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The Solutions of Some Riemann–Liouville Fractional Integral Equations

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Abstract: In this paper, we propose the solutions of nonhomogeneous fractional integral equations of the form $I^{3\sigma}_{0+}y(t) + a \cdot I^{2\sigma}_{0+}y(t) + b \cdot I^{\sigma}_{0+}y(t) + c \cdot y(t) = f(t)$, where I^{σ}_{0+} is the Riemann–Liouville fractional integral of order $\sigma = 1/3, 1, f(t) = t^n, t^n e^t, n \in \mathbb{N} \cup \{0\}, t \in \mathbb{R}^+$, and a, b, c are constants, by using the Laplace transform technique. We obtain solutions in the form of Mellin–Ross function and of exponential function. To illustrate our findings, some examples are exhibited.

Keywords: Laplace transform; fractional differential equations; fractional integral equations; Riemann–Liouville fractional integral

MSC: 26A33; 34A08; 46F10; 46A12

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1. Introduction

Fractional calculus is the theory of derivatives and integrals of arbitrary complex or real order. It began in 1695 when G.F.A. L'Hôpital asked G.W. Leibniz to give the meaning of $d^n y/dx^n$, where $n = 1/2$. In his predictive answer, G.W. Leibniz expected the beginning of the area that is presently named fractional calculus. Since that time, fractional calculus interested many mathematicians such as L. Euler, H. Laurent, P.S. Laplace, J.B.J. Fourier, N.H. Abel, J. Liouville, and G.F.B. Riemann, etc. Moreover, it is shown to be very useful and active in various mathematical areas.

Fractional derivatives are a part of fractional calculus that plays a key role in modeling real-world phenomena within different branches of engineering and science, see [1–22] for more details. With the help of fractional calculus, many mathematical models of real problems were produced in various fields of engineering and science, such as dielectric polarization, viscoelastic, electromagnetic waves, and electrode-electrolyte polarization, which can be found in [23–30].

In addition, in these years, the theory of fractional integral attracted many researchers, see [31–38]. In 1812, P.S. Laplace defined a fractional derivative through an integral and developed it as a mere mathematical exercise, generalizing from the case of integer order. Later, in 1832, J. Liouville recommended a definition based on the formula for differentiating the exponential function known as the first Liouville definition. Next, he presented the second definition formula in terms of an integral, called Liouville integral, to integrate, with respect, a noninteger order. After that, J. Liouville and G.F.B. Riemann developed an approach to noninteger order derivatives in terms of convergent series, conversely to the Riemann–Liouville approach, that was given as an integral. Many researchers focused on developing the theoretical aspects, methods of solution, and applications of fractional integral equations, see [37–45].

In 2005, T. Morita [6] studied the initial value problem of fractional differential equations by using the Laplace transform. As a result, he obtained the solutions to the fractional differential equations with Riemann–Liouville fractional derivative and Caputo fractional derivative or its modification. In 2010, T. Morita and K. Sato [8] studied the initial value problem of fractional differential equations with constant coefficients. Moreover, T. Morita and K. Sato obtained solutions in terms of Green's function and distribution theory.

In 1996, A.A. Kilbas and M. Saigo [39] discovered the connections of the Mittag–Leffler type function with the Riemann–Liouville fractional integrals and derivatives and solved the linear Abel–Volterra integral equations as applications.

In 2015, R. Agarwal et al. [40] studied the solutions of fractional Volterra integral equation with Caputo fractional derivative using the integral transform of pathway type. More interestingly, they discussed the solution of the nonhomogeneous time-fractional heat equation in a spherical domain.

In 2017, C. Li et al. [41] studied a generalized Abel's integral equation and its variant in the distributional (Schwartz) sense based on fractional calculus of distributions. Particularly, in 2018, C. Li and K. Clarkson [42] studied Abel's integral equation of the second kind of the form

$$y(t) + \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-\lambda)^{\alpha-1} y(\tau) d\tau = f(t), \quad t > 0, \tag{1}$$

where Γ is the gamma function, λ is a constant, and $\alpha \in \mathbb{R}$. Equation (1) can be written in the form

$$(1 + \lambda I_{0+}^{\alpha}) y(t) = f(t),$$

where I_{0+}^{α} is the Riemann–Liouville fractional integral. Finally, C. Li and K. Clarkson applied Babenko's method and fractional integral for solving the equation above.

In [35], S.G. Samko et al. considered the linear fractional order integral equations with constant coefficients of the form

$$c_1 I_{a^+}^{\alpha_1} y(t) + c_2 I_{a^+}^{\alpha_2} y(t) + \cdots + c_n I_{a^+}^{\alpha_n} y(t) = f(t), \tag{2}$$

where $a \in \mathbb{R}, \alpha_i \in \mathbb{R}, \alpha_1 > \alpha_2 > \cdots > \alpha_n \geq 0, c_i \in \mathbb{C}$, for $i \in \{1, 2, \ldots, n\}$, and f is assumed to be a real valued function of real variable defined on an interval (a, b). The general solution of (2) can be found in the space S'_+ of tempered distributions with support in $[0, \infty)$.

In 2017, D.C. Labora and R. Rodriguez–Lopez [44] showed a new method by applying a suitable fractional integral operator for solving some fractional order integral equations with constant coefficients, and all the integration orders involved are rational. Next, they applied and extended ideas presented in [44] for solving fractional integral equations with Riemann–Liouville definition; see [38] for more details.

In 2021, K. Karuna et al. [46] studied the solutions of nonhomogeneous fractional integral equations of the form

$$I_{0^+}^{2\sigma} y(t) + a \cdot I_{0^+}^{\sigma} y(t) + b \cdot y(t) = f(t),$$

where $I_{0^+}^{\sigma}$ is the Riemann–Liouville fractional integral of order $\sigma = 1/2, 1, f(t) = t^n, t^n e^t$, $n \in \mathbb{N} \cup \{0\}, t \in \mathbb{R}^+$, and a, b are constants, by using the Laplace transform technique. They obtained solutions in the form of Mellin–Ross functions and of exponential functions.

