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Oscillation Criteria of Solutions of Fourth-Order Neutral Differential Equations

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Abstract: In this paper, we study the oscillation of solutions of fourth-order neutral delay differential equations in non-canonical form. By using Riccati transformation, we establish some new oscillation conditions. We provide some examples to examine the applicability of our results.

Keywords: oscillation criteria; fourth-order differential equations; neutral delay

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1. Introduction

In this work, we obtain some oscillation conditions of equation

$$\left(b(t)\left(y'''(t)\right)^{\alpha}\right)' + \sum_{i=1}^{j} \nu_i(t)x^{\alpha}(g_i(t)) = 0. \tag{1}$$

where j is a positive integer and

$$y(t) = x(t) + \beta(t)x(\tilde{z}(t)). \tag{2}$$

Equation (1) is said to be in canonical form if $\int_{t_0}^{\infty} b^{-1/\alpha}(s)\mathrm{d}s = \infty$; otherwise, it is called noncanonical. Throughout this work, we suppose the hypotheses as follows:

$$\begin{cases} b \in C^1([t_0,\infty)), b(t) > 0, b'(t) \geq 0, \beta, \nu \in C([t_0,\infty)), \nu(t) > 0, 0 \leq \beta(t) < \beta_0 < \infty, \\ \tilde{z} \in C^1([t_0,\infty)), g_i \in C([t_0,\infty)), \tilde{z}'(t) > 0, \tilde{z}(t) \leq t \text{ and } \lim_{t\to\infty} \tilde{z}(t) = \lim_{t\to\infty} g_i(t) = \infty, \\ \alpha \text{ is quotient of odd positive integers}, \alpha > 0 \end{cases}$$

and

$$\xi(t_0) := \int_{t_0}^{\infty} b^{-1/\alpha}(s)\mathrm{d}s < \infty. \tag{3}$$

Neutral/delay differential equations are used in a variety of problems in economics, biology, medicine, engineering and physics, including lossless transmission lines, vibration of bridges, as well as vibrational motion in flight, and as the Euler equation in some variational problems, see [1,2]. In particular, fourth-order neutral delay differential Equation (1) find application in explaining human self-balancing. With regard to their practical importance, oscillation of fourth-order neutral differential equations has been studied extensively during recent decades, see [3–9].

As a result, there is an ongoing interest in obtaining several sufficient conditions for the oscillatory behavior of the solutions of different kinds of differential equations, especially their the oscillation and asymptotic. Baculikova [10], Dzurina and Jadlovska [11], and Bohner et al. [12] developed approaches and techniques for studying oscillatory properties in order to improve the oscillation criteria of second-order differential equations with delay/advanced terms. Xing et al. [13] and Moaaz et al. [14] also extended this evolution to differential equations of the neutral type. Therefore, there are many studies on the oscillatory properties of different orders of some differential equations in noncanonical form, see [15–25].

The qualitative theory of differential equations as well as analytical methods for qualitative behavior of solutions have contributed to the development of many new mathematical ideas and methodologies for solving ordinary and fractional differential equations as well as systems of differential equations. From the viewpoint of applications, differential equations are crucially important for modeling any kind of dynamical systems or processes in real life. So, in this work, we study the oscillatory behavior of solutions of the fourth-order neutral delay differential equations in noncanonical form. However, to the best of our knowledge, only a few papers have studied the oscillation and qualitative behavior of fourth-order neutral delay differential equations in noncanonical form.

2. Mathematical Background

In this section, we collect some relevant facts and auxiliary results from the existing literature. Furthermore, we fix the notations.

Definition 1. *A solution of (1) is said to be non-oscillatory if it is positive or negative, ultimately; otherwise, it is said to be oscillatory.*

Definition 2. *Equation (1) is said to be oscillatory if every solution of it is oscillatory.*

For convenience, we denote:

$$\begin{aligned} P_k(t) \quad &: \; = \frac{1}{\beta(\tilde{z}^{-1}(t))}\left(1 - \frac{\left(\left(\tilde{z}^{-1}(t)\right)^{k-1}\beta\left(\tilde{z}^{-1}\left(\tilde{z}^{-1}(t)\right)\right)\right)^{-1}}{\left(\tilde{z}^{-1}\left(\tilde{z}^{-1}(t)\right)\right)^{1-k}}\right), \quad \text{for } k = 2, \ldots, n, \\ \Theta(s) \quad &: \; = \frac{\alpha^{\alpha+1}}{(\alpha+1)^{\alpha+1}} \frac{b^{\alpha}\left(\tilde{z}^{-1}(g_i(s))\right)}{b^{1/\alpha}(s)\xi(s)\left(\left(\tilde{z}^{-1}(g_i(s))\right)'\right)^{\alpha}} \end{aligned}$$

and

$$\widetilde{\Theta}(s) = \frac{\alpha^{b+1}}{(\alpha+1)^{\alpha+1}} \frac{2^{\alpha} b^{\alpha}\left(\tilde{z}^{-1}(g_i(s))\right)}{b^{1/\alpha}(s)\xi(s)\mu_1\left(\left(\tilde{z}^{-1}(g_i(s))\right)'\left(\tilde{z}^{-1}(g_i(s))\right)^2\right)^{\alpha}}.$$

