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Some New Extensions on Fractional Differential and Integral Properties for Mittag-Leffler Confluent Hypergeometric Function

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Abstract: This article uses fractional calculus to create novel links between the well-known Mittag-Leffler functions of one, two, three, and four parameters. Hence, this paper studies several new analytical properties using fractional integration and differentiation for the Mittag-Leffler function formulated by confluent hypergeometric functions. We construct a four-parameter integral expression in terms of one-parameter. The paper explains the significance and applications of each of the four Mittag-Leffler functions, with the goal of using our findings to make analyzing specific kinds of experimental results considerably simpler.

Keywords: mittag-leffler function; laplace transform; confluent hypergeometric function; fractional calculus; integral operator



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1. Introduction

The regular integration and differentiation in calculus operations are extended to instructions beyond the integers in fractional calculus: to determine the order of differentiation, real and complex numbers may be used [1–3]. This topic is more than 400 years old, although it has been greatly developed over the previous century, uncovering applications in a wide range of scientific and engineering areas [4–6].

Riemann-Liouville is the most important concept of fractional derivatives and integrals, where the meanings are provided by the following definitions:

$${}^RL D_x^\alpha f(x) = \frac{1}{\Gamma(-\alpha)} \int_a^x (x-t)^{-\alpha-1} f(t) dt, \quad \Re(\alpha) < 0, \quad (1)$$

and

$${}^RL D_x^\alpha f(x) = \frac{d^m}{dx^m} \left({}^RL D_x^{\alpha-m} f(x) \right), \quad \Re(\alpha) \geq 0, m := [\Re(\alpha)] + 1, \quad (2)$$

where $D^\alpha f$ refers to the derivative of order α for the function f , and a is the differentiation constant.

It is essential to keep in mind that derivatives and integrals in fractional calculus are detected by the use of an arbitrary constant a . The value of a is usually set to one of two values: $a = 0$ or $a = -\infty$. We establish the following lemma's validity as a source of two "natural" differentiation formulae, $a = 0$ and $a = -\infty$; both are viable choices to understand the advantages of both choices. Neither option may be excluded from the set of potential values for a .

Lemma 1 ([7]). With differintegration constants $a = 0$ and $a = -\infty$, the Riemann-Liouville (RL) differintegrals of exponential and power functions are as follows:

$${}^RL_0 D_x^\alpha f(x^\gamma) = {}^RL D_x^\alpha f(x^\gamma) = \frac{\Gamma(\gamma + 1)}{\Gamma(\gamma - \alpha + 1)} x^{\gamma - \alpha}, \quad \alpha, \gamma \in \mathbb{C}, \Re(\gamma) > -1; \quad (3)$$

$${}^RL_{-\infty} D_x^\alpha (e^{\gamma x}) = \gamma^\alpha e^{\gamma x}, \quad \alpha, \gamma \in \mathbb{C}, \gamma \notin \mathbb{R}_0^-. \quad (4)$$

To describe complex power functions, we utilize the main branch with values ranging from $-\pi$ to π in both equations; also see Reference [1–3].

Many recent definitions and investigations of fractional differintegrals have been considered recently. Part of them was aided by a number of well recognized systems that can be formed by various structures based on fractional calculus. As an example to better demonstrate certain processes in dynamical systems, in Equation (1), another function is used in lieu of the power function [8,9]. Other processes were created by integrating more generalization levels and parameters into formulas and functions [10–12].

The Mittag-Leffler function is a precise function that appears often in the study of fractional integrals and derivatives; see, for example, Ghanim and Al-Janaby [13,14], Ghanim et al. [15], Oros [16,17], Haubold et al. [18], Paneva-Konovska [19], Mainardi and Gorenflo [20], Mathai and Haubold [21], Srivastava [22,23], and Srivastava et al. [24].

In 1903, Magnus Gustaf (Gösta) Mittag-Leffler (1846–1927) [25] (also see Mittag-Leffler [26]), a Swedish mathematician, invented and studied the well-known Mittag-Leffler function $E_\alpha(z)$ given by

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)} \quad (z \in \mathbb{C}; \Re(\alpha) > 0). \quad (5)$$

Wiman [27] and Reference [28] then proposed a generalization $E_\alpha(z)$ of $E_{\alpha,\gamma}(z)$ provided by

$$E_{\alpha,\gamma}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \gamma)} \quad (\Re(\alpha) > 0; \gamma, \alpha \in \mathbb{C}). \quad (6)$$