In this paper, we provide the solution of nonhomogeneous fractional integral equations of the form

$$I_{0^+}^{3\sigma} y(t) + a \cdot I_{0^+}^{2\sigma} y(t) + b \cdot I_{0^+}^{\sigma} y(t) + c \cdot y(t) = f(t),$$

where $I_{0^+}^{\sigma}$ is the Riemann–Liouville fractional integral of order $\sigma = 1/3, 1, f(t) = t^n, t^n e^t$, $n \in \mathbb{N} \cup \{0\}, t \in \mathbb{R}^+$, and a, b are constants by using the Laplace transform technique and its variants in the classical sense.

Section 2 introduces definitions of the Riemann–Liouville fractional integral and the Laplace transform, which helps us obtain our main results. In Section 3, we establish our main results and some examples due to our main results. Finally, we give the conclusions in Section 4.

2. Preliminaries

Before we proceed to the main results, the following definitions, lemmas, and concepts are required.

Definition 1 ([30])**.** *Let α be a constant, v a real number and t a positive real number. The Mellin–Ross function $E_t(v,\alpha)$ is defined by*

$$E_t(v,\alpha) = t^v e^{\alpha t}\Gamma^*(v,\alpha t),$$

where Γ^* is the incomplete gamma function, which is defined by

$$\Gamma^*(v,t) = e^{-t}\sum_{k=0}^{\infty}\frac{t^k}{\Gamma(v+k+1)},$$

and Γ is the gamma function.

In addition, if $v > 0$, then $E_t(v,\alpha)$ has an integral representation as

$$E_t(v,\alpha) = \frac{1}{\Gamma(v)}\int_0^t x^{v-1}e^{\alpha(t-x)}dx.$$

Example 1. *Let α be a constant, v a real number, and t a positive real number. We recall some special values and recursion relations of the Mellin–Ross function that are needed for our calculations:*

(i) $E_t(0,\alpha) = e^{\alpha t}$;

(ii) $E_t(v,0) = \dfrac{t^v}{\Gamma(v+1)}$;

(iii) $E_t(1,\alpha) = \dfrac{E_t(0,\alpha)-1}{\alpha}, \alpha \neq 0$;

(iv) $E_t(v,\alpha) = \alpha E_t(v+1,\alpha) + \dfrac{t^v}{\Gamma(v+1)}$.

Definition 2 ([30])**.** *Let $f(t)$ be piece-wise continuous on $(0,\infty)$ and integrable on any finite subinterval of $[0,\infty)$. Then, the Riemann–Liouville fractional integral of $f(t)$ of order v is defined by*

$$I_{0^+}^v f(t) = \frac{1}{\Gamma(v)}\int_0^t (t-x)^{v-1}f(x)dx,$$

where $v \in \mathbb{R}^+$.

Example 2. *Let α be a constant, μ a real number, v and t positive real numbers. Then, the following Riemann–Liouville fractional integrals hold:*

(i) $I_{0^+}^v t^\mu = \dfrac{\Gamma(\mu+1)}{\Gamma(\mu+v+1)}t^{\mu+v}, \quad \mu > -1$;

(ii) $I_{0^+}^v e^{\alpha t} = E_t(v,\alpha)$;

(iii) $I_{0^+}^v \left[te^{\alpha t}\right] = tE_t(v,\alpha) - vE_t(v+1,\alpha)$;

(iv) $I_{0^+}^v [E_t(\mu,\alpha)] = E_t(\mu+v,\alpha), \quad \mu > -1$;

(v) $I_{0^+}^v [tE_t(\mu,\alpha)] = tE_t(\mu+v,\alpha) - vE_t(\mu+v+1,\alpha), \quad \mu > -2$.

Definition 3 ([30])**.** *Let $f(t)$ be piece-wise continuous on $(0,\infty)$ and integrable on any finite subinterval of $[0,\infty)$. Let n be the smallest integer that exceeds β. Then, Riemann–Liouville fractional derivative ${}_0D_t^{\beta} f(t)$ is defined by*

$$ {}_0D_t^{\beta} f(t) = \frac{1}{\Gamma(n-\beta)} \frac{d^n}{dt^n} \int_0^t (t-x)^{n-\beta-1} f(x) dx, $$

where $\beta \in \mathbb{R}^+$ and $n-\beta > 0$.

Definition 4 ([30])**.** *Let $f(t)$ be a function satisfying the conditions in Definition 2 and of exponential order v where $v \in \mathbb{R}^+$. The Laplace transform of $f(t)$ is defined by*

$$ F(s) = \mathscr{L}\{f(t)\} = \int_0^{\infty} f(t) e^{-st} dt, $$

where $Re(s) > v$.

Example 3. *Let α be a constant, n a real number, v and t positive real numbers. Then, the following Laplace transforms hold:*

(i) $\mathscr{L}\{1\} = \frac{1}{s}, \quad s > 0;$

(ii) $\mathscr{L}\{t^n\} = \frac{\Gamma(n+1)}{s^{n+1}}, \quad s > 0, n > -1;$

(iii) $\mathscr{L}\{e^{\alpha t}\} = \frac{1}{s-\alpha}, \quad s > \alpha;$

(iv) $\mathscr{L}\{t^n e^{\alpha t}\} = \frac{\Gamma(n+1)}{(s-\alpha)^{n+1}}, \quad s > \alpha, n > 0;$

(v) $\mathscr{L}\{E_t(v,\alpha)\} = \frac{1}{s^v(s-\alpha)}, \quad s > \alpha.$

Lemma 1 ([30])**.** *Let $f(t)$ be a function satisfying the conditions in Definition 2 and of exponential order v where $v \in \mathbb{R}^+$. Then*

$$ \mathscr{L}\left[I_{0+}^{v} f(t)\right] = s^{-v} \mathscr{L}[f(t)]. $$

Definition 5. *Let $f(t)$ be a function satisfying the conditions in Definition 4 and $\mathscr{L}\{f(t)\} = F(s)$. The inverse Laplace transform of $F(s)$ is defined by*

$$ f(t) = \mathscr{L}^{-1}\{F(s)\} = \frac{1}{2\pi i} \lim_{\omega\to\infty} \int_{c-i\omega}^{c+i\omega} F(s) e^{st} ds, $$

where $Re(s) > \sigma_a, \sigma_a$ is an abscissa of absolute convergence for $\mathscr{L}\{f(t)\}$.

