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New Estimations of Hermite–Hadamard Type Integral Inequalities for Special Functions

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Abstract: In this paper, we propose some generalized integral inequalities of the Raina type depicting the Mittag–Leffler function. We introduce and explore the idea of generalized s-type convex function of Raina type. Based on this, we discuss its algebraic properties and establish the novel version of Hermite–Hadamard inequality. Furthermore, to improve our results, we explore two new equalities, and employing these we present some refinements of the Hermite–Hadamard-type inequality. A few remarkable cases are discussed, which can be seen as valuable applications. Applications of some of our presented results to special means are given as well. An endeavor is made to introduce an almost thorough rundown of references concerning the Mittag–Leffler functions and the Raina functions to make the readers acquainted with the current pattern of emerging research in various fields including Mittag–Leffler and Raina type functions. Results established in this paper can be viewed as a significant improvement of previously known results.

Keywords: convex function; Hölder's inequality; Hölder-Íscan inequality; improved power-mean integral inequality; s-type convexity

MSC: 26A51; 26A33; 26D07; 26D10; 26D15

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1. Introduction

The Hermite–Hadamard inequality, which is the primary consequence of convex functions having a beautiful geometrical understanding and broad use, has stood out with incredible interest in fundamental mathematics. Numerous mathematicians have given their endeavors to normalization, refining, impersonation, and extension of the Hermite–Hadamard inequality using different types of novel convexities.

The theory of convexity is an incredible and viable methodology for contemplating the huge greatness of issues that emerges in different fields of pure and applied sciences. Many new structures have been presented and researched concerning convex sets and convex mappings. A few scientists have inferred new variations related with convex mappings, see references [1–5]. Integral inequalities on the Raina function have additionally been a subject of discussion for a significant length of time. Because of their possibilities to be extended, a few variations have been set up by many mathematicians, see references [6,7].

The theory of convex mappings has a wide scope of possible applications in many interesting and captivating fields of exploration. Moreover, this theory likewise assumes an eminent part in different areas, such as information theory, coding theory, engineering,

optimization, and inequality theory. Guessab et al. [8–10] worked on the error estimations and multivariate approximation theory. Presently, this hypothesis has an amazing commitment to the expansions and enhancements of various areas of numerical and applied sciences. Many mathematicians inspected, celebrated, and played out their work on the thoughts of convexity and expanded its various forms in helpful manners utilizing productive techniques and imaginative thought. For some of the recent considerations, we refer to the references [11–14].

2. Preliminaries

In this section we recall some known concepts.

Definition 1 ([5]). *Let* $\mathsf{F} : I \to \mathbb{R}$ *be a real valued function. A function* F *is said to be convex, if*

$$\mathsf{F}(\ell\wp_1 + (1-\ell)\wp_2) \leq \ell\mathsf{F}(\wp_1) + (1-\ell)\mathsf{F}(\wp_2), \tag{1}$$

holds for all $\wp_1, \wp_2 \in I$ *and* $\ell \in [0,1]$.

The most important inequality concerning convex functions is the Hermite–Hadamard inequality given as:

Theorem 1. *If* $\mathsf{F} : [\wp_1, \wp_2] \to \mathbb{R}$ *is a convex function, then*

$$\mathsf{F}\left(\frac{\wp_1+\wp_2}{2}\right) \leq \frac{1}{\wp_2-\wp_1}\int_{\wp_1}^{\wp_2}\mathsf{F}(x)dx \leq \frac{\mathsf{F}(\wp_1)+\mathsf{F}(\wp_2)}{2}. \tag{2}$$

The double inequality (2) holds in reverse direction if F is a concave function on $[\wp_1, \wp_2]$ and the constant $\frac{1}{2}$ is sharp and cannot be replaced by a similar constant. Since then, various papers with new proof, various speculations, and advancements have been proposed in the literature. This type of inequality has remained an area of great interest due to its widespread view and applications in the field of mathematical analysis.
In 2005, Raina [15] introduced a class of functions defined formally by

$$\mathcal{F}^{\sigma}_{\rho,\lambda}(z) = \mathcal{F}^{\sigma(0),\sigma(1),\cdots}_{\rho,\lambda}(z) = \sum_{k=0}^{+\infty}\frac{\sigma(k)}{\Gamma(\rho k+\lambda)}z^k, \tag{3}$$

where $\sigma = (\sigma(0), \ldots, \sigma(k), \ldots)$ and $\rho, \lambda > 0, |z| < R$. The above class of function is the generalization of classical Mittag–Leffler function and the Kummer function.

If $\rho = 1, \lambda = 0$ and $\sigma(k) = \frac{(\alpha)_k(\beta)_k}{(\gamma)_k}$ for $k = 0, 1, 2, \ldots$, where α, β and γ are parameters which can take arbitrary real or complex values (provided that $\gamma \neq 0, -1, -2, \ldots$), and the symbol α_k denotes the quantity

$$(\alpha)_k = \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)} = \alpha(\alpha+1)\ldots(\alpha+k-1), \quad k = 0, 1, 2, \ldots,$$

and restricts its domain to $|z| \leq 1$ (with $z \in \mathbb{C}$), then we have the classical hypergeometric function, that is

$$\mathcal{F}(\alpha, \beta; \gamma; z) = \sum_{k=0}^{+\infty}\frac{(\alpha)_k(\beta)_k}{k!(\gamma)_k}z^k.$$

Moreover, if $\sigma = (1, 1, \ldots)$ with $\rho = \alpha$, $(Re(\alpha) > 0), \lambda = 1$, then

$$E_\alpha(z) = \sum_{k=0}^{+\infty}\frac{z^k}{\Gamma(1+\alpha k)}.$$

The above exact function, which intermittently appears in the investigation of fractional integrals and derivatives is called a classical Mittag–Leffler function, and was first

considered by Magnus Gustaf (Gösta) Mittag–Leffler (1846–1927) in 1903 and Anders Wiman (1865–1959) in 1905.

From that point forward, the Mittag–Leffler function has been broadened and explored in numerous elective ways and settings. Mittag–Leffler type functions with all of their applications have reached out in different investigations like those in science, physics, engineering, statistics, and mathematics.

The Mittag–Leffler function emerges normally in the arrangement of fractional order integral equations and particularly in the investigations of the fractional speculation of the kinetic equation, random walks, Lévy flights, superdiffusive transport, and in the investigations of complex frameworks. In numerous new research articles, the interest in the group of Mittag–Leffler type functions has become impressive due primarily to their potential for applications in reaction–diffusion and other applied issues and their different speculations show up in the arrangements of fractional order differential and integral equations, see the references [16–18].

Cortez established the new class of set and function involving Raina's function in [6,7], which is said to be generalized convex set and convex function.

Definition 2 ([7]). *Let $\sigma = (\sigma(0), \ldots, \sigma(k), \ldots)$ and $\rho, \lambda > 0$. A set $X \neq \emptyset$ is said to be generalized convex, if*

$$\wp_2 + \ell \, \mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1 - \wp_2) \in X, \tag{4}$$

for all $\wp_1, \wp_2 \in X$ and $\ell \in [0,1]$.

Definition 3 ([7]). *Let σ denote a bounded sequence then $\sigma = (\sigma(0), \ldots, \sigma(k), \ldots)$ and $\rho, \lambda > 0$. If $\mathsf{F} : X \to \mathbb{R}$ satisfies the following inequality*

$$\mathsf{F}\Big(\wp_2 + \ell \, \mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1 - \wp_2)\Big) \leq \ell \mathsf{F}(\wp_1) + (1-\ell)\mathsf{F}(\wp_2), \tag{5}$$

for all $\wp_1, \wp_2 \in X$, where $\wp_1 < \wp_2$ and $\ell \in [0,1]$, then F is called generalized convex function.

Remark 1. *We have $\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1 - \wp_2) = \wp_1 - \wp_2 > 0$, and so we obtain Definition 1.*

Condition 1. *Let $X \subseteq \mathbb{R}$ be an open generalized convex subset with respect to $\mathcal{F}^{\sigma}_{\rho,\lambda}(\cdot)$. For any $\wp_1, \wp_2 \in X$ and $\ell \in [0,1]$,*

$$\mathcal{F}^{\sigma}_{\rho,\lambda}\Big(\wp_2 - (\wp_2 + \ell \, \mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1 - \wp_2))\Big) = -\ell \, \mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1 - \wp_2),$$

$$\mathcal{F}^{\sigma}_{\rho,\lambda}\Big(\wp_1 - \Big(\wp_2 + \ell \, \mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1 - \wp_2)\Big)\Big) = (1-\ell) \, \mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1 - \wp_2).$$

Note that, for every $\wp_1$, $\wp_2 \in X$ and for all ℓ_1, $\ell_2 \in [0,1]$ from Condition 1, we have

$$\mathcal{F}^{\sigma}_{\rho,\lambda}\Big(\wp_2 + \ell_2 \, \mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1 - \wp_2) - (\wp_2 + \ell_1 \, \mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1 - \wp_2))\Big) = (\ell_2 - \ell_1) \, \mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1 - \wp_2). \tag{6}$$

Definition 4 ([19]). *A nonnegative function $\mathsf{F} : \mathbb{X} \to \mathbb{R}$ is called s-type convex function if for every $\wp_1, \wp_2 \in \mathbb{X}$, $s \in [0,1]$ and $\ell \in [0,1]$, if*

$$\mathsf{F}(\ell\wp_1 + (1-\ell)\wp_2) \leq [1 - (s(1-\ell))]\mathsf{F}(\wp_1) + [1 - s\ell]\mathsf{F}(\wp_2). \tag{7}$$

Definition 5 ([20]). *Two functions F and G are said to be similarly ordered, if*

$$(\mathsf{F}(\wp_1) - \mathsf{F}(\wp_2))(\mathsf{G}(\wp_1) - \mathsf{G}(\wp_2)) \geq 0, \quad \forall \wp_1, \wp_2 \in \mathbb{R}.$$

Theorem 2 (Hölder-İşcan inequality [21]). *Let $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. If F and G are real functions defined on interval $[\wp_1, \wp_2]$ and if $|\mathsf{F}|^p, |\mathsf{G}|^q$ are integrable functions on $[\wp_1, \wp_2]$, then*

$$\int_{\wp_1}^{\wp_2} |\mathsf{F}(x)\mathsf{G}(x)|dx \leq \frac{1}{\wp_2 - \wp_1}\left\{ \left(\int_{\wp_1}^{\wp_2} (\wp_2 - x)|\mathsf{F}(x)|^p dx \right)^{\frac{1}{p}} \left(\int_{\wp_1}^{\wp_2} (\wp_2 - x)|\mathsf{G}(x)|^q dx \right)^{\frac{1}{q}} + \left(\int_{\wp_1}^{\wp_2} (x - \wp_1)|\mathsf{F}(x)|^p dx \right)^{\frac{1}{p}} \left(\int_{\wp_1}^{\wp_2} (x - \wp_1)|\mathsf{G}(x)|^q dx \right)^{\frac{1}{q}} \right\}.$$

Theorem 3 (Improved power-mean integral inequality [22]). *Let $q \geq 1$. If F and G are real functions defined on interval $[\wp_1, \wp_2]$ and if $|\mathsf{F}|, |\mathsf{G}|^q$ are integrable functions on $[\wp_1, \wp_2]$, then*

$$\begin{aligned} &\int_{\wp_1}^{b} |\mathsf{F}(x)\mathsf{G}(x)|dx \\ &\leq \frac{1}{\wp_2 - \wp_1}\left\{ \left(\int_{\wp_1}^{\wp_2} (\wp_2 - x)|\mathsf{F}(x)|dx \right)^{1-\frac{1}{q}} \left(\int_{\wp_1}^{\wp_2} (\wp_2 - x)|\mathsf{F}(x)||\mathsf{G}(x)|^q dx \right)^{\frac{1}{q}} \right. \\ &\left. + \left(\int_{\wp_1}^{\wp_2} (x - \wp_1)|\mathsf{F}(x)|dx \right)^{1-\frac{1}{q}} \left(\int_{\wp_1}^{\wp_2} (x - \wp_1)|\mathsf{F}(x)||\mathsf{G}(x)|^q dx \right)^{\frac{1}{q}} \right\} \end{aligned}$$

Owing to the aforementioned trend and inspired by the ongoing activities, the rest of this paper is organized as follows. First of all, in Section 3, we define and explore the newly introduced idea about generalized convex functions and their algebraic properties. In Section 4, we present the novel version of Hermite–Hadamard type inequality. In Section 5, we establish two new equalities and employing these new equalities and with the help of newly introduced definition, we present some refinements of Hermite–Hadamard type inequality. Finally, we give some applications in support of the newly introduced idea and a brief conclusion.

3. Generalized s–Type Convex Function of Raina Type and Its Properties

In this section, we are to add and introduce a new notion for a new family of convex functions namely generalized s-type convex function of Raina type.

Definition 6. *Let $\mathbb{X}$ be a nonempty generalized convex set with respect to $\mathcal{F}^{\sigma}_{\rho,\lambda} : \mathbb{X} \times \mathbb{X} \to \mathbb{R}$. Then the nonnegative function $\mathsf{F} : \mathbb{X} \to \mathbb{R}$ is said to be generalized s-type convex function of Raina type, if*

$$\mathsf{F}(\wp_2 + \ell\, \mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1 - \wp_2)) \leq [1 - (s(1-\ell))]\mathsf{F}(\wp_1) + [1 - s\ell]\mathsf{F}(\wp_2), \tag{8}$$

holds for every $\wp_1, \wp_2 \in \mathbb{X}$, $\sigma = (\sigma(0), \ldots, \sigma(k), \ldots)$, $\rho, \lambda > 0$, $s \in [0,1]$, and $\ell \in [0,1]$.

Remark 2. *(i) Taking $s = 1$ in Definition 6, then we attain a definition which is called generalized convex function which is first time explored by Cortez [6,7].*
(ii) Taking $\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1 - \wp_2) = \wp_1 - \wp_2$ in Definition 6, then we attain s-type convex function which is explored by İşcan et al. [19].
(iii) Taking $s = 1$ and $\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1 - \wp_2) = \wp_1 - \wp_2$ in Definition 6, then we obtain the convex function which is investigated by Niculescu et al. [5].

Lemma 1. *The following inequalities*

$$\ell \leq [1 - (s(1-\ell))] \quad \text{and} \quad 1 - \ell \leq [1 - s\ell]$$

are holds, if for all $\ell \in [0,1]$ and $s \in [0,1]$.

Proof. The rest of the proof is clearly seen. □

Proposition 1. *Every nonnegative generalized convex function of Raina type is s-type preinvex function for* $s \in [0,1]$.