As a result, the functions of Mittag-Leffler $E_\alpha(z)$ and $E_{\alpha,\gamma}(z)$ in (5) and (6), respectively, have been studied and expanded in a variety of different ways and applications. The implementation in the physical model has succeeded in recent decades, the generalized Mittag-Leffler functions were also used in mathematical and physical issues, as the solutions of the fractional integral and differential equations were naturally presented. Fractional order calculus is associated with practical endeavors, and it is widely used in nanotechnology [29], chaos theory [30], optics [31], human diseases [32], and other fields [33,34]. In fact, the authors are collaborating with a group of college of engineering researchers on several recent engineering applications involving generalized multi-parameter Mittag-Leffler functions and their extended types, such as noise measurement and heat transfer in asphalt concrete. Wright [35] investigated these functions and their associated disseminations. Pillai [36] also established links between generalized Mittag-Leffler type functions and route models. Ref. [37,38] are two studies which have utilized the function (5) with parameters α and γ in more general functions linked to one or more parameters.

$$E_{\alpha,\gamma}^\lambda(z) = \sum_{n=0}^{\infty} \frac{\Gamma(n + \lambda) z^n}{\Gamma(\lambda) \Gamma(\alpha n + \gamma) n!}, \quad (\Re(\alpha) > 0; \lambda, \gamma, \alpha \in \mathbb{C}). \quad (7)$$

$$E_{\alpha,\gamma}^{\beta,\lambda}(z) = \sum_{n=0}^{\infty} \frac{\Gamma(\beta n + \lambda) z^n}{\Gamma(\lambda) \Gamma(\alpha n + \gamma) n!}, \quad (\Re(\alpha) > 0; \lambda, \gamma, \alpha \in \mathbb{C}). \quad (8)$$

A variety of physical disciplines have shown the application for the functions of confluent hypergeometric. It has been employed in cases where both sedimentation and diffusion are related; in ultracentrifuges, for example, isotope separation and protein molecular weight measurements are applications. Such functions are often used to express the formula for the velocity distribution of electrons in high-frequency gas discharges.

The solution of the second-order linear homogeneous differential equation for the confluent hypergeometric function $M(\lambda; \gamma; z)$ is as follows:

$$z \frac{d^2 M}{dz^2} + (\gamma - z) \frac{dM}{dz} - \lambda M = 0, \quad (\lambda, \gamma, z \in \mathbb{C}). \quad (9)$$

Equation (9) contains a sporadic singularity at infinity, as well as a normal initial singularity [39].

A second solution of Equation (9) may be found if γ is not integral.

$$W(\lambda; \gamma; z) = z^{1-\gamma} M(\lambda - \gamma + 1; 2 - \gamma; z). \quad (10)$$

If γ is integral, then

$$\begin{aligned} W(\alpha; \gamma; z) = M(\alpha; \gamma; z) \{ \ln z + \Omega(1 - \alpha) - \Omega(\gamma) + C \} + \sum_{n=1}^{\infty} \frac{\Gamma(n + \lambda) \Gamma(\gamma) B_n z^n}{\Gamma(\lambda) \Gamma(n + \gamma) n!} \\ + (-1)^\gamma \sum_{n=0}^{\infty} \frac{\Gamma(\gamma) \Gamma(n + \lambda - \gamma + 1) \Gamma(\gamma - n - 1) (-1)^n}{\Gamma(\lambda) n! z} \end{aligned} \quad (11)$$

may be used to provide a second solution, where

$$\Omega(\lambda) = \frac{\Gamma'(\lambda)}{\Gamma(\lambda)},$$

where the Euler's constant C is equal to 0.577216..., and

$$\begin{aligned} B_n = \left(\frac{1}{\lambda} + \frac{1}{\lambda+1} + \dots + \frac{1}{\lambda+n-1} \right) \\ - \left(\frac{1}{\gamma} + \frac{1}{\gamma+1} + \dots + \frac{1}{\gamma+n-1} \right) - \left(1 + \frac{1}{2} + \dots + \frac{1}{n} \right). \end{aligned}$$

The Ω function's elaboration tables are given in Reference [40].

Furthermore, $M(\lambda; \gamma; z)$ has a series of representation in

$$M(\alpha; \gamma; z) = 1 + \frac{\lambda}{\gamma} z + \frac{\lambda(\lambda+1)}{\gamma(\gamma+1)} \frac{z^2}{2} + \dots = \sum_{n=0}^{\infty} \frac{\Gamma(\gamma) \Gamma(n + \lambda) z^n}{\Gamma(\lambda) \Gamma(n + \gamma) n!}. \quad (12)$$

For all values of z , it is obvious that the previous series absolutely converges.

The following are guaranteed by both M and W :

$$\frac{d}{dz} M(\lambda; \gamma; z) = \frac{\lambda}{\gamma} M(\lambda + 1; \gamma + 1; z), \quad (13)$$

$$\lambda M(\lambda + 1; \gamma + 1; z) = (\lambda - \gamma) M(\lambda; \gamma + 1; z) + \gamma M(\lambda; \gamma; z), \quad (14)$$

$$\lambda M(\lambda + 1; \gamma; z) = (z + 2\lambda - \gamma) M(\lambda; \gamma; z) + (\gamma - \lambda) M(\lambda - 1; \gamma; z). \quad (15)$$

Equation (13) follows directly, since the differentiation of Equations (12), (14) and (15) may be obtained subsequently from the differential equation. Using the Wronskian of Equation (9), the functional connection between a Wronskian's value at any point in a plane and its value at a specific place may be utilized to demonstrate a relationship between $M(\lambda; \gamma; z)$ and $W(\alpha; \gamma; z)$. For additional information, see References [39–41].