Example 4. *Let α be a constant, v a real number, n and t positive real numbers. Then, the following inverse Laplace transforms hold:*

(i) $\mathscr{L}^{-1}\left\{\frac{1}{s^{v+1}}\right\} = \frac{t^v}{\Gamma(v+1)}, \quad v > 0;$

(ii) $\mathscr{L}^{-1}\left\{\frac{1}{s-\alpha}\right\} = E_t(0,\alpha) = e^{\alpha t};$

(iii) $\mathscr{L}^{-1}\left\{\frac{\Gamma(n+1)}{(s-\alpha)^{n+1}}\right\} = t^n e^{\alpha t};$

(iv) $\mathscr{L}^{-1}\left\{\frac{1}{s^{1/3}-\alpha}\right\} = E_t\left(-\frac{2}{3},\alpha^3\right) + \alpha E_t\left(-\frac{1}{3},\alpha^3\right) + \alpha^2 E_t\left(0,\alpha^3\right);$

(v) $\mathscr{L}^{-1}\left\{\frac{1}{s^v(s-\alpha)^3}\right\} = \frac{1}{2} t^2 E_t(v,\alpha) - vt E_t(v+1,\alpha) + \frac{1}{2} v(v+1) E_t(v+2,\alpha), \quad v > -3.$

We turn now to the problem of founded inverse transforms of slightly more complicated functions in Example 4 (iv) and (v) are particular cases of the following Lemmas.

Lemma 2 ([30]). *Let n be a positive integer, α be a constant, v be a real number, and t be a positive real number, then*

$$\mathscr{L}^{-1}\left\{\frac{1}{s^{v}(s-\alpha)^{n}}\right\} = \frac{1}{(n-1)!\,\Gamma(v)}\sum_{i=0}^{n-1}(-1)^{i}\binom{n-1}{i}\Gamma(v+i)t^{n-1-i}E_{t}(v+i,\alpha),$$

where $v > -n$.

Lemma 3 ([30]). *Let n and q be positive integers, α be a constant, u be a real number, and t be a positive real number, then*

$$\mathscr{L}^{-1}\left\{\frac{1}{s^{u}(s^{v}-\alpha)^{n}}\right\} = \sum_{r_1=1}^{q}\cdots\sum_{r_n=1}^{q}\frac{\alpha^{m-n}}{(n-1)!\,\Gamma(u-n+mv)}\sum_{k=0}^{n-1}(-1)^{k}\binom{n-1}{k}$$
$$\times\,\Gamma(u-n+mv+k)t^{n-1-k}E_{t}(u-n+mv+k,\alpha^{q}),$$

where $u > -n, v = \frac{1}{q}$ *and* $m = \sum_{i=1}^{n} r_i$

3. Main Results

In this section, we state our main results and give their proofs.

Theorem 1. *Consider that the nonhomogeneous fractional integral equation is given by*

$$I_{0^+}^{3\sigma}y(t) + a\cdot I_{0^+}^{2\sigma}y(t) + b\cdot I_{0^+}^{\sigma}y(t) + c\cdot y(t) = t^{n}, \tag{3}$$

where $I_{0^+}^{\sigma}$ *is the Riemann–Liouville fractional integral of order* $\sigma = 1/3, \sigma = 1, n \in \mathbb{N}\cup\{0\}$, $t \in \mathbb{R}^{+}$, *and* a, b, c *are constants. Then, the solution of* (3) *as the following:*

(i) *If* $\sigma = 1/3$, *and* $j,k,l \in \mathbb{R}\setminus\{0\}$ *with* j,k,l *are different such that* $a = j+k+l$, $b = jk+jl+kl$, *and* $c = jkl$ *then*

$$\begin{aligned} y(t) = &\sum_{i=0}^{3n}(-1)^{n-i}n!\left[\frac{j^{3n-i+2}}{(j-k)(j-l)} - \frac{k^{3n-i+2}}{(j-k)(k-l)} + \frac{l^{3n-i+2}}{(j-l)(k-l)}\right]\frac{t^{(i-3)/3}}{\Gamma(i/3)} \\ &+ \frac{(-1)^{n}n!\,j^{3n+1}}{(j-k)(j-l)}\left[E_t\left(-\frac{2}{3},-\frac{1}{j^3}\right) - \frac{1}{j}E_t\left(-\frac{1}{3},-\frac{1}{j^3}\right) + \frac{1}{j^2}E_t\left(0,-\frac{1}{j^3}\right)\right] \\ &+ \frac{(-1)^{n+1}n!\,k^{3n+1}}{(j-k)(k-l)}\left[E_t\left(-\frac{2}{3},-\frac{1}{k^3}\right) - \frac{1}{k}E_t\left(-\frac{1}{3},-\frac{1}{k^3}\right) + \frac{1}{k^2}E_t\left(0,-\frac{1}{k^3}\right)\right] \\ &+ \frac{(-1)^{n+2}n!\,l^{3n+1}}{(j-l)(k-l)}\left[E_t\left(-\frac{2}{3},-\frac{1}{l^3}\right) - \frac{1}{l}E_t\left(-\frac{1}{3},-\frac{1}{l^3}\right) + \frac{1}{l^2}E_t\left(0,-\frac{1}{l^3}\right)\right]. \end{aligned} \tag{4}$$

as the solution to (3).

