*fractal and fractional*

Article

# Abstract Fractional Monotone Approximation with Applications

**George A. Anastassiou**

Department of Mathematical Sciences, University of Memphis, Memphis, TN 38152, USA; ganastss@memphis.edu

**Abstract:** Here we extended our earlier fractional monotone approximation theory to abstract fractional monotone approximation, with applications to Prabhakar fractional calculus and non-singular kernel fractional calculi. We cover both the left and right sides of this constrained approximation. Let $f \in C^p([-1,1])$, $p \geq 0$ and let $L$ be a linear abstract left or right fractional differential operator such that $L(f) \geq 0$ over $[0,1]$ or $[-1,0]$, respectively. We can find a sequence of polynomials $Q_n$ of degree $\leq n$ such that $L(Q_n) \geq 0$ over $[0,1]$ or $[-1,0]$, respectively. Additionally $f$ is approximated quantitatively with rates uniformly by $Q_n$ with the use of first modulus of continuity of $f^{(p)}$.

**Keywords:** monotone fractional approximation; abstract fractional calculus; fractional linear differential operator; Prabhakar fractional calculus; non-singular kernel fractional calculi

**AMS 2020 Mathematics Subject Classification:** 41A10; 41A17; 41A25; 41A28; 41A29

check for updates

**Citation:** Anastassiou, G.A. Abstract Fractional Monotone Approximation with Applications. *Fractal Fract.* **2021**, *5*, 158. https://doi.org/10.3390/fractalfract5040158

Academic Editor: Vassili Kolokoltsov

Received: 19 August 2021
Accepted: 27 September 2021
Published: 9 October 2021



## 1. Introduction

The topic of monotone approximaton initiated in 1965, Ref. [12] by O. Shisha, and became a major trend in approximation theory. The original problem was: given a positive integer $k$, approximate with rates a given function $f$ whose $k$th derivative is $\geq 0$ by polynomials $(Q_n)_{n\in\mathbb{N}}$ having the same property.

In 1985, Ref. [2], the author and O. Shisha continued this study by replacing the $k$th derivative with a linear differential operator of order $k$ involving ordinary derivatives, again the approximation was with rates.

Later, in 1991, Ref. [3], the author extended this kind of study in two dimensions, etc.

In 2015, Ref. [4] (see chapters 1–8) went a step further, by starting the fractional monotone approximation, in that the linear differential operator is a fractional one, involving left or right side Caputo fractional derivatives.

To give a flavor of it, we need:

**Definition 1** ([5], p. 50). *Let $\alpha > 0$ and $\lceil \alpha \rceil = m \in \mathbb{N}$ ($\lceil \cdot \rceil$ is the ceiling of the number). Consider $f \in C^m([-1,1])$. We define the left side Caputo fractional derivative of $f$ of order $\alpha$ as follows:*

$$\left(D_{*-1}^{\alpha} f\right)(x) = \frac{1}{\Gamma(m-\alpha)} \int_{-1}^{x} (x-t)^{m-\alpha-1} f^{(m)}(t)dt, \tag{1}$$

*for any $x \in [-1,1]$, where $\Gamma$ is the gamma function. We set*

$$D_{*-1}^{0} f(x) = f(x), \;\; D_{*-1}^{m} f(x) = f^{(m)}(x), \;\; \forall\, x \in [-1,1].$$

In addition, to motivate our work, we mention:

**Theorem 1** ([4], p. 2). *Let $h, k, p$ be integers, $0 \leq h \leq k \leq p$ and let $f$ be a real function, with $f^{(p)}$ continuous in $[-1,1]$ and first modulus of continuity $\omega_1\left(f^{(p)}, \delta\right)$, where $\delta > 0$. Let $\alpha_j(x)$, $j = h, h+1, \ldots, k$ be real functions, defined and bounded on $[-1,1]$ and assume for $x \in [0,1]$*

*that $\alpha_h(x)$ is either $\geq$ some number $\alpha > 0$ or $\leq$ some number $\beta < 0$. Let the real numbers $\alpha_0 = 0 < \alpha_1 \leq 1 < \alpha_2 \leq 2 < \ldots < \alpha_p \leq p$. Here $D_{*-1}^{\alpha_j} f$ stands for the left Caputo fractional derivative of $f$ of order $\alpha_j$ anchored at $-1$. Consider the linear left fractional differential operator*

$$L := \sum_{j=h}^{k} \alpha_j(x) \left[ D_{*-1}^{\alpha_j} \right] \tag{2}$$

*and suppose, throughout $[0,1]$,*

$$L(f) \geq 0. \tag{3}$$

*Then, for any $n \in \mathbb{N}$, there exists a real polynomial $Q_n(x)$ of degree $\leq n$ such that*

$$L(Q_n) \geq 0 \text{ throughout } [0,1], \tag{4}$$

*and*

$$\max_{-1 \leq x \leq 1} |f(x) - Q_n(x)| \leq C n^{k-p} \omega_1 \left( f^{(p)}, \frac{1}{n} \right), \tag{5}$$

*where $C$ is a constant independent of $n$ or $f$.*

*As you see the monotonicity property here is true only on the critical interval $[0,1]$.*

We will use the following important result:

**Theorem 2** (see: [6] by S.A. Teljakovskii and [7] by R.M. Trigub)**.** *Let $n \in \mathbb{N}$. Be given a real function $g$, with $g^{(p)}$ continuous in $[-1,1]$, there exists a real polynomial $q_n(x)$ of degree $\leq n$ such that*

$$\max_{-1 \leq x \leq 1} \left| g^{(j)}(x) - q_n^{(j)}(x) \right| \leq R_p n^{j-p} \omega_1 \left( g^{(p)}, \frac{1}{n} \right), \tag{6}$$

*$j = 0, 1, \ldots, p$, where $R_p$ is a constant independent of $n$ or $g$.*

In this article, we perform abstract fractional calculus, left and right monotone approximation theory of Caputo type, and then we apply our results to Prabhakar fractional Calculus, generalized non-singular fractional calculus, and parametrized Caputo-Fabrizio non-singular fractional calculus.

Next, we build the related necessary fractional calculi background.

## 2. Fractional Calculi

Here, we need to be very specific in preparation for our main results.

### *2.1. Abstract Fractional Calculus*

Let $h, k \in \mathbb{Z}_+$, $p \in \mathbb{N} : 0 \leq h \leq k \leq p$. Let also $\mathbb{N} \not\ni \alpha_j > 0$, $j = 1, \ldots, p$, such that $\alpha_0 = 0 < \alpha_1 < 1 < \alpha_2 < 2 < \alpha_3 < 3 < \ldots < \ldots < \alpha_p < p$. That is $\lceil \alpha_j \rceil = j$, $j = 1, \ldots, p$; $\lceil \alpha_0 \rceil = 0$.

Consider the integrable functions $k_j := K_{\alpha_j} : [0,2] \to \mathbb{R}_+$, $j = 0, 1, \ldots, p$. Here, $g \in C^p([-1,1])$.

We consider the following abstract left side Caputo type fractional derivatives:

$$\left( {}^{k_j}D_{*-1}^{\alpha_j} g \right)(x) := \int_{-1}^{x} k_j(x-t) g^{(j)}(t) dt, \tag{7}$$

$j = 1, \ldots, p$; $\forall\, x \in [-1,1]$.

Similarly, we define the corresponding right side generalized Caputo type fractional derivatives:

$$\left( {}^{k_j}D_{1-}^{\alpha_j} g \right)(x) := (-1)^j \int_{x}^{1} k_j(t-x) g^{(j)}(t) dt, \tag{8}$$

$j = 1, \ldots, p$; $\forall\, x \in [-1,1]$.