The confluent hypergeometric Mittag-Leffler function $M_{\alpha, \gamma}^{\beta, \lambda}(z)$ is now presented as follows:

$$M_{\alpha, \gamma}^{\beta, \lambda}(z) = \sum_{n=0}^{\infty} \frac{\Gamma(\gamma)\Gamma(\beta n + \lambda)z^n}{\Gamma(\lambda)\Gamma(\alpha n + \gamma)n!}, \quad (\alpha, \gamma, \beta, \lambda \in \mathbb{C}; \Re(\alpha) > 0). \quad (16)$$

The following are some special instances of the confluent hypergeometric Mittag-Leffler function $M_{\alpha, \gamma}^{\beta, \lambda}(z)$:

$$M_{0, \gamma}^{0, \lambda}(z) = e^z, \quad M_{0, 1}^{1, 1}(z) = \frac{1}{1-z}, \quad M_{1, 2}^{1, 1}(z) = \frac{e^z - 1}{z},$$

$$M_{2, 1}^{1, 1}(z^2) = \cosh z, \quad M_{2, 1}^{1, 1}(-z^2) = \cos z$$

$$M_{1, \gamma}^{1, \lambda}(z) = M(\alpha; \gamma; z),$$

$$M_{\alpha, 1}^{1, 1}(z) = E_{\alpha}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}.$$

When α is a positive integer, such as n , then:

$$M_{\alpha, \gamma}^{1, \lambda}(z) = {}_1F_n\left(\lambda; \frac{\gamma}{n}, \frac{\gamma+1}{n}, \dots, \frac{\gamma+n-1}{n}; \frac{z}{n^n}\right);$$

in addition,

$$H_i(x, n) = x^{i-1}M_{n, i}^{1, 1}(x^n),$$

$$T_i(x, n) = x^{i-1}M_{n, i}^{1, 1}(-x^n).$$

H_i and T_i are generalized hyperbolic and trigonometrical functions, respectively [42]. Furthermore,

$$\left(\frac{d}{dz}\right)^m M_{\alpha, \gamma}^{\beta, \lambda}(z) = (\lambda)_m M_{\alpha, \gamma+m}^{\beta, \lambda+m}(z), \quad (17)$$

$$\left(\frac{d}{dz}\right)^m z^{\gamma-1} M_{\alpha, \gamma}^{\beta, \lambda}(z^{\alpha}) = z^{\gamma-m-1} M_{\alpha, \gamma}^{\beta, \lambda}(z^{\alpha}), \quad (18)$$

$$\left(z \frac{d}{dz} + \lambda\right) M_{\alpha, \gamma}^{\beta, \lambda}(z^{\alpha}) = \lambda M_{\alpha, \gamma}^{\beta, \lambda+1}(z), \quad (19)$$

$$(\gamma - \alpha\lambda - 1)M_{\alpha, \gamma}^{\beta, \lambda}(z) = M_{\alpha, \gamma-1}^{\beta, \lambda}(z) - \alpha\lambda M_{\alpha, \gamma}^{\beta, \lambda+1}(z), \quad (20)$$

and

$$\mathcal{L}\{t^{\gamma-1}M_{\alpha,\gamma}^{\beta,\lambda}(\mu t^\alpha)\} = \rho^{-\gamma}(1 - \mu\rho^{-\alpha})^{-\lambda} \quad |\rho| > |\mu|^{\frac{1}{\Re(\alpha)}}, \Re(\rho) > 0, \Re(\gamma) > 0. \quad (21)$$

In this case, the Laplace transform of $f(t)$ is represented by $\mathcal{L}\{f(t)\}$.

Several scholars (see, for example, References [43–48]) have utilized the Laplace transform to solve convolution equations, which are special instances of

$$\int_0^x \frac{(x-t)^{b-1}}{\Gamma(b)} {}_1F_1(a; b; c(x-t))f(t)dt = g(x) \quad \Re(b) > 0, \quad (22)$$

described in fractional integration [49]. The target of this research is to look into an integral equation

$$\int_a^x (t-x)^{\gamma-1}M_{\alpha,\gamma}^{\beta,\lambda}\mu(t-x)^\alpha f(t)dt = g(x) \quad \Re(\gamma) > 0, \quad (23)$$

for any real number $a > 0$, where the function $M_{\alpha,\gamma}^{\beta,\lambda}(z)$ in (16) is an analytic function of order α that includes many well-known unique functions.