(ii) *If* $\sigma = 1$, *and* $j,k,l \in \mathbb{R}\setminus\{0\}$ *with* j,k,l *are different such that* $a = j+k+l$, $b = jk+jl+kl$ *and* $c = jkl$ *then*

$$\begin{aligned} y(t) = &\sum_{i=0}^{n}(-1)^{n-i}n!\left[\frac{j^{n-i}(k-l) - k^{n-i}(j-l) + l^{n-i}(j-k)}{(j-k)(j-l)(k-l)}\right]\frac{t^{i-1}}{\Gamma(i)} \\ &+ \frac{(-1)^{n}n!\,j^{n-1}e^{-t/j}}{(j-k)(j-l)} + \frac{(-1)^{n+1}n!\,k^{n-1}e^{-t/k}}{(j-k)(k-l)} + \frac{(-1)^{n+2}n!\,l^{n-1}e^{-t/l}}{(j-l)(k-l)}. \end{aligned} \tag{5}$$

as the solution to (3).

Proof. Applying the Laplace transform to both sides of (3), we have

$$\mathscr{L}\{I_{0^+}^{3\sigma}y(t)\} + a\cdot\mathscr{L}\{I_{0^+}^{2\sigma}y(t)\} + b\cdot\mathscr{L}\{I_{0^+}^{\sigma}y(t)\} + c\cdot\mathscr{L}\{y(t)\} = \mathscr{L}\{t^n\}. \tag{6}$$

Using Lemma 1, Example 3 (ii) on (6) and denoting the Laplace transform $\mathscr{L}\{y(t)\} = Y(s)$, we obtain

$$Y(s) = \frac{n!\,s^{3\sigma}}{s^{n+1}(cs^{3\sigma} + bs^{2\sigma} + as^{\sigma} + 1)}. \tag{7}$$

For $\sigma = 1/3$, Equation (7) becomes

$$Y(s) = \frac{n!}{s^n\left(cs + bs^{2/3} + as^{1/3} + 1\right)},$$

and turns into

$$Y(s) = \frac{n!}{u^{3n}(cu^3 + bu^2 + au + 1)},$$

with a substitution of $u = s^{1/3}$. Using partial fractions with explicit values of a, b, c, we can rewrite it as

$$\begin{aligned} Y(s) = \sum_{i=1}^{3n}(-1)^{n-i}n!&\left[\frac{j^{3n-i+2}}{(j-k)(j-l)} - \frac{k^{3n-i+2}}{(j-k)(k-l)} + \frac{l^{3n-i+2}}{(j-l)(k-l)}\right]\frac{1}{u^i} \\ &+ \frac{(-1)^n n!\,j^{3n+1}}{(j-k)(j-l)}\left(\frac{1}{u+1/j}\right) + \frac{(-1)^{n+1}n!\,k^{3n+1}}{(j-k)(k-l)}\left(\frac{1}{u+1/k}\right) \\ &+ \frac{(-1)^{n+2}n!\,l^{3n+1}}{(j-l)(k-l)}\left(\frac{1}{u+1/l}\right). \end{aligned} \tag{8}$$

Finally, resubstituting $u = s^{1/3}$ and taking the inverse Laplace transform to (8) with the help of Example 4 (i), (iv), we obtain (4) as the solution to (3).

For $\sigma = 1$, Equation (7) becomes

$$Y(s) = \frac{n!}{s^{n-2}(cs^3 + bs^2 + as + 1)},$$

Using partial fractions with explicit values of a, b, c, we can rewrite the above equation as

$$\begin{aligned} Y(s) = \sum_{i=1}^{n}(-1)^{n-i}n!&\left[\frac{j^{n-i}(k-l) - k^{n-i}(j-l) + l^{n-i}(j-k)}{(j-k)(j-l)(k-l)}\right]\left(\frac{1}{s^i}\right) \\ &+ \frac{(-1)^n n!\,j^{n-1}}{(j-k)(j-l)}\left(\frac{1}{\left(s+\frac{1}{j}\right)}\right) + \frac{(-1)^{n+1}n!\,k^{n-1}}{(j-k)(k-l)}\left(\frac{1}{\left(s+\frac{1}{k}\right)}\right) \\ &+ \frac{(-1)^{n+2}n!\,l^{n-1}}{(j-l)(k-l)}\left(\frac{1}{\left(s+\frac{1}{l}\right)}\right). \end{aligned} \tag{9}$$

Applying the inverse Laplace transform to (9) and using Example 4 (i) and (ii) yields (5) as the solution to (3). To include the case $n = 0$ into the solution formulas of both cases, we adopt the notation $1/\Gamma(0) = 0$. The proof is completed. □

Remark 1. *Let $n \in \mathbb{N} \cup \{0\}$, and a, b, c satisfy condition in Theorem 1, then (5) is a solution of*

$$c\cdot y'''(t) + b\cdot y''(t) + a\cdot y'(t) + y(t) = n(n-1)(n-2)t^{n-3},$$

see [44] for more details.

Example 5. *Letting $a = \frac{1}{2}, b = -1, c = -\frac{1}{2}$ and $\sigma = 1/3$ then (3) changes to*

$$I_{0^+}y(t) + \frac{1}{2} \cdot I_{0^+}^{2/3}y(t) - I_{0^+}^{1/3}y(t) - \frac{1}{2} \cdot y(t) = t^n. \tag{10}$$

From Theorem 1, Equation (10) has solution

$$\begin{aligned} y(t) = &\sum_{i=0}^{3n}(-1)^{n-i+1}n!\left[\frac{(1/2)^{3n-i} - 3 + (-1)^{3n-i+1}}{3}\right]\frac{t^{(i-3)/3}}{\Gamma(i)} \\ &+ \frac{(-1)^{n+1}n!\,(1/2)^{3n-1}}{3}\left[E_t\left(-\frac{2}{3}, -8\right) - 2E_t\left(-\frac{1}{3}, -8\right) + 4E_t(0, -8)\right] \\ &+ (-1)^{n+2}n!\left[E_t\left(-\frac{2}{3}, -1\right) - E_t\left(-\frac{1}{3}, -1\right) + E_t(0, -1)\right] \\ &+ \frac{(-1)^{4n+1}n!}{3}\left[E_t\left(-\frac{2}{3}, 1\right) + E_t\left(-\frac{1}{3}, 1\right) + E_t(0, 1)\right]. \end{aligned} \tag{11}$$

By applying Example 2 (i), (ii), and (iv), it is not difficult to verify that (11) satisfies (10).

Moreover, if $n = 1$, then Equation (10) becomes

$$I_{0^+}y(t) + \frac{1}{2} \cdot I_{0^+}^{2/3}y(t) - I_{0^+}^{1/3}y(t) - \frac{1}{2} \cdot y(t) = t. \tag{12}$$

From (11), it follows that (12) has solution

$$\begin{aligned} y(t) = &\frac{5t^{-2/3}}{4\,\Gamma(1/3)} + 1 - \frac{t^{-1/3}}{2\,\Gamma(2/3)} + \frac{1}{12}\left[E_t\left(-\frac{2}{3}, -8\right) - 2E_t\left(-\frac{1}{3}, -8\right) + 4E_t(0, -8)\right] \\ &- \left[E_t\left(-\frac{2}{3}, -1\right) - E_t\left(-\frac{1}{3}, -1\right) + E_t(0, -1)\right] \\ &- \frac{1}{3}\left[E_t\left(-\frac{2}{3}, 1\right) + E_t\left(-\frac{1}{3}, 1\right) + E_t(0, 1).\right] \end{aligned} \tag{13}$$

It is not difficult to verify that (13) satisfies (12).

Example 6. *Letting $a = \frac{1}{2}, b = -1, c = -\frac{1}{2}$ and $\sigma = 1$ then (3) changes to*

$$I_{0^+}^3y(t) + \frac{1}{2} \cdot I_{0^+}^2y(t) - I_{0^+}^1y(t) - \frac{1}{2} \cdot y(t) = t^n. \tag{14}$$

From Theorem 1, Equation (14) has solution

$$\begin{aligned} y(t) \quad &= \textstyle\sum_{i=0}^{n}(-1)^{n-i+1}n!\left[\frac{(1/2)^{n-i-2} - 3 + (-1)^{n-i+1}}{3}\right]\frac{t^{i-1}}{\Gamma(i)} + \frac{(-1)^{n+1}n!(1/2)^{n-3}e^{-2t}}{3} \\ &\qquad + (-1)^{n+2}n!e^{-t} + \frac{(-1)^{2n+1}n!e^t}{3}. \end{aligned} \tag{15}$$

By applying Example 2 (ii), it is not difficult to verify that (15) satisfies (14).

Moreover, if $n = 3$, then Equation (14) becomes

$$I_{0^+}^3y(t) + \frac{1}{2} \cdot I_{0^+}^2y(t) - I_{0^+}^1y(t) - \frac{1}{2} \cdot y(t) = t^3. \tag{16}$$

From (15), it follows that (16) has solution

$$y(t) = 6 + 2e^{-2t} - 6e^{-t} - 2e^t. \tag{17}$$

It is not difficult to verify that (17) satisfies (16).

According to Remark 1, function (17) is solution of $\frac{1}{2}y'''(t) + y''(t) - \frac{1}{2}y'(t) - y(t) = -6$.

Theorem 2. *Consider the nonhomogeneous fractional integral equation is given by*

$$I_{0^+}^{3\sigma} y(t) + a \cdot I_{0^+}^{2\sigma} y(t) + b \cdot I_{0^+}^{\sigma} y(t) + c \cdot y(t) = t^n e^t, \tag{18}$$

where $I_{0^+}^{\sigma}$ is the Riemann–Liouville fractional integral of order $\sigma = 1/3, \sigma = 1, n \in \mathbb{N} \cup \{-1, 0\}$, $t \in \mathbb{R}^+$, and a, b, c are constants. Then:

(i) *If $\sigma = 1/3$, and $j, k, l \in \mathbb{R} \setminus \{-1, 0\}$ with j, k, l are different such that $a = j + k + l$, $b = jk + jl + kl$, and $c = jkl$ then*

$$\begin{aligned} y(t) \quad = {} & n! \textstyle\sum_{i=0}^{n} (-1)^{n+1} \Big[\frac{j^{3n+1-2i}}{(j^2-j+1)^{n-i+1}(j+1)^{n+1}(j-k)(j-l)} \\ & - \frac{k^{3n+1-2i}}{(k^2-k+1)^{n-i+1}(k+1)^{n+1}(j-k)(k-l)} + \frac{l^{3n+1-2i}}{(l^2-l+1)^{n-i+1}(l+1)^{n+1}(j-k)(k-l)} \\ & - \textstyle\sum_{r=0}^{n} \frac{\Gamma(n+r-i+1)}{3^{n+r-i+1}\Gamma(n+r-i)} \Big[\frac{1}{jkl} + \sum_{m=0}^{r} \frac{1}{\Gamma(m+1)} \Big[\frac{j^{n-r-1}}{(j+1)^{n-r+1}(j-k)(j-l)} \\ & - \frac{k^{n-r-1}}{(k+1)^{n-r+1}(j-k)(k-l)} + \frac{l^{n-r-1}}{(l+1)^{n-r+1}(j-k)(k-l)} \Big] \Big] \Big] \textstyle\sum_{p=0}^{i+1} (-1)^p \binom{i+1}{p} \\ & \times \frac{1}{i!\Gamma(-(i-p+2)/3)} \textstyle\sum_{q=0}^{i} (-1)^q \binom{i}{q} \Gamma(-(i-p+2)/3+q) t^{i-q} E_t \Big(-\frac{i-p+2}{3} + q, 1 \Big) \\ & + n! \textstyle\sum_{i=0}^{n} (-1)^{n+1} \Big[\frac{j^{3n-2i}(j-1)}{(j^2-j+1)^{n-i+1}(j+1)^{n+1}(j-k)(j-l)} \\ & - \frac{k^{3n-2i}(k-1)}{(k^2-k+1)^{n-i+1}(k+1)^{n+1}(j-k)(k-l)} + \frac{l^{3n-2i}(l-1)}{(l^2-l+1)^{n-i+1}(l+1)^{n+1}(j-k)(k-l)} \\ & - \textstyle\sum_{r=0}^{n} \frac{2\Gamma(n+r-i+1)}{3^{n+r-i+1}\Gamma(n+r-i)} \Big[\frac{1}{jkl} + \sum_{m=0}^{r} \frac{1}{\Gamma(m+1)} \Big[\frac{j^{n-r-1}}{(j+1)^{n-r+1}(j-k)(j-l)} \\ & - \frac{k^{n-r-1}}{(k+1)^{n-r+1}(j-k)(k-l)} + \frac{l^{n-r-1}}{(l+1)^{n-r+1}(j-k)(k-l)} \Big] \Big] \Big] \textstyle\sum_{p=0}^{i+1} (-1)^p \binom{i+1}{p} \\ & \times \frac{1}{i!\Gamma(-(i-p+1)/3)} \textstyle\sum_{q=0}^{i} (-1)^q \binom{i}{q} \Gamma(-(i-p+1)/3+q) t^{i-q} E_t \Big(-\frac{i-p+1}{3} + q, 1 \Big) \\ & + n! \textstyle\sum_{i=1}^{n} \frac{(-1)^{n-i+1}(n+1)}{3^{n-i+1}} \sum_{r=0}^{n-i} \Big[\frac{j^{n-r-1}}{(j+1)^{n-r+1}(j-k)(j-l)} - \frac{k^{n-r-1}}{(k+1)^{n-r+1}(j-k)(k-l)} \\ & + \frac{l^{n-r-1}}{(l+1)^{n-r+1}(j-k)(k-l)} - \frac{1}{(j+1)(k+1)(l+1)} \Big] \textstyle\sum_{r_1=1}^{3} \cdots \sum_{r_n=1}^{3} \frac{1}{(i-1)!\,\Gamma(M/3-i+1)} \\ & \times \textstyle\sum_{p=0}^{i-1} (-1)^p \binom{i-1}{p} \Gamma(M/3+p-i+1) t^{i-1-p} E_t(M/3+p-i+1, 1) \\ & + \frac{n!}{3^{n+1}(j+1)(k+1)(l+1)} \textstyle\sum_{r_1=1}^{3} \cdots \sum_{r_{n+1}=1}^{3} \frac{1}{(n)!\,\Gamma(M/3-n)} \sum_{p=0}^{n} (-1)^p \times \\ & \binom{n}{p} \Gamma(M/3+p-n) t^{n-p} E_t(M/3+p-n, 1) + \frac{(-1)^n n! j^{3n-1} e^{-t/j}}{(j^3+1)^{n+1}(j-k)(j-l)} \\ & + \frac{(-1)^{n+1} n! k^{3n-1} e^{-t/k}}{(k^3+1)^{n+1}(j-k)(k-l)} + \frac{(-1)^{n+2} n! l^{3n-1} e^{-t/l}}{(l^3+1)^{n+1}(j-l)(k-l)}. \end{aligned} \tag{19}$$

as the solution to (18), where $M = \sum_{i=1}^{n+1} r_i$.

(ii) *If $\sigma = 1$, and $j, k, l \in \mathbb{R} \setminus \{-1, 0\}$ with j, k, l are different such that $a = j + k + l$, $b = jk + jl + kl$, and $c = jkl$ then*

$$\begin{aligned} y(t) = {} & \sum_{i=0}^{n} (-1)^{n-i} n! \Big[\frac{j^{n-i}}{(j+1)^{n-i+2}(j-k)(j-l)} - \frac{k^{n-i}}{(k+1)^{n-i+2}(j-k)(k-l)} \\ & + \frac{l^{n-i}}{(l+1)^{n-i+2}(j-l)(k-l)} \Big] \frac{t^{i-1} e^t}{\Gamma(i)} + \frac{t^n e^t}{(j+1)(k+1)(l+1)} \\ & + \frac{(-1)^n n! j^{n-1} e^{-t/j}}{(j+1)^{n+1}(j-k)(j-l)} + \frac{(-1)^{n+1} n! k^{n-1} e^{-t/k}}{(k+1)^{n+1}(j-k)(k-l)} + \frac{(-1)^{n+2} n! l^{n-1} e^{-t/l}}{(l+1)^{n+1}(j-l)(k-l)}. \end{aligned} \tag{20}$$

as the solution to (18).

Proof. Applying the Laplace transform to both sides of (18), we have

$$\mathscr{L}\{I_{0^+}^{3\sigma}y(t)\} + a\mathscr{L}\{I_{0^+}^{2\sigma}y(t)\} + b\mathscr{L}\{I_{0^+}^{\sigma}y(t)\} + c\mathscr{L}\{y(t)\} = \mathscr{L}\{t^n e^t\}. \tag{21}$$

Using Lemma 1, Example 3 (iv), on (21) and denoting the Laplace transform $\mathscr{L}\{y(t)\} = Y(s)$, we obtain

$$Y(s) = \frac{n!s^{3\sigma}}{(s-1)^{n+1}(cs^{3\sigma} + bs^{2\sigma} + as^{\sigma} + 1)}. \tag{22}$$

For $\sigma = 1/3$, Equation (22) becomes

$$Y(s) = \frac{n!s}{(s-1)^{n+1}(cs + bs^{2/3} + as^{1/3} + 1)},$$

and turns into

$$Y(s) = \frac{n!\, u^3}{(u^3-1)^{n+1}(cu^3 + bu^2 + au + 1)}, \tag{23}$$

with a substitution of $u = s^{1/3}$. Using partial fractions with explicit values of a, b, c, we can rewrite it as

$$\begin{aligned}
Y(s) = n!\sum_{i=0}^{n}(-1)^{n+1}\Bigg[&\frac{j^{3n+1-2i}}{(j^2-j+1)^{n-i+1}(j+1)^{n+1}(j-k)(j-l)} \\
&- \frac{k^{3n+1-2i}}{(k^2-k+1)^{n-i+1}(k+1)^{n+1}(j-k)(k-l)} + \frac{l^{3n+1-2i}}{(l^2-l+1)^{n-i+1}(l+1)^{n+1}(j-k)(k-l)} \\
&- \sum_{r=0}^{n}\frac{\Gamma(n+r-i+1)}{3^{n+r-i+1}\Gamma(n+r-i)}\Bigg[\frac{1}{jkl} + \sum_{m=0}^{r}\frac{1}{\Gamma(m+1)}\Bigg[\frac{j^{n-r-1}}{(j+1)^{n-r+1}(j-k)(j-l)} \\
&- \frac{k^{n-r-1}}{(k+1)^{n-r+1}(j-k)(k-l)} + \frac{l^{n-r-1}}{(l+1)^{n-r+1}(j-k)(k-l)}\Bigg]\Bigg]\Bigg]\frac{u}{(u^2+u+1)^{i+1}} \\
&+ n!\sum_{i=0}^{n}(-1)^{n+1}\Bigg[\frac{j^{3n-2i}(j-1)}{(j^2-j+1)^{n-i+1}(j+1)^{n+1}(j-k)(j-l)} \\
&- \frac{k^{3n-2i}(k-1)}{(k^2-k+1)^{n-i+1}(k+1)^{n+1}(j-k)(k-l)} + \frac{l^{3n-2i}(l-1)}{(l^2-l+1)^{n-i+1}(l+1)^{n+1}(j-k)(k-l)} \\
&- \sum_{r=0}^{n}\frac{2\Gamma(n+r-i+1)}{3^{n+r-i+1}\Gamma(n+r-i)}\Bigg[\frac{1}{jkl} + \sum_{m=0}^{r}\frac{1}{\Gamma(m+1)}\Bigg[\frac{j^{n-r-1}}{(j+1)^{n-r+1}(j-k)(j-l)} \\
&- \frac{k^{n-r-1}}{(k+1)^{n-r+1}(j-k)(k-l)} + \frac{l^{n-r-1}}{(l+1)^{n-r+1}(j-k)(k-l)}\Bigg]\Bigg]\Bigg]\frac{1}{(u^2+u+1)^{i+1}} \\
&+ n!\sum_{i=1}^{n}\frac{(-1)^{n-i+1}(n+1)}{3^{n-i+1}}\sum_{r=0}^{n-i}\Bigg[\frac{j^{n-r-1}}{(j+1)^{n-r+1}(j-k)(j-l)} - \frac{k^{n-r-1}}{(k+1)^{n-r+1}(j-k)(k-l)} \\
&+ \frac{l^{n-r-1}}{(l+1)^{n-r+1}(j-k)(k-l)} - \frac{1}{(j+1)(k+1)(l+1)}\Bigg]\frac{1}{(u-1)^i} \\
&+ \Bigg[\frac{n!}{3^{n+1}(j+1)(k+1)(l+1)}\Bigg]\frac{1}{(u-1)^{n+1}} + \Bigg[\frac{(-1)^n n!\, j^{3n-1}}{(j^3+1)^{n+1}(j-k)(j-l)}\Bigg]\frac{1}{u+1/j} \\
&+ \Bigg[\frac{(-1)^{n+1} n!\, k^{3n-1}}{(k^3+1)^{n+1}(j-k)(k-l)}\Bigg]\frac{1}{u+1/k} + \Bigg[\frac{(-1)^{n+2} n!\, l^{3n-1}}{(l^3+1)^{n+1}(j-l)(k-l)}\Bigg]\frac{1}{u+1/k},
\end{aligned}$$

equivalently,

$$
\begin{aligned}
Y(s) = n!\sum_{i=0}^{n}(-1)^{n+1}\Bigg[&\frac{j^{3n+1-2i}}{(j^2-j+1)^{n-i+1}(j+1)^{n+1}(j-k)(j-l)}\\
&-\frac{k^{3n+1-2i}}{(k^2-k+1)^{n-i+1}(k+1)^{n+1}(j-k)(k-l)}+\frac{l^{3n+1-2i}}{(l^2-l+1)^{n-i+1}(l+1)^{n+1}(j-k)(k-l)}\\
&-\sum_{r=0}^{n}\frac{\Gamma(n+r-i+1)}{3^{n+r-i+1}\Gamma(n+r-i)}\Bigg[\frac{1}{jkl}+\sum_{m=0}^{r}\frac{1}{\Gamma(m+1)}\Bigg[\frac{j^{n-r-1}}{(j+1)^{n-r+1}(j-k)(j-l)}\\
&-\frac{k^{n-r-1}}{(k+1)^{n-r+1}(j-k)(k-l)}+\frac{l^{n-r-1}}{(l+1)^{n-r+1}(j-k)(k-l)}\Bigg]\Bigg]\Bigg]\sum_{p=0}^{i+1}\frac{\binom{i+1}{p}}{u^{-(i-p+2)}(u^3-1)^{i+1}}\\
&+n!\sum_{i=0}^{n}(-1)^{n+1}\Bigg[\frac{j^{3n-2i}(j-1)}{(j^2-j+1)^{n-i+1}(j+1)^{n+1}(j-k)(j-l)}\\
&-\frac{k^{3n-2i}(k-1)}{(k^2-k+1)^{n-i+1}(k+1)^{n+1}(j-k)(k-l)}+\frac{l^{3n-2i}(l-1)}{(l^2-l+1)^{n-i+1}(l+1)^{n+1}(j-k)(k-l)}\\
&-\sum_{r=0}^{n}\frac{2\Gamma(n+r-i+1)}{3^{n+r-i+1}\Gamma(n+r-i)}\Bigg[\frac{1}{jkl}+\sum_{m=0}^{r}\frac{1}{\Gamma(m+1)}\Bigg[\frac{j^{n-r-1}}{(j+1)^{n-r+1}(j-k)(j-l)}\\
&-\frac{k^{n-r-1}}{(k+1)^{n-r+1}(j-k)(k-l)}+\frac{l^{n-r-1}}{(l+1)^{n-r+1}(j-k)(k-l)}\Bigg]\Bigg]\Bigg]\sum_{p=0}^{i+1}\frac{\binom{i+1}{p}}{u^{-(i-p+1)}(u^3-1)^{i+1}}\\
&+n!\sum_{i=1}^{n}\frac{(-1)^{n-i+1}(n+1)}{3^{n-i+1}}\sum_{i=0}^{n-1}\Bigg[\frac{j^{n-r-1}}{(j+1)^{n-r+1}(j-k)(j-l)}-\frac{k^{n-r-1}}{(k+1)^{n-r+1}(j-k)(k-l)}\\
&+\frac{l^{n-r-1}}{(l+1)^{n-r+1}(j-k)(k-l)}-\frac{1}{(j+1)(k+1)(l+1)}\Bigg]\frac{1}{(u-1)^i}\\
&+\left[\frac{n!}{3^{n+1}(j+1)(k+1)(l+1)}\right]\frac{1}{(u-1)^{n+1}}+\left[\frac{(-1)^n n!\, j^{3n-1}}{(j^3+1)^{n+1}(j-k)(j-l)}\right]\frac{1}{u+1/j}\\
&+\left[\frac{(-1)^{n+1}n!\, k^{3n-1}}{(k^3+1)^{n+1}(j-k)(k-l)}\right]\frac{1}{u+1/k}+\left[\frac{(-1)^{n+2}n!\, l^{3n-1}}{(l^3+1)^{n+1}(j-l)(k-l)}\right]\frac{1}{u+1/k}
\end{aligned}
\tag{24}
$$