Furthermore,

$$\omega(t) := \frac{b(t)\left(y'''(t)\right)^{\alpha}}{\left(y''\left(\tilde{z}^{-1}(g_i(t))\right)\right)^{\alpha}} \tag{4}$$

and

$$\zeta(t) := \frac{b(t)\left(y'''(t)\right)^{\alpha}}{y^{\alpha}\left(\tilde{z}^{-1}(g_i(t))\right)}. \tag{5}$$

The motivation for this article is to complement the results reported in [13,26], which discussed the oscillatory properties of equation in a canonical form.

Xing et al. [13] discussed the equation

$$\left(b(t)y^{(m-1)}(t)\right)' + \nu(t)\varphi(x(g(t))) = 0.$$

Moreover, the authors used the comparison method to obtain oscillation conditions for this equation.

Agarwal et al. [26] investigated the oscillation of equation

$$\left(b(t)\left(y'''(t)\right)^{\alpha}\right)' + \nu(t)\varphi(x(g(t))) = 0.$$

The authors used the integral averaging technique to obtain oscillatory properties for this equation.

Moaaz et al. [14] established some criteria of (1) under condition

$$\liminf_{t\to\infty} \int_{\tilde{z}^{-1}(\eta(t))}^{t} \left(\frac{\left(\tilde{z}^{-1}(\eta(s))\right)^{n-1}}{b^{1/\alpha}\left(\tilde{z}^{-1}(\eta(s))\right)} \right)^{\alpha} \nu(s) P_n^{\alpha}(g(s)) \mathrm{d}s > \frac{((n-1)!)^{\alpha}}{\mathrm{e}}. \tag{6}$$

Tang et al. [27] presented oscillation results for (1) under

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left(\nu(s)(1-\beta(\tilde{z}(s)))^{\alpha} \left(\frac{\lambda_1 \tilde{z}^{n-2}(s)\tilde{z}(s)}{(n-2)!} \right)^{\alpha} - \frac{\alpha^{b+1}}{(\alpha+1)^{\alpha+1}\tilde{z}(s)b^{1/\alpha}(s)} \right) \mathrm{d}s = \infty.$$

In [18], the authors established asymptotic behavior for neutral equation

$$\left(b(t)\left(y'''(t)\right)^{\alpha}\right)' + \nu(t)x^{\alpha}(g(t)) = 0,$$

under condition

$$\int_{t_0}^{\infty} b^{-1/\alpha}(s)\mathrm{d}s = \infty. \tag{7}$$

The authors in [13,26] used the comparison technique that differs from the one we used in this article. Their approach is based on using these mentioned methods to reduce Equation (1) into a first-order equation, while in our article, we discuss the oscillation and asymptotic properties of differential equations in a noncanonical form of the neutral-type, and we employ a different approach based on using the Riccati technique to reduce the main equation into a first-order inequality to obtain more effective oscillation conditions for Equation (1).

Motivated by these reasons mentioned above, in this paper, we extend the results using Riccati transformation under (3). These results contribute to adding some important conditions that were previously studied in the subject of oscillation of differential equations with neutral term. To prove our main results, we give some examples.

To prove the main results, we present some lemmas:

Lemma 1 ([16]). *If the function x satisfies $x^{(i)}(t) > 0$, $i = 0,1,\ldots,n$, and $x^{(n+1)}(t) < 0$ eventually. Then, for every $\varepsilon \in (0,1)$, $x(t)/x'(t) \geq \varepsilon t/n$ eventually.*

Lemma 2 ([17]). *Let $x \in C^n([t_0,\infty),(0,\infty))$ and $x^{(n-1)}(t)x^{(n)}(t) \leq 0$ for all $t \geq t_1$. If $\lim_{t\to\infty} x(t) \neq 0$, then for every $\mu \in (0,1)$ there exists $t_\mu \geq t_1$ such that*

$$x(t) \geq \frac{\mu}{(n-1)!} t^{n-1}\left|x^{(n-1)}(t)\right| \text{ for } t \geq t_\mu.$$

Lemma 3 ([24]). *Let $A_2 > 0$. Then*

$$A_2 w - A_1 w^{(r+1)/r} \leq \frac{r^r}{(r+1)^{r+1}} \frac{A_1^{r+1}}{A_2^r},\ A_1.$$

Lemma 4. *Let*

$$x \text{ be a positive solution of } (1), \tag{8}$$

Then, $b(z)(y'''(z))^{\alpha}$ is non-increasing. Furthermore, the following cases are possible:

$$\begin{array}{ll} (S_1): & y'(t) > 0, y''(t) > 0, y'''(t) > 0 \text{ and } y^{(4)}(t) < 0; \\ (S_2): & y'(t) > 0, y''(t) < 0, y'''(t) > 0 \text{ and } y^{(4)}(t) < 0; \\ (S_3): & y'(t) > 0, y''(t) > 0 \text{ and } y'''(t) < 0; \\ (S_4): & y'(t) < 0, y''(t) > 0 \text{ and } y'''(t) < 0. \end{array}$$

3. Oscillation Criteria

Lemma 5. *Let (8) hold with property (S_1) or (S_2). Then*

$$w'(t) + (1-\beta_0)^{\alpha} \frac{\sum_{i=1}^{j} \nu_i(t)}{b(g_i(t))} \left(\frac{\mu}{6} g_i^3(t)\right)^{\alpha} w(g_i(t)) = 0 \tag{9}$$

has a non-oscillatory solution for every constant $\mu \in (0,1)$.

Proof. Let (8) hold with property (S_1) or (S_2). Then, we have that

$$y'(t) > 0,\ y'''(t) > 0 \text{ and } y^{(4)}(t) < 0.$$

Using Lemma 2, we find

$$y(t) \geq \frac{\mu}{6} t^3 y'''(t). \tag{10}$$

From definition of y, we get that $x(t) \geq (1-\beta_0) y(t)$, which with (1) gives

$$\left(b(t)\left(y'''(t)\right)^{\alpha}\right)' + (1-\beta_0)^{\alpha} \sum_{i=1}^{j} \nu_i(t) y^{\alpha}(g_i(t)) \leq 0. \tag{11}$$

Hence, from (10), if we set $w := b(y''')^{\alpha} > 0$, then

$$w'(t) + (1-\beta_0)^{\alpha} \frac{\sum_{i=1}^{j} \nu_i(t)}{b(g_i(t))} \left(\frac{\mu}{6} g_i^3(t)\right)^{\alpha} w(g_i(t)) \leq 0.$$

From [19] (Corollary 1), we find (9) also has a positive solution. Thus, Lemma 5 is proved. □

Lemma 6. *Let (8) hold with property (S_3). Then the equation*

$$\left(b(t)\left(\omega'(t)\right)^{\alpha}\right)' + (1-\beta_0)^{\alpha} \sum_{i=1}^{j} \nu_i(t) \left(\frac{\mu}{2} g_i^2(t)\right)^{\alpha} \omega^{\alpha}(t) = 0, \tag{12}$$

has a non-oscillatory solution for every constant $\mu \in (0,1)$.

Proof. Let (8) hold with property (S_3). Using Lemma 2, we obtain

$$y(t) \geq \frac{\mu}{2} t^2 y''(t). \tag{13}$$

As in the proof of Lemma 6, we find (11). Next, if we set $G := b(y'''/y'')^{\alpha} < 0$, then we find

$$G'(t) \leq -(1-\beta_0)^{\alpha} \sum_{i=1}^{j} \nu_i(t) \frac{y^{\alpha}(g_i(t))}{(y''(t))^{\alpha}} - \alpha b^{-1/\alpha}(t) G^{1+1/\alpha}(t).$$

Hence, from the fact that $y''' < 0$ and (13), we get

$$G'(t) + (1-\beta_0)^{\alpha} \sum_{i=1}^{j} \nu_i(t)\Big(\frac{\mu}{2} g_i^2(t)\Big)^{\alpha} + \alpha b^{-1/\alpha}(t) G^{1+1/\alpha}(t) \leq 0. \tag{14}$$

Thus, we get that (14) holds. It follow from [19] that (12) has a non-oscillatory solution. Lemma 6 is proved. □

Theorem 1. *Let (9) and (12) be oscillatory. If*

$$\int_{t_0}^{\infty} \left(\frac{1}{b(u)} \int_{t_0}^{t} \nu(s) \mathrm{d}s\right)^{1/\alpha} \mathrm{d}u = \infty, \tag{15}$$

then every non-oscillatory solution of (1) tends to zero.

Proof. Let (8) hold with property $\lim_{t\to\infty} x(t) \neq 0$. From Lemma 4, we have cases (S_1)–(S_4). Using Lemmas 5 and 6 with the fact that (9) and (12) are oscillatory, we get that x satisfies case (S_4). Then, we find $\lim_{t\to\infty} y(t) = c \geq 0$. Let $c > 0$. Thus, for all $\varepsilon > 0$ and t enough large, we have $c \leq y(t) < c + \varepsilon$. Set $\varepsilon < (1-\beta_0)(c/\beta_0)$, we find

$$\begin{aligned} x(t) &= y(t) - \beta_0(t) x(\tilde{z}(t)) > c - \beta_0 y(\tilde{z}(t)) \\ &> L(\gamma + \varepsilon) > Ly(t), \end{aligned} \tag{16}$$

where $L = (c - \beta_0(c+\varepsilon))/(c+\varepsilon) > 0$. So, from (1), we see

$$\begin{aligned} \Big(b(t)\big(y'''(t)\big)^{\alpha}\Big)' &= -\sum_{i=1}^{j} \nu_i(t) x^{\alpha}(g_i(t)) \leq -L^{\alpha} \sum_{i=1}^{j} \nu_i(t) y^{\alpha}(g_i(t)) \\ &\leq -L^{\alpha}\varepsilon^{\alpha} \sum_{i=1}^{j} \nu_i(t). \end{aligned}$$

Integrating this inequality from t_1 to t, we get

$$y'''(t) \leq -L\varepsilon \left(\frac{1}{b(t)} \int_{t_1}^{t} \nu(s)\mathrm{d}s\right)^{1/\alpha}.$$

By integrating from t_1 to t, we obtain

$$y''(t) \leq y''(t_1) - L\varepsilon \int_{t_1}^{t} \left(\frac{1}{b(u)} \int_{t_1}^{t} \nu(s)\mathrm{d}s\right)^{1/\alpha} \mathrm{d}u.$$

Letting $t \to \infty$ and taking into account (15), we get that $\lim_{t\to\infty} y''(t) = -\infty$. This contradicts the fact that $y''(t) > 0$. Therefore, $c = 0$; moreover the fact $x(t) \leq y(t)$ implies $\lim_{t\to\infty} x(t) = 0$, a contradiction. Theorem 1 is proved. □

Corollary 1. *Assume that (15) holds. If $\int_{t_0}^{\infty} \nu(s)\mathrm{d}s = \infty$, and*

$$\liminf_{t\to\infty} \int_{g(t)}^{t} \frac{\nu(s) g_i^{3\alpha}(s)}{b(g_i(s))} \mathrm{d}s > \frac{6^{\alpha}}{\mathrm{e}\mu^{\alpha}(1-\beta_0)^{\alpha}} \tag{17}$$

and

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left((1-\beta_0)^{\alpha} \xi^{\alpha}(s)\nu(s)\Big(\frac{\mu}{2} g_i^2(s)\Big)^{\alpha} - \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1} \frac{1}{b^{1/\alpha}(s)\xi(s)}\right)\mathrm{d}s > 0, \tag{18}$$

for every $\mu \in (0,1)$, then every non-oscillatory solution of (1) tends to zero.

Corollary 2. *Assume that (15) holds. Then every non-oscillatory solution of (1) tends to zero if* $\int_{t_0}^{\infty} \nu(s)\mathrm{d}s = \infty$,

$$\liminf_{t\to\infty} \int_{g(t)}^{t} \left(1-\widehat{\beta}(g_i(s))\right)^{\alpha} \frac{\nu(s)g_i^{3\alpha}(s)}{b(g_i(s))}\mathrm{d}s > \frac{6^{\alpha}}{\mu^{\alpha}\mathrm{e}}$$

and

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left(\left(1-\widehat{\beta}(g_i(s))\right)^{\alpha} \xi^{\alpha}(s)\nu(s)\left(\frac{\mu}{2}g_i^2(s)\right)^{\alpha} - \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1} \frac{1}{b^{1/\alpha}(s)\xi(s)} \right)\mathrm{d}s > 0,$$

for every constant $\mu \in (0,1)$.

Lemma 7. *Assume that (8) holds and*

$$\left(\tilde{z}^{-1}\left(\tilde{z}^{-1}(t)\right)\right)^{n-1} < \left(\tilde{z}^{-1}(t)\right)^{n-1}\beta\left(\tilde{z}^{-1}\left(\tilde{z}^{-1}(t)\right)\right). \tag{19}$$

Then

$$x(t) \geq \frac{y(\tilde{z}^{-1}(t))}{\beta(\tilde{z}^{-1}(t))} - \frac{1}{\beta(\tilde{z}^{-1}(t))}\frac{y(\tilde{z}^{-1}(\tilde{z}^{-1}(t)))}{\beta(\tilde{z}^{-1}(\tilde{z}^{-1}(t)))}. \tag{20}$$

Proof. Let (8) hold. From the definition of $y(t)$, we get

$$\beta(t)x(\tilde{z}(t)) = y(t) - x(t)$$

and so

$$\beta\left(\tilde{z}^{-1}(t)\right)x(t) = y\left(\tilde{z}^{-1}(t)\right) - y\left(\tilde{z}^{-1}(t)\right).$$

Repeating the same process, we find

$$x(t) = \frac{1}{\beta(\tilde{z}^{-1}(t))}\left(y\left(\tilde{z}^{-1}(t)\right) - \left(\frac{y(\tilde{z}^{-1}(\tilde{z}^{-1}(t)))}{\beta(\tilde{z}^{-1}(\tilde{z}^{-1}(t)))} - \frac{x(\tilde{z}^{-1}(\tilde{z}^{-1}(t)))}{\beta(\tilde{z}^{-1}(\tilde{z}^{-1}(t)))} \right)\right),$$

which yields

$$x(t) \geq \frac{y(\tilde{z}^{-1}(t))}{\beta(\tilde{z}^{-1}(t))} - \frac{1}{\beta(\tilde{z}^{-1}(t))}\frac{y(\tilde{z}^{-1}(\tilde{z}^{-1}(t)))}{\beta(\tilde{z}^{-1}(\tilde{z}^{-1}(t)))}.$$

Thus, (20) holds. Lemma 7 is proved. □

Lemma 8. *Suppose that (8) holds. If y satisfies* (S_3), *then*

$$\left(b(t)\left(y'''(t)\right)^{\alpha}\right)' \leq -\sum_{i=1}^{j} \nu_i(t)P_1^{\alpha}(g_i(t))y^{\alpha}\left(\tilde{z}^{-1}(g_i(t))\right) \tag{21}$$

and if y satisfies (S_4), *then*

$$\left(b(t)\left(y'''(t)\right)^{\alpha}\right)' + \sum_{i=1}^{j} \nu_i(t)P_2^{\alpha}(g_i(t))y^{\alpha}\left(\tilde{z}^{-1}(g_i(t))\right) \leq 0. \tag{22}$$

Proof. Suppose that case $(\mathbf{S}_3)$ holds. Using Lemma 1, we find $y(t) \geq \varepsilon t y'(t)$ and hence the function $t^{-1}y(t)$ is nonincreasing, which with the fact that $\tilde{z}(t) \leq t$ gives

$$\left(\tilde{z}^{-1}(t)\right)y\left(\tilde{z}^{-1}\left(\tilde{z}^{-1}(t)\right)\right) \leq \left(\tilde{z}^{-1}\left(\tilde{z}^{-1}(t)\right)\right)y\left(\tilde{z}^{-1}(t)\right). \tag{23}$$

Combining (20) and (23), we see that

$$\begin{aligned} x(t) &\geq \frac{1}{\beta(\tilde{z}^{-1}(t))}\left(1-\frac{(\tilde{z}^{-1}(\tilde{z}^{-1}(t)))}{\tilde{z}^{-1}(t)\beta(\tilde{z}^{-1}(\tilde{z}^{-1}(t)))}\right)y\left(\tilde{z}^{-1}(t)\right) \\ &= P_2(t)y\left(\tilde{z}^{-1}(t)\right). \end{aligned} \tag{24}$$

From (1) and (24), we obtain

$$\left(b(t)(y'''(t))^{\alpha}\right)' \leq -\sum_{i=1}^{j} \nu_i(t)P_n^{\alpha}(g_i(t))y^{\alpha}\left(\tilde{z}^{-1}(g_i(t))\right). \tag{25}$$

Thus, (21) holds.
Let $(\mathbf{S}_4)$ holds. Since $\tilde{z}^{-1}(t) \leq \tilde{z}^{-1}(\tilde{z}^{-1}(t))$. From (20), we see that

$$\begin{aligned} x(t) &\geq \frac{1}{\beta(\tilde{z}^{-1}(t))}\left(1-\frac{1}{\beta(\tilde{z}^{-1}(\tilde{z}^{-1}(t)))}\right)y\left(\tilde{z}^{-1}(t)\right) \\ &= P_2(t)y\left(\tilde{z}^{-1}(t)\right). \end{aligned} \tag{26}$$

which with (1) yields

$$\left(b(t)(y'''(t))^{\alpha}\right)' + \sum_{i=1}^{j} \nu_i(t)P_2^{\alpha}(g_i(t))y^{\alpha}\left(\tilde{z}^{-1}(g_i(t))\right) \leq 0.$$

Thus, (22) holds. This completes the proof. □

Lemma 9. *Let (8) and (S_3) hold. If $\omega \in C^1[t,\infty)$ defined as (4), then*

$$\omega'(t) \leq -\sum_{i=1}^{j} \nu_i(t)P_1^{\alpha}(g_i(t))\left(\frac{\lambda}{2}\left(\tilde{z}^{-1}(g_i(t))\right)^2\right)^{\alpha} - \alpha\frac{\left(\tilde{z}^{-1}(g_i(t))\right)'}{b^{1/\alpha}(t)b(\tilde{z}^{-1}(g_i(t)))}\omega^{\frac{\alpha+1}{\alpha}}(t), \tag{27}$$

for all $t > t_1$and $\lambda \in (0,\ 1)$, where t_1 large enough.

Proof. Let (8) hold. From Lemma (2), we get

$$y\left(\tilde{z}^{-1}(g_i(t))\right) \geq \frac{\lambda}{2}\left(\tilde{z}^{-1}(g_i(t))\right)^2 y''\left(\tilde{z}^{-1}(g_i(t))\right). \tag{28}$$

Recalling that $b(t)(y'''(t))^{\alpha}$ is decreasing, we get

$$b\left(\tilde{z}^{-1}(g_i(t))\right)\left(y'''\left(\tilde{z}^{-1}(g_i(t))\right)\right)^{\alpha} \geq b(t)(y'''(t))^{\alpha}.$$

This yields

$$\left(y'''\left(\tilde{z}^{-1}(g_i(t))\right)\right)^{\alpha} \geq \frac{b(t)}{b(\tilde{z}^{-1}(g_i(t)))}(y'''(t))^{\alpha}. \tag{29}$$

From (4), we obtain

$$\omega'(t) = \frac{(b(t)(y'''(t))^{\alpha})'}{(y''(\tilde{z}^{-1}(g_i(t))))^{\alpha}} - \alpha\frac{b(t)(y'''(t))^{\alpha}y'''(\tilde{z}^{-1}(g_i(t)))\left(\tilde{z}^{-1}(g_i(t))\right)'}{(y''(\tilde{z}^{-1}(g_i(t))))^{\alpha+1}}.$$

From (4), (28) and (29), we get

$$\begin{aligned}
\omega'(t) &\leq -\sum_{i=1}^{j} \nu_i(t) P_1^{\alpha}(g_i(t)) \frac{y^{\alpha}(\tilde{z}^{-1}(g_i(t)))}{(y''(\tilde{z}^{-1}(g_i(t))))^{\alpha}} \\
&- \alpha \frac{b(t)(\tilde{z}^{-1}(g_i(t)))'}{b(\tilde{z}^{-1}(g_i(t)))} \frac{(y'''(t))^{\alpha+1}}{(y''(\tilde{z}^{-1}(g_i(t))))^{\alpha+1}} \\
&\leq -\sum_{i=1}^{j} \nu_i(t) P_1^{\alpha}(g_i(t)) \left(\frac{\lambda}{2}\left(\tilde{z}^{-1}(g_i(t))\right)^2\right)^{\alpha} - \alpha \frac{(\tilde{z}^{-1}(g_i(t)))'}{b^{1/\alpha}(t) b(\tilde{z}^{-1}(g_i(t)))} \omega^{\frac{\alpha+1}{\alpha}}(t).
\end{aligned}$$

The proof is complete. □

Lemma 10. *Let (8) and (S_4) hold. If $\zeta \in C^1[t,\infty)$ defined as (5), then*

$$\zeta'(t) \leq -\sum_{i=1}^{j} \nu_i(t) P_2^{\alpha}(g_i(t)) - \alpha \frac{\mu_1 (\tilde{z}^{-1}(g_i(t)))' (\tilde{z}^{-1}(g_i(t)))^2}{2 b^{1/\alpha}(\tilde{z}^{-1}(g_i(t)))} \zeta^{\alpha+1}(t), \tag{30}$$

for all $\mu_1 \in (0,1)$ and $t > t_1$, where t_1 large enough.

Proof. Let (8) hold. From $\xi(t)$, we find $\xi(t) < 0$. By differentiating, we see

$$\zeta'(t) \leq -\sum_{i=1}^{j} \nu_i(t) P_2^{\alpha}(g_i(t)) - \alpha \frac{b(t)(y'''(t))^{\alpha} (\tilde{z}^{-1}(g_i(t)))' y'(\tilde{z}^{-1}(g_i(t)))}{y^{\alpha+1}(\tilde{z}^{-1}(g_i(t)))}. \tag{31}$$

From Lemma 2 and (29), we get

$$y'\left(\tilde{z}^{-1}(g_i(t))\right) \geq \frac{\mu_1}{2}\left(\tilde{z}^{-1}(g_i(t))\right)^2 \left(\frac{b(t)}{b(\tilde{z}^{-1}(g_i(t)))}\right)^{1/\alpha} y'''(t), \tag{32}$$

for all $\mu_1 \in (0,1)$. Thus, by (5), (31) and (32), we get

$$\zeta'(t) \leq -\sum_{i=1}^{j} \nu_i(t) P_2^{\alpha}(g_i(t)) - \alpha \frac{\mu_1 (\tilde{z}^{-1}(g_i(t)))' (\tilde{z}^{-1}(g_i(t)))^2}{2 b^{1/\alpha}(\tilde{z}^{-1}(g_i(t)))} \zeta^{\alpha+1}(t).$$

The proof is complete. □

Theorem 2. *Suppose that (6) holds. If*

$$\int_{t_0}^{\infty} \left(\nu(s) P_1^{\alpha}(g_i(s)) \left(\frac{\lambda}{2}\left(\tilde{z}^{-1}(g_i(t))\right)^2\right)^{\alpha} \xi^{\alpha}(s) ds - \Theta(s)\right) \mathrm{d}s = \infty \tag{33}$$

and

$$\int_{t_0}^{\infty} \left(\nu(s) P_2^{\alpha}(g_i(s)) \xi^{\alpha}(s) ds - \widetilde{\Theta}(s)\right) \mathrm{d}s = \infty, \tag{34}$$

then (1) is oscillatory.

Proof. Let (8) hold. From Lemma 4, we have cases (S_1)–(S_4). Let $(\mathbf{S}_3)$ holds. From Lemma 9, we find (27) holds.

When we multiply this inequality by $\xi^{\alpha}(t)$ and then integrating from t_1 to t, we find

$$\begin{aligned}\xi^{\alpha}(t)\omega(t)-\xi^{\alpha}(t_1)\omega(t_1)+\alpha\int_{t_1}^{t} b^{\frac{-1}{\alpha}}(s)\xi^{\alpha-1}(s)\omega(s)ds \\ \leq -\int_{t_1}^{t}\nu(s)P_1^{\alpha}(g_i(s))\left(\frac{\lambda}{2}\left(\tilde{z}^{-1}(g_i(t))\right)^2\right)^{\alpha}\xi^{\alpha}(s)ds \\ -\alpha\int_{t_1}^{t}\frac{\xi^{\alpha}(s)\left(\tilde{z}^{-1}(g_i(s))\right)'g_i'(s)}{b^{1/\alpha}(s)b(\tilde{z}^{-1}(g_i(s)))}\omega^{\frac{\alpha+1}{\alpha}}(s)ds. \end{aligned} \tag{35}$$

We set

$$A_2=b^{\frac{-1}{\alpha}}(s)\xi^{\alpha-1}(s),\ A_1=\frac{\xi^{\alpha}(s)\left(\tilde{z}^{-1}(g_i(s))\right)'g_i'(s)}{b^{1/\alpha}(s)b(\tilde{z}^{-1}(g_i(s)))},\ w:=-\omega(s).$$

Using Lemma 10, we find

$$\begin{aligned} & b^{\frac{-1}{\alpha}}(s)\xi^{\alpha-1}(s)\omega(s)-\frac{\xi^{\alpha}(s)\left(\tilde{z}^{-1}(g_i(s))\right)'g_i'(s)}{b^{1/\alpha}(s)b(\tilde{z}^{-1}(g_i(s)))}\omega^{\frac{\alpha+1}{\alpha}} \\ \leq\ & \frac{\alpha^{b+1}}{(\alpha+1)^{\alpha+1}}\frac{b^{\alpha}\left(\tilde{z}^{-1}(g_i(s))\right)}{b^{1/\alpha}(s)\xi(s)\left(\left(\tilde{z}^{-1}(g_i(s))\right)'\right)^{\alpha}}. \end{aligned}$$

From (35), we get

$$\int_{t_1}^{t}\left(\nu(s)P_1^{\alpha}(g_i(s))\left(\frac{\lambda}{2}\left(\tilde{z}^{-1}(g_i(t))\right)^2\right)^{\alpha}\xi^{\alpha}(s)ds-\Theta(s)\right)ds\leq\xi^{\alpha}(t_1)g_i(t_1)+1,$$

but this contradicts (33).

Suppose that case $(\mathbf{S}_3)$ holds. By Lemma 10, we find (30) holds.

When we multiply this inequality by $\xi^{\alpha}(t)$ and then integrating from t_1 to t, we get

$$\begin{aligned} & \xi^{\alpha}(t)\,\zeta(t)-\xi^{\alpha}(t_1)\zeta(t_1)+\alpha\int_{t_1}^{t} b^{\frac{-1}{\alpha}}(s)\xi^{\alpha-1}(s)\zeta(s)ds \\ \leq\ & -\int_{t_1}^{t}\nu(s)P_2^{\alpha}(g_i(s))\xi^{\alpha}(s)ds \\ -\ & \alpha\int_{t_1}^{t}\frac{\mu_1\xi^{\alpha}(s)\left(\tilde{z}^{-1}(g_i(s))\right)'g_i'(s)\left(\tilde{z}^{-1}(g_i(s))\right)^2}{2b^{1/\alpha}(s)b(\tilde{z}^{-1}(g_i(s)))}\zeta^{\frac{\alpha+1}{\alpha}}(s)ds. \end{aligned}$$

We set

$$A_2=b^{\frac{-1}{\alpha}}(s)\xi^{\alpha-1}(s),\ A_1=\frac{\mu_1\xi^{\alpha}(s)\left(\tilde{z}^{-1}(g_i(s))\right)'g_i'(s)\left(\tilde{z}^{-1}(g_i(s))\right)^2}{2b^{1/\alpha}(s)b(\tilde{z}^{-1}(g_i(s)))},\ w:=-\zeta(s).$$

Applying Lemma 3, for every $\mu_1\in(0,\ 1)$, we obtain

$$\begin{aligned} & b^{\frac{-1}{\alpha}}(s)\xi^{\alpha-1}(s)\zeta(s)-\frac{\mu_1\xi^{\alpha}(s)\left(\tilde{z}^{-1}(g_i(s))\right)'g_i'(s)\left(\tilde{z}^{-1}(g_i(s))\right)^2}{2b^{1/\alpha}(s)b(\tilde{z}^{-1}(g_i(s)))}\zeta^{\frac{\alpha+1}{\alpha}} \\ \leq\ & \frac{\alpha^{b+1}}{(\alpha+1)^{\alpha+1}}\frac{2^{\alpha}b^{\alpha}\left(\tilde{z}^{-1}(g_i(s))\right)}{b^{1/\alpha}(s)\xi(s)\mu_1\left(\left(\tilde{z}^{-1}(g_i(s))\right)'\left(\tilde{z}^{-1}(g_i(s))\right)^2\right)^{\alpha}}, \end{aligned}$$

which implies that

$$\int_{t_1}^{t}\left(\nu(s)P_2^{\alpha}(g_i(s))\xi^{\alpha}(s)ds-\widetilde{\Theta}(s)\right)ds\leq\xi^{\alpha}(t_1)g_i(t_1)+1,$$

but this contradicts (34). Theorem 2 is proved. □

Example 1. *Consider the equation*

$$\left(t^2\left(x(t)+4x\left(\frac{t}{2}\right)\right)'''\right)'+\nu_0 x\left(\frac{t}{2}\right)=0, t\geq 1,\ \nu_0>0. \tag{36}$$

Let $\alpha=1, b(t)=t^2, \beta(t)=4,\ \tilde{z}(t)=g(t)=t/2$ and $\nu(t)=\nu_0$. Furthermore, we see

$$P_1(t)=\frac{1}{8},\ P_2(t)=\frac{3}{16}.$$

Hence, Conditions (33) and (34) become

$$\nu_0>4$$

and

$$\nu_0>\frac{8}{3},$$

By using Theorem 2, Equation (36) is oscillatory if $\nu_0>4$.

Example 2. *Consider the equation*

$$\left(t^2\left(x(t)+16x\left(\frac{t}{2}\right)\right)'''\right)'+\nu_0 x\left(\frac{t}{2}\right)=0, t\geq 1,\ \nu_0>0, \tag{37}$$

let $\alpha=1, b(t)=t^2, \beta(t)=16,\ \tilde{z}(t)=g(t)=t/2$ and $\nu(t)=\nu_0$. Moreover, we find

$$P_1(t)=\frac{7}{128},\ P_2(t)=\frac{1}{32},\ \xi(t)=\frac{1}{t},\ \Theta(t)=\frac{t}{4}$$

and

$$\widetilde{\Theta}(t)=\frac{1}{2t}.$$

So, we obtain

$$\begin{aligned} &\int_{t_0}^{\infty}\left(\nu(s)P_1^{\alpha}(g_i(s))\left(\frac{\lambda}{2}\left(\tilde{z}^{-1}(g_i(t))\right)^2\right)^{\alpha}\xi^{\alpha}(s)ds-\Theta(s)\right)\mathrm{d}s \\ =&\left(\frac{7\nu_0}{256}-\frac{1}{4}\right)\int_{t_0}^{\infty}s\mathrm{d}s \\ =&\ \infty \quad \text{if } \nu_0>9.14 \end{aligned}$$

and

$$\begin{aligned} &\int_{t_0}^{\infty}\left(\nu(s)P_2^{\alpha}(g_i(s))\xi^{\alpha}(s)ds-\widetilde{\Theta}(s)\right)\mathrm{d}s \\ =&\left(\frac{\nu_0}{32}-\frac{1}{2}\right)\int_{t_0}^{\infty}\frac{1}{s}\mathrm{d}s \\ =&\ \infty \quad \text{if } \nu_0>16. \end{aligned}$$

From Theorem 2, Equation (37) is oscillatory if $\nu_0>16$.

4. Conclusions

In this paper, we study the qualitative and oscillatory properties of solutions to a class of fourth-order neutral delay differential equations with noncanonical operators. Via the Riccati transformation, we offer new criteria for the oscillation of all solutions to a given differential equation. Our technique essentially simplifies the process of investigation and reduces the number of conditions required in previously known results. We may say that, in future work, we will study oscillatory properties of Equation (1) with p-Laplacian like operators and under the condition

$$\int_{t_0}^{\infty} b^{-1/p-1}(s)\mathrm{d}s < \infty. \tag{38}$$

An interesting problem is to extend our results to even-order damped differential equations with p-Laplacian like operators

$$\left(b(t)\left(y^{(n-1)}(t)\right)^{p-1}\right)' + q(t)\left(y^{(n-1)}(t)\right)^{p-1} + \sum_{i=1}^{j} \nu_i(t)x^{p-1}(g_i(t)) = 0,$$

under the condition

$$\int_{t_0}^{\infty}\left[\frac{1}{b(s)}\exp\left(-\int_{t_0}^{t}\frac{q(z)}{b(z)}dz\right)\right]^{1/p-1} ds < \infty.$$

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