Ghanim and Al-Janaby [50] introduced (23) by using an integral operator $\Sigma_{\alpha,\gamma}^{\beta,\lambda}(\mu)$ on a space Ψ of functions and a fractional integration operator $\Delta^\delta : \Psi \rightarrow \Psi$ to prove results on $\Sigma_{\alpha,\gamma}^{\beta,\lambda}(\mu)$. Theorems on the solutions of (23) were then discussed using these results. This technique may be used to obtain comparable results on the integral equation

$$\mathfrak{E}_{\alpha,\gamma}^{\beta,\lambda}(\mu)f(x) \equiv \int_a^x (t-x)^{\gamma-1}M_{\alpha,\gamma}^{\beta,\lambda}\delta(t-x)^\alpha f(t)dt = g(x) \quad \Re(\gamma) > 0, \quad (24)$$

which consists of the equations given in References [51,52] as special cases. The linear space of complex-valued functions f that are Ψ -integrable on a finite $[a, b]$, $a \geq 0$ with $\|f\| = \int_a^b |f(t)|dt$ is denoted by Ψ . $\Delta^\delta : \Psi \rightarrow \Psi$ is a fractional operator defined by the fractional integral for complex δ with $\Re(\delta) > 0$

$$\Delta^\delta f(x) = \int_a^x \frac{(x-t)^{\delta-1}}{\Gamma(\delta)} f(t)dt. \quad (25)$$

The fact that Δ^δ is bounded and that $\Delta^\delta f = 0 \rightarrow f = 0$ is a standard result means that the inverse operator exists on the Ψ_δ subspace of Ψ . If $0 < \Re(\delta) < \Re(\xi)$, it is easily demonstrated that $\Psi_\xi \subset \Psi_\delta \subset \Psi$, and the inclusion is appropriate. For $\Re(\delta) < 0$, Δ^δ , the inverse of $\Delta^{-\delta}$ is defined. If $\Re(\delta) \neq 0$ and $\Re(\xi) \neq 0$, then $\Delta^\delta \Delta^\xi f = \Delta^{\delta+\xi}$ for appropriate functions f . Δ^δ is defined on Ψ_μ with $\Re(\mu) > 0$ as $\Delta^{-1}\Delta^{1+\delta}$ for $\Re(\delta) = 0$. For complex $\alpha, \gamma, \lambda, \mu$ with $f \in \Psi$ and $\Re(\gamma) > 0$, the fractional operator $\Sigma_{\alpha,\gamma}^{\beta,\lambda}$ on Ψ into itself is defined by

$$\Sigma_{\alpha,\gamma}^{\beta,\lambda}(\mu)f(x) = \int_a^x (t-x)^{\gamma-1}M_{\alpha,\gamma}^{\beta,\lambda}\mu(t-x)^\alpha f(t)dt \quad a < x < b.$$

Lemma 2 ([53]). Assume f is a function defined by

$$f(x) = \sum_{n=1}^{\infty} f_n(x)$$

that uniformly converges on $|x-a| \leq L$, where $L > 0$ and $a \in \mathbb{C}$ are fixed constants. For a fixed order of differintegration $\alpha \in \mathbb{C}$, if $1. \Re(\alpha) < 0$ (fractional integration), then

$${}^RL D_x^\alpha f(x) = \sum_{n=1}^{\infty} {}^RL D_x^\alpha f_n(x), \quad |x-a| \leq L,$$

and the right side of the series is uniformly convergent on $|x - a| \leq L$.

2. $\Re(\alpha) \geq 0$ (fractional differentiation) and $\sum_{n=1}^{\infty} {}^{RL}D_x^\alpha f_n(x)$ is uniformly convergent series on the given region, then

$${}^{RL}D_x^\alpha f(x) = \sum_{n=1}^{\infty} {}^{RL}D_x^\alpha f_n(x), \quad |x - a| \leq L.$$

2. Results

Theorem 1. Let $\alpha, \beta, \lambda, \gamma \in \mathbb{C}$ with $\Re(\alpha), \Re(\gamma) > 0$; then:

$$M_{\alpha, \gamma}^{\beta, \lambda}(z) = \frac{\Gamma(\gamma)}{\Gamma(\lambda)} {}^0RLD_z^{n\beta + \lambda - n - 1} \left[z^{\lambda - 1} E_{\alpha, \gamma}(z^\beta) \right], \quad z \in \mathbb{C}. \quad (26)$$