Proof. By using Lemma 1 and definition of generalized convex function of Raina type for $s \in [0,1]$, we have

$$\mathsf{F}(\wp_2 + \ell\, \mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1 - \wp_2)) \leq \ell \mathsf{F}(\wp_1) + (1-\ell)\mathsf{F}(\wp_2)$$

$$\leq [1-(s(1-\ell))]\mathsf{F}(\wp_1) + [1-s\ell]\mathsf{F}(\wp_2).$$

□

Proposition 2. *Every non-negative generalized s-type convex function of Raina type for* $s \in [0,1]$ *is an generalized h–convex function of Raina type with* $h(\ell) = [1-(s(1-\ell))]$.

Proof. Using the definition of generalized s-type convex function of Raina type for $s \in [0,1]$ and mention condition $h(\ell) = [1-(s(1-\ell))]$, we have

$$\mathsf{F}(\wp_2 + \ell\, \mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1 - \wp_2)) \leq [1-(s(1-\ell))]\mathsf{F}(\wp_1) + [1-s\ell]\mathsf{F}(\wp_2),$$

$$\leq h(\ell)\mathsf{F}(\wp_1) + h(1-\ell)\mathsf{F}(\wp_2).$$

□

This means that, the new class of generalized s-type convex function of Raina type is very larger with respect to the known class of functions, like generalized convex functions and convex functions. This is the beauty of the proposed new Definition 6.

Now, we will discuss and explore the some properties in the support of the newly introduced idea.

Theorem 4. *Let* $\mathsf{F}, \mathsf{G} : \mathbb{X} = [\wp_1, \wp_2] \to \mathbb{R}$. *If* F, G *be two generalized s-type convex function of Raina type with respect to same* $\mathcal{F}^{\sigma}_{\rho,\lambda}$, *then*
(i) $\mathsf{F}+\mathsf{G}$ *is a generalized s-type convex function of Raina type with respect to* $\mathcal{F}^{\sigma}_{\rho,\lambda}$.
(ii) *For* $c \in \mathbb{R}(c \geq 0)$, *then* $c\mathsf{F}$ *is a generalized s-type convex function of Raina type with respect to* $\mathcal{F}^{\sigma}_{\rho,\lambda}$.

Proof. (i) Let F, G be generalized s-type convex function of Raina type with respect to same $\mathcal{F}^{\sigma}_{\rho,\lambda}$, then for all $\wp_1, \wp_2 \in \mathbb{X}$, $s \in [0,1]$ and $\ell \in [0,1]$, we have

$$\begin{aligned}
&(\mathsf{F}+\mathsf{G})(\wp_2 + \ell\, \mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1 - \wp_2)) \\
&= \mathsf{F}(\wp_2 + \ell\, \mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1 - \wp_2)) + \mathsf{G}(\wp_2 + \ell\, \mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1 - \wp_2)) \\
&\leq [1-(s(1-\ell))]\mathsf{F}(\wp_1) + [1-s\ell]\mathsf{F}(\wp_2) \\
&+ [1-(s(1-\ell))]\mathsf{G}(\wp_1) + [1-s\ell]\mathsf{G}(\wp_2) \\
&= [1-(s(1-\ell))][\mathsf{F}(\wp_1) + \mathsf{G}(\wp_1)] + [1-s\ell][\mathsf{F}(\wp_2) + \mathsf{G}(\wp_2)] \\
&= [1-(s(1-\ell))](\mathsf{F}+\mathsf{G})(\wp_1) + [1-s\ell](\mathsf{F}+\mathsf{G})(\wp_2).
\end{aligned}$$

(ii) Let F be a generalized s-type convex function of Raina type with respect to $\mathcal{F}^{\sigma}_{\rho,\lambda}$, then for all $\wp_1, \wp_2 \in \mathbb{X}$, $s \in [0,1]$, $c \in \mathbb{R}(c \geq 0)$ and $\ell \in [0,1]$, we have

$$\begin{aligned}
&(c\mathsf{F})(\wp_2 + \ell\, \mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1 - \wp_2)) \\
&\leq c\left[[1-(s(1-\ell))]\mathsf{F}(\wp_1) + [1-s\ell]\mathsf{F}(\wp_2)\right] \\
&= [1-(s(1-\ell))]c\mathsf{F}(\wp_1) + [1-s\ell]c\mathsf{F}(\wp_2) \\
&= [1-(s(1-\ell))](c\mathsf{F})(\wp_1) + [1-s\ell](c\mathsf{F})(\wp_2).
\end{aligned}$$

This is the required proof. □

Remark 3. (i) *Choosing $s = 1$ in Theorem 4, then we get the $\mathsf{F} + \mathsf{G}$ and $c\mathsf{F}$ are generalized convex functions of Raina type.*
(ii) *Choosing $\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1 - \wp_2) = \wp_1 - \wp_2$ in Theorem 4, then we get the $\mathsf{F} + \mathsf{G}$ and $c\mathsf{F}$ are s-type convex functions.*
(iii) *Choosing $\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1 - \wp_2) = \wp_1 - \wp_2$ and $s = 1$ in Theorem 4, then we get the $\mathsf{F} + \mathsf{G}$ and $c\mathsf{F}$ are convex functions.*

Theorem 5. *Let $\mathsf{F} : \mathbb{X} \to \mathbb{J}$ be a generalized s-type convex function of Raina type with respect to $\mathcal{F}^{\sigma}_{\rho,\lambda}$ and $\mathsf{G} : \mathbb{J} \to \mathbb{R}$ is non-decreasing function. Then the function $\mathsf{G} \circ \mathsf{F}$ is a generalized s-type convex function of Raina type with respect to same $\mathcal{F}^{\sigma}_{\rho,\lambda}$.*

Proof. For all $\wp_1, \wp_2 \in \mathbb{X}$, $s \in [0,1]$ and $\ell \in [0,1]$, we have

$$\begin{aligned}
&(\mathsf{G} \circ \mathsf{F})(\wp_2 + \ell\, \mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1 - \wp_2)) \\
&= \mathsf{G}(\mathsf{F}(\wp_2 + \ell\, \mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1 - \wp_2))) \\
&\leq \mathsf{G}\left[[1-(s(1-\ell))]\mathsf{F}(\wp_1) + [1-s\ell]\mathsf{F}(\wp_2)\right] \\
&\leq [1-(s(1-\ell))]\mathsf{G}(\mathsf{F}(\wp_1)) + [1-s\ell]\mathsf{G}(\mathsf{F}(\wp_2)) \\
&= [1-(s(1-\ell))](\mathsf{G} \circ \mathsf{F})(\wp_1) + [1-s\ell](\mathsf{G} \circ \mathsf{F})(\wp_2).
\end{aligned}$$

This is the required proof. □

Remark 4. (i) *If $s = 1$ in Theorem 5, then*

$$(\mathsf{G} \circ \mathsf{F})(\wp_2 + \ell\, \mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1 - \wp_2)) \leq \ell(\mathsf{G} \circ \mathsf{F})(\wp_1) + (1-\ell)(\mathsf{G} \circ \mathsf{F})(\wp_2).$$

(ii) *If we put $\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1 - \wp_2) = \wp_1 - \wp_2$ in Theorem 5, then*

$$(\mathsf{G} \circ \mathsf{F})(\ell\wp_1 + (1-\ell)\wp_2) \leq [1-(s(1-\ell))](\mathsf{G} \circ \mathsf{F})(\wp_1) + [1-s\ell](\mathsf{G} \circ \mathsf{F})(\wp_2).$$

Theorem 6. *Let $0 < \wp_1 < \wp_2$, $\mathsf{F}_j : \mathbb{X} = [\wp_1, \wp_2] \to [0, +\infty)$ be a class of generalized s-type convex function of Raina type with respect to same $\mathcal{F}^{\sigma}_{\rho,\lambda}$ and $\mathsf{F}(u) = \sup_j \mathsf{F}_j(u)$. Then F is a generalized s-type convex function of Raina type with respect to $\mathcal{F}^{\sigma}_{\rho,\lambda}$ and $\mathcal{U} = \{\nu \in [\wp_1, \wp_2] : \mathsf{F}(\nu_i) < \infty\}$ is an interval.*

Proof. Let $\wp_1, \wp_2 \in U$, $s \in [0,1]$ and $\ell \in [0,1]$, then

$$\begin{aligned}
&\mathsf{F}(\wp_2 + \ell\, \mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1 - \wp_2)) \\
&= \sup_j \mathsf{F}_j(\wp_2 + \ell\, \mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1 - \wp_2)) \\
&\leq [1-(s(1-\ell))] \sup_j \mathsf{F}_j(\wp_1) + [1-s\ell] \sup_j \mathsf{F}_j(\wp_2) \\
&= [1-(s(1-\ell))]\mathsf{F}(\wp_1) + [1-s\ell]\mathsf{F}(\wp_2) < \infty.
\end{aligned}$$

This is the required proof. □

Theorem 7. *Let* $\mathsf{F}, \mathsf{G} : \mathbb{X} = [\wp_1, \wp_2] \to \mathbb{R}$. *If* F, G *be two generalized s-type convex function of Raina type with respect to same* $\mathcal{F}^{\sigma}_{\rho,\lambda}$ *and* F, G *are similarly ordered functions and* $[1-(s(1-\ell))] + [1-s\ell] \leq 1$, *then the product* FG *is a generalized s-type convex function of Raina type with respect to* $\mathcal{F}^{\sigma}_{\rho,\lambda}$.

Proof. Let F, G be a generalized s-type convex function of Raina type with respect to same $\mathcal{F}^{\sigma}_{\rho,\lambda}$, $s \in [0,1]$ and $\ell \in [0,1]$, then

$$\begin{aligned}
&\mathsf{F}(\wp_2 + \ell\, \mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1 - \wp_2))\mathsf{G}(\wp_2 + \ell\, \mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1 - \wp_2)) \\
&\leq \Big[[1-(s(1-\ell))]\mathsf{F}(\wp_1) + [1-s\ell]\mathsf{F}(\wp_2)\Big] \\
&\times \Big[[1-(s(1-\ell))]\mathsf{G}(\wp_1) + [1-s\ell]\mathsf{G}(\wp_2).\Big] \\
&\leq [1-(s(1-\ell))]^2\mathsf{F}(\wp_1)\mathsf{G}(\wp_1) + [1-s\ell]^2\mathsf{F}(\wp_2)\mathsf{G}(\wp_2) \\
&+ [1-(s(1-\ell))][1-s\ell][\mathsf{F}(\wp_1)\mathsf{G}(\wp_2) + \mathsf{F}(\wp_2)\mathsf{G}(\wp_1)] \\
&\leq [1-(s(1-\ell))]^2\mathsf{F}(\wp_1)\mathsf{G}(\wp_1) + [1-s\ell]^2\mathsf{F}(\wp_2)\mathsf{G}(\wp_2) \\
&+ [1-(s(1-\ell))][1-(s\ell)][\mathsf{F}(\wp_1)\mathsf{G}(\wp_1) + \mathsf{F}(\wp_2)\mathsf{G}(\wp_2)] \\
&= \Big[[1-(s(1-\ell))]\mathsf{F}(\wp_1)\mathsf{G}(\wp_1) + [1-s\ell]\mathsf{F}(\wp_2)\mathsf{G}(\wp_2)\Big] \\
&\times \Big([1-(s(1-\ell))] + [1-s\ell]\Big) \\
&\leq [1-(s(1-\ell))]\mathsf{F}(\wp_1)\mathsf{G}(\wp_1) + [1-s\ell]\mathsf{F}(\wp_2)\mathsf{G}(\wp_2).
\end{aligned}$$

This shows that the product of two generalized s-type convex function of Raina type with respect to same $\mathcal{F}^{\sigma}_{\rho,\lambda}$ is again a generalized s-type convex function of Raina type with respect to $\mathcal{F}^{\sigma}_{\rho,\lambda}$. □

Remark 5. *Taking* $\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1 - \wp_2) = \wp_1 - \wp_2$ *in Theorem 7, then we attain the new inequality namely the product of s-type convex functions*

$$\mathsf{F}(\ell\wp_1 + (1-\ell)\wp_2)\mathsf{G}(\ell\wp_1 + (1-\ell)\wp_2) \leq [1-(s(1-\ell))]\mathsf{F}(\wp_1)\mathsf{G}(\wp_1) + [1-s\ell]\mathsf{F}(\wp_2)\mathsf{G}(\wp_2).$$

4. Hermite–Hadamard Type Inequality via Generalized *s*–Type Convex Function of Raina Type

The principal intention and main aim of this section is to establish novel version of Hermite–Hadamard type inequality in the mode of newly discussed concept namely generalized s-type convex function of Raina type.

Theorem 8. *Let* $\mathsf{F} : [\wp_1, \wp_2] \in \mathbb{R}$ *be a generalized s-type convex function of Raina type, if* $\wp_1 < \wp_2$ *and* $\mathsf{F} \in L[\wp_1, \wp_2]$ *and satisfies Condition 1 then the following Hermite–Hadamard type inequalities hold*

$$\frac{1}{2-s}\mathsf{F}(\wp_2 + \frac{1}{2}\,\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1 - \wp_2)) \leq \frac{1}{\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1 - \wp_2)} \int_{\wp_2}^{\wp_2 + \mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1 - \wp_2)} \mathsf{F}(x)dx \leq \frac{\mathsf{F}(\wp_1) + \mathsf{F}(\wp_2)}{2}(2-s).$$

Proof. Since $\wp_1, \wp_2 \in \mathbb{X}^{\circ}$ and $\mathbb{X}^{\circ}$ is a generalized convex set with respect to $\mathcal{F}^{\sigma}_{\rho,\lambda}$, for every and $\ell \in [0,1]$, we have $\wp_2 + \ell\,\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1 - \wp_2) \in \mathbb{X}^{\circ}$. From the definition of s-type preinvex function of F, we have

$$\mathsf{F}(\wp_2 + \ell\,\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1 - \wp_2)) \leq [1 - (s(1-\ell))]\mathsf{F}(\wp_1) + [1 - s\ell]\mathsf{F}(\wp_2)$$

$$\int_0^1 \mathsf{F}(\wp_2 + \ell\,\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1 - \wp_2))d\ell \leq \mathsf{F}(\wp_1)\int_0^1 [1 - (s(1-\ell))]d\ell + \mathsf{F}(\wp_2)\int_0^1 [1 - s\ell]d\ell$$

but,

$$\int_0^1 \mathsf{F}(\wp_2 + \ell\,\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1 - \wp_2))d\ell = \frac{1}{\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1 - \wp_2)} \int_{\wp_2}^{\wp_2 + \mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1 - \wp_2)} \mathsf{F}(x)dx$$

so,

$$\frac{1}{\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1 - \wp_2)} \int_{\wp_2}^{\wp_2 + \mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1 - \wp_2)} \mathsf{F}(x)dx \leq \frac{\mathsf{F}(\wp_1) + \mathsf{F}(\wp_2)}{2}(2-s).$$