Finally, resubstituting $u = s^{1/3}$ and taking the inverse Laplace transform to (24) with the help of Example 4 in section (iii) and (iv), and Lemma 2 and Lemma 3, we obtain (19) as the solution to (18).

For $\sigma = 1$, Equation (22) becomes

$$
Y(s) = \frac{n!s^3}{(s-1)^{n+1}(cs^3+bs^2+as+1)}.
$$

Using partial fractions with explicit values of a, b, c, we can rewrite the above equation as

$$
\begin{aligned}
Y(s) &= \textstyle\sum_{i=1}^{n}(-1)^{n-i}n!\Big[\frac{j^{n-i}}{(j+1)^{n-i+2}(j-k)(j-l)}-\frac{k^{n-i}}{(k+1)^{n-i+2}(j-k)(k-l)}\\
&\quad+\frac{l^{n-i}}{(l+1)^{n-i+2}(j-l)(k-l)}\Big]\left(\frac{1}{(s-1)^i}\right)+\frac{n!}{(j+1)(k+1)(l+1)}\left(\frac{1}{(s-1)^{n+1}}\right)\\
&\quad+\frac{(-1)^n n! j^{n-1}}{(j+1)^{n+1}(j-k)(j-l)}\left(\frac{1}{s+1/j}\right)+\frac{(-1)^{n+1}n!k^{n-1}}{(k+1)^{n+1}(j-k)(k-l)}\left(\frac{1}{s+1/k}\right)\\
&\quad+\frac{(-1)^{n+2}n!l^{n-1}}{(l+1)^{n+1}(j-l)(k-l)}\left(\frac{1}{s+1/l}\right).
\end{aligned}
\tag{25}
$$

Applying the inverse Laplace transform to (25) with the help of Example 4 in section (iii) and (iv) yields (20) as the solution of (18). To include the case $n = 0$ into the solution formulas of both cases, we adopt the notation $1/\Gamma(0) = 0$. The proof is completed. $\square$

Example 7. *Let $a = \frac{11}{6}, b = 1, c = \frac{1}{6}$ and $\sigma = 1$ then (22) becomes*

$$I^3_{0+}y(t) + \frac{11}{6} \cdot I^2_{0+}y(t) + I_{0+}y(t) + \frac{1}{6} \cdot y(t) = t^n e^t. \quad (26)$$

From Theorem 2, Equation (26) has solution

$$\begin{aligned} y(t) &= \textstyle\sum_{i=0}^{n} (-1)^{n-i-1} n! \left[\frac{(1/2)^{n-i-4}}{(3/2)^{n-i}} + \frac{(1/3)^{n-i-3}}{(4/3)^{n-i+1}} + \frac{3}{2^{n-i+1}} \right] \frac{t^i e^t}{i!} + t^n e^t \\ &\quad + \frac{(-1)^n 3 \cdot n! j^{n-1} e^{-t}}{2^{n+1}} + \frac{(-1)^{n+2} n! (1/3)^{n-3} e^{-3t}}{(4/3)^{n+1}} + \frac{(-1)^{n+3} n! (1/2)^{n-4} e^{-2t}}{(3/2)^n}. \end{aligned} \quad (27)$$

By applying Example 2 in section (i), (iii), and (iv), it is not difficult to verify that (27) satisfies (26).

Moreover, if $n = 0$, then (26) becomes

$$I^3_{0+}y(t) + \frac{11}{6} \cdot I^2_{0+}y(t) + I_{0+}y(t) + \frac{1}{6} \cdot y(t) = e^t. \quad (28)$$

From (27), it follows that (28) has solution

$$y(t) = \frac{1}{4}e^t + \frac{81}{4}e^{-3t} + \frac{3}{2}e^{-t} - 16e^{-2t}. \quad (29)$$

It is not difficult to verify that (29) satisfies (28).

4. Conclusions

We used the Laplace transform technique to find the solutions of nonhomogeneous fractional integral equation of the form

$$I^{3\sigma}_{0+}y(t) + a \cdot I^{2\sigma}_{0+}y(t) + b \cdot I^{\sigma}_{0+}y(t) + c \cdot y(t) = f(t),$$

where I^{σ}_{0+} is the Riemann–Liouville fractional integral of order $\sigma = 1/3, 1, f(t) = t^n, t^n e^t$, $n \in \mathbb{N} \cup \{0\}, t \in \mathbb{R}^+$, and a, b, c are constants. Moreover, example are given to demonstrate the effectiveness of these results. It is expected that our findings may encourage further research in this field.

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