Proof. It is clear from Lemma 1 that most often a quotient of gamma functions can be defined as originating from a fractional power function differintegral. The expression $\frac{\Gamma(\beta n + \lambda)}{n!}$ in the coefficients of the series Equation (16) yields the following:

$$\begin{aligned} M_{\alpha, \gamma}^{\beta, \lambda}(z) &= \sum_{n=0}^{\infty} \frac{\Gamma(\gamma)\Gamma(\beta n + \lambda)z^n}{\Gamma(\lambda)\Gamma(\alpha n + \gamma)n!} = \sum_{n=0}^{\infty} \frac{\Gamma(\gamma)}{\Gamma(\lambda)\Gamma(\alpha n + \gamma)} \frac{\Gamma(\beta n + \lambda)}{\Gamma(n + 1)} z^n \\ &= \sum_{n=0}^{\infty} \frac{\Gamma(\gamma)}{\Gamma(\lambda)\Gamma(\alpha n + \gamma)} {}^0RLD_z^{n\beta + \lambda - n - 1} \left[z^{n\beta + \lambda - 1} \right] \\ &= \sum_{n=0}^{\infty} \frac{\Gamma(\gamma)}{\Gamma(\lambda)} {}^0RLD_z^{n\beta + \lambda - n - 1} \left[\frac{z^{n\beta + \lambda - 1}}{\Gamma(\alpha n + \gamma)} \right]. \end{aligned}$$

We may utilize the Lemma 2 result to replace the summation with fractional differential integration since the series is uniformly convergent. Then, we obtain:

$$\begin{aligned} M_{\alpha, \gamma}^{\beta, \lambda}(z) &= \frac{\Gamma(\gamma)}{\Gamma(\lambda)} \sum_{n=0}^{\infty} {}^0RLD_z^{n\beta + \lambda - n - 1} \left[\frac{z^{n\beta + \lambda - 1}}{\Gamma(\alpha n + \gamma)} \right] \\ &= \frac{\Gamma(\gamma)}{\Gamma(\lambda)} {}^0RLD_z^{n\beta + \lambda - n - 1} \left[\sum_{n=0}^{\infty} \frac{z^{n\beta + \lambda - 1}}{\Gamma(\alpha n + \gamma)} \right] \\ &= \frac{\Gamma(\gamma)}{\Gamma(\lambda)} {}^0RLD_z^{n\beta + \lambda - n - 1} \left[z^{\lambda - 1} \sum_{n=0}^{\infty} \frac{z^{n\beta}}{\Gamma(\alpha n + \gamma)} \right] \\ &= \frac{\Gamma(\gamma)}{\Gamma(\lambda)} {}^0RLD_z^{n\beta + \lambda - n - 1} \left[z^{\lambda - 1} E_{\alpha, \gamma}(z^\beta) \right]. \end{aligned}$$

Hence, we have the result. \square

Corollary 1. For any $\alpha, \lambda \in \mathbb{C}$ with $\Re(\alpha) > 0$, we have:

$$M_{\alpha}^{\beta, \lambda}(z) = \frac{\Gamma(\gamma)}{\Gamma(\lambda)} {}^0RLD_z^{n\beta + \lambda - n - 1} \left[z^{\lambda - 1} E_{\alpha}(z^\beta) \right]. \quad (27)$$

Proof. The proof is immediately followed by replacing γ with 1 in Theorem 1. \square

It is obvious that, if we recover the basic identity, $M_{\alpha,1}^{1,1}(z) = E_{\alpha}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}$ by establishing $\beta = \gamma = \lambda = 1$ in Equation (26). Using the exact same logic as in Theorem 1, we can demonstrate this.

Corollary 2. Let $\alpha, \beta, \lambda, \gamma, \tau \in \mathbb{C}$ with $\Re(\alpha) > 0, \Re(\gamma) > 0$. We obtain:

$$M_{\alpha,\gamma}^{\beta,\lambda}(\tau z) = \frac{\Gamma(\gamma)}{\Gamma(\lambda)} {}_0^R D_z^{n\beta+\lambda-n-1} \left[z^{\lambda-1} E_{\alpha,\gamma}(\tau z^{\beta}) \right], \quad z \in \mathbb{C}. \quad (28)$$

Proof. The proof is as proven above in Theorem 1 with an additional element of τ^n contained in each of the sum terms. \square

Theorem 2. For $\alpha, \beta, \gamma, \tau \in \mathbb{C}$ and $\lambda = 1$ with $\Re(\alpha) > 0, \Re(\gamma) > 0$, we have:

$$M_{\alpha,\gamma}^{\beta}(\tau z^{\alpha}) = \frac{z^{1-\gamma}}{\Gamma(\gamma)} {}_0^R D_z^{n\beta-\alpha n-\gamma+1} \left[E_{\alpha,\gamma}(\tau z^{\alpha\beta}) \right], \quad z \in \mathbb{C}. \quad (29)$$

Proof. We start this time from the right side of the appropriate identity and then use the definition in Equation (6) of the function $E_{\alpha,\gamma}$:

$$\begin{aligned} \frac{z^{1-\gamma}}{\Gamma(\gamma)} {}_0^R D_z^{n\beta-\alpha n-\gamma+1} \left[E_{\alpha,\gamma}(\tau z^{\alpha\beta}) \right] &= \frac{z^{1-\gamma}}{\Gamma(\gamma)} {}_0^R D_z^{n\beta-\alpha n-\gamma+1} \left[\sum_{n=0}^{\infty} \frac{(\tau z^{\alpha\beta})^n}{\Gamma(\alpha n + \gamma)} \right] \\ &= \frac{z^{1-\gamma}}{\Gamma(\gamma)} {}_0^R D_z^{n\beta-\alpha n-\gamma+1} \left[\sum_{n=0}^{\infty} \frac{\tau^n z^{n\alpha\beta}}{\Gamma(\alpha n + \gamma)} \right]. \end{aligned}$$

The series is uniformly convergent, so we can switch the summation and fractional differentiation by Lemma 2 given that the resulting series always converges uniformly (at least in the case of $1 > \Re(\gamma) > 0$). Then, we switch the operations and end with the appropriate illustration.

$$\begin{aligned} \frac{z^{1-\gamma}}{\Gamma(\gamma)} {}_0^R D_z^{n\beta-\alpha n-\gamma+1} \left[E_{\alpha,\gamma}(\tau z^{\alpha\beta}) \right] &= \frac{z^{1-\gamma}}{\Gamma(\gamma)} \sum_{n=0}^{\infty} {}_0^R D_z^{n\beta-\alpha n-\gamma+1} \left[\frac{\tau^n z^{n\alpha\beta}}{\Gamma(\alpha n + \gamma)} \right] \\ &= \frac{z^{1-\gamma}}{\Gamma(\gamma)} \sum_{n=0}^{\infty} \frac{\Gamma(\beta n + 1)}{\Gamma(\alpha n + \gamma)} \frac{\tau^n z^{n\alpha\beta}}{\Gamma(\beta n + 1)} = \sum_{n=0}^{\infty} \frac{\tau^n z^{n\alpha\beta}}{\Gamma(\beta n + 1)}. \end{aligned}$$

The series converges uniformly, being explicitly in the Equation (6) series expression for $E_{\alpha,\gamma}(\tau z^{\beta})$. Thus, our aforementioned switching of operations was appropriate, and the proof is now true. \square

Corollary 3. For $\alpha, \beta, \tau \in \mathbb{C}$ and $\gamma = \lambda = 1$ with $\Re(\alpha) > 0$, we have:

$$M_{\alpha}^{\beta}(\tau z) = z^{1-\gamma} {}_0^R D_z^{n(\beta-\alpha)} \left[E_{\alpha}(\tau z^{\alpha\beta}) \right]. \quad (30)$$

Proof. The proof is immediately followed by replacing γ with 1 in Theorem 2. \square

By combining Theorems 1 and 2, a composite expression can be obtained in Equation (6) for the four-parameter Mittag-Leffler function, as defined in the following theorem, in terms of fractional integrals.

Theorem 3. Let $\alpha, \beta, \lambda, \gamma, \tau \in \mathbb{C}$ and $\Re(\alpha) > 0$, $\Re(\gamma) > 0$ with $\Re(\lambda) < 1$, and we obtain:

$$M_{\alpha, \gamma}^{\beta, \lambda}(\tau z^\alpha) = \frac{\alpha \sin(\pi \lambda) B(1 - \lambda, n - n\beta)}{\pi \Gamma(n - n\beta)} \times \int_0^z (z^\alpha - u^\alpha)^{-n\beta + n - \lambda} u^{\alpha(n\beta + \lambda - n) - \gamma} {}_0^{\text{RL}} D_u^{n\beta - \alpha n - \gamma + 1} [E_{\alpha, \gamma}(\tau u^{\alpha\beta})] du, \quad (31)$$

where $z \in \mathbb{C}$ and B is beta function.

Proof. For $\Re(\lambda) < 1$, the fractional differintegrals showing up in Theorem 1 and Corollary 1 are integrals, so Equation (28) can be modified as follows:

$$\begin{aligned} M_{\alpha, \gamma}^{\beta, \lambda}(\tau z) &= \frac{\Gamma(\gamma)}{\Gamma(\lambda)} {}_0^{\text{RL}} D_z^{n\beta + \lambda - n - 1} [z^{\lambda - 1} E_{\alpha, \gamma}(\tau z^\beta)] \\ &= \frac{\Gamma(\gamma)}{\Gamma(\lambda) \Gamma(n + 1 - n\beta - \lambda)} \int_0^z (z - y)^{-n\beta + n - \lambda} y^{n\beta + \lambda - n - 1} [E_{\alpha, \gamma}(\tau y^\beta)] dy. \end{aligned}$$