This completes the right hand side of above inequality. For the left hand side we use the Definition 6, put $\ell = \frac{1}{2}$ and condition C for $\mathcal{F}^{\sigma}_{\rho,\lambda}$ and integrating over $[0,1]$,

$$\mathsf{F}(y + \ell\,\mathcal{F}^{\sigma}_{\rho,\lambda}(x-y)) \leq [1 - (s(1-\ell))]\mathsf{F}(x) + [1 - (s\ell)]\mathsf{F}(y)$$

$$\mathsf{F}(y + \frac{1}{2}\,\mathcal{F}^{\sigma}_{\rho,\lambda}(x-y)) \leq \left[1 - (\frac{s}{2})\right]\left[\mathsf{F}(x) + \mathsf{F}(y)\right].$$

Putting $x = \wp_2 + (1-\ell)\ \mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1 - \wp_2)$ and $y = \wp_2 + \ell\ \mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1 - \wp_2)$ in above inequality, we prove the L.H.S of above inequality

$$\mathsf{F}(y + \frac{1}{2}\,\mathcal{F}^{\sigma}_{\rho,\lambda}(x-y)) = \mathsf{F}(\wp_2 + \frac{1}{2}\,\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1 - \wp_2)).$$

So after putting the value of x and y, we get

$$\mathsf{F}(y + \frac{1}{2}\,\mathcal{F}^{\sigma}_{\rho,\lambda}(x-y))$$
$$= \mathsf{F}(\wp_2 + \ell\,\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1 - \wp_2) + \frac{1}{2}\,\mathcal{F}^{\sigma}_{\rho,\lambda}((\wp_2 + (1-\ell)\,\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1 - \wp_2)) - (\wp_2 + \ell\,\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1 - \wp_2))). \quad (9)$$

Now by using Condition 1, we have

$$\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2 + (1-\ell)\,\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1 - \wp_2) - \wp_2 + \ell\,\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1 - \wp_2)) = (1 - \ell - \ell)\,\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1 - \wp_2).$$

$$\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2 + (1-\ell)\,\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1 - \wp_2) - \wp_2 + \ell\,\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1 - \wp_2)) = (1 - 2\ell)\,\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1 - \wp_2).$$

Now we put the value of $\mathcal{F}^{\sigma}_{\rho,\lambda}$ in (9), then as a result, we get

$$\mathsf{F}(y + \frac{1}{2}\,\mathcal{F}^{\sigma}_{\rho,\lambda}(x-y)) = \mathsf{F}(\wp_2 + \ell\,\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1 - \wp_2) + \frac{1}{2}(1 - 2\ell)\,\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1 - \wp_2)).$$

$$\mathsf{F}(y+\frac{1}{2}\,\mathcal{F}^{\sigma}_{\rho,\lambda}(x-y)) = \mathsf{F}(\wp_2+(\ell+\frac{1}{2}-\ell)\,\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1-\wp_2)).$$

$$\mathsf{F}(y+\frac{1}{2}\,\mathcal{F}^{\sigma}_{\rho,\lambda}(x-y)) = \mathsf{F}(\wp_2+\frac{1}{2}\,\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1-\wp_2)).$$

Now

$$\begin{aligned}
&\mathsf{F}(\wp_2+\frac{1}{2}\,\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1-\wp_2))\\
&\leq \left[1-(\frac{s}{2})\right]\left[\int_0^1 \mathsf{F}(\wp_2+(1-\ell)\,\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1-\wp_2))d\ell+\int_0^1 \mathsf{F}(\wp_2+\ell\,\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1-\wp_2))d\ell\right]\\
&\leq \left[1-(\frac{s}{2})\right]\frac{2}{\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1-\wp_2)}\int_{\wp_2}^{\wp_2+\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1-\wp_2)}\mathsf{F}(x)dx\\
&\leq 2\left[1-(\frac{s}{2})\right]\frac{1}{\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1-\wp_2)}\int_{\wp_2}^{\wp_2+\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1-\wp_2)}\mathsf{F}(x)dx.
\end{aligned}$$

This is the required proof. □

Corollary 1. *If we put* $s=1$ *and* $\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1-\wp_2)=\wp_1-\wp_2$ *in Theorem 8, then we get Hermite–Hadamard inequality in [23].*

Remark 6. *Under the assumption of Theorem 8, if we take* $\sigma=(1,1,\ldots)$ *with* $\rho=\alpha,\ \lambda=1$, *we get the following inequality involving classical Mittag–Leffler function*

$$\begin{aligned}
\frac{1}{2-s}\mathsf{F}(\wp_2+\frac{1}{2}E_{\alpha}(\wp_1-\wp_2)) &\leq \frac{1}{E_{\alpha}(\wp_1-\wp_2)}\int_{\wp_2}^{\wp_2+E_{\alpha}(\wp_1-\wp_2)}\mathsf{F}(x)dx\\
&\leq \frac{\mathsf{F}(\wp_1)+\mathsf{F}(\wp_2)}{2}(2-s).
\end{aligned}$$

5. Refinements of Hermite–Hadamard Type Inequality

The aim of this section is to investigate the refinements of Hermite–Hadamard type inequality by using the newly introduced definition. In order to attain the refinements of Hermite–Hadamard inequality, we need the following lemmas.

Lemma 2. *Let* $X\subseteq\mathbb{R}$ *be a generalized convex subset with respect to* $\mathcal{F}^{\sigma}_{\rho,\lambda}: X\times X\to\mathbb{R}$ *and* $\wp_1,\wp_2\in X$ *with* $\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1-\wp_2)\neq 0$. *Suppose that* $\mathsf{F}: X\to\mathbb{R}$ *is a differentiable function. If* F *is integrable on the* $\mathcal{F}^{\sigma}_{\rho,\lambda}$, *then the following equality holds:*

$$\begin{aligned}
&\frac{\mathsf{F}(\wp_1)+\mathsf{F}(\wp_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\frac{\wp_2}{c}-\wp_1))}{2}-\frac{c}{\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-c\wp_1)}\int_{\wp_1}^{\wp_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\frac{\wp_2}{c}-\wp_1)}\mathsf{F}(x)dx\\
&=\frac{\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-c\wp_1)}{2c}\int_0^1(1-2\ell)\mathsf{F}'\Big(\frac{\wp_2}{c}+\ell\,\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1-\frac{\wp_2}{c})\Big)d\ell.
\end{aligned}$$

Proof. Suppose that $\wp_1,\wp_2\in X$. Since X is generalized convex set with respect to $\mathcal{F}^{\sigma}_{\rho,\lambda}$, for every $\ell\in[0,1]$, we have $\wp_2+\ell\,\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1-\wp_2)\in X$. Integrating by parts implies that

$$\begin{aligned}
&\frac{\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-c\wp_1)}{2c}\int_0^1(1-2\ell)\mathsf{F}'\Big(\frac{\wp_2}{c}+\ell\,\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1-\frac{\wp_2}{c})\Big)d\ell\\
&=\frac{\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-c\wp_1)}{2c}\left[\frac{(1-2\ell)\mathsf{F}(\frac{\wp_2}{c}+\ell\,\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1-\frac{\wp_2}{c}))}{\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1-\frac{\wp_2}{c})}\Big|_0^1+2\int_0^1\frac{\mathsf{F}(\frac{\wp_2}{c}+\ell\,\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1-\frac{\wp_2}{c}))}{\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1-\frac{\wp_2}{c})}d\ell\right]\\
&=\frac{\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-c\wp_1)}{2c}\left[\frac{c(\mathsf{F}(\wp_1)+\mathsf{F}(\frac{\wp_2}{c}))}{\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-c\wp_1)}-\frac{2c}{\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-c\wp_1)}\int_0^1\mathsf{F}(\frac{\wp_2}{c}+\ell\,\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1-\frac{\wp_2}{c}))d\ell\right]\\
&=\frac{\mathsf{F}(\wp_1)+\mathsf{F}(\wp_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\frac{\wp_2}{c}-\wp_1))}{2}-\frac{c}{\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-c\wp_1)}\int_{\wp_1}^{\wp_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\frac{\wp_2}{c}-\wp_1)}\mathsf{F}(x)dx.
\end{aligned}$$

Which completes the proof. □

Lemma 3. *Let $X\subseteq\mathbb{R}$ be a generalized convex subset with respect to $\mathcal{F}^{\sigma}_{\rho,\lambda}:X\times X\to\mathbb{R}$ and $\wp_1,\wp_2\in X$ with $\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1-\wp_2)\neq 0$. Suppose that $\mathsf{F}:X\to\mathbb{R}$ is a differentiable function. If F is integrable on the $\mathcal{F}^{\sigma}_{\rho,\lambda}$, then the following equality holds:*

$$\begin{aligned}
&\frac{c}{\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-c\wp_1)}\int_{\wp_1}^{\wp_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\frac{\wp_2}{c}-\wp_1)}\mathsf{F}(x)dx-\mathsf{F}\left(\frac{2\wp_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-\wp_1)}{2c}\right)\\
&=\frac{\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-c\wp_1)}{c}\left\{\int_0^1\ell\mathsf{F}'\Big(\frac{\wp_2}{c}+\ell\,\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1-\frac{\wp_2}{c})\Big)d\ell-\int_{1/2}^1\mathsf{F}'\Big(\frac{\wp_2}{c}+\ell\,\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1-\frac{\wp_2}{c})\Big)d\ell\right\}.
\end{aligned}$$

Proof. Suppose that $\wp_1,\wp_2\in X$. Since X is generalized convex set with respect to $\mathcal{F}^{\sigma}_{\rho,\lambda}$, for every $\ell\in[0,1]$, we have $\wp_2+\ell\,\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1-\wp_2)\in X$. Integrating by parts implies that

$$\begin{aligned}
&\frac{\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-c\wp_1)}{c}\left\{\int_0^1\ell\mathsf{F}'\Big(\frac{\wp_2}{c}+\ell\,\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1-\frac{\wp_2}{c})\Big)d\ell-\int_{1/2}^1\mathsf{F}'\Big(\frac{\wp_2}{c}+\ell\,\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1-\frac{\wp_2}{c})\Big)d\ell\right\}\\
&=\frac{\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-c\wp_1)}{c}\times\left[\frac{\ell\mathsf{F}(\frac{\wp_2}{c}+\ell\,\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1-\frac{\wp_2}{c}))}{\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1-\frac{\wp_2}{c})}\Big|_0^1-\int_0^1\frac{\mathsf{F}(\frac{\wp_2}{c}+\ell\,\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1-\frac{\wp_2}{c}))}{\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1-\frac{\wp_2}{c})}d\ell\right.\\
&\qquad\left.-\frac{\mathsf{F}(\frac{\wp_2}{c}+\ell\,\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1-\frac{\wp_2}{c}))}{\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1-\frac{\wp_2}{c})}d\ell\Big|_{\frac{1}{2}}^1\right]\\
&=\frac{\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-c\wp_1)}{c}\left[\frac{c\mathsf{F}(\wp_1)}{\mathcal{F}^{\sigma}_{\rho,\lambda}(c\wp_1-\wp_2)}-\frac{c}{\mathcal{F}^{\sigma}_{\rho,\lambda}(c\wp_1-\wp_2)}\int_0^1\mathsf{F}(\frac{\wp_2}{c}+\ell\,\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1-\frac{\wp_2}{c}))d\ell\right.\\
&\qquad\left.-\frac{c}{\mathcal{F}^{\sigma}_{\rho,\lambda}(c\wp_1-\wp_2)}\left(\mathsf{F}(\wp_1)-\mathsf{F}\left(\frac{(2\wp_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-\wp_1))}{2c}\right)\right)\right]\\
&=\frac{c}{\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-c\wp_1)}\int_{\wp_1}^{\wp_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\frac{\wp_2}{c}-\wp_1)}\mathsf{F}(x)dx-\mathsf{F}\left(\frac{2\wp_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-\wp_1)}{2c}\right).
\end{aligned}$$

In this way the proof is completed. □

Theorem 9. *Suppose I° is a generalized convex set with respect to $\mathcal{F}^\sigma_{\rho,\lambda}$ and $\mathsf{F} : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $\wp_1, \wp_2 \in I^\circ$ with $\wp_1 < \wp_2$ and suppose that $\mathsf{F}' \in L[\wp_1, \wp_2]$. If $|\mathsf{F}'|$ is a generalized s-type convex function of Raina type on $L[\wp_1, \wp_2]$, then*

$$\left|\frac{\mathsf{F}(\wp_1)+\mathsf{F}(\wp_1+\ \mathcal{F}^\sigma_{\rho,\lambda}(\frac{\wp_2}{c}-\wp_1))}{2}-\frac{c}{\mathcal{F}^\sigma_{\rho,\lambda}(\wp_2-c\wp_1)}\int_{\wp_1}^{\wp_1+\mathcal{F}^\sigma_{\rho,\lambda}(\frac{\wp_2}{c}-\wp_1)}\mathsf{F}(x)dx\right| \leq \frac{\mathcal{F}^\sigma_{\rho,\lambda}(\wp_2-c\wp_1)}{2c}\left\{\frac{2-s}{4}\left[|\mathsf{F}'(\wp_1)|+|\mathsf{F}'\Big(\frac{\wp_2}{c}\Big)|\right]\right\}.$$

holds.

Proof. Suppose that $\wp_1, \wp_2 \in I^\circ$. Since I° is a generalized convex set with respect to $\mathcal{F}^\sigma_{\rho,\lambda}$, for any $\ell \in [0,1]$, we have $\wp_2 + \ell\ \mathcal{F}^\sigma_{\rho,\lambda}(\wp_1 - \wp_2) \in I^\circ$.

Using Lemma 2, one has

$$\left|\frac{\mathsf{F}(\wp_1)+\mathsf{F}(\wp_1+\ \mathcal{F}^\sigma_{\rho,\lambda}(\frac{\wp_2}{c}-\wp_1))}{2}-\frac{c}{\mathcal{F}^\sigma_{\rho,\lambda}(\wp_2-c\wp_1)}\int_{\wp_1}^{\wp_1+\mathcal{F}^\sigma_{\rho,\lambda}(\frac{\wp_2}{c}-\wp_1)}\mathsf{F}(x)dx\right| = \frac{\mathcal{F}^\sigma_{\rho,\lambda}(\wp_2-c\wp_1)}{2c}\int_0^1 |1-2\ell||\mathsf{F}'\Big(\frac{\wp_2}{c}+\ell\ \mathcal{F}^\sigma_{\rho,\lambda}(\wp_1-\frac{\wp_2}{c})\Big)|d\ell$$

Since, $|\mathsf{F}'|$ is generalized s-type convex function of Raina type on $(\wp_1, \wp_1 + \mathcal{F}^\sigma_{\rho,\lambda}(\wp_2 - \wp_1))$, we have

$$\begin{aligned}
&\left|\frac{\mathsf{F}(\wp_1)+\mathsf{F}(\wp_1+\ \mathcal{F}^\sigma_{\rho,\lambda}(\frac{\wp_2}{c}-\wp_1))}{2}-\frac{c}{\mathcal{F}^\sigma_{\rho,\lambda}(\wp_2-c\wp_1)}\int_{\wp_1}^{\wp_1+\mathcal{F}^\sigma_{\rho,\lambda}(\frac{\wp_2}{c}-\wp_1)}\mathsf{F}(x)dx\right| \\
&\leq \frac{\mathcal{F}^\sigma_{\rho,\lambda}(\wp_2-c\wp_1)}{2c}\int_0^1 |1-2\ell|\Big[(1-s(1-\ell))\mathsf{F}'(\wp_1)|+(1-s\ell)|\mathsf{F}'\Big(\frac{\wp_2}{c}\Big)|\Big]d\ell \\
&\leq \frac{\mathcal{F}^\sigma_{\rho,\lambda}(\wp_2-c\wp_1)}{2c}\left\{|\mathsf{F}'(\wp_1)|\int_0^1|1-2\ell|1-s(1-\ell)d\ell+|\mathsf{F}'\Big(\frac{\wp_2}{c}\Big)|\int_0^1|1-2\ell|(1-s\ell)d\ell\right\}. \qquad (10)
\end{aligned}$$

Since,

$$\int_0^1(1-s(1-\ell))|1-2\ell|d\ell=\int_0^1(1-s\ell)|1-2\ell|d\ell=-\frac{s-2}{4}$$

The proof of the theorem is completed by using the above computations in (10). □

Corollary 2. *If we choose $s = 1$, then we attain the following inequality*

$$\left|\frac{\mathsf{F}(\wp_1)+\mathsf{F}(\wp_1+\ \mathcal{F}^\sigma_{\rho,\lambda}(\frac{\wp_2}{c}-\wp_1))}{2}-\frac{c}{\mathcal{F}^\sigma_{\rho,\lambda}(\wp_2-c\wp_1)}\int_{\wp_1}^{\wp_1+\mathcal{F}^\sigma_{\rho,\lambda}(\frac{\wp_2}{c}-\wp_1)}\mathsf{F}(x)dx\right| \leq \frac{\mathcal{F}^\sigma_{\rho,\lambda}(\wp_2-c\wp_1)}{8c}\left\{\left[|\mathsf{F}'(\wp_1)|+4|\mathsf{F}'\Big(\frac{\wp_2}{c}\Big)|\right]\right\}.$$

Corollary 3. *If we choose $\mathcal{F}^\sigma_{\rho,\lambda}(\wp_1 - \wp_2) = \wp_1 - \wp_2$, then we attain the following inequality*

$$\left|\frac{\mathsf{F}(\wp_1)+\mathsf{F}(\frac{\wp_2}{c})}{2}-\frac{c}{(\wp_2-c\wp_1)}\int_{\wp_1}^{\frac{\wp_2}{c}}\mathsf{F}(x)dx\right| \leq \frac{(\wp_2-c\wp_1)}{2c}\left\{\frac{2-s}{4}\left[|\mathsf{F}'(\wp_1)|+|\mathsf{F}'\Big(\frac{\wp_2}{c}\Big)|\right]\right\}.$$

Corollary 4. *If we choose* $s = 1$ *and* $\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1 - \wp_2) = \wp_1 - \wp_2$, *then we attain the following inequality*

$$\left|\frac{\mathsf{F}(\wp_1)+\mathsf{F}(\frac{\wp_2}{c})}{2} - \frac{c}{(\wp_2 - c\wp_1)}\int_{\wp_1}^{\frac{\wp_2}{c}} \mathsf{F}(x)dx\right| \leq \frac{(\wp_2 - c\wp_1)}{8c}\left\{\left[|\mathsf{F}'(\wp_1)| + 4|\mathsf{F}'\left(\frac{\wp_2}{c}\right)|\right]\right\}.$$

Remark 7. *Under the assumption of Theorem 9, if we take* $\sigma = (1, 1, \ldots)$ *with* $\rho = \alpha,\ \lambda = 1$, *we get the following inequality involving classical Mittag–Leffler function*

$$\begin{aligned}&\left|\frac{\mathsf{F}(\wp_1)+\mathsf{F}(\wp_1 + \mathcal{F}^{\sigma}_{\rho,\lambda}(\frac{\wp_2}{c} - \wp_1))}{2} - \frac{c}{E_{\alpha}(\wp_2 - c\wp_1)}\int_{\wp_1}^{\wp_1 + E_{\alpha}(\frac{\wp_2}{c} - \wp_1)} \mathsf{F}(x)dx\right| \\ &\leq \frac{E_{\alpha}(\wp_2 - c\wp_1)}{2c}\left\{\frac{2-s}{4}\left[|\mathsf{F}'(\wp_1)| + |\mathsf{F}'\left(\frac{\wp_2}{c}\right)|\right]\right\}.\end{aligned}$$

Theorem 10. *Suppose* I° *is a generalized convex set with respect to* $\mathcal{F}^{\sigma}_{\rho,\lambda}$ *and* $\mathsf{F} : I^{\circ} \subseteq \mathbb{R} \to \mathbb{R}$ *be a differentiable mapping on* I°, $\wp_1, \wp_2 \in I^{\circ}$ *with* $\wp_1 < \wp_2, q > 1, \frac{1}{p} + \frac{1}{q} = 1$ *and suppose that* $\mathsf{F}' \in L[\wp_1, \wp_2]$. *If* $|\mathsf{F}'|^q$ *is generalized s-type convex function of Raina type on* $L[\wp_1, \wp_2]$, *then*

$$\begin{aligned}&\left|\frac{\mathsf{F}(\wp_1)+\mathsf{F}(\wp_1 + \mathcal{F}^{\sigma}_{\rho,\lambda}(\frac{\wp_2}{c} - \wp_1))}{2} - \frac{c}{\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2 - c\wp_1)}\int_{\wp_1}^{\wp_1 + \mathcal{F}^{\sigma}_{\rho,\lambda}(\frac{\wp_2}{c} - \wp_1)} \mathsf{F}(x)dx\right| \\ &\leq \frac{\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2 - c\wp_1)}{2c}\left[\frac{1}{p+1}\right]^{1/p}\left\{\frac{2-s}{2}\left[|\mathsf{F}'(\wp_1)|^q + |\mathsf{F}'\left(\frac{\wp_2}{c}\right)|^q\right]\right\}^{1/q}.\end{aligned}$$

Proof. Suppose that $\wp_1, \wp_2 \in I^{\circ}$. Since I° is a generalized convex set with respect to $\mathcal{F}^{\sigma}_{\rho,\lambda}$, for any $\ell \in [0, 1]$, we have $\wp_2 + \ell\, \mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1 - \wp_2) \in I^{\circ}$.

Using Lemma 2 and Hölder's inequality, one has

$$\begin{aligned}&\left|\frac{\mathsf{F}(\wp_1)+\mathsf{F}(\wp_1 + \mathcal{F}^{\sigma}_{\rho,\lambda}(\frac{\wp_2}{c} - \wp_1))}{2} - \frac{c}{\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2 - c\wp_1)}\int_{\wp_1}^{\wp_1 + \mathcal{F}^{\sigma}_{\rho,\lambda}(\frac{\wp_2}{c} - \wp_1)} \mathsf{F}(x)dx\right| \\ &= \frac{\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2 - c\wp_1)}{2c}\int_0^1 |1-2\ell||\mathsf{F}'\Big(\frac{\wp_2}{c} + \ell\, \mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1 - \frac{\wp_2}{c})\Big)|d\ell \\ &\leq \frac{\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2 - c\wp_1)}{2c}\left(\int_0^1 |1-2\ell|^p d\ell\right)^{1/p}\left(\int_0^1 |\mathsf{F}'\Big(\frac{\wp_2}{c} + \ell\, \mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1 - \frac{\wp_2}{c})\Big)|^q d\ell\right)^{1/q} \qquad (11)\end{aligned}$$

Since, $|\mathsf{F}'|^q$ is generalized s-type convex function of Raina type on $(\wp_1, \wp_1 + \mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2 - \wp_1))$, we have

$$\int_0^1 |\mathsf{F}'\Big(\frac{\wp_2}{c} + \ell\, \mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1 - \frac{\wp_2}{c})\Big)|^q d\ell = |\mathsf{F}'(\wp_1)|^q \int_0^1 (1 - s(1-\ell))d\ell + |\mathsf{F}'\left(\frac{\wp_2}{c}\right)|^q \int_0^1 (1-s\ell)d\ell.$$

Now, Equation (11) becomes

$$\begin{aligned}&\left|\frac{\mathsf{F}(\wp_1)+\mathsf{F}(\wp_1 + \mathcal{F}^{\sigma}_{\rho,\lambda}(\frac{\wp_2}{c} - \wp_1))}{2} - \frac{c}{\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2 - c\wp_1)}\int_{\wp_1}^{\wp_1 + \mathcal{F}^{\sigma}_{\rho,\lambda}(\frac{\wp_2}{c} - \wp_1)} \mathsf{F}(x)dx\right| \\ &\leq \frac{\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2 - c\wp_1)}{2c}\left[\frac{1}{p+1}\right]^{1/p}\left(|\mathsf{F}'(\wp_1)|^q \int_0^1 (1 - s(1-\ell))d\ell + |\mathsf{F}'\left(\frac{\wp_2}{c}\right)|^q \int_0^1 (1-s\ell)d\ell\right)^{1/q} \qquad (12)\end{aligned}$$

Since,

$$\int_0^1 (1-s(1-\ell))d\ell = \int_0^1 (1-s\ell)d\ell = -\frac{s-2}{2}$$
$$\int_0^1 |1-2\ell|^p d\ell = \left[\frac{1}{p+1}\right]$$

The proof of the Theorem is completed by using the above computations in (12). □

Corollary 5. *If we choose $s=1$, then we attain the following inequality:*

$$\left|\frac{\mathsf{F}(\wp_1)+\mathsf{F}(\wp_1+\ \mathcal{F}^{\sigma}_{\rho,\lambda}(\frac{\wp_2}{c}-\wp_1))}{2}-\frac{c}{\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-c\wp_1)}\int_{\wp_1}^{\wp_1+\ \mathcal{F}^{\sigma}_{\rho,\lambda}(\frac{\wp_2}{c}-\wp_1)}\mathsf{F}(x)dx\right|$$
$$\leq \frac{\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-c\wp_1)}{2^{\frac{1+q}{q}}c}\left[\frac{1}{p+1}\right]^{1/p}\left\{\left[|\mathsf{F}'(\wp_1)|^q+|\mathsf{F}'\Big(\frac{\wp_2}{c}\Big)|^q\right]\right\}^{1/q}.$$

Corollary 6. *If we choose $\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1-\wp_2)=\wp_1-\wp_2$, then we attain the following inequality:*

$$\left|\frac{\mathsf{F}(\wp_1)+\mathsf{F}(\frac{\wp_2}{c})}{2}-\frac{c}{(\wp_2-c\wp_1)}\int_{\wp_1}^{\frac{\wp_2}{c}}\mathsf{F}(x)dx\right|$$
$$\leq \frac{(\wp_2-c\wp_1)}{2c}\left[\frac{1}{p+1}\right]^{1/p}\left\{\frac{2-s}{2}\left[|\mathsf{F}'(\wp_1)|^q+|\mathsf{F}'\Big(\frac{\wp_2}{c}\Big)|^q\right]\right\}^{1/q}.$$

Corollary 7. *If we choose $s=1$ and $\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1-\wp_2)=\wp_1-\wp_2$, then we attain the following inequality:*

$$\left|\frac{\mathsf{F}(\wp_1)+\mathsf{F}(\frac{\wp_2}{c})}{2}-\frac{c}{(\wp_2-c\wp_1)}\int_{\wp_1}^{\frac{\wp_2}{c}}\mathsf{F}(x)dx\right|$$
$$\leq \frac{(\wp_2-c\wp_1)}{2^{\frac{1+q}{q}}c}\left[\frac{1}{p+1}\right]^{1/p}\left\{\left[|\mathsf{F}'(\wp_1)|^q+|\mathsf{F}'\Big(\frac{\wp_2}{c}\Big)|^q\right]\right\}^{1/q}.$$

Remark 8. *Under the assumptions of Theorem 10, if we take $\sigma=(1,1,\ldots)$ with $\rho=\alpha,\ \lambda=1$, we get the following inequality involving classical Mittag–Leffler function:*

$$\left|\frac{\mathsf{F}(\wp_1)+\mathsf{F}(\wp_1+E_{\alpha}(\frac{\wp_2}{c}-\wp_1))}{2}-\frac{c}{E_{\alpha}(\wp_2-c\wp_1)}\int_{\wp_1}^{\wp_1+E_{\alpha}(\frac{\wp_2}{c}-\wp_1)}\mathsf{F}(x)dx\right|$$
$$\leq \frac{E_{\alpha}(\wp_2-c\wp_1)}{2c}\left[\frac{1}{p+1}\right]^{1/p}\left\{\frac{2-s}{2}\left[|\mathsf{F}'(\wp_1)|^q+|\mathsf{F}'\Big(\frac{\wp_2}{c}\Big)|^q\right]\right\}^{1/q}.$$

Theorem 11. *Suppose I° is a generalized convex set with respect to $\mathcal{F}^{\sigma}_{\rho,\lambda}$ and $\mathsf{F}: I^{\circ}\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I°, $\wp_1,\wp_2\in I^{\circ}$ with $\wp_1<\wp_2, q>1, \frac{1}{p}+\frac{1}{q}=1$ and suppose that $\mathsf{F}'\in L[\wp_1,\wp_2]$. If $|\mathsf{F}'|^q$ is generalized s-type convex function of Raina type on $L[\wp_1,\wp_2]$, then*

$$\left|\frac{\mathsf{F}(\wp_1)+\mathsf{F}(\wp_1+\ \mathcal{F}^{\sigma}_{\rho,\lambda}(\frac{\wp_2}{c}-\wp_1))}{2}-\frac{c}{\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-c\wp_1)}\int_{\wp_1}^{\wp_1+\ \mathcal{F}^{\sigma}_{\rho,\lambda}(\frac{\wp_2}{c}-\wp_1)}\mathsf{F}(x)dx\right|$$
$$\leq \frac{\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-c\wp_1)}{c}\left[\frac{1}{2^{\left(\frac{p+1}{p}\right)}}\right]\left\{\frac{2-s}{4}\left[|\mathsf{F}'(\wp_1)|^q+|\mathsf{F}'\Big(\frac{\wp_2}{c}\Big)|^q\right]\right\}^{1/q}.$$

Proof. Suppose that $\wp_1,\wp_2\in I^{\circ}$. Since I° is a generalized convex set with respect to $\mathcal{F}^{\sigma}_{\rho,\lambda}$, for any $\ell\in[0,1]$, we have $\wp_2+\ell\ \mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1-\wp_2)\in I^{\circ}$.