We set

$$\left({}^{k_j}D^j_{*-1}g\right)(x) := g^{(j)}(x); \ \left({}^{k_j}D^j_{1-}g\right)(x) := (-1)^j g^{(j)}(x), \tag{9}$$

for $j = 1, \ldots, p$, and also we set

$$\left({}^{k_0}D^0_{*-1}g\right)(x) := \left({}^{k_0}D^0_{1-}g\right)(x) := g(x), \tag{10}$$

$\forall\ x \in [-1, 1]$.

We will assume that

$$\int_0^1 k_h(z)dz \geq 1, \text{ when } h \neq 0. \tag{11}$$

In the usual Caputo fractional derivatives case, it is

$$k_j(z) = \frac{z^{j-\alpha_j-1}}{\Gamma(j-\alpha_j)}, \ \ j = 1, \ldots, p; \ \forall\ z \in [0, 2], \tag{12}$$

and (11) is fulfilled, by the fact that $\Gamma(h - \alpha_h + 1) \leq 1$, see [4], p. 6.

### *2.2. About Prabhakar Fractional Calculus*

Here, we follow [8,9].

We consider the Prabhakar function (also known as the three parameter Mittag-Leffler function), (see [10], p. 97; [11])

$$E^\gamma_{\alpha,\beta}(z) = \sum_{k=0}^\infty \frac{(\gamma)_k}{k!\Gamma(\alpha k + \beta)} z^k, \tag{13}$$

where $\Gamma$ is the gamma function; $\alpha, \beta > 0$, $\gamma \in \mathbb{R}$, $z \in \mathbb{R}$, and $(\gamma)_k = \gamma(\gamma+1)...(\gamma+k-1)$. It is $E^0_{\alpha,\beta}(z) = \frac{1}{\Gamma(\beta)}$.

Let $a, b \in \mathbb{R}$, $a < b$ and $x \in [a, b]$; $f \in C([a, b])$. The left and right Prabhakar fractional integrals are defined ([8,9]) as follows:

$$\left(e^\gamma_{\rho,\mu,\omega,a+}f\right)(x) = \int_a^x (x-t)^{\mu-1} E^\gamma_{\rho,\mu}\left[\omega(x-t)^\rho\right] f(t)dt, \tag{14}$$

and

$$\left(e^\gamma_{\rho,\mu,\omega,b-}f\right)(x) = \int_x^b (t-x)^{\mu-1} E^\gamma_{\rho,\mu}\left[\omega(t-x)^\rho\right] f(t)dt, \tag{15}$$

where $\rho, \mu > 0$; $\gamma, \omega \in \mathbb{R}$.

Functions (14) and (15) are continuous, see [8].

Next, let $f \in C^N([a, b])$, where $N = \lceil \mu \rceil$, ($\lceil \cdot \rceil$ is the ceiling of the number), $0 < \mu \notin \mathbb{N}$. We define the Prabhakar-Caputo left and right fractional derivatives of order $\mu$ ([8,9]) as follows ($x \in [a, b]$):

$$\left({}^C D^\gamma_{\rho,\mu,\omega,a+}f\right)(x) = \int_a^x (x-t)^{N-\mu-1} E^{-\gamma}_{\rho,N-\mu}\left[\omega(x-t)^\rho\right] f^{(N)}(t)dt, \tag{16}$$

with ${}^C D^\gamma_{\rho,0,\omega,a+}f := f$; ${}^C D^\gamma_{\rho,N,\omega,a+}f := f^{(N)}$, $N \in \mathbb{N}$, and

$$\left({}^C D^\gamma_{\rho,\mu,\omega,b-}f\right)(x) = (-1)^N \int_x^b (t-x)^{N-\mu-1} E^{-\gamma}_{\rho,N-\mu}\left[\omega(t-x)^\rho\right] f^{(N)}(t)dt, \tag{17}$$

with ${}^C D^\gamma_{\rho,0,\omega,b-}f := f$; ${}^C D^\gamma_{\rho,N,\omega,b-}f := (-1)^N f^{(N)}$, $N \in \mathbb{N}$.

One can rewrite (16) and (17) as

$$\left({}^C D^\gamma_{\rho,\mu,\omega,a+}f\right)(x) = \left(e^{-\gamma}_{\rho,N-\mu,\omega,a+}f^{[N]}\right)(x), \tag{18}$$

and

$$\left({}^{C}D^{\gamma}_{\rho,\mu,\omega,b-}f\right)(x) = (-1)^{N}\left(e^{-\gamma}_{\rho,N-\mu,\omega,b-}f^{[N]}\right)(x), \tag{19}$$

$\forall\, x \in [a,b]$.

Clearly, the functions (18) and (19) are continuous.

By [8], we have that

$$\begin{aligned}\left(e^{\gamma}_{\rho,\mu,\omega,a+}f\right)(x) &= \int_a^x (x-t)^{\mu-1}\left(\sum_{k=0}^{\infty}\frac{(\gamma)_k}{k!\Gamma(\rho k+\mu)}\left(\omega(x-t)^{\rho}\right)^k\right)f(t)dt \\ &= \sum_{k=0}^{\infty}\frac{(\gamma)_k\omega^k}{k!\Gamma(\rho k+\mu)}\int_a^x (x-t)^{(\rho k+\mu)-1}f(t)dt,\end{aligned} \tag{20}$$

$\forall\, x \in [a,b]$.

That is

$$\begin{gathered}\left(e^{\gamma}_{\rho,\mu,\omega,a+}1\right)(x) = \sum_{k=0}^{\infty}\frac{(\gamma)_k\omega^k}{k!\Gamma(\rho k+\mu)}\int_a^x (x-t)^{(\rho k+\mu)-1}dt = \\ \sum_{k=0}^{\infty}\frac{(\gamma)_k\omega^k}{k!\Gamma(\rho k+\mu)}\frac{(x-a)^{(\rho k+\mu)}}{(\rho k+\mu)} = \sum_{k=0}^{\infty}\frac{(\gamma)_k\omega^k}{k!\Gamma(\rho k+\mu+1)}(x-a)^{(\rho k+\mu)} = \\ (x-a)^{\mu}\sum_{k=0}^{\infty}\frac{(\gamma)_k}{k!\Gamma(\rho k+\mu+1)}\left[\omega(x-a)^{\rho}\right]^k = (x-a)^{\mu}E^{\gamma}_{\rho,\mu+1}\left(\omega(x-a)^{\rho}\right).\end{gathered} \tag{21}$$

So, we have proved that

$$\left(e^{\gamma}_{\rho,\mu,\omega,a+}1\right)(x) = (x-a)^{\mu}E^{\gamma}_{\rho,\mu+1}\left(\omega(x-a)^{\rho}\right), \tag{22}$$

$\forall\, x \in [a,b]$.

Similarly, we have

$$\begin{gathered}\left(e^{\gamma}_{\rho,\mu,\omega,b-}1\right)(x) = \sum_{k=0}^{\infty}\frac{(\gamma)_k\omega^k}{k!\Gamma(\rho k+\mu)}\int_x^b (t-x)^{(\rho k+\mu)-1}dt = \\ \sum_{k=0}^{\infty}\frac{(\gamma)_k\omega^k}{k!\Gamma(\rho k+\mu)}\frac{(b-x)^{(\rho k+\mu)}}{(\rho k+\mu)} = \sum_{k=0}^{\infty}\frac{(\gamma)_k\omega^k}{k!\Gamma(\rho k+\mu+1)}(b-x)^{(\rho k+\mu)} = \\ (b-x)^{\mu}\sum_{k=0}^{\infty}\frac{(\gamma)_k}{k!\Gamma(\rho k+\mu+1)}\left[\omega(b-x)^{\rho}\right]^k = (b-x)^{\mu}E^{\gamma}_{\rho,\mu+1}\left(\omega(b-x)^{\rho}\right).\end{gathered} \tag{23}$$

So, it holds

$$\left(e^{\gamma}_{\rho,\mu,\omega,b-}1\right)(x) = (b-x)^{\mu}E^{\gamma}_{\rho,\mu+1}\left(\omega(b-x)^{\rho}\right), \tag{24}$$

$\forall\, x \in [a,b]$.