Then, we shift variables y with u^α and use the reflection formula for the gamma function with the definition of beta function to have

$$\begin{aligned} M_{\alpha, \gamma}^{\beta, \lambda}(\tau z) &= \frac{\Gamma(\gamma) \sin(\pi \lambda) B(1 - \lambda, n - n\beta)}{\pi \Gamma(n - n\beta)} \\ &\times \int_0^{z^{1/\alpha}} (z - u^\alpha)^{-n\beta + n - \lambda} u^{\alpha(n\beta + \lambda - n - 1)} [E_{\alpha, \gamma}(\tau u^{\alpha\beta})] \alpha u^{\alpha - 1} du \\ &= \frac{\alpha \Gamma(\gamma) \sin(\pi \lambda) B(1 - \lambda, n - n\beta)}{\pi \Gamma(n - n\beta)} \\ &\times \int_0^{z^{1/\alpha}} (z - u^\alpha)^{-n\beta + n - \lambda} u^{\alpha(n\beta + \lambda - n) - 1} [E_{\alpha, \gamma}(\tau u^{\alpha\beta})] du. \end{aligned}$$

Applying the result of Theorem 2, we have

$$\begin{aligned} M_{\alpha, \gamma}^{\beta, \lambda}(\tau z) &= \frac{\alpha \Gamma(\gamma) \sin(\pi \lambda) B(1 - \lambda, n - n\beta)}{\pi \Gamma(n - n\beta)} \\ &\times \int_0^{z^{1/\alpha}} (z - u^\alpha)^{-n\beta + n - \lambda} u^{\alpha(n\beta + \lambda - n) - 1} \left[\frac{u^{1 - \gamma}}{\Gamma(\gamma)} {}_0^{\text{RL}} D_z^{n\beta - \alpha n - \gamma + 1} [E_{\alpha, \gamma}(\tau u^{\alpha\beta})] \right] du \\ &= \frac{\alpha \sin(\pi \lambda) B(1 - \lambda, n - n\beta)}{\pi \Gamma(n - n\beta)} \\ &\times \int_0^{z^{1/\alpha}} (z - u^\alpha)^{-n\beta + n - \lambda} u^{\alpha(n\beta + \lambda - n) - \gamma} {}_0^{\text{RL}} D_z^{n\beta - \alpha n - \gamma + 1} [E_{\alpha, \gamma}(\tau u^{\alpha\beta})] du. \end{aligned}$$

Substituting z^α for z , we obtain:

$$M_{\alpha, \gamma}^{\beta, \lambda}(\tau z^\alpha) = \frac{\alpha \sin(\pi \lambda) B(1 - \lambda, n - n\beta)}{\pi \Gamma(n - n\beta)}$$

$$\times \int_0^z (z^\alpha - u^\alpha)^{-n\beta+n-\lambda} u^{\alpha(n\beta+\lambda-n)-\gamma} {}^{RL}D_u^{n\beta-\alpha n-\gamma+1} [E_{\alpha,\gamma}(\tau u^{\alpha\beta})] du.$$

Hence, we have the result. \square

Corollary 4. Let $\alpha, \beta, \lambda, \tau \in \mathbb{C}$ and $\Re(\alpha) > 0$ with $\Re(\lambda) < 1$, and we have:

$$M_{\alpha}^{\beta,\lambda}(\tau z^\alpha) = \frac{\alpha \sin(\pi\lambda) B(1-\lambda, n-n\beta)}{\pi \Gamma(n-n\beta)} \times \int_0^z (z^\alpha - u^\alpha)^{-n\beta+n-\lambda} u^{\alpha(n\beta+\lambda-n)-1} {}^{RL}D_u^{n(\beta-\alpha)} [E_{\alpha}(\tau u^{\alpha\beta})] du, \quad (32)$$

where $z \in \mathbb{C}$ and B is beta function.

Proof. The proof is immediately followed by replacing γ with 1 in Theorem 3. \square

Corollary 5. Let $\alpha, \beta, \lambda, \gamma, \tau \in \mathbb{C}$ and $\Re(\alpha) > 0$, $\Re(\gamma) > 0$ with $\Re(\lambda) < 1$, we have:

$$M_{\alpha,\gamma}^{\beta,\lambda}(\tau z^\alpha) = \frac{\alpha \sin(\pi\lambda) B(1-\lambda, n-n\beta)}{\pi \Gamma(n-n\beta) \Gamma\alpha n + \gamma - n\beta} \times \int_0^z (z^\alpha - u^\alpha)^{-n\beta+n-\lambda} u^{\alpha(n\beta+\lambda-n)-\gamma} \int_0^u (u-x)^{\alpha n + \gamma - n\beta - 1} E_{\alpha,\gamma}(\tau x^{\alpha\beta}) dx du, \quad (33)$$

$z \in \mathbb{C}$.

Proof. Here, we have $\Re(1-\gamma) < 0$; hence, the fractional diffintegral that occurs in the Equation (31) is an integral component. Thus, we obtain:

$${}^{RL}D_u^{\alpha n + \gamma - n\beta - 1} [E_{\alpha,\gamma}(\tau u^{\alpha\beta})] = \frac{1}{\Gamma\alpha n + \gamma - n\beta} \int_0^u (u-x)^{\alpha n + \gamma - n\beta - 1} E_{\alpha,\gamma}(\tau x^{\alpha\beta}) dx,$$

and substitute this into Equation (31) to find:

$$M_{\alpha,\gamma}^{\beta,\lambda}(\tau z^\alpha) = \frac{\alpha \sin(\pi\lambda) B(1-\lambda, n-n\beta)}{\pi \Gamma(n-n\beta)} \times \int_0^z (z^\alpha - u^\alpha)^{-n\beta+n-\lambda} u^{\alpha(n\beta+\lambda-n)-\gamma} \left[\frac{1}{\Gamma\alpha n + \gamma - n\beta} \int_0^u (u-x)^{\alpha n + \gamma - n\beta - 1} E_{\alpha,\gamma}(\tau x^{\alpha\beta}) dx \right] du,$$

which gives the required proof. \square

Theorem 4. The Mittag-Leffler Four-Parameter function in Equation (16) can be given as an integral transformation of the following form:

$$M_{\alpha,\gamma}^{\beta,\lambda}(\tau z^\alpha) = \frac{\alpha \sin(\pi\lambda) B(1-\lambda, n-n\beta)}{\pi \Gamma(n-n\beta) \Gamma\alpha n + \gamma - n\beta} \int_0^z \Omega_{\alpha,\gamma,\lambda}(x; z) E_{\alpha,\gamma}(\tau x^{\alpha\beta}) dx, \quad z \in \mathbb{C}, \quad (34)$$

where $\alpha, \beta, \lambda, \gamma, \tau \in \mathbb{C}$ and $\Re(\alpha) > 0$, $\Re(\gamma) > 1$ with $\Re(\lambda) < 1$, and Ω is a function defined as

$$\Omega_{\alpha,\gamma,\lambda}(x; z) = \int_x^z (z^\alpha - u^\alpha)^{-n\beta+n-\lambda} u^{\alpha(n\beta+\lambda-n)-\gamma} (u-x)^{\alpha n + \gamma - n\beta - 1} du. \quad (35)$$

Proof. Fubini's theorem implies that the order of the integrals can be exchanged in Equation (33). We have $0 \leq u \leq z$ and $0 \leq x \leq u$, which is equivalent to $0 \leq x \leq z$ and $x \leq u \leq z$ after swapping. Thereby, we have (33):

$$M_{\alpha,\gamma}^{\beta,\lambda}(\tau z^\alpha) = \frac{\alpha \sin(\pi\lambda) B(1-\lambda, n-n\beta)}{\pi \Gamma(n-n\beta) \Gamma\alpha n + \gamma - n\beta} \times \int_0^z \int_0^u (z^\alpha - u^\alpha)^{-n\beta+n-\lambda} u^{\alpha(n\beta+\lambda-n)-\gamma} (u-x)^{\alpha n + \gamma - n\beta - 1} E_{\alpha,\gamma}(\tau u^{\alpha\beta}) dx du$$

$$\begin{aligned}
&= \frac{\alpha \sin(\pi\lambda) B(1-\lambda, n-n\beta)}{\pi \Gamma(n-n\beta) \Gamma\alpha n + \gamma - n\beta} \\
&\times \int_0^z \int_x^z (z^\alpha - u^\alpha)^{-n\beta+n-\lambda} u^\alpha (n\beta+\lambda-n)^{-\gamma} (u-x)^{\alpha n + \gamma - n\beta - 1} E_{\alpha, \gamma}(\tau u^{\alpha\beta}) du dx \\
&= \frac{\alpha \sin(\pi\lambda) B(1-\lambda, n-n\beta)}{\pi \Gamma(n-n\beta) \Gamma\alpha n + \gamma - n\beta} \\
&\times \int_0^z E_{\alpha, \gamma}(\tau u^{\alpha\beta}) \int_x^z (z^\alpha - u^\alpha)^{-n\beta+n-\lambda} u^\alpha (n\beta+\lambda-n)^{-\gamma} (u-x)^{\alpha n + \gamma - n\beta - 1} du dx \\
&= \frac{\alpha \sin(\pi\lambda) B(1-\lambda, n-n\beta)}{\pi \Gamma(n-n\beta) \Gamma\alpha n + \gamma - n\beta} \int_0^z E_{\alpha, \gamma}(\tau x^{\alpha\beta}) \Omega_{\alpha, \gamma, \lambda}(x; z) dx,
\end{aligned}$$

as required. \square

3. Conclusions

The study uses Riemann-Liouville fractional calculus to establish new links between the Mittag-Leffler functions of one to four parameters. Most probably, those findings can be extended in the future to simplify certain important physical models that utilize four parameters or more of Mittag-Leffler functions, or to offer more productive mathematical models for these functions, as the original Mittag-Leffler function is increasingly popular and studied in greater detail.

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