Using Lemma 2 and Hölder's inequality, one has

$$\begin{aligned}
&\left|\frac{\mathsf{F}(\wp_1)+\mathsf{F}(\wp_1+\ \mathcal{F}^{\sigma}_{\rho,\lambda}(\frac{\wp_2}{c}-\wp_1))}{2}-\frac{c}{\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-c\wp_1)}\int_{\wp_1}^{\wp_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\frac{\wp_2}{c}-\wp_1)}\mathsf{F}(x)dx\right|\\
&\leq\frac{\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-c\wp_1)}{2c}\int_0^1|1-2\ell||\mathsf{F}'\Big(\frac{\wp_2}{c}+\ell\ \mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1-\frac{\wp_2}{c})\Big)|d\ell\\
&=\frac{\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-c\wp_1)}{2c}\int_0^1|1-2\ell|^{1/p}|1-2\ell|^{1/q}|\mathsf{F}'\Big(\frac{\wp_2}{c}+\ell\ \mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1-\frac{\wp_2}{c})\Big)|d\ell\\
&\leq\frac{\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-c\wp_1)}{2c}\left(\int_0^1|1-2\ell|d\ell\right)^{1/p}\left(\int_0^1|1-2\ell||\mathsf{F}'\Big(\frac{\wp_2}{c}+\ell\ \mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1-\frac{\wp_2}{c})\Big)|^q d\ell\right)^{1/q} \qquad (13)
\end{aligned}$$

Since $|\mathsf{F}'|^q$ is generalized s-type convex function of Raina type on $(\wp_1,\wp_1+\ \mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-\wp_1))$, we have

$$\int_0^1|\mathsf{F}'\Big(\frac{\wp_2}{c}+\ell\ \mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1-\frac{\wp_2}{c})\Big)|^q d\ell=|\mathsf{F}'(\wp_1)|^q\int_0^1(1-s(1-\ell))d\ell+|\mathsf{F}'\Big(\frac{\wp_2}{c}\Big)|^q\int_0^1(1-s\ell)d\ell$$

Now, Equation (13) becomes

$$\begin{aligned}
&\left|\frac{\mathsf{F}(\wp_1)+\mathsf{F}(\wp_1+\ \mathcal{F}^{\sigma}_{\rho,\lambda}(\frac{\wp_2}{c}-\wp_1))}{2}-\frac{c}{\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-c\wp_1)}\int_{\wp_1}^{\wp_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\frac{\wp_2}{c}-\wp_1)}\mathsf{F}(x)dx\right|\\
&\leq\frac{\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-c\wp_1)}{2c}\left(\int_0^1|1-2\ell|d\ell\right)^{1/p}\left(|\mathsf{F}'(\wp_1)|^q\int_0^1|1-2\ell|(1-s(1-\ell))d\ell\right.\\
&\qquad\left.+|\mathsf{F}'\Big(\frac{\wp_2}{c}\Big)|^q\int_0^1|1-2\ell|(1-s\ell)d\ell\right)^{1/q} \qquad (14)
\end{aligned}$$

Since,

$$\int_0^1|1-2\ell|(1-s(1-\ell))d\ell=\int_0^1|1-2\ell|(1-s\ell)d\ell=-\frac{s-2}{4}$$
$$\int_0^1|1-2\ell|d\ell=\frac{1}{2}$$

The proof of the theorem gets completed by using the above computations in (14). □

Corollary 8. *If we choose $s=1$, then we attain the following inequality*

$$\begin{aligned}
&\left|\frac{\mathsf{F}(\wp_1)+\mathsf{F}(\wp_1+\ \mathcal{F}^{\sigma}_{\rho,\lambda}(\frac{\wp_2}{c}-\wp_1))}{2}-\frac{c}{\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-c\wp_1)}\int_{\wp_1}^{\wp_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\frac{\wp_2}{c}-\wp_1)}\mathsf{F}(x)dx\right|\\
&\leq\frac{\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-c\wp_1)}{c}\left[\frac{1}{2^{\left(\frac{2q+1}{q}\right)}}\right]\left\{\left[|\mathsf{F}'(\wp_1)|^q+|\mathsf{F}'\Big(\frac{\wp_2}{c}\Big)|^q\right]\right\}^{1/q}.
\end{aligned}$$

Corollary 9. *If we choose $\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1-\wp_2)=\wp_1-\wp_2$, then we attain the following inequality,*

$$\begin{aligned}
&\left|\frac{\mathsf{F}(\wp_1)+\mathsf{F}(\frac{\wp_2}{c})}{2}-\frac{c}{(\wp_2-c\wp_1)}\int_{\wp_1}^{\frac{\wp_2}{c}}\mathsf{F}(x)dx\right|\\
&\leq\frac{(\wp_2-c\wp_1)}{c}\left[\frac{1}{2^{\left(\frac{p+1}{p}\right)}}\right]\left\{\frac{2-s}{4}\left[|\mathsf{F}'(\wp_1)|^q+|\mathsf{F}'\Big(\frac{\wp_2}{c}\Big)|^q\right]\right\}^{1/q}.
\end{aligned}$$

Corollary 10. *If we choose $s=1$ and $\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1-\wp_2)=\wp_1-\wp_2$, then we attain the following inequality,*

$$\left|\frac{\mathsf{F}(\wp_1)+\mathsf{F}(\frac{\wp_2}{c})}{2}-\frac{c}{(\wp_2-c\wp_1)}\int_{\wp_1}^{\frac{\wp_2}{c}}\mathsf{F}(x)dx\right|$$
$$\leq\frac{(\wp_2-c\wp_1)}{c}\left[\frac{1}{2^{\left(\frac{2q+1}{q}\right)}}\right]\left\{\left[|\mathsf{F}'(\wp_1)|^q+|\mathsf{F}'\left(\frac{\wp_2}{c}\right)|^q\right]\right\}^{1/q}.$$

Remark 9. *Under the assumptions of Theorem 11, if we take $\sigma=(1,1,\ldots)$ with $\rho=\alpha,\ \lambda=1$, we get the following inequality involving classical Mittag–Leffler function,*

$$\left|\frac{\mathsf{F}(\wp_1)+\mathsf{F}(\wp_1+E_\alpha(\frac{\wp_2}{c}-\wp_1))}{2}-\frac{c}{E_\alpha(\wp_2-c\wp_1)}\int_{\wp_1}^{\wp_1+E_\alpha(\frac{\wp_2}{c}-\wp_1)}\mathsf{F}(x)dx\right|$$
$$\leq\frac{E_\alpha(\wp_2-c\wp_1)}{c}\left[\frac{1}{2^{\left(\frac{p+1}{p}\right)}}\right]\left\{\frac{2-s}{4}\left[|\mathsf{F}'(\wp_1)|^q+|\mathsf{F}'\left(\frac{\wp_2}{c}\right)|^q\right]\right\}^{1/q}.$$

Theorem 12. *Suppose I° is a generalized convex set with respect to $\mathcal{F}^\sigma_{\rho,\lambda}$ and $\mathsf{F}:I^\circ\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I°, $\wp_1,\wp_2\in I^\circ$ with $\wp_1<\wp_2, q>1, \frac{1}{p}+\frac{1}{q}=1$ and suppose that $\mathsf{F}'\in L[\wp_1,\wp_2]$. If $|\mathsf{F}'|^q$ is generalized s-type convex function of Raina type on $L[\wp_1,\wp_2]$, then*

$$\left|\frac{\mathsf{F}(\wp_1)+\mathsf{F}(\wp_1+\mathcal{F}^\sigma_{\rho,\lambda}(\frac{\wp_2}{c}-\wp_1))}{2}-\frac{c}{\mathcal{F}^\sigma_{\rho,\lambda}(\wp_2-c\wp_1)}\int_{\wp_1}^{\wp_1+\mathcal{F}^\sigma_{\rho,\lambda}(\frac{\wp_2}{c}-\wp_1)}\mathsf{F}(x)dx\right|$$
$$\leq\frac{\mathcal{F}^\sigma_{\rho,\lambda}(\wp_2-c\wp_1)}{2c}\left[\frac{1}{2(p+1)}\right]^{1/p}\left[\left\{\frac{3-2s}{6}|\mathsf{F}'(\wp_1)|^q+\frac{3-s}{6}|\mathsf{F}'\left(\frac{\wp_2}{c}\right)|^q\right\}^{1/q}\right.$$
$$\left.+\left\{\frac{3-s}{6}|\mathsf{F}'(\wp_1)|^q+\frac{3-2s}{6}|\mathsf{F}'\left(\frac{\wp_2}{c}\right)|^q\right\}^{1/q}\right].$$

Proof. Suppose that $\wp_1,\wp_2\in I^\circ$. Since, I° is a generalized convex set with respect to $\mathcal{F}^\sigma_{\rho,\lambda}$, for any $\ell\in[0,1]$, we have $\wp_2+\ell\,\mathcal{F}^\sigma_{\rho,\lambda}(\wp_1-\wp_2)\in I^\circ$.

Using Lemma 2 and Hölder-Íscan inequality, one has

$$\left|\frac{\mathsf{F}(\wp_1)+\mathsf{F}(\wp_1+\mathcal{F}^\sigma_{\rho,\lambda}(\frac{\wp_2}{c}-\wp_1))}{2}-\frac{c}{\mathcal{F}^\sigma_{\rho,\lambda}(\wp_2-c\wp_1)}\int_{\wp_1}^{\wp_1+\mathcal{F}^\sigma_{\rho,\lambda}(\frac{\wp_2}{c}-\wp_1)}\mathsf{F}(x)dx\right|$$
$$\leq\frac{\mathcal{F}^\sigma_{\rho,\lambda}(\wp_2-c\wp_1)}{2c}\left[\left(\int_0^1(1-\ell)|1-2\ell|^pd\ell\right)^{1/p}\left(\int_0^1(1-\ell)|\mathsf{F}'\left(\frac{\wp_2}{c}+\ell\,\mathcal{F}^\sigma_{\rho,\lambda}(\wp_1-\frac{\wp_2}{c})\right)|^qd\ell\right)^{1/q}\right.$$
$$\left.+\left(\int_0^1\ell|1-2\ell|^pd\ell\right)^{1/p}\left(\int_0^1\ell|\mathsf{F}'\left(\frac{\wp_2}{c}+\ell\,\mathcal{F}^\sigma_{\rho,\lambda}(\wp_1-\frac{\wp_2}{c})\right)|^qd\ell\right)^{1/q}\right]. \tag{15}$$

Since, $|\mathsf{F}'|^q$ is generalized s-type convex function of Raina type on $(\wp_1,\wp_1+\mathcal{F}^\sigma_{\rho,\lambda}(\wp_2-\wp_1))$, we have

$$\int_0^1|\mathsf{F}'\left(\frac{\wp_2}{c}+\ell\,\mathcal{F}^\sigma_{\rho,\lambda}(\wp_1-\frac{\wp_2}{c})\right)|^qd\ell$$
$$=|\mathsf{F}'(\wp_1)|^q\int_0^1(1-s(1-\ell))d\ell+|\mathsf{F}'\left(\frac{\wp_2}{c}\right)|^q\int_0^1(1-s\ell)d\ell$$

Now, Equation (15) becomes

$$\left|\frac{\mathsf{F}(\wp_1)+\mathsf{F}(\wp_1+\ \mathcal{F}^{\sigma}_{\rho,\lambda}(\frac{\wp_2}{c}-\wp_1))}{2}-\frac{c}{\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-c\wp_1)}\int_{\wp_1}^{\wp_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\frac{\wp_2}{c}-\wp_1)}\mathsf{F}(x)dx\right|$$
$$\leq\frac{\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-c\wp_1)}{2c}\left[\left(\int_0^1(1-\ell)|1-2\ell|^p d\ell\right)^{1/p}\left(|\mathsf{F}'(\wp_1)|^q\int_0^1(1-\ell)(1-s(1-\ell))d\ell\right.\right.$$
$$\left.+|\mathsf{F}'\Big(\frac{\wp_2}{c}\Big)|^q\int_0^1(1-\ell)(1-s\ell)d\ell\right)^{1/q}$$
$$\left.+\left(\int_0^1\ell|1-2\ell|^p d\ell\right)^{1/p}\left(|\mathsf{F}'(\wp_1)|^q\int_0^1\ell(1-s(1-\ell))d\ell+|\mathsf{F}'\Big(\frac{\wp_2}{c}\Big)|^q\int_0^1\ell(1-s\ell)d\ell\right)^{1/q}\right]. \tag{16}$$

Since,

$$\int_0^1(1-\ell)(1-s(1-\ell))d\ell=\int_0^1\ell(1-s\ell)d\ell=-\frac{2s-3}{6}.$$
$$\int_0^1\ell(1-s(1-\ell))d\ell=\int_0^1(1-\ell)(1-s\ell)d\ell=-\frac{s-3}{6}.$$
$$\int_0^1\ell|1-2\ell|^p d\ell=\int_0^1(1-\ell)|1-2\ell|^p d\ell=\left[\frac{1}{2(p+1)}\right].$$

The proof of the Theorem is completed by using the above computations in (16). □

Corollary 11. *If we choose $s=1$, then we attain the following inequality,*

$$\left|\frac{\mathsf{F}(\wp_1)+\mathsf{F}(\wp_1+\ \mathcal{F}^{\sigma}_{\rho,\lambda}(\frac{\wp_2}{c}-\wp_1))}{2}-\frac{c}{\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-c\wp_1)}\int_{\wp_1}^{\wp_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\frac{\wp_2}{c}-\wp_1)}\mathsf{F}(x)dx\right|$$
$$\leq\frac{\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-c\wp_1)}{2c}\left[\frac{1}{2(p+1)}\right]^{1/p}\left[\left\{\frac{1}{6}|\mathsf{F}'(\wp_1)|^q+\frac{1}{3}|\mathsf{F}'\Big(\frac{\wp_2}{c}\Big)|^q\right\}^{1/q}\right.$$
$$\left.+\left\{\frac{1}{3}|\mathsf{F}'(\wp_1)|^q+\frac{1}{6}|\mathsf{F}'\Big(\frac{\wp_2}{c}\Big)|^q\right\}^{1/q}\right].$$

Corollary 12. *If we choose $\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1-\wp_2)=\wp_1-\wp_2$, then we attain the following inequality:*

$$\left|\frac{\mathsf{F}(\wp_1)+\mathsf{F}(\frac{\wp_2}{c})}{2}-\frac{c}{(\wp_2-c\wp_1)}\int_{\wp_1}^{\frac{\wp_2}{c}}\mathsf{F}(x)dx\right|\leq\frac{(\wp_2-c\wp_1)}{2c}\left[\frac{1}{2(p+1)}\right]^{1/p}$$
$$\times\left[\left\{\frac{3-2s}{6}|\mathsf{F}'(\wp_1)|^q+\frac{3-s}{6}|\mathsf{F}'\Big(\frac{\wp_2}{c}\Big)|^q\right\}^{1/q}+\left\{\frac{3-s}{6}|\mathsf{F}'(\wp_1)|^q+\frac{3-2s}{6}|\mathsf{F}'\Big(\frac{\wp_2}{c}\Big)|^q\right\}^{1/q}\right].$$