Next, we take $[a,b] = [-1,1]$.

Thus,

$$\left(e^{\gamma}_{\rho,\mu,\omega,-1+}1\right)(x) = (x+1)^{\mu}E^{\gamma}_{\rho,\mu+1}\left(\omega(x+1)^{\rho}\right), \tag{25}$$

and

$$\left(e^{\gamma}_{\rho,\mu,\omega,1-}1\right)(x) = (1-x)^{\mu}E^{\gamma}_{\rho,\mu+1}\left(\omega(1-x)^{\rho}\right), \tag{26}$$

$\forall\, x \in [-1,1]$.

Clearly, then we get

$$\left(e^{-\gamma}_{\rho,N-\mu,\omega,-1+}1\right)(x) = (x+1)^{N-\mu}E^{-\gamma}_{\rho,N-\mu+1}\left(\omega(x+1)^{\rho}\right), \tag{27}$$

and

$$\left(e^{-\gamma}_{\rho,N-\mu,\omega,1-}1\right)(x) = (1-x)^{N-\mu}E^{-\gamma}_{\rho,N-\mu+1}\left(\omega(1-x)^{\rho}\right), \tag{28}$$

$\forall\, x \in [-1,1]$.

Here, it is $N-\mu > 0$. By assumption we take $\rho > 0$, $\gamma < 0$ and, for convenience, we consider only $\omega > 0$.

Therefore, we derive the basic Hardy type inequalities:

$$\left\| e^{-\gamma}_{\rho,N-\mu,\omega,-1+}1 \right\|_{\infty,[-1,1]} \leq 2^{N-\mu}E^{-\gamma}_{\rho,N-\mu+1}(2^{\rho}\omega), \tag{29}$$

and

$$\left\| e^{-\gamma}_{\rho,N-\mu,\omega,1-}1 \right\|_{\infty,[-1,1]} \leq 2^{N-\mu}E^{-\gamma}_{\rho,N-\mu+1}(2^{\rho}\omega). \tag{30}$$

### *2.3. From Generalized Non-Singular Fractional Calculus*

We need

**Definition 2.** *Here, we use the multivariate analogue of generalized Mittag-Leffler function, see [12], defined for $\lambda, \gamma_j, \rho_j, z_j \in \mathbb{C}$, $Re(\rho_j) > 0$ ($j = 1, \ldots, m$) in terms of a multiple series of the form:*

$$E^{(\gamma_j)}_{(\rho_j),\lambda}(z_1,\ldots,z_m) = E^{(\gamma_1,\ldots,\gamma_m)}_{(\rho_1,\ldots,\rho_m),\lambda}(z_1,\ldots,z_m) =$$

$$\sum_{k_1,\ldots,k_m=0}^{\infty} \frac{(\gamma_1)_{k_1}\cdots(\gamma_m)_{k_m}}{\Gamma\left(\lambda + \sum\limits_{j=1}^{m} k_j\rho_j\right)} \frac{z_1^{k_1}\ldots z_m^{k_m}}{k_1!\ldots k_m!}, \tag{31}$$

*where $(\gamma_j)_{k_j}$ is the Pochhammer symbol, $\Gamma$ is the gamma function. By [13], p. 157, (31) converges for $Re(\rho_j) > 0$, $j = 1, \ldots, m$.*

In what follows, we will use the particular case of $E^{(\gamma_1,\ldots,\gamma_m)}_{(\rho,\ldots,\rho),\lambda}[\omega_1 t^{\rho},\ldots,\omega_m t^{\rho}]$, denoted by $E^{(\gamma_j)}_{(\rho),\lambda}[\omega_1 t^{\rho},\ldots,\omega_m t^{\rho}]$, where $0 < \rho < 1$, $t \geq 0$, $\lambda > 0$, $\gamma_j \in \mathbb{R}$ with $(\gamma_j)_{k_j} := \gamma_j(\gamma_j+1)\ldots(\gamma_j+k_j-1)$, $\omega_j \in \mathbb{R} - \{0\}$, for $j = 1, \ldots, m$.

Let now $f \in C^{n+1}([a,b])$, $n \in \mathbb{Z}_+$.

We define the Caputo type generalized left fractional derivative with non-singular kernel of order $n+\rho$, as

$$D^{n+\rho}_{a*}f(x) := {}^{CA}_{(\gamma_j)(\omega_j)}D^{n+\rho,\lambda}_{a*}f(x) :=$$

$$\frac{A(\rho)}{1-\rho}\int_a^x E^{(\gamma_j)}_{(\rho),\lambda}\left[\frac{-\omega_1\rho}{1-\rho}(x-t)^{\rho},\ldots,\frac{-\omega_m\rho}{1-\rho}(x-t)^{\rho}\right] f^{(n+1)}(t)dt, \tag{32}$$

$\forall\, x \in [a,b]$.

Similarly, we define the Caputo type generalized right fractional derivative with non-singular kernel of order $n+\rho$, as

$$D^{n+\rho}_{b-}f(x) := {}^{CA}_{(\gamma_j)(\omega_j)}D^{n+\rho,\lambda}_{b-}f(x) :=$$

$$(-1)^{n+1}\frac{A(\rho)}{1-\rho}\int_x^b E^{(\gamma_j)}_{(\rho),\lambda}\left[\frac{-\omega_1\rho}{1-\rho}(t-x)^{\rho},\ldots,\frac{-\omega_m\rho}{1-\rho}(t-x)^{\rho}\right] f^{(n+1)}(t)dt, \tag{33}$$

$\forall\, x \in [a,b]$.

Above $A(\rho)$ is a normalizing constant.

The above derivatives (32), (33) generalize the Atangana-Baleanu fractional derivatives [14].

We mention the following Hardy type inequalities:

**Theorem 3** ([15])**.** *All as above with $\gamma_j > 0$, $j = 1, \dots, m$; $\lambda = 1$. Then*

$$\left\{ \left\| D_{a*}^{n+\rho} f \right\|_{\infty}, \left\| D_{b-}^{n+\rho} f \right\|_{\infty} \right\} \leq \frac{(b-a)|A(\rho)|}{1-\rho}$$

$$E_{(\rho),2}^{(\gamma_j)} \left[ \frac{|\omega_1|\rho}{1-\rho}(b-a)^{\rho}, \dots, \frac{|\omega_m|\rho}{1-\rho}(b-a)^{\rho} \right] \left\| f^{(n+1)} \right\|_{\infty} < \infty, \tag{34}$$

*where $n \in \mathbb{Z}_+$.*

We also mention:

**Theorem 4** ([15])**.** *All as above with $\gamma_j > 0$, $j = 1, \dots, m$, and $\lambda > 0$, $0 < \rho < 1$, etc. Then*

$$D_{a*}^{n+\rho} f,\ D_{b-}^{n+\rho} f \in C([a,b]),\ n \in \mathbb{Z}_+.$$

We rewrite (32) and (33), and for $[a,b] = [-1,1]$.

Let $\mu > 0$ with $\mu \notin \mathbb{N}$ and $\lceil \mu \rceil = n \in \mathbb{N}$. That is $0 < 1 - n + \mu < 1$, and let $f \in C^n([-1,1])$. Then, we have

$$D_{-1*}^{\mu} f(x) := {}_{(\gamma_j)(\omega_j)}^{CA} D_{-1*}^{\mu,\lambda} f(x) :=$$

$$\frac{A(1-n+\mu)}{n-\mu} \int_{-1}^{x} E_{(1-n+\mu),\lambda}^{(\gamma_j)} \left[ \frac{-\omega_1(1-n+\mu)}{n-\mu}(x-t)^{1-n+\mu}, \dots, \right. \tag{35}$$

$$\left. \frac{-\omega_m(1-n+\mu)}{n-\mu}(x-t)^{1-n+\mu} \right] f^{(n)}(t)dt,$$

and

$$D_{1-}^{\mu} f(x) := {}_{(\gamma_j)(\omega_j)}^{CA} D_{1-}^{\mu,\lambda} f(x) :=$$

$$(-1)^n \frac{A(1-n+\mu)}{n-\mu} \int_{x}^{1} E_{(1-n+\mu),\lambda}^{(\gamma_j)} \left[ \frac{-\omega_1(1-n+\mu)}{n-\mu}(t-x)^{1-n+\mu}, \dots, \right. \tag{36}$$

$$\left. \frac{-\omega_m(1-n+\mu)}{n-\mu}(t-x)^{1-n+\mu} \right] f^{(n)}(t)dt,$$

$\forall\, x \in [-1,1]$.

We will set $D_{-1*}^{0} f = f$, $D_{1-}^{0} f = f$, and $D_{-1*}^{m} f = f^{(m)}$, $D_{1-}^{m} f = (-1)^m f^{(m)}$, when $m \in \mathbb{N}$.

We make

**Remark 1.** *Fractional Calculi of Sections 2.2 and 2.3 are special cases of abstract fractional calculus, see Section 2.1. In particular, the important condition (11) is fulfilled.*

So, we will verify $\int_0^1 k_h(z)dz \geq 1$, $h \neq 0$.

(I) First, for Section 2.2:

We notice that

$$\int_0^1 z^{N-\mu-1} E_{\rho,N-\mu}^{-\gamma}(\omega z^{\rho})dz =$$

(here $\rho, N-\mu > 0$, $\gamma < 0$, $\omega > 0$)

$$\int_0^1 z^{N-\mu-1} \sum_{k=0}^{\infty} \frac{(-\gamma)_k}{k!\Gamma(\rho k + N - \mu)} (\omega z^{\rho})^k dz =$$

$$\sum_{k=0}^{\infty} \frac{(-\gamma)_k \omega^k}{k!\Gamma(\rho k + N - \mu)} \int_0^1 z^{N-\mu-1} z^{\rho k} dz = \tag{37}$$

$$\sum_{k=0}^{\infty} \frac{(-\gamma)_k \omega^k}{k!\Gamma(\rho k + N - \mu)} \int_0^1 z^{(\rho k + N)-\mu-1} dz =$$

$$\sum_{k=0}^{\infty} \frac{(-\gamma)_k \omega^k}{k!\Gamma(\rho k + N - \mu + 1)} = E^{-\gamma}_{\rho, N-\mu+1}(\omega) \geq 1,$$

for suitable $\omega > 0$.

(II) Next, for Section 2.3:

Here $\gamma_j > 0$, $j = 1, \ldots, m$; $\lambda = 1$; $\mathbb{N} \not\ni \mu > 0$, $\lceil \mu \rceil = n \in \mathbb{N}$, $\omega_j < 0$, $j = 1, \ldots, m$.

Without loss of generality we assume that $A(1-n+\mu) > 0$.

We have that

$$\frac{A(1-n+\mu)}{n-\mu} \int_0^1 E^{(\gamma_j)}_{(1-n+\mu),1} \Bigg[ \frac{-\omega_1(1-n+\mu)}{n-\mu} z^{1-n+\mu}, \ldots, \frac{-\omega_m(1-n+\mu)}{n-\mu} z^{1-n+\mu} \Bigg] dz =$$

(here $0 < 1-(n-\mu) = 1-n+\mu < 1$)

$$\frac{A(1-n+\mu)}{n-\mu} \int_0^1 \Bigg[ \sum_{k_1,\ldots,k_m=0}^{\infty} \frac{(\gamma_1)_{k_1} \ldots (\gamma_m)_{k_m}}{\Gamma\left(1 + \left(\sum_{j=1}^m k_j\right)(1-n+\mu)\right)} \frac{\prod_{j=1}^m \left( \left( \frac{-\omega_j(1-n+\mu)}{n-\mu} \right)^{k_j} z^{(1-n+\mu) \sum_{j=1}^m k_j} \right)}{k_1! \ldots k_m!} \Bigg] dz$$

$$= \left( \frac{A(1-n+\mu)}{n-\mu} \right)$$

$$\sum_{k_1,\ldots,k_m=0}^{\infty} \frac{(\gamma_1)_{k_1} \ldots (\gamma_m)_{k_m}}{\Gamma\left(1 + \left(\sum_{j=1}^m k_j\right)(1-n+\mu)\right)} \frac{\prod_{j=1}^m \left( \frac{-\omega_j(1-n+\mu)}{n-\mu} \right)^{k_j}}{k_1! \ldots k_m!} \int_0^1 z^{(1-n+\mu) \sum_{j=1}^m k_j} dz \tag{38}$$

$$= \left( \frac{A(1-n+\mu)}{n-\mu} \right) \sum_{k_1,\ldots,k_m=0}^{\infty} \frac{(\gamma_1)_{k_1} \ldots (\gamma_m)_{k_m}}{\Gamma\left(2 + \left(\sum_{j=1}^m k_j\right)(1-n+\mu)\right)} \frac{\prod_{j=1}^m \left( \frac{-\omega_j(1-n+\mu)}{n-\mu} \right)^{k_j}}{k_1! \ldots k_m!}$$

$$= \frac{A(1-n+\mu)}{n-\mu} E^{(\gamma_j)}_{(1-n+\mu),2} \left( \frac{-\omega_1(1-n+\mu)}{n-\mu}, \ldots, \frac{-\omega_m(1-n+\mu)}{n-\mu} \right) \geq 1, \tag{39}$$

for suitable $\omega_j < 0$, for $j = 1, \ldots, m$.

We also need

**Definition 3.** *Let $f \in C^n([-1,1])$, $\mathbb{N} \not\ni \mu > 0$, $\lceil \mu \rceil = n \in \mathbb{N}$; $\omega < 0$. That is $0 < 1 - n + \mu < 1$. The parametrized Caputo-Fabrizio non-singular kernel fractional derivatives, left and right of order $\mu$, respectively, are given as follows (also see [16]):*

$$ {}_{\omega}^{CF}D_{-1+}^{\mu} f(x) := \frac{1}{n-\mu} \int_{-1}^{x} \exp\left( -\frac{(1-n+\mu)\omega}{n-\mu}(x-t) \right) f^{(n)}(t)dt, \tag{40} $$

$$ {}_{\omega}^{CF}D_{1-}^{\mu} f(x) := \frac{(-1)^n}{n-\mu} \int_{x}^{1} \exp\left( -\frac{(1-n+\mu)\omega}{n-\mu}(t-x) \right) f^{(n)}(t)dt, \tag{41} $$

$\forall\, x \in [-1,1]$.

*Equations (40) and (41) are special cases of (7) and (8).*

We make

**Remark 2.** *We want to evaluate*

$$ \infty > \int_0^1 \exp\left( -\frac{(1-n+\mu)\omega}{n-\mu} z \right) dz $$

*(call $\delta := -\frac{(1-n+\mu)\omega}{n-\mu}$)*

$$ = \int_0^1 e^{\delta z} dz = \frac{1}{\delta} e^{\delta z}|_0^1 = \frac{1}{\delta}\left(e^{\delta} - 1\right) = \frac{e^{\delta}}{\delta} - \frac{1}{\delta} = \frac{\sum\limits_{k=0}^{\infty} \frac{\delta^k}{k!}}{\delta} - \frac{1}{\delta} \tag{42} $$

$$ = \sum_{k=0}^{\infty} \frac{\delta^{k-1}}{k!} = \sum_{k=0}^{\infty} \left( \frac{\left(\frac{1-n+\mu}{n-\mu}\right)^{k-1} (-\omega)^{k-1}}{k!} \right) \geq 1, $$

*for suitable $\omega < 0$.*

So, again condition (11) is fulfilled.

## 3. Main Results

We give

**Theorem 5.** *Let $h, k, p$ be integers, $0 \leq h \leq k \leq p \in \mathbb{N}$ and let $f$ be a real function, $f^{(p)}$ is continuous in $[-1,1]$ with modulus of continuity $\omega_1\left(f^{(p)}, \delta\right)$, $\delta > 0$. Let $\alpha_j(x)$, $j = h, h+1, \ldots, k$ be real functions, defined and bounded on $[-1,1]$ and assume for $x \in [0,1]$ that $\alpha_h(x)$ is either $\geq$ some number $\alpha > 0$ or $\leq$ some number $\beta < 0$. Let the real numbers $\alpha_0 = 0 < \alpha_1 < 1 < \alpha_2 < 2 < \ldots < \alpha_p < p$. Here, we adopt the abstract fractional calculus terminology and assumptions from above. So, ${}^{k_j}D_{*-1}^{\alpha_j} f$ stands for the abstract left Caputo type fractional derivative of order $\alpha_j$ anchored at $-1$. We consider the linear abstract left fractional differential operator*

$$ L := \sum_{j=h}^{k} \alpha_j(x) \left[ {}^{k_j}D_{*-1}^{\alpha_j} \right] \tag{43} $$

*and suppose, throughout $[0,1]$,*

$$ L(f) \geq 0. $$

*Then, for any $n \in \mathbb{N}$, there exists a real polynomial $Q_n(x)$ of degree $\leq n$ such that*

$$ L(Q_n) \geq 0 \text{ throughout } [0,1], \tag{44} $$

*and*

$$\max_{-1\leq x\leq 1}|f(x)-Q_n(x)|\leq Cn^{k-p}\omega_1\left(f^{(p)},\frac{1}{n}\right), \tag{45}$$

*where C is independent of n or f.*

**Proof.** Let $n\in\mathbb{N}$. By Theorem 2 given a real function $g$, with $g^{(p)}$ continuous in $[-1,1]$, there exists a real polynomial $q_n(x)$ of degree $\leq n$ such that

$$\max_{-1\leq x\leq 1}\left|g^{(j)}(x)-q_n^{(j)}(x)\right|\leq R_p n^{j-p}\omega_1\left(g^{(p)},\frac{1}{n}\right), \tag{46}$$

$j=0,1,\dots,p$, where $R_p$ is independent of $n$ or $g$.

We notice that ($x\in[-1,1]$)

$$\begin{gathered}\left|\left({}^{k_j}D_{*-1}^{\alpha_j}g\right)(x)-\left({}^{k_j}D_{*-1}^{\alpha_j}q_n\right)(x)\right|=\\ \left|\int_{-1}^{x}k_j(x-t)g^{(j)}(t)dt-\int_{-1}^{x}k_j(x-t)q_n^{(j)}(t)dt\right|=\\ \left|\int_{-1}^{x}k_j(x-t)\left(g^{(j)}(t)-q_n^{(j)}(t)\right)dt\right|\leq\\ \int_{-1}^{x}k_j(x-t)\left|g^{(j)}(t)-q_n^{(j)}(t)\right|dt\overset{(46)}{\leq}\\ \left(\int_{-1}^{x}k_j(x-t)dt\right)R_p n^{j-p}\omega_1\left(g^{(p)},\frac{1}{n}\right)=\\ \left(\int_{0}^{x+1}k_j(z)dz\right)R_p n^{j-p}\omega_1\left(g^{(p)},\frac{1}{n}\right)\leq\left(\int_0^2 k_j(z)dz\right)R_p n^{j-p}\omega_1\left(g^{(p)},\frac{1}{n}\right).\end{gathered} \tag{47}$$

We have proved that

$$\begin{gathered}\left|\left({}^{k_j}D_{*-1}^{\alpha_j}g\right)(x)-\left({}^{k_j}D_{*-1}^{\alpha_j}q_n\right)(x)\right|\leq\\ \left(\int_0^2 k_j(z)dz\right)R_p n^{j-p}\omega_1\left(g^{(p)},\frac{1}{n}\right),\ \forall\, x\in[-1,1].\end{gathered} \tag{48}$$

That is:

$$\begin{gathered}\max_{-1\leq x\leq 1}\left|\left({}^{k_j}D_{*-1}^{\alpha_j}g\right)(x)-\left({}^{k_j}D_{*-1}^{\alpha_j}q_n\right)(x)\right|\leq\\ \left(\int_0^2 k_j(z)dz\right)R_p n^{j-p}\omega_1\left(g^{(p)},\frac{1}{n}\right),\ j=1,\dots,p.\end{gathered}$$

So, we have

$$\max_{-1\leq x\leq 1}\left|\left({}^{k_j}D_{*-1}^{\alpha_j}g\right)(x)-\left({}^{k_j}D_{*-1}^{\alpha_j}q_n\right)(x)\right|\leq\lambda_j R_p n^{j-p}\omega_1\left(g^{(p)},\frac{1}{n}\right), \tag{49}$$

where

$$\lambda_j:=\int_0^2 k_j(z)dz,\quad j=1,\dots,p. \tag{50}$$

Inequality (49) is valid when $j=0$ by (46), and we can set $\lambda_0=1$.

Put

$$s_j\equiv\sup_{-1\leq x\leq 1}\left|\alpha_h^{-1}(x)\alpha_j(x)\right|,\ j=h,\dots,k, \tag{51}$$

and

$$\eta_n:=R_p\omega_1\left(f^{(p)},\frac{1}{n}\right)\left(\sum_{j=h}^{k}s_j\lambda_j n^{j-p}\right). \tag{52}$$

I. Suppose, throughout $[0,1]$, $\alpha_h(x) \geq \alpha > 0$. Let $Q_n(x)$, $x \in [-1,1]$, be a real polynomial of degree $\leq n$ so that

$$\max_{-1\leq x\leq 1}\left|{}^{k_j}D^{\alpha_j}_{*-1}\left(f(x)+\eta_n(h!)^{-1}x^h\right)-\left({}^{k_j}D^{\alpha_j}_{*-1}Q_n\right)(x)\right| \overset{(49)}{\leq}$$

$$\lambda_j R_p n^{j-p}\omega_1\left(f^{(p)},\frac{1}{n}\right), \tag{53}$$

$j=0,1,\ldots,p$.

In particular, ($j=0$) holds

$$\max_{-1\leq x\leq 1}\left|\left(f(x)+\eta_n(h!)^{-1}x^h\right)-Q_n(x)\right| \overset{(53)}{\leq} R_p n^{-p}\omega_1\left(f^{(p)},\frac{1}{n}\right), \tag{54}$$

and

$$\max_{-1\leq x\leq 1}|f(x)-Q_n(x)| \leq \eta_n(h!)^{-1}+R_p n^{-p}\omega_1\left(f^{(p)},\frac{1}{n}\right)=$$

$$(h!)^{-1}R_p\omega_1\left(f^{(p)},\frac{1}{n}\right)\left(\sum_{j=h}^{k}s_j\lambda_j n^{j-p}\right)+R_p n^{-p}\omega_1\left(f^{(p)},\frac{1}{n}\right)\leq \tag{55}$$

$$R_p\omega_1\left(f^{(p)},\frac{1}{n}\right)n^{k-p}\left(1+(h!)^{-1}\sum_{j=h}^{k}s_j\lambda_j\right).$$

That is:

$$\max_{-1\leq x\leq 1}|f(x)-Q_n(x)| \leq R_p\left(1+(h!)^{-1}\sum_{j=h}^{k}s_j\lambda_j\right)n^{k-p}\omega_1\left(f^{(p)},\frac{1}{n}\right), \tag{56}$$

proving (45).

Here,

$$L=\sum_{j=h}^{k}\alpha_j(x)\left[{}^{k_j}D^{\alpha_j}_{*-1}\right],$$

and suppose, throughout $[0,1]$, $Lf \geq 0$. So over $0 \leq x \leq 1$, using (52) and (53), we have

$$\alpha_h^{-1}(x)L(Q_n(x))=\alpha_h^{-1}(x)L(f(x))+\frac{\eta_n}{h!}\,{}^{k_h}D^{\alpha_h}_{*-1}x^h+$$

$$\sum_{j=h}^{k}\alpha_h^{-1}(x)\alpha_j(x)\left[{}^{k_j}D^{\alpha_j}_{*-1}Q_n(x)-{}^{k_j}D^{\alpha_j}_{*-1}f(x)-\frac{\eta_n}{h!}\,{}^{k_j}D^{\alpha_j}_{*-1}x^h\right]\geq \tag{57}$$

$$\frac{\eta_n}{h!}\,{}^{k_h}D^{\alpha_h}_{*-1}x^h-\left(\sum_{j=h}^{k}s_j\lambda_j n^{j-p}\right)R_p\omega_1\left(f^{(p)},\frac{1}{n}\right)=\frac{\eta_n}{h!}\,{}^{k_h}D^{\alpha_h}_{*-1}x^h-\eta_n=:\varphi$$

(if $h=0$, then $\alpha_h=0$, and $\varphi=0$).

If $h\neq 0$, then

$$\varphi=\eta_n\left(\frac{{}^{k_h}D^{\alpha_h}_{*-1}x^h}{h!}-1\right)=\eta_n\left(\int_{-1}^{x}k_h(x-t)dt\ -1\right)=$$

$$\eta_n\left(\int_0^{x+1}k_h(z)dz\ -1\right)\geq\eta_n\left(\int_0^{1}k_h(z)dz\ -1\right)\geq 0 \tag{58}$$

by the assumption (11): $\int_0^1 k_h(z)dz \geq 1$, when $h\neq 0$.

Hence, in both cases, we get

$$L(Q_n(x)) \geq 0, \ x \in [0,1]. \tag{59}$$

II. Suppose, throughout $[0,1]$, $\alpha_h(x) \leq \beta < 0$. In this case let $Q_n(x)$, $x \in [-1,1]$, be a real polynomial of degree $\leq n$ such that

$$\max_{-1 \leq x \leq 1} \left| {}^{k_j}D_{*-1}^{\alpha_j}\left(f(x) - \eta_n (h!)^{-1} x^h\right) - \left({}^{k_j}D_{*-1}^{\alpha_j} Q_n\right)(x)\right| \overset{(49)}{\leq}$$

$$\lambda_j R_p n^{j-p} \omega_1\left(f^{(p)}, \frac{1}{n}\right), \tag{60}$$

$j = 0, 1, \ldots, p$.

In particular, ($j = 0$) holds

$$\max_{-1 \leq x \leq 1} \left| \left(f(x) - \eta_n (h!)^{-1} x^h\right) - Q_n(x)\right| \overset{(60)}{\leq} R_p n^{-p} \omega_1\left(f^{(p)}, \frac{1}{n}\right), \tag{61}$$

and

$$\max_{-1 \leq x \leq 1} |f(x) - Q_n(x)| \leq \eta_n (h!)^{-1} + R_p n^{-p} \omega_1\left(f^{(p)}, \frac{1}{n}\right) \overset{\text{(as before)}}{\leq} \tag{62}$$

$$R_p \omega_1\left(f^{(p)}, \frac{1}{n}\right) n^{k-p} \left(1 + (h!)^{-1} \sum_{j=h}^{k} s_j \lambda_j\right).$$

That is, (45) is again true.

Again suppose, throughout $[0,1]$, $Lf \geq 0$. Also if $0 \leq x \leq 1$, then

$$\alpha_h^{-1}(x) L(Q_n(x)) = \alpha_h^{-1}(x) L(f(x)) - \frac{\eta_n}{h!} \, {}^{k_h}D_{*-1}^{\alpha_h} x^h +$$

$$\sum_{j=h}^{k} \alpha_h^{-1}(x) \alpha_j(x) \left[{}^{k_j}D_{*-1}^{\alpha_j} Q_n(x) - {}^{k_j}D_{*-1}^{\alpha_j} f(x) + \frac{\eta_n}{h!} \, {}^{k_j}D_{*-1}^{\alpha_j} x^h\right] \overset{(60)}{\leq}$$

$$-\frac{\eta_n}{h!} \, {}^{k_h}D_{*-1}^{\alpha_h} x^h + \left(\sum_{j=h}^{k} s_j \lambda_j n^{j-p}\right) R_p \omega_1\left(f^{(p)}, \frac{1}{n}\right) = -\frac{\eta_n}{h!} \, {}^{k_h}D_{*-1}^{\alpha_h} x^h + \eta_n =: \psi$$

(if $h = 0$, then $\alpha_h = 0$, and $\psi = 0$).

If $h \neq 0$, then

$$\psi = \eta_n \left[1 - \frac{{}^{k_h}D_{*-1}^{\alpha_h} x^h}{h!}\right] = \eta_n \left[1 - \int_{-1}^{x} k_h(x-t)dt\right] = \tag{63}$$

$$\eta_n \left[1 - \int_0^{x+1} k_h(z)dz\right] \leq \eta_n \left[1 - \int_0^1 k_h(z)dz\right] \leq 0.$$

Hence, again, in both cases

$$L(Q_n(x)) \geq 0, \ \forall \, x \in [0,1]. \tag{64}$$

□

We also present

**Theorem 6.** *Let $h, k, p$ be integers, $0 \leq h \leq k \leq p \in \mathbb{N}$, where $h$ is even, and let $f$ be a real function, $f^{(p)}$ is continuous in $[-1,1]$ with modulus of continuity $\omega_1\left(f^{(p)}, \delta\right)$, $\delta > 0$. Let $\alpha_j(x)$, $j = h, h+1, \ldots, k$ be real functions, defined and bounded on $[-1,1]$ and assume for $x \in [-1,0]$*

*that $\alpha_h(x)$ is either $\geq$ some number $\alpha > 0$ or $\leq$ some number $\beta < 0$. Let the real numbers $\alpha_0 = 0 < \alpha_1 < 1 < \alpha_2 < 2 < \ldots < \alpha_p < p$. Here, we adopt the abstract fractional calculus terminology and assumptions from above. So ${}^{k_j}D_{1-}^{\alpha_j} f$ stands for the abstract right Caputo type fractional derivative of order $\alpha_j$ anchored at 1. We consider the linear abstract right fractional differential operator*

$$L := \sum_{j=h}^{k} \alpha_j(x) \left[ {}^{k_j}D_{1-}^{\alpha_j} \right] \tag{65}$$

*and suppose, throughout $[-1, 0]$,*

$$L(f) \geq 0.$$

*Then, for any $n \in \mathbb{N}$, there exists a real polynomial $Q_n(x)$ of degree $\leq n$ such that*

$$L(Q_n) \geq 0 \ \text{ throughout } [-1, 0], \tag{66}$$

*and*

$$\max_{-1 \leq x \leq 1} |f(x) - Q_n(x)| \leq C n^{k-p} \omega_1 \left( f^{(p)}, \frac{1}{n} \right), \tag{67}$$

*where C is independent of n or f.*

**Proof.** Let $x \in [-1, 1]$, we observe that

$$\begin{gathered}
\left| \left( {}^{k_j}D_{1-}^{\alpha_j} g \right)(x) - \left( {}^{k_j}D_{1-}^{\alpha_j} q_n \right)(x) \right| = \\
\left| \int_x^1 k_j(t-x) g^{(j)}(t) dt - \int_x^1 k_j(t-x) q_n^{(j)}(t) dt \right| = \\
\left| \int_x^1 k_j(t-x) \left( g^{(j)}(t) - q_n^{(j)}(t) \right) dt \right| \leq \\
\int_x^1 k_j(t-x) \left| g^{(j)}(t) - q_n^{(j)}(t) \right| dt \overset{(6)}{\leq} \\
\left( \int_x^1 k_j(t-x) dt \right) R_p n^{j-p} \omega_1 \left( g^{(p)}, \frac{1}{n} \right) = \\
\left( \int_0^{1-x} k_j(z) dz \right) R_p n^{j-p} \omega_1 \left( g^{(p)}, \frac{1}{n} \right) \leq \left( \int_0^2 k_j(z) dz \right) R_p n^{j-p} \omega_1 \left( g^{(p)}, \frac{1}{n} \right).
\end{gathered} \tag{68}$$

That is, we have derived

$$\begin{gathered}
\max_{-1 \leq x \leq 1} \left| \left( {}^{k_j}D_{1-}^{\alpha_j} g \right)(x) - \left( {}^{k_j}D_{1-}^{\alpha_j} q_n \right)(x) \right| \leq \\
\left( \int_0^2 k_j(z) dz \right) R_p n^{j-p} \omega_1 \left( g^{(p)}, \frac{1}{n} \right), \ \ j = 1, \ldots, p.
\end{gathered} \tag{69}$$

We call

$$\lambda_j := \int_0^2 k_j(z) dz, \quad j = 1, \ldots, p. \tag{70}$$

Therefore, we can write

$$\max_{-1 \leq x \leq 1} \left| \left( {}^{k_j}D_{1-}^{\alpha_j} g \right)(x) - \left( {}^{k_j}D_{1-}^{\alpha_j} q_n \right)(x) \right| \leq \lambda_j R_p n^{j-p} \omega_1 \left( g^{(p)}, \frac{1}{n} \right), \tag{71}$$

for $j = 1, \ldots, p$.

Inequality (71) is valid when $j = 0$ by (6), so we can set $\lambda_0 = 1$.

Put

$$s_j \equiv \sup_{-1 \leq x \leq 1} \left| \alpha_h^{-1}(x) \alpha_j(x) \right|, \ \ j = h, \ldots, k, \tag{72}$$

and
$$\eta_n := R_p \omega_1 \left( f^{(p)}, \frac{1}{n} \right) \left( \sum_{j=h}^{k} s_j \lambda_j n^{j-p} \right). \tag{73}$$

I. Suppose, throughout $[-1,0]$, $\alpha_h(x) \geq \alpha > 0$. Let $Q_n(x)$, $x \in [-1,1]$, be a real polynomial of degree $\leq n$ so that

$$\max_{-1 \leq x \leq 1} \left| {}^{k_j}D_{1-}^{\alpha_j} \left( f(x) + \eta_n (h!)^{-1} x^h \right) - \left( {}^{k_j}D_{1-}^{\alpha_j} Q_n \right)(x) \right| \overset{(71)}{\leq}$$
$$\lambda_j R_p n^{j-p} \omega_1 \left( f^{(p)}, \frac{1}{n} \right), \tag{74}$$

$j = 0, 1, \ldots, p$.

In particular ($j = 0$) holds

$$\max_{-1 \leq x \leq 1} \left| \left( f(x) + \eta_n (h!)^{-1} x^h \right) - Q_n(x) \right| \overset{(74)}{\leq} R_p n^{-p} \omega_1 \left( f^{(p)}, \frac{1}{n} \right), \tag{75}$$

and, as earlier,

$$\max_{-1 \leq x \leq 1} |f(x) - Q_n(x)| \leq R_p \left( 1 + (h!)^{-1} \sum_{j=h}^{k} s_j \lambda_j \right) n^{k-p} \omega_1 \left( f^{(p)}, \frac{1}{n} \right), \tag{76}$$

proving (67).

Here,
$$L = \sum_{j=h}^{k} \alpha_j(x) \left[ {}^{k_j}D_{1-}^{\alpha_j} \right],$$

and suppose, throughout $[-1,0]$, $Lf \geq 0$. So over $-1 \leq x \leq 0$, we get

$$\alpha_h^{-1}(x) L(Q_n(x)) = \alpha_h^{-1}(x) L(f(x)) + \frac{\eta_n}{h!} \, {}^{k_h}D_{1-}^{\alpha_h} x^h +$$
$$\sum_{j=h}^{k} \alpha_h^{-1}(x) \alpha_j(x) \left[ {}^{k_j}D_{1-}^{\alpha_j} Q_n(x) - {}^{k_j}D_{1-}^{\alpha_j} f(x) - \frac{\eta_n}{h!} \, {}^{k_j}D_{1-}^{\alpha_j} x^h \right] \overset{(74)}{\geq} \tag{77}$$
$$\frac{\eta_n}{h!} \, {}^{k_h}D_{1-}^{\alpha_h} x^h - \left( \sum_{j=h}^{k} s_j \lambda_j n^{j-p} \right) R_p \omega_1 \left( f^{(p)}, \frac{1}{n} \right) = \frac{\eta_n}{h!} \, {}^{k_h}D_{1-}^{\alpha_h} x^h - \eta_n =: \xi$$

(if $h = 0$, then $\alpha_h = 0$, and $\xi = 0$).

If $h \neq 0$, then

$$\xi = \eta_n \left( \frac{{}^{k_h}D_{1-}^{\alpha_h} x^h}{h!} - 1 \right) = \eta_n \left( (-1)^h \int_x^1 k_h(t-x) dt - 1 \right) =$$

($h$ is even)
$$\eta_n \left( \int_x^1 k_h(t-x) dt - 1 \right) = \eta_n \left( \int_0^{1-x} k_h(z) dz - 1 \right) \geq$$
$$\eta_n \left( \int_0^1 k_h(z) dz - 1 \right) \geq 0, \tag{78}$$

by the assumption (11): $\int_0^1 k_h(z) dz \geq 1$, when $h \neq 0$.

Hence, in both cases, we get

$$L(Q_n(x)) \geq 0, \; x \in [-1,0]. \tag{79}$$

II. Suppose, throughout $[-1,0]$, $\alpha_h(x) \leq \beta < 0$. Let $Q_n(x)$, $x \in [-1,1]$, be a real polynomial of degree $\leq n$ so that

$$\max_{-1\leq x\leq 1}\left|{}^{k_j}D_{1-}^{\alpha_j}\left(f(x)-\eta_n(h!)^{-1}x^h\right)-\left({}^{k_j}D_{1-}^{\alpha_j}Q_n\right)(x)\right| \overset{(71)}{\leq}$$

$$\lambda_j R_p n^{j-p}\omega_1\left(f^{(p)},\frac{1}{n}\right), \tag{80}$$

$j = 0, 1, \ldots, p$.

In particular ($j = 0$) holds

$$\max_{-1\leq x\leq 1}\left|\left(f(x)-\eta_n(h!)^{-1}x^h\right)-Q_n(x)\right| \overset{(80)}{\leq} R_p n^{-p}\omega_1\left(f^{(p)},\frac{1}{n}\right), \tag{81}$$

and, as earlier,

$$\max_{-1\leq x\leq 1}|f(x)-Q_n(x)| \leq R_p\left(1+(h!)^{-1}\sum_{j=h}^{k}s_j\lambda_j\right)n^{k-p}\omega_1\left(f^{(p)},\frac{1}{n}\right), \tag{82}$$

proving (67).

Again, suppose, throughout $[-1,0]$, $Lf \geq 0$. Also if $-1 \leq x \leq 0$, then

$$\alpha_h^{-1}(x)L(Q_n(x)) = \alpha_h^{-1}(x)L(f(x)) - \frac{\eta_n}{h!}\,{}^{k_h}D_{1-}^{\alpha_h}x^h +$$

$$\sum_{j=h}^{k}\alpha_h^{-1}(x)\alpha_j(x)\left[{}^{k_j}D_{1-}^{\alpha_j}Q_n(x) - {}^{k_j}D_{1-}^{\alpha_j}f(x) + \frac{\eta_n}{h!}\,{}^{k_j}D_{1-}^{\alpha_j}x^h\right] \overset{(80)}{\leq} \tag{83}$$

$$-\frac{\eta_n}{h!}\,{}^{k_h}D_{1-}^{\alpha_h}x^h + \left(\sum_{j=h}^{k}s_j\lambda_j n^{j-p}\right)R_p\omega_1\left(f^{(p)},\frac{1}{n}\right) = -\frac{\eta_n}{h!}\,{}^{k_h}D_{1-}^{\alpha_h}x^h + \eta_n$$

$$= \eta_n\left(1-\frac{{}^{k_h}D_{1-}^{\alpha_h}x^h}{h!}\right) =: \rho$$

(if $h = 0$, then $\alpha_h = 0$, and $\rho = 0$).

If $h \neq 0$, then

$$\rho = \eta_n\left(1-\int_x^1 k_h(t-x)dt\right) = \eta_n\left(1-\int_0^{1-x}k_h(z)dz\right) \leq \tag{84}$$

$$\eta_n\left(1-\int_0^1 k_h(z)dz\right) \leq 0.$$

Hence, in both cases, we get,again

$$L(Q_n(x)) \geq 0,\ \forall\, x \in [-1,0]. \tag{85}$$

□

**Conclusion 1.** *Clearly Theorem 5 generalizes Theorem 1, and Theorem 6 generalizes Theorem 2.2, p. 12 of [4]. Furthermore, there, the approximating polynomial $Q_n$ depends on $f, \eta_n, h$; which $\eta_n$ depends on $n$, $R_p, n, k, s_j, \lambda_j$; which $\lambda_j$ depends on $k_j$. I.e. polynomial $Q_n$ among others depends on the type of fractional calculus we use.*

*Consequently, Theorem 5 is valid for the following left fractional linear differential operators:*

*(1)*

$$L_1 := \sum_{j=h}^{k}\alpha_j(x)\left[{}^{C}D^{\gamma}_{\rho,\alpha_j,\omega,-1+}\right], \tag{86}$$

*where $\rho > 0$, $\gamma < 0$, and $\omega > 0$ large enough (from Prabhakar fractional calculus, see (16));*

*(2)*

$$L_2 := \sum_{j=h}^{k} \alpha_j(x) \left[ D_{-1*}^{\alpha_j} \right], \tag{87}$$

*(see (35)) where $\gamma_j > 0$, $j = 1, \ldots, m$; $\lambda = 1$; and small enough $\omega_j < 0$, $j = 1, \ldots, m$ (from generalized non-singular fractional calculus);*

*and (3)*

$$L_3 := \sum_{j=h}^{k} \alpha_j(x) \left[ {}_{\omega}^{CF} D_{-1+}^{\alpha_j} \right], \tag{88}$$

*with $\omega < 0$, sufficiently small (from parametrized Caputo-Fabrizio non-singular kernel fractional calculus).*

*Similarly, Theorem 6 is valid for the following right fractional linear differential operators:*

*(1)**

$$L_1^* := \sum_{j=h}^{k} \alpha_j(x) \left[ {}^{C} D_{\rho, \alpha_j, \omega, 1-}^{\gamma} \right], \tag{89}$$

*where $\rho > 0$, $\gamma < 0$, and $\omega > 0$ large enough (from Prabhakar fractional calculus, see (17));*

*(2)**

$$L_2^* := \sum_{j=h}^{k} \alpha_j(x) \left[ D_{1-}^{\alpha_j} \right], \tag{90}$$

*(see (36)) where $\gamma_j > 0$, $j = 1, \ldots, m$; $\lambda = 1$; and small enough $\omega_j < 0$, $j = 1, \ldots, m$ (from generalized non-singular fractional calculus);*

*and (3)**

$$L_3^* := \sum_{j=h}^{k} \alpha_j(x) \left[ {}_{\omega}^{CF} D_{1-}^{\alpha_j} \right], \tag{91}$$

*with $\omega < 0$, sufficiently small (from parametrized Caputo-Fabrizio non-singular kernel fractional calculus).*

*Our developed abstract fractional monotone approximation theory with its applications involves weaker conditions than the one with ordinary derivatives ([2]) and can cover many diverse general cases in a multitude of complex settings and environments.*

**Funding:** This research received no external funding.

**Institutional Review Board Statement:** Not applicable.

**Informed Consent Statement:** Not applicable.

**Conflicts of Interest:** The authors declare no conflict of interest.

## References

1. Shisha, O. Monotone approximation. *Pac. J. Math.* **1965**, *15*, 667–671. [CrossRef]
2. Anastassiou, G.A.; Shisha, O. Monotone approximation with linear differential operators. *J. Approx. Theory***1985**, *44*, 391–393. [CrossRef]
3. Anastassiou, G.A. Bivariate Monotone Approximation. *Proc. Am. Math. Soc.* **1991**, *112*, 959–964. [CrossRef]
4. Anastassiou, G.A. *Frontiers in Approximation Theory*; World Scientific Publ. Corp.: Hackensack, NJ, USA; Singapore, 2015.
5. Diethelm, K. *The Analysis of Fractional Differential Equations*, 1st ed.; Lecture Notes in Mathematics; Springer: New York, NY, USA; Heidelberg, Germany, 2010; Volume 2004.
6. Teljakovskii, S.A. Two theorems on the approximation of functions by algebraic polynomials. *Mat. Sb.* **1966**, *70*, 252–265. (In Russian); *Am. Math. Soc. Trans.* **1968**, *77*, 163–178. (In English)
7. Trigub, R.M. Approximation of functions by polynomials with integer coefficients. *Izv. Akad. Nauk SSSR Ser. Mat.* **1962**, *26*, 261–280. (In Russian)
8. Anastassiou, G.A. Foundations of Generalized Prabhakar-Hilfer fractional Calculus with Applications. **2021**, submitted.
9. Polito, F.; Tomovski, Z. Some properties of Prabhakar-type fractional calculus operators. *Fract. Differ.* **2016**, *1*, 73–94. [CrossRef]

10. Gorenflo, R.; Kilbas, A.; Mainardi, F.; Rogosin, S. *Mittag-Leffler Functions, Related Topics and Applications*; Springer: Heidelberg, Germany; New York, NY, USA, 2014.
11. Giusti, A.; Colombaro, I.; Garra, R.; Garrappa, R.; Polito, F.; Popolizio, M.; Mainardi, F. A practical Guide to Prabhakar Fractional Calculus. *Fract. Calc. Appl. Anal.* **2020**, *23*, 9–54. [CrossRef]
12. Saxena, R.K.; Kalla, S.L.; Saxena, R. Multivariate analogue of generalized Mittag-Leffler function. *Integral Transform Special Funct.* **2011**, *22*, 533–548. [CrossRef]
13. Srivastava, H.M.; Daoust, M.C. A note on the convergence of Kompe' de Feriet's double hypergeometric series. *Math. Nachr.* **1972**, *53*, 151–159. [CrossRef]
14. Atangana, A.; Baleanu, D. New fractional derivatives with nonlocal and non-singular kernel: Theory and application to heat transfer model. *Therm. Sci.* **2016**, *20*, 763–769. [CrossRef]
15. Anastassiou, G.A. Multiparameter Fractional Differentiation with non singular kernel. **2021**, submitted.
16. Losada, J.; Nieto, J.J. Properties of a New Fractional Derivative without Singular Kernel. *Progr. Fract. Differ. Appl.* **2015**, