_n(s)ds\right. \\
&\quad \left. - \int_a^b \frac{\psi'(s)(\psi(b)-\psi(s))^{p-1}}{\Gamma(p)}\mathcal{G}x_n(s)\mathcal{F}x_n(s)ds\right] \\
&= \int_a^t \frac{\psi'(s)(\psi(t)-\psi(s))^{p-1}}{\Gamma(p)}\lim_{n\to\infty}\mathcal{G}x_n(s)\mathcal{F}x_n(s)ds \\
&\quad + \frac{(\psi(t)-\psi(a))^{\gamma-1}}{\Lambda}\left[\theta\int_a^\xi \frac{\psi'(s)(\psi(\xi)-\psi(s))^{p-1}}{\Gamma(p)}\lim_{n\to\infty}\mathcal{G}x_n(s)\mathcal{F}x_n(s)ds\right. \\
&\quad \left. - \int_a^b \frac{\psi'(s)(\psi(b)-\psi(s))^{p-1}}{\Gamma(p)}\lim_{n\to\infty}\mathcal{G}x_n(s)\mathcal{F}x_n(s)ds\right] \\
&= \int_a^t \frac{\psi'(s)(\psi(t)-\psi(s))^{p-1}}{\Gamma(p)}\mathcal{G}x(s)\mathcal{F}x(s)ds \\
&\quad + \frac{(\psi(t)-\psi(a))^{\gamma-1}}{\Lambda}\left[\theta\int_a^\xi \frac{\psi'(s)(\psi(\xi)-\psi(s))^{p-1}}{\Gamma(p)}\mathcal{G}x(s)\mathcal{F}x(s)ds\right. \\
&\quad \left. - \int_a^b \frac{\psi'(s)(\psi(b)-\psi(s))^{p-1}}{\Gamma(p)}\mathcal{G}x(s)\mathcal{F}x(s)ds\right] \\
&= \mathcal{B}x(t).
\end{aligned}$$

Therefore, the operator $\mathcal{B}$ is a continuous operator on $S$. Next, we show that the operator $\mathcal{B}$ is uniformly bounded on $S$. For any $x \in S$, we have

$$\begin{aligned}
|\mathcal{B}x(t)| &\leq \int_a^t \frac{\psi'(s)(\psi(t)-\psi(s))^{p-1}}{\Gamma(p)}|\mathcal{G}x(s)||\mathcal{F}x(s)|ds \\
&\quad + \frac{(\psi(b)-\psi(a))^{\gamma-1}}{|\Lambda|}\left[|\theta|\int_a^\xi \frac{\psi'(s)(\psi(\xi)-\psi(s))^{p-1}}{\Gamma(p)}|\mathcal{G}x(s)||\mathcal{F}x(s)|ds\right. \\
&\quad \left. + \int_a^b \frac{\psi'(s)(\psi(b)-\psi(s))^{p-1}}{\Gamma(p)}|\mathcal{G}x(s)||\mathcal{F}x(s)|ds\right] \\
&\leq \left[1+(|\theta|+1)\frac{(\psi(b)-\psi(a))^{\gamma-1}}{|\Lambda|}\right]\frac{(\psi(b)-\psi(a))^{p}}{\Gamma(p+1)}\|\nu\|\frac{(\psi(b)-\psi(a))^{\alpha}}{\Gamma(\alpha+1)}\|\mu\| := M.
\end{aligned}$$

Hence, $\|\mathcal{B}x\| \leq M, x \in S$, which shows that the operator $\mathcal{B}$ is uniformly bounded on $S$. Finally, we show that the operator $\mathcal{B}$ is equicontinuous. Let $t_1 < t_2$ and $x \in S$. Then, we have

$$\begin{aligned}
&|\mathcal{B}x(t_2)-\mathcal{B}x(t_1)| \\
\leq &\left|\frac{1}{\Gamma(p)}\int_a^{t_1}\psi'(s)\left[(\psi(t_2)-\psi(s))^{p-1}-(\psi(t_1)-\psi(s))^{p-1}\right]|\mathcal{G}x(s)||\mathcal{F}x(s)|ds\right. \\
&\left. + \frac{1}{\Gamma(p)}\int_{t_1}^{t_2}\psi'(s)(\psi(t_2)-\psi(s))^{p-1}|\mathcal{G}x(s)||\mathcal{F}x(s)|ds\right| \\
&+ \frac{\left|(\psi(t_2)-\psi(a))^{\gamma-1}-(\psi(t_1)-\psi(a))^{\gamma-1}\right|}{|\Lambda|}\left[|\theta|\int_a^\xi \frac{\psi'(s)(\psi(\xi)-\psi(s))^{p-1}}{\Gamma(p)}|\mathcal{G}x(s)||\mathcal{F}x(s)|ds\right.
\end{aligned}$$

$$
\begin{aligned}
&+\int_a^b \frac{\psi'(s)(\psi(b)-\psi(s))^{p-1}}{\Gamma(p)}|\mathcal{G}x(s)||\mathcal{F}x(s)|ds\Bigg] \\
\leq\ & \frac{\|\nu\|\|\mu\|}{\Gamma(p+1)}\frac{(\psi(b)-\psi(a))^{\alpha}}{\Gamma(\alpha+1)}\left|(\psi(t_2)-\psi(a))^{p}-(\psi(t_1)-\psi(a))^{p}\right| \\
&+\frac{\left|(\psi(t_2)-\psi(a))^{\gamma_1-1}-(\psi(t_1)-\psi(a))^{\gamma_1-1}\right|}{|\Lambda|}\left[(|\theta|+1)\frac{(\psi(b)-\psi(a))^{p}}{\Gamma(p+1)}\right] \\
&\times\|\nu\|\|\mu\|\frac{(\psi(b)-\psi(a))^{\alpha}}{\Gamma(\alpha+1)}.
\end{aligned}
$$

As $t_2 - t_1 \to 0$, the right-hand side of the above inequality tends to zero, independently of $x$. Thus, $\mathcal{B}$ is equicontinuous. Therefore, it follows by Aezelà–Ascoli theorem that $\mathcal{B}$ is a completely continuous operator on $S$.

**Step 3.** We show that the third condition (iii) of Lemma 5 is fulfilled. For any $y \in S$, we have

$$
\begin{aligned}
|x(t)| \ =\ & |\mathcal{A}x(t)+\mathcal{B}y(t)| \leq |\mathcal{A}x(t)|+|\mathcal{B}y(t)| \\
\leq\ & \int_a^t \frac{\psi'(s)(\psi(t)-\psi(s))^{p-1}}{\Gamma(p)}|\mathcal{G}x(s)||\mathcal{H}x(s)|ds \\
&+\frac{(\psi(t)-\psi(a))^{\gamma-1}}{|\Lambda|}\Bigg[|\theta|\int_a^{\xi}\frac{\psi'(s)(\psi(\xi)-\psi(s))^{p-1}}{\Gamma(p)}|\mathcal{G}x(s)||\mathcal{H}x(s)|ds \\
&+\int_a^b \frac{\psi'(s)(\psi(b)-\psi(s))^{p-1}}{\Gamma(p)}|\mathcal{G}x(s)||\mathcal{H}x(s)|ds\Bigg] \\
&+\int_a^t \frac{\psi'(s)(\psi(t)-\psi(s))^{p-1}}{\Gamma(p)}|\mathcal{G}y(s)||\mathcal{F}y(s)|ds \\
&+\frac{(\psi(b)-\psi(a))^{\gamma-1}}{|\Lambda|}\Bigg[|\theta|\int_a^{\xi}\frac{\psi'(s)(\psi(\xi)-\psi(s))^{p-1}}{\Gamma(p)}|\mathcal{G}y(s)||\mathcal{F}y(s)|ds \\
&+\int_a^b \frac{\psi'(s)(\psi(b)-\psi(s))^{p-1}}{\Gamma(p)}|\mathcal{G}y(s)||\mathcal{F}y(s)|ds\Bigg] \\
\leq\ & \left[1+(|\theta|+1)\frac{(\psi(b)-\psi(a))^{\gamma-1}}{|\Lambda|}\right]\frac{(\psi(b)-\psi(a))^{p}}{\Gamma(p+1)}\|\nu\|\sum_{i=1}^{n}\frac{(\psi(b)-\psi(a))^{\beta_i}}{\Gamma(\beta_i+1)}\|\lambda_i\| \\
&+\left[1+(|\theta|+1)\frac{(\psi(b)-\psi(a))^{\gamma-1}}{|\Lambda|}\right]\frac{(\psi(b)-\psi(a))^{p}}{\Gamma(p+1)}\|\nu\|\frac{(\psi(b)-\psi(a))^{\alpha}}{\Gamma(\alpha+1)}\|\mu\| = r,
\end{aligned}
$$

which implies $\|x\| \leq r$, and so $x \in S$.

Hence, all the conditions of Lemma 5 are satisfied, and consequently the operator equation $x(t) = \mathcal{A}x(t) + \mathcal{B}x(t)$ has at least one solution in $S$. Therefore, there exists a solution of the $\psi$-Hilfer hybrid fractional integro-differential boundary value problem (6) in $[a, b]$. The proof is finished. □

## 4. An Example

Now, we are in the position to present an example of a $\psi$-Hilfer hybrid fractional integro-differential boundary value problem to illustrate our main result.

**Example 1.** *Consider the boundary value problem of the form*

$$\begin{cases} {}^H\mathfrak{D}_{\frac{1}{4}}^{\frac{1}{2},\frac{3}{4};\log_e(t^2+1)}\left[\dfrac{{}^H\mathfrak{D}_{\frac{1}{4}}^{\frac{3}{2},\frac{3}{4};\log_e(t^2+1)}x(t)}{g(t,x(t))} - \mathcal{I}_{\frac{1}{4}}^{\frac{3}{4};\log_e(t^2+1)}h_1(t,x(t)) - \mathcal{I}_{\frac{1}{4}}^{\frac{5}{4};\log_e(t^2+1))}h_2(t,x(t)) \right. \\ \left. \quad -\mathcal{I}_{\frac{1}{4}}^{\frac{7}{4};\log_e(t^2+1))}h_3(t,x(t))\right] = \left(\dfrac{2}{7}t+1\right)\left[\dfrac{|x(t)|}{1+|x(t)|}+\sin x(t)\right], \quad t \in \left[\dfrac{1}{4},\dfrac{7}{4}\right], \\ x\left(\dfrac{1}{4}\right) = 0, \quad {}^H\mathfrak{D}_{\frac{1}{4}}^{\frac{3}{2},\frac{3}{4};\log_e(t^2+1)}x\left(\dfrac{1}{4}\right) = 0 \quad x\left(\dfrac{7}{4}\right) = \dfrac{2}{11}x\left(\dfrac{5}{4}\right), \end{cases} \tag{25}$$

*where*

$$\begin{aligned} g(t,x) &= \frac{1}{8t+5}\left(\frac{|x|}{1+|x|}\right) + \frac{2}{40t+25}, \quad h_1(t,x) = \frac{\sin|x|}{4t+2} + \frac{1}{4t+5}, \\ h_2(t,x) &= \frac{|x|}{(8t+3)(1+|x|)} + \frac{1}{20}, \quad h_3(t,x) = \frac{|x|}{(12t+4)(1+|x|)} + \frac{1}{42}. \end{aligned} \tag{26}$$

In the above problem, $\alpha = 1/2$, $\rho = 3/4$, $p = 3/2$, $\beta_1 = 3/4$, $\beta_2 = 5/4$, $\beta_3 = 7/4$, $\psi(t) = \log_e(t^2+1)$, $a = 1/4$, $b = 7/4$, $\theta = 2/11$, $\xi = 5/4$. Then, we find that $\gamma = 15/8$, $\gamma_1 = 7/8$, $\Lambda \approx 1.130218751$. In addition, we compute that

$$|g(t,x) - g(t,y)| \le \frac{1}{8t+5}|x-y|,$$

and

$$|f(t,x) - f(t,y)| \le \left(\frac{4}{7}t+2\right)|x-y|,$$

and $|h_1(t,x) - h_1(t,y)| \le (1/(4t+2))|x-y|$, $|h_2(t,x) - h_2(t,y)| \le (1/(8t+3))|x-y|$, $|h_3(t,x) - h_3(t,y)| \le (1/(12t+4))|x-y|$, where $\phi(t) = 1/(8t+5)$, $\|\phi\| = 1/7$, $\omega(t) = ((4t/7)+2)$, $\|\omega\| = 3$, $\chi_1(t) = 1/(4t+2)$, $\|\chi_1\| = 1/3$, $\chi_2(t) = 1/(8t+3)$, $\|\chi_2\| = 1/5$, $\chi_3(t) = 1/(12t+4)$, $\|\chi_3\| = 1/7$. The bounds can be computed as $|f(t,x)| \le \mu(t) = (4t/7)+2$, $\|\mu\| = 3$, $|g(t,x)| \le \nu(t) = (1/(8t+5)) + (2/(40t+25))$, $\|\nu\| = 1/5$, $|h_1(t,x)| \le (1/(4t+2)) + (1/(4t+5)) = \lambda_1(t)$, $\|\lambda_1\| = 1/2$, $|h_2(t,x)| \le (1/(8t+3)) + (1/20) = \lambda_2(t)$, $\|\lambda_2\| = 1/4$, $|h_3(t,x)| \le (1/(12t+4)) + (1/42) = \lambda_3(t)$, $\|\lambda_3\| = 1/6$. Then, from all information, the condition $(A_3)$ is satisfied with

$$\begin{aligned} &\left[1 + (|\theta|+1)\frac{(\psi(b)-\psi(a))^{\gamma-1}}{|\Lambda|}\right]\frac{(\psi(b)-\psi(a))^p}{\Gamma(p+1)}\left(\|\nu\|\sum_{i=1}^{n}\frac{(\psi(b)-\psi(a))^{\beta_i}}{\Gamma(\beta_i+1)}\|\chi_i\|\right. \\ &\left. +\|\phi\|\sum_{i=1}^{n}\frac{(\psi(b)-\psi(a))^{\beta_i}}{\Gamma(\beta_i+1)}\|\lambda_i\|\right) \approx 0.9292073979 < 1. \end{aligned}$$

Therefore, all assumptions of Theorem 1 are satisfied and we can use the conclusion that the problem (25)–(26) has at least one solution $x(t)$ on $[1/4, 7/4]$.

## 5. Special Cases

The problem (6) considered in the present work is general in the sense that it includes the following classes of new boundary value problems of $\psi$-Hilfer fractional differential equations.

(I) Let $g(t,x) = 1$ and $h_i(t,x) = 0$, $i = 1,2,\ldots,n$ for all $t \in [a,b]$ and $x \in \mathbb{R}$. Then, the problem (6) reduces to the following $\psi$-Hilfer fractional boundary value problem:

$$\begin{cases} {}^H\mathfrak{D}_a^{\alpha,\rho;\psi}\left({}^H\mathfrak{D}_a^{p,\rho;\psi}x(t)\right) = f(t,x(t)), \quad t \in [a,b], \\ x(a) = 0, \quad {}^H\mathfrak{D}_a^{p,\rho;\psi}x(a) = 0, \quad x(b) = \theta x(\xi). \end{cases}$$

(II) Let $h_i(t,x) = 0, i = 1,2,\ldots,n$ for all $t \in [a,b]$ and $x \in \mathbb{R}$. Then, the problem (6) reduces to the following $\psi$-Hilfer fractional boundary value problem:

$$\begin{cases} {}^H\mathfrak{D}_a^{\alpha,\rho;\psi}\left[\dfrac{{}^H\mathfrak{D}_a^{p,\rho;\psi}x(t)}{g(t,x(t))}\right] = f(t,x(t)), \quad t \in [a,b], \\ x(a) = 0, \ {}^H\mathfrak{D}_a^{p,\rho;\psi}x(a) = 0, \ x(b) = \theta x(\xi). \end{cases}$$

(III) Let $g(t,x) = 1$ for all $t \in [a,b]$ and $x \in \mathbb{R}$. Then, the problem (6) reduces to the following $\psi$-Hilfer fractional boundary value problem:

$$\begin{cases} {}^H\mathfrak{D}_a^{\alpha,\rho;\psi}\left[{}^H\mathfrak{D}_a^{p,\rho;\psi}x(t) - \sum_{i=1}^{m}\mathcal{I}_a^{\beta_i;\psi}h_i(t,x(t))\right] = f(t,x(t)), \quad t \in [a,b], \\ x(a) = 0, \ {}^H\mathfrak{D}_a^{p,\rho;\psi}x(a) = 0, \ x(b) = \theta x(\xi). \end{cases}$$

Therefore, the main result of this paper also includes the existence results for the solutions of the abovementioned $\psi$-Hilfer boundary value problems of fractional differential equations as special cases.

## 6. Conclusions

In this paper, we studied a new class of $\psi$-Hilfer hybrid fractional integro-differential boundary value problems with three-point boundary conditions. By using a generalization of Krasnosel'skiĭ's fixed point theorem, we proved an existence result. An example is presented to illustrate our main result. Some special cases are also discussed.

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