Corollary 13. *If we choose $s=1$ and $\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1-\wp_2)=\wp_1-\wp_2$, then we attain the following inequality:*

$$\left|\frac{\mathsf{F}(\wp_1)+\mathsf{F}(\frac{\wp_2}{c})}{2}-\frac{c}{(\wp_2-c\wp_1)}\int_{\wp_1}^{\frac{\wp_2}{c}}\mathsf{F}(x)dx\right|$$
$$\leq\frac{(\wp_2-c\wp_1)}{2c}\left[\frac{1}{2(p+1)}\right]^{1/p}\left[\left\{\frac{1}{6}|\mathsf{F}'(\wp_1)|^q+\frac{1}{3}|\mathsf{F}'\Big(\frac{\wp_2}{c}\Big)|^q\right\}^{1/q}\right.$$
$$\left.+\left\{\frac{1}{3}|\mathsf{F}'(\wp_1)|^q+\frac{1}{6}|\mathsf{F}'\Big(\frac{\wp_2}{c}\Big)|^q\right\}^{1/q}\right].$$

Remark 10. *Under the assumptions of Theorem 12, if we take $\sigma = (1, 1, \ldots)$ with $\rho = \alpha,\ \lambda = 1$, we get the following inequality involving classical Mittag–Leffler function,*

$$\left|\frac{\mathsf{F}(\wp_1)+\mathsf{F}(\wp_1+E_\alpha(\frac{\wp_2}{c}-\wp_1))}{2}-\frac{c}{E_\alpha(\wp_2-c\wp_1)}\int_{\wp_1}^{\wp_1+E_\alpha(\frac{\wp_2}{c}-\wp_1)}\mathsf{F}(x)dx\right|$$
$$\leq \frac{E_\alpha(\wp_2-c\wp_1)}{2c}\left[\frac{1}{2(p+1)}\right]^{1/p}\left[\left\{\frac{3-2s}{6}|\mathsf{F}'(\wp_1)|^q+\frac{3-s}{6}|\mathsf{F}'\Big(\frac{\wp_2}{c}\Big)|^q\right\}^{1/q}\right.$$
$$\left.+\left\{\frac{3-s}{6}|\mathsf{F}'(\wp_1)|^q+\frac{3-2s}{6}|\mathsf{F}'\Big(\frac{\wp_2}{c}\Big)|^q\right\}^{1/q}\right].$$

Theorem 13. *Suppose I° is a generalized convex set with respect to $\mathcal{F}^\sigma_{\rho,\lambda}$ and $\mathsf{F}: I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $\wp_1, \wp_2 \in I^\circ$ with $\wp_1 < \wp_2, q \geq 1$ and suppose that $\mathsf{F}' \in L[\wp_1, \wp_2]$. If $|\mathsf{F}'|^q$ is a generalized s-type convex function of Raina type on $L[\wp_1, \wp_2]$, then*

$$\left|\frac{\mathsf{F}(\wp_1)+\mathsf{F}(\wp_1+\mathcal{F}^\sigma_{\rho,\lambda}(\frac{\wp_2}{c}-\wp_1))}{2}-\frac{c}{\mathcal{F}^\sigma_{\rho,\lambda}(\wp_2-c\wp_1)}\int_{\wp_1}^{\wp_1+\mathcal{F}^\sigma_{\rho,\lambda}(\frac{\wp_2}{c}-\wp_1)}\mathsf{F}(x)dx\right|$$
$$\leq \frac{\mathcal{F}^\sigma_{\rho,\lambda}(\wp_2-c\wp_1)}{2c}\left(\frac{1}{4}\right)^{1-1/q}\left[\left\{\frac{4-3s}{16}|\mathsf{F}'(\wp_1)|^q+\frac{4-s}{16}|\mathsf{F}'\Big(\frac{\wp_2}{c}\Big)|^q\right\}^{1/q}\right.$$
$$\left.+\left\{\frac{4-s}{16}|\mathsf{F}'(\wp_1)|^q+\frac{4-3s}{16}|\mathsf{F}'\Big(\frac{\wp_2}{c}\Big)|^q\right\}^{1/q}\right].$$

Proof. Suppose that $\wp_1, \wp_2 \in I^\circ$. Since I° is a generalized convex set with respect to $\mathcal{F}^\sigma_{\rho,\lambda}$, for any $\ell \in [0, 1]$, we have $\wp_2 + \ell\, \mathcal{F}^\sigma_{\rho,\lambda}(\wp_1 - \wp_2) \in I^\circ$.

Using Lemma 2 and Improved power-mean inequality, one has

$$\left|\frac{\mathsf{F}(\wp_1)+\mathsf{F}(\wp_1+\mathcal{F}^\sigma_{\rho,\lambda}(\frac{\wp_2}{c}-\wp_1))}{2}-\frac{c}{\mathcal{F}^\sigma_{\rho,\lambda}(\wp_2-c\wp_1)}\int_{\wp_1}^{\wp_1+\mathcal{F}^\sigma_{\rho,\lambda}(\frac{\wp_2}{c}-\wp_1)}\mathsf{F}(x)dx\right|$$
$$\leq \frac{\mathcal{F}^\sigma_{\rho,\lambda}(\wp_2-c\wp_1)}{2c}\left[\left(\int_0^1(1-\ell)|1-2\ell|d\ell\right)^{1-1/q}\left(\int_0^1(1-\ell)|1-2\ell||\mathsf{F}'\Big(\frac{\wp_2}{c}+\ell\,\mathcal{F}^\sigma_{\rho,\lambda}(\wp_1-\frac{\wp_2}{c})\Big)|^q d\ell\right)^{1/q}\right.$$
$$\left.+\left(\int_0^1 \ell|1-2\ell|d\ell\right)^{1-1/q}\left(\int_0^1 \ell|1-2\ell||\mathsf{F}'\Big(\frac{\wp_2}{c}+\ell\,\mathcal{F}^\sigma_{\rho,\lambda}(\wp_1-\frac{\wp_2}{c})\Big)|^q d\ell\right)^{1/q}\right]. \tag{17}$$

Since $|\mathsf{F}'|^q$ is generalized s-type convex function of Raina type on $(\wp_1, \wp_1 + \mathcal{F}^\sigma_{\rho,\lambda}(\wp_2 - \wp_1))$, we have

$$\int_0^1 |\mathsf{F}'\Big(\frac{\wp_2}{c}+\ell\,\mathcal{F}^\sigma_{\rho,\lambda}(\wp_1-\frac{\wp_2}{c})\Big)|^q d\ell$$
$$= |\mathsf{F}'(\wp_1)|^q\int_0^1(1-s(1-\ell))d\ell + |\mathsf{F}'\Big(\frac{\wp_2}{c}\Big)|^q\int_0^1(1-s\ell)d\ell.$$

Now, Equation (17) becomes

$$\left|\frac{\mathsf{F}(\wp_1)+\mathsf{F}(\wp_1+\ \mathcal{F}^{\sigma}_{\rho,\lambda}(\frac{\wp_2}{c}-\wp_1))}{2}-\frac{c}{\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-c\wp_1)}\int_{\wp_1}^{\wp_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\frac{\wp_2}{c}-\wp_1)}\mathsf{F}(x)dx\right|$$
$$\leq\frac{\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-c\wp_1)}{2c}\left[\left(\int_0^1(1-\ell)|1-2\ell|d\ell\right)^{1-1/q}\left(|\mathsf{F}'(\wp_1)|^q\int_0^1(1-\ell)|1-2\ell|(1-s(1-\ell))d\ell\right.\right.$$
$$\left.+|\mathsf{F}'\Big(\frac{\wp_2}{c}\Big)|^q\int_0^1(1-\ell)|1-2\ell|(1-s\ell)d\ell\right)^{1/q}$$
$$+\left(\int_0^1\ell|1-2\ell|d\ell\right)^{1-1/q}$$
$$\left.\times\left(|\mathsf{F}'(\wp_1)|^q\int_0^1\ell|1-2\ell|(1-s(1-\ell))d\ell+|\mathsf{F}'\Big(\frac{\wp_2}{c}\Big)|^q\int_0^1\ell|1-2\ell|(1-s\ell)d\ell\right)^{1/q}\right]. \tag{18}$$

Since,

$$\int_0^1(1-\ell)|1-2\ell|(1-s(1-\ell))d\ell=\int_0^1\ell|1-2\ell|(1-s\ell)d\ell=-\frac{3s-4}{16}.$$
$$\int_0^1\ell|1-2\ell|(1-s(1-\ell))d\ell=\int_0^1(1-\ell)|1-2\ell|(1-s\ell)d\ell=-\frac{s-4}{16}.$$
$$\int_0^1\ell|1-2\ell|d\ell=\int_0^1(1-\ell)|1-2\ell|d\ell=\left[\frac{1}{4}\right].$$

The proof of the theorem is completed by using the above computations in (18). □

Corollary 14. *If we choose $s=1$, then we attain the following inequality:*

$$\left|\frac{\mathsf{F}(\wp_1)+\mathsf{F}(\wp_1+\ \mathcal{F}^{\sigma}_{\rho,\lambda}(\frac{\wp_2}{c}-\wp_1))}{2}-\frac{c}{\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-c\wp_1)}\int_{\wp_1}^{\wp_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\frac{\wp_2}{c}-\wp_1)}\mathsf{F}(x)dx\right|$$
$$\leq\frac{\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-c\wp_1)}{2c}\left[\frac{1}{4}\right]^{1-1/q}\left[\left\{\frac{1}{16}|\mathsf{F}'(\wp_1)|^q+\frac{3}{16}|\mathsf{F}'\Big(\frac{\wp_2}{c}\Big)|^q\right\}^{1/q}\right.$$
$$\left.+\left\{\frac{3}{16}|\mathsf{F}'(\wp_1)|^q+\frac{1}{16}|\mathsf{F}'\Big(\frac{\wp_2}{c}\Big)|^q\right\}^{1/q}\right].$$

Corollary 15. *If we choose $\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1-\wp_2)=\wp_1-\wp_2$, then we attain the following inequality:*

$$\left|\frac{\mathsf{F}(\wp_1)+\mathsf{F}(\frac{\wp_2}{c})}{2}-\frac{c}{(\wp_2-c\wp_1)}\int_{\wp_1}^{\frac{\wp_2}{c}}\mathsf{F}(x)dx\right|$$
$$\leq\frac{(\wp_2-c\wp_1)}{2c}\left[\frac{1}{4}\right]^{1-1/q}\left[\left\{\frac{4-3s}{16}|\mathsf{F}'(\wp_1)|^q+\frac{4-s}{16}|\mathsf{F}'\Big(\frac{\wp_2}{c}\Big)|^q\right\}^{1/q}\right.$$
$$\left.+\left\{\frac{4-s}{16}|\mathsf{F}'(\wp_1)|^q+\frac{4-3s}{16}|\mathsf{F}'\Big(\frac{\wp_2}{c}\Big)|^q\right\}^{1/q}\right].$$

Corollary 16. *If we choose* $s = 1$ *and* $\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1 - \wp_2) = \wp_1 - \wp_2$, *then we attain the following inequality:*

$$\begin{aligned}&\left|\frac{\mathsf{F}(\wp_1)+\mathsf{F}(\frac{\wp_2}{c})}{2}-\frac{c}{(\wp_2-c\wp_1)}\int_{\wp_1}^{\frac{\wp_2}{c}}\mathsf{F}(x)dx\right|\\&\leq\frac{(\wp_2-c\wp_1)}{2c}\left[\frac{1}{4}\right]^{1-1/q}\left[\left\{\frac{1}{16}|\mathsf{F}'(\wp_1)|^q+\frac{3}{16}|\mathsf{F}'\Big(\frac{\wp_2}{c}\Big)|^q\right\}^{1/q}\right.\\&\left.+\left\{\frac{3}{16}|\mathsf{F}'(\wp_1)|^q+\frac{1}{16}|\mathsf{F}'\Big(\frac{\wp_2}{c}\Big)|^q\right\}^{1/q}\right].\end{aligned}$$

Remark 11. *Under the assumptions of Theorem 13, if we take* $\sigma = (1, 1, \ldots)$ *with* $\rho = \alpha,\ \lambda = 1$, *we get the following inequality involving classical Mittag–Leffler function,*

$$\begin{aligned}&\left|\frac{\mathsf{F}(\wp_1)+\mathsf{F}(\wp_1+E_\alpha(\frac{\wp_2}{c}-\wp_1))}{2}-\frac{c}{E_\alpha(\wp_2-c\wp_1)}\int_{\wp_1}^{\wp_1+E_\alpha(\frac{\wp_2}{c}-\wp_1)}\mathsf{F}(x)dx\right|\\&\leq\frac{E_\alpha(\wp_2-c\wp_1)}{2c}\left[\frac{1}{4}\right]^{1-1/q}\left[\left\{\frac{4-3s}{16}|\mathsf{F}'(\wp_1)|^q+\frac{4-s}{16}|\mathsf{F}'\Big(\frac{\wp_2}{c}\Big)|^q\right\}^{1/q}\right.\\&\left.+\left\{\frac{4-s}{16}|\mathsf{F}'(\wp_1)|^q+\frac{4-3s}{16}|\mathsf{F}'\Big(\frac{\wp_2}{c}\Big)|^q\right\}^{1/q}\right].\end{aligned}$$

Theorem 14. *Suppose* I° *is a generalized convex set with respect to* $\mathcal{F}^{\sigma}_{\rho,\lambda}$ *and* $\mathsf{F}: I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ *be a differentiable mapping on* I°, $\wp_1, \wp_2 \in I^\circ$ *with* $\wp_1 < \wp_2, q \geq 1$, *and suppose that* $\mathsf{F}' \in L[\wp_1, \wp_2]$. *If* $|\mathsf{F}'|^q$ *is generalized s-type convex function of Raina type on* $L[\wp_1, \wp_2]$, *then*

$$\begin{aligned}&\left|\frac{\mathsf{F}(\wp_1)+\mathsf{F}(\wp_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\frac{\wp_2}{c}-\wp_1))}{2}-\frac{c}{\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-c\wp_1)}\int_{\wp_1}^{\wp_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\frac{\wp_2}{c}-\wp_1)}\mathsf{F}(x)dx\right|\\&\leq\frac{\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-c\wp_1)}{2c}\left[\frac{1}{2}\right]^{1-\frac{1}{q}}\left\{\frac{2-s}{4}\left[|\mathsf{F}'(\wp_1)|^q+|\mathsf{F}'\Big(\frac{\wp_2}{c}\Big)|^q\right]\right\}^{1/q}.\end{aligned}$$

Proof. Suppose that $\wp_1, \wp_2 \in I^\circ$. Since I° is a generalized convex set with respect to $\mathcal{F}^{\sigma}_{\rho,\lambda}$, for any $\ell \in [0, 1]$, we have $\wp_2 + \ell\, \mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1 - \wp_2) \in I^\circ$.

Using Lemma 2 and Power-mean inequality, one has

$$\begin{aligned}&\left|\frac{\mathsf{F}(\wp_1)+\mathsf{F}(\wp_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\frac{\wp_2}{c}-\wp_1))}{2}-\frac{c}{\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-c\wp_1)}\int_{\wp_1}^{\wp_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\frac{\wp_2}{c}-\wp_1)}\mathsf{F}(x)dx\right|\\&\leq\frac{\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-c\wp_1)}{2c}\int_0^1|1-2\ell||\mathsf{F}'\Big(\frac{\wp_2}{c}+\ell\,\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1-\frac{\wp_2}{c})\Big)|d\ell\\&\leq\frac{\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-c\wp_1)}{2c}\left(\int_0^1|1-2\ell|d\ell\right)^{1-1/q}\left(\int_0^1|1-2\ell||\mathsf{F}'\Big(\frac{\wp_2}{c}+\ell\,\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1-\frac{\wp_2}{c})\Big)|^q d\ell\right)^{1/q}\end{aligned}\tag{19}$$

Since, $|\mathsf{F}'|^q$ is generalized s-type convex function of Raina type on $(\wp_1, \wp_1 + \mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2 - \wp_1))$, we have

$$\begin{aligned}&\int_0^1|\mathsf{F}'\Big(\frac{\wp_2}{c}+\ell\,\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1-\frac{\wp_2}{c})\Big)|^q d\ell\\&=|\mathsf{F}'(\wp_1)|^q\int_0^1(1-s(1-\ell))d\ell+|\mathsf{F}'\Big(\frac{\wp_2}{c}\Big)|^q\int_0^1(1-s\ell)d\ell\end{aligned}$$

Now, Equation (19) becomes

$$\left|\frac{\mathsf{F}(\wp_1)+\mathsf{F}(\wp_1+\ \mathcal{F}^{\sigma}_{\rho,\lambda}(\frac{\wp_2}{c}-\wp_1))}{2}-\frac{c}{\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-c\wp_1)}\int_{\wp_1}^{\wp_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\frac{\wp_2}{c}-\wp_1)}\mathsf{F}(x)dx\right|$$
$$\leq\frac{\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-c\wp_1)}{2c}\left(\int_0^1|1-2\ell|d\ell\right)^{1-1/q}\left(|\mathsf{F}'(\wp_1)|^q\int_0^1|1-2\ell|(1-s(1-\ell))d\ell\right.$$
$$\left.+|\mathsf{F}'\Big(\frac{\wp_2}{c}\Big)|^q\int_0^1|1-2\ell|(1-s\ell)d\ell\right)^{1/q} \tag{20}$$

Since,

$$\int_0^1|1-2\ell|(1-s(1-\ell))d\ell=\int_0^1|1-2\ell|(1-s\ell)d\ell=-\frac{s-2}{4}$$
$$\int_0^1|1-2\ell|d\ell=\frac{1}{2}$$

The proof of the theorem gets completed by using the above computations in (20). □

Corollary 17. *If we choose $s=1$, then we attain the following inequality:*

$$\left|\frac{\mathsf{F}(\wp_1)+\mathsf{F}(\wp_1+\ \mathcal{F}^{\sigma}_{\rho,\lambda}(\frac{\wp_2}{c}-\wp_1))}{2}-\frac{c}{\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-c\wp_1)}\int_{\wp_1}^{\wp_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\frac{\wp_2}{c}-\wp_1)}\mathsf{F}(x)dx\right|$$
$$\leq\frac{\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-c\wp_1)}{2^{\frac{2q+1}{q}}c}\left\{\left[|\mathsf{F}'(\wp_1)|^q+|\mathsf{F}'\Big(\frac{\wp_2}{c}\Big)|^q\right]\right\}^{1/q}.$$

Corollary 18. *If we choose $\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1-\wp_2)=\wp_1-\wp_2$, then we attain the following inequality:*

$$\left|\frac{\mathsf{F}(\wp_1)+\mathsf{F}(\frac{\wp_2}{c})}{2}-\frac{c}{(\wp_2-c\wp_1)}\int_{\wp_1}^{\frac{\wp_2}{c}}\mathsf{F}(x)dx\right|$$
$$\leq\frac{(\wp_2-c\wp_1)}{2c}\left[\frac{1}{2}\right]^{1-\frac{1}{q}}\left\{\frac{2-s}{4}\left[|\mathsf{F}'(\wp_1)|^q+|\mathsf{F}'\Big(\frac{\wp_2}{c}\Big)|^q\right]\right\}^{1/q}.$$

Corollary 19. *If we choose $s=1$ and $\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1-\wp_2)=\wp_1-\wp_2$, then we attain the following inequality:*

$$\left|\frac{\mathsf{F}(\wp_1)+\mathsf{F}(\frac{\wp_2}{c})}{2}-\frac{c}{(\wp_2-c\wp_1)}\int_{\wp_1}^{\frac{\wp_2}{c}}\mathsf{F}(x)dx\right|\leq\frac{(\wp_2-c\wp_1)}{2^{\frac{2q+1}{q}}c}\left\{\left[|\mathsf{F}'(\wp_1)|^q+|\mathsf{F}'\Big(\frac{\wp_2}{c}\Big)|^q\right]\right\}^{1/q}.$$

Remark 12. *Under the assumptions of Theorem 14, if we take $\sigma=(1,1,\ldots)$ with $\rho=\alpha,\ \lambda=1$, we get the following inequality involving classical Mittag–Leffler function:*

$$\left|\frac{\mathsf{F}(\wp_1)+\mathsf{F}(\wp_1+E_{\alpha}(\frac{\wp_2}{c}-\wp_1))}{2}-\frac{c}{E_{\alpha}(\wp_2-c\wp_1)}\int_{\wp_1}^{\wp_1+E_{\alpha}(\frac{\wp_2}{c}-\wp_1)}\mathsf{F}(x)dx\right|$$
$$\leq\frac{E_{\alpha}(\wp_2-c\wp_1)}{2c}\left[\frac{1}{2}\right]^{1-\frac{1}{q}}\left\{\frac{2-s}{4}\left[|\mathsf{F}'(\wp_1)|^q+|\mathsf{F}'\Big(\frac{\wp_2}{c}\Big)|^q\right]\right\}^{1/q}.$$

Theorem 15. *Suppose I° is a generalized convex set with respect to $\mathcal{F}^\sigma_{\rho,\lambda}$ and $\mathsf{F} : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $\wp_1, \wp_2 \in I^\circ$ with $\wp_1 < \wp_2, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and suppose that $\mathsf{F}' \in L[\wp_1, \wp_2]$. If $|\mathsf{F}'|^q$ is generalized s-type convex function of Raina type on $L[\wp_1, \wp_2]$, then*

$$\begin{aligned}
&\left| \frac{c}{\mathcal{F}^\sigma_{\rho,\lambda}(\wp_2 - c\wp_1)} \int_{\wp_1}^{\wp_1 + \mathcal{F}^\sigma_{\rho,\lambda}(\frac{\wp_2}{c} - \wp_1)} \mathsf{F}(x)dx - \mathsf{F}\left(\frac{2\wp_1 + \mathcal{F}^\sigma_{\rho,\lambda}(\wp_2 - \wp_1)}{2c} \right) \right| \\
&\leq \frac{\mathcal{F}^\sigma_{\rho,\lambda}(\wp_2 - c\wp_1)}{c} \left[\left(\frac{1}{p+1} \right)^{1/p} \left\{ \frac{2-s}{2} [|\mathsf{F}'(\wp_1)|^q + |\mathsf{F}'(\frac{\wp_2}{c})|^q] \right\}^{1/q} \right. \\
&\left. + \left\{ \frac{4-3s}{8} [|\mathsf{F}'(\wp_1)|^q + |\mathsf{F}'(\frac{\wp_2}{c})|^q] \right\}^{1/q} \right].
\end{aligned}$$

Proof. Suppose that $\wp_1, \wp_2 \in I^\circ$. Since I° is a generalized convex set with respect to $\mathcal{F}^\sigma_{\rho,\lambda}$, for any $\ell \in [0,1]$, we have $\wp_2 + \ell\, \mathcal{F}^\sigma_{\rho,\lambda}(\wp_1 - \wp_2) \in I^\circ$.

Using Lemma 3 and Hölder's inequality, one has

$$\begin{aligned}
&\left| \frac{c}{\mathcal{F}^\sigma_{\rho,\lambda}(\wp_2 - c\wp_1)} \int_{\wp_1}^{\wp_1 + \mathcal{F}^\sigma_{\rho,\lambda}(\frac{\wp_2}{c} - \wp_1)} \mathsf{F}(x)dx - \mathsf{F}\left(\frac{2\wp_1 + \mathcal{F}^\sigma_{\rho,\lambda}(\wp_2 - \wp_1)}{2c} \right) \right| \\
&= \frac{\mathcal{F}^\sigma_{\rho,\lambda}(\wp_2 - c\wp_1)}{c} \left\{ \int_0^1 \ell \mathsf{F}'\Big(\frac{\wp_2}{c} + \ell\, \mathcal{F}^\sigma_{\rho,\lambda}(\wp_1 - \frac{\wp_2}{c}) \Big) d\ell - \int_{1/2}^1 \mathsf{F}'\Big(\frac{\wp_2}{c} + \ell\, \mathcal{F}^\sigma_{\rho,\lambda}(\wp_1 - \frac{\wp_2}{c}) \Big) d\ell \right\} \\
&\leq \frac{\mathcal{F}^\sigma_{\rho,\lambda}(\wp_2 - c\wp_1)}{c} \left[\left(\int_0^1 \ell^p d\ell \right)^{1/p} \left\{ \int_0^1 |\mathsf{F}'\Big(\frac{\wp_2}{c} + \ell\, \mathcal{F}^\sigma_{\rho,\lambda}(\wp_1 - \frac{\wp_2}{c}) \Big)|^q d\ell \right\}^{1/q} \right. \\
&\left. + \left\{ \int_{1/2}^1 |\mathsf{F}'\Big(\frac{\wp_2}{c} + \ell\, \mathcal{F}^\sigma_{\rho,\lambda}(\wp_1 - \frac{\wp_2}{c}) \Big)|^q d\ell \right\}^{1/q} \right] \\
&\leq \frac{\mathcal{F}^\sigma_{\rho,\lambda}(\wp_2 - c\wp_1)}{c} \left[\left(\frac{1}{p+1} \right)^{1/p} \left\{ \int_0^1 [1 - s(1-\ell)] |\mathsf{F}'(\wp_1)|^q d\ell + \int_0^1 [1 - s\ell] |\mathsf{F}'\Big(\frac{\wp_2}{c} \Big)|^q d\ell \right\}^{1/q} \right. \\
&\left. + \left\{ \int_{1/2}^1 [1 - s(1-\ell)] |\mathsf{F}'(\wp_1)|^q d\ell + \int_{1/2}^1 [1 - s\ell] |\mathsf{F}'\Big(\frac{\wp_2}{c} \Big)|^q d\ell \right\}^{1/q} \right] \\
&= \frac{\mathcal{F}^\sigma_{\rho,\lambda}(\wp_2 - c\wp_1)}{c} \left[\left(\frac{1}{p+1} \right)^{1/p} \left\{ \frac{2-s}{2} [|\mathsf{F}'(\wp_1)|^q + |\mathsf{F}'(\frac{\wp_2}{c})|^q] \right\}^{1/q} \right. \\
&\left. + \left\{ \frac{4-3s}{8} [|\mathsf{F}'(\wp_1)|^q + |\mathsf{F}'(\frac{\wp_2}{c})|^q] \right\}^{1/q} \right].
\end{aligned}$$

□

Corollary 20. *If we choose $s = 1$, then we attain the following inequality:*

$$\begin{aligned}
&\left| \frac{c}{\mathcal{F}^\sigma_{\rho,\lambda}(\wp_2 - c\wp_1)} \int_{\wp_1}^{\wp_1 + \mathcal{F}^\sigma_{\rho,\lambda}(\frac{\wp_2}{c} - \wp_1)} \mathsf{F}(x)dx - \mathsf{F}\left(\frac{2\wp_1 + \mathcal{F}^\sigma_{\rho,\lambda}(\wp_2 - \wp_1)}{2c} \right) \right| \\
&\leq \frac{\mathcal{F}^\sigma_{\rho,\lambda}(\wp_2 - c\wp_1)}{c} \left[\left(\frac{1}{p+1} \right)^{1/p} \left\{ \frac{1}{2} [|\mathsf{F}'(\wp_1)|^q + |\mathsf{F}'(\frac{\wp_2}{c})|^q] \right\}^{1/q} + \left\{ \frac{1}{8} [|\mathsf{F}'(\wp_1)|^q + |\mathsf{F}'(\frac{\wp_2}{c})|^q] \right\}^{1/q} \right].
\end{aligned}$$

Corollary 21. *If we choose* $\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1-\wp_2)=\wp_1-\wp_2$, *then we attain the following inequality:*

$$\begin{aligned}&\left|\frac{c}{(\wp_2-c\wp_1)}\int_{\wp_1}^{\frac{\wp_2}{c}}\mathsf{F}(x)dx-\mathsf{F}\left(\frac{\wp_1+\wp_2)}{2c}\right)\right|\\&\leq\frac{\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-c\wp_1)}{c}\left[\left(\frac{1}{p+1}\right)^{1/p}\left\{\frac{2-s}{2}[|\mathsf{F}'(\wp_1)|^q+|\mathsf{F}'(\frac{\wp_2}{c})|^q]\right\}^{1/q}\right.\\&\left.+\left\{\frac{4-3s}{8}[|\mathsf{F}'(\wp_1)|^q+|\mathsf{F}'(\frac{\wp_2}{c})|^q]\right\}^{1/q}\right].\end{aligned}$$

Corollary 22. *If we choose* $s=1$ *and* $\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1-\wp_2)=\wp_1-\wp_2$, *then we attain the following inequality:*

$$\begin{aligned}&\left|\frac{c}{(\wp_2-c\wp_1)}\int_{\wp_1}^{\frac{\wp_2}{c}}\mathsf{F}(x)dx-\mathsf{F}\left(\frac{\wp_1+\wp_2)}{2c}\right)\right|\\&\leq\frac{(\wp_2-c\wp_1)}{c}\left[\left(\frac{1}{p+1}\right)^{1/p}\left\{\frac{1}{2}[|\mathsf{F}'(\wp_1)|^q+|\mathsf{F}'(\frac{\wp_2}{c})|^q]\right\}^{1/q}+\left\{\frac{1}{8}[|\mathsf{F}'(\wp_1)|^q+|\mathsf{F}'(\frac{\wp_2}{c})|^q]\right\}^{1/q}\right].\end{aligned}$$

Remark 13. *Under the assumption of Theorem 15, if we take* $\sigma=(1,1,\ldots)$ *with* $\rho=\alpha,\ \lambda=1$, *we get the following inequality involving classical Mittag–Leffler function,*

$$\begin{aligned}&\left|\frac{c}{E_{\alpha}(\wp_2-c\wp_1)}\int_{\wp_1}^{\wp_1+E_{\alpha}(\frac{\wp_2}{c}-\wp_1)}\mathsf{F}(x)dx-\mathsf{F}\left(\frac{2\wp_1+E_{\alpha}(\wp_2-\wp_1)}{2c}\right)\right|\\&\leq\frac{E_{\alpha}(\wp_2-c\wp_1)}{c}\left[\left(\frac{1}{p+1}\right)^{1/p}\left\{\frac{2-s}{2}[|\mathsf{F}'(\wp_1)|^q+|\mathsf{F}'(\frac{\wp_2}{c})|^q]\right\}^{1/q}\right.\\&\left.+\left\{\frac{4-3s}{8}[|\mathsf{F}'(\wp_1)|^q+|\mathsf{F}'(\frac{\wp_2}{c})|^q]\right\}^{1/q}\right].\end{aligned}$$

Theorem 16. *Suppose* I° *is a generalized convex set with respect to* $\mathcal{F}^{\sigma}_{\rho,\lambda}$ *and* $\mathsf{F}:I^{\circ}\subseteq\mathbb{R}\to\mathbb{R}$ *be a differentiable mapping on* I°, $\wp_1,\wp_2\in I^{\circ}$ *with* $\wp_1<\wp_2, q\geq 1$, *and suppose that* $\mathsf{F}'\in L[\wp_1,\wp_2]$. *If* $|\mathsf{F}'|^q$ *is generalized s-type convex function of Raina type on* $L[\wp_1,\wp_2]$, *then*

$$\begin{aligned}&\left|\frac{c}{\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-c\wp_1)}\int_{\wp_1}^{\wp_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\frac{\wp_2}{c}-\wp_1)}\mathsf{F}(x)dx-\mathsf{F}\left(\frac{2\wp_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-\wp_1)}{2c}\right)\right|\\&\leq\frac{\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-c\wp_1)}{c}\left[\left(\frac{1}{2}\right)^{1-1/q}\right.\\&\left.\left\{\frac{3-s}{6}|\mathsf{F}'(\wp_1)|^q+\frac{3-2s}{6}|\mathsf{F}'(\frac{\wp_2}{c})|^q\right\}^{1/q}+\left\{\frac{4-3s}{8}[|\mathsf{F}'(\wp_1)|^q+|\mathsf{F}'(\frac{\wp_2}{c})|^q]\right\}^{1/q}\right].\end{aligned}$$

Proof. Suppose that $\wp_1,\wp_2\in I^{\circ}$. Since I° is a generalized convex set with respect to $\mathcal{F}^{\sigma}_{\rho,\lambda}$, for any $\ell\in[0,1]$, we have $\wp_2+\ell\,\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1-\wp_2)\in I^{\circ}$.

Using Lemma 3 and power-mean inequality, one has

$$
\begin{aligned}
&\left|\frac{c}{\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-c\wp_1)}\int_{\wp_1}^{\wp_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\frac{\wp_2}{c}-\wp_1)}\mathsf{F}(x)dx-\mathsf{F}\left(\frac{2\wp_1+\ \mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-\wp_1)}{2c}\right)\right|\\
&=\frac{\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-c\wp_1)}{c}\left\{\int_0^1\ell\mathsf{F}'\Big(\frac{\wp_2}{c}+\ell\ \mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1-\frac{\wp_2}{c})\Big)d\ell-\int_{1/2}^1\mathsf{F}'\Big(\frac{\wp_2}{c}+\ell\ \mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1-\frac{\wp_2}{c})\Big)d\ell\right\}\\
&\leq\frac{\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-c\wp_1)}{c}\left[\left(\int_0^1\ell d\ell\right)^{1-1/q}\left\{\int_0^1\ell|\mathsf{F}'\Big(\frac{\wp_2}{c}+\ell\ \mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1-\frac{\wp_2}{c})\Big)|^q d\ell\right\}^{1/q}\right.\\
&\left.+\left\{\int_{1/2}^1|\mathsf{F}'\Big(\frac{\wp_2}{c}+\ell\ \mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1-\frac{\wp_2}{c})\Big)|^q d\ell\right\}^{1/q}\right]\\
&\leq\frac{\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-c\wp_1)}{c}\left[\left(\frac{1}{2}\right)^{1-1/q}\left\{\int_0^1\ell[1-s(1-\ell)]|\mathsf{F}'(\wp_1)|^q d\ell+\int_0^1\ell[1-s\ell]|\mathsf{F}'\Big(\frac{\wp_2}{c}\Big)|^q d\ell\right\}^{1/q}\right.\\
&\left.+\left\{\int_{1/2}^1[1-s(1-\ell)]|\mathsf{F}'(\wp_1)|^q d\ell+\int_{1/2}^1[1-s\ell]|\mathsf{F}'\Big(\frac{\wp_2}{c}\Big)|^q d\ell\right\}^{1/q}\right]\\
&=\frac{\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-c\wp_1)}{c}\left[\left(\frac{1}{2}\right)^{1-1/q}\left\{\frac{3-s}{6}|\mathsf{F}'(\wp_1)|^q+\frac{3-2s}{6}|\mathsf{F}'(\frac{\wp_2}{c})|^q\right\}^{1/q}\right.\\
&\left.+\left\{\frac{4-3s}{8}[|\mathsf{F}'(\wp_1)|^q+|\mathsf{F}'(\frac{\wp_2}{c})|^q]\right\}^{1/q}\right].
\end{aligned}
$$

□

Corollary 23. *If we choose $s=1$, then we attain the following inequality:*

$$
\begin{aligned}
&\left|\frac{c}{\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-c\wp_1)}\int_{\wp_1}^{\wp_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\frac{\wp_2}{c}-\wp_1)}\mathsf{F}(x)dx-\mathsf{F}\left(\frac{2\wp_1+\ \mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-\wp_1)}{2c}\right)\right|\\
&\leq\frac{\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_2-c\wp_1)}{c}\left[\left(\frac{1}{2}\right)^{1-1/q}\left\{\frac{1}{3}|\mathsf{F}'(\wp_1)|^q+\frac{1}{6}|\mathsf{F}'(\frac{\wp_2}{c})|^q\right\}^{1/q}\right.\\
&\left.+\left\{\frac{1}{8}[|\mathsf{F}'(\wp_1)|^q+|\mathsf{F}'(\frac{\wp_2}{c})|^q]\right\}^{1/q}\right].
\end{aligned}
$$

Corollary 24. *If we choose $\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1-\wp_2)=\wp_1-\wp_2$, then we attain the following inequality:*

$$
\begin{aligned}
&\left|\frac{c}{(\wp_2-c\wp_1)}\int_{\wp_1}^{\frac{\wp_2}{c}}\mathsf{F}(x)dx-\mathsf{F}\left(\frac{\wp_1+\wp_2)}{2c}\right)\right|\\
&\leq\frac{(\wp_2-c\wp_1)}{c}\left[\left(\frac{1}{2}\right)^{1-1/q}\left\{\frac{3-s}{6}|\mathsf{F}'(\wp_1)|^q+\frac{3-2s}{6}|\mathsf{F}'(\frac{\wp_2}{c})|^q\right\}^{1/q}\right.\\
&\left.+\left\{\frac{4-3s}{8}[|\mathsf{F}'(\wp_1)|^q+|\mathsf{F}'(\frac{\wp_2}{c})|^q]\right\}^{1/q}\right].
\end{aligned}
$$

Corollary 25. *If we choose $s=1$ and $\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1-\wp_2)=\wp_1-\wp_2$, then we attain the following inequality:*

$$
\begin{aligned}
&\left|\frac{c}{(\wp_2-c\wp_1)}\int_{\wp_1}^{\frac{\wp_2}{c}}\mathsf{F}(x)dx-\mathsf{F}\left(\frac{\wp_1+\wp_2)}{2c}\right)\right|\\
&\leq\frac{(\wp_2-c\wp_1)}{c}\left[\left(\frac{1}{2}\right)^{1-1/q}\left\{\frac{1}{3}|\mathsf{F}'(\wp_1)|^q+\frac{1}{6}|\mathsf{F}'(\frac{\wp_2}{c})|^q\right\}^{1/q}+\left\{\frac{1}{8}[|\mathsf{F}'(\wp_1)|^q+|\mathsf{F}'(\frac{\wp_2}{c})|^q]\right\}^{1/q}\right].
\end{aligned}
$$

Remark 14. *Under the assumption of Theorem 16, if we take* $\sigma = (1, 1, \ldots)$ *with* $\rho = \alpha,\ \lambda = 1$, *we get the following inequality involving classical Mittag–Leffler function:*

$$\left| \frac{c}{E_\alpha(\wp_2 - c\wp_1)} \int_{\wp_1}^{\wp_1 + E_\alpha(\frac{\wp_2}{c} - \wp_1)} \mathsf{F}(x)dx - \mathsf{F}\left(\frac{2\wp_1 + E_\alpha(\wp_2 - \wp_1)}{2c} \right) \right|$$
$$\leq \frac{E_\alpha(\wp_2 - c\wp_1)}{c} \left[\left(\frac{1}{2} \right)^{1-1/q} \left\{ \frac{3-s}{6} |\mathsf{F}'(\wp_1)|^q + \frac{3-2s}{6} |\mathsf{F}'(\frac{\wp_2}{c})|^q \right\}^{1/q} \right.$$
$$\left. + \left\{ \frac{4-3s}{8} [|\mathsf{F}'(\wp_1)|^q + |\mathsf{F}'(\frac{\wp_2}{c})|^q] \right\}^{1/q} \right].$$

6. Applications to Special Means

In this section, we recall the following special means of two positive numbers $\wp_1, \wp_2$ with $\wp_1 < \wp_2$:

(1) The arithmetic mean

$$A = A(\wp_1, \wp_2) = \frac{\wp_1 + \wp_2}{2}.$$

(2) The geometric mean

$$G = G(\wp_1, \wp_2) = \sqrt{\wp_1 \wp_2}.$$

(3) The harmonic mean

$$H = H(\wp_1, \wp_2) = \frac{2\wp_1\wp_2}{\wp_1 + \wp_2}.$$

The following relationship is well-known in the literature.

$$H(\wp_1, \wp_2) \leq G(\wp_1, \wp_2) \leq A(\wp_1, \wp_2).$$

Proposition 3. *Let* $0 < \wp_1 < \wp_2$ *and* $s \in [0, 1]$. *then*

$$\frac{1}{2-s}(\wp_2 + \frac{1}{2}\, \mathcal{F}^\sigma_{\rho,\lambda}(\wp_1 - \wp_2)) \leq \frac{\mathcal{F}^\sigma_{\rho,\lambda}(\wp_1 - \wp_2) + 2\wp_2)}{2} \leq A(\wp_1, \wp_2)(2-s). \tag{21}$$

Proof. We attain the above inequality (21), if we put $\mathsf{F}(\nu) = \nu$ for $\nu > 0$ in Theorem 8. □

Proposition 4. *Let* $\wp_1, \wp_2 \in (0, 1]$ *with* $\wp_1 < \wp_2$ *and* $s \in [0, 1]$, *then*

$$\frac{1}{2-s} \ln G(\wp_1, \wp_2) \leq \frac{1}{\wp_2(\wp_2 + \mathcal{F}^\sigma_{\rho,\lambda}(\wp_1 - \wp_2))} \leq \ln(\wp_2 + \frac{1}{2}\, \mathcal{F}^\sigma_{\rho,\lambda}(\wp_1 - \wp_2)) \frac{2-s}{2}. \tag{22}$$

Proof. We attain the above inequality (22), if we put $\mathsf{F}(\nu) = -\ln \nu$ for $\nu \in (0, 1]$ in Theorem 8. □

Remark 15. *Under the assumption of Proposition 4, if we take* $\sigma = (1, 1, \ldots)$ *with* $\rho = \alpha,\ \lambda = 1$, *we get the following inequality involving classical Mittag–Leffler function:*

$$\frac{1}{2-s} \ln G(\wp_1, \wp_2) \leq \frac{1}{\wp_2(\wp_2 + E_\alpha(\wp_1 - \wp_2))} \leq \ln(\wp_2 + \frac{1}{2} E_\alpha(\wp_1 - \wp_2)) \frac{2-s}{2}. \tag{23}$$

Proposition 5. *Let* $\wp_1, \wp_2 \in (0, \infty)$ *with* $\wp_1 < \wp_2$ *and* $s \in [0, 1]$, *then*

$$\frac{1}{(2-s)(\wp_2 + \frac{1}{2}\,\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1 - \wp_2))} \leq \frac{\ln\left(1 + \frac{\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1 - \wp_2)}{\wp_2}\right)}{\mathcal{F}^{\sigma}_{\rho,\lambda}(\wp_1 - \wp_2)} \leq \frac{2-s}{H(\wp_1, \wp_2)}. \tag{24}$$

Proof. We attain the above inequality (24), if we put $\mathsf{F}(\nu) = \frac{1}{\nu}$ for $\nu > 0$ in Theorem 8. □

Remark 16. *Under the assumption of Proposition 5, if we take* $\sigma = (1, 1, \ldots)$ *with* $\rho = \alpha,\ \lambda = 1$, *we get the following inequality involving classical Mittag–Leffler function:*

$$\frac{1}{(2-s)(\wp_2 + \frac{1}{2}E_{\alpha}(\wp_1 - \wp_2))} \leq \frac{\ln\left(1 + \frac{E_{\alpha}(\wp_1 - \wp_2)}{\wp_2}\right)}{E_{\alpha}(\wp_1 - \wp_2)} \leq \frac{2-s}{H(\wp_1, \wp_2)}. \tag{25}$$

7. Conclusions

In this article, we addressed a novel idea for the generalized preinvex function, namely the *s*-type preinvex function. Some algebraic properties were examined concerning the proposed definition. In the manner of the newly proposed definition, we described the novel version of Hermite–Hadamard type inequality. Further, we made two new lemmas. Our attained results in the order of new lemmas can be considered as refinements and remarkable extensions to the new family of preinvex functions. Our novel results can be deduced from the previously known results. Applications to special means were considered. In addition we made some comments; the above estimations on the mentioned lemmas need an interesting and amazing comparison. On Lemma 2, we examined three Theorems 10–12, in which we used the Hölder and Hölder-İ scan inequality. In comparison, Theorem 12 gives a better result as compared to the other Theorems 10 and 11. Similarly, On Lemma 2, we examined two Theorems 13 and 14, in which we used power mean and improved power mean inequality. In comparison, Theorem 13 gives a better result as compared to the other Theorem 14. We hope the consequences and techniques of this article will energize and inspire researchers to explore a more interesting sequel in this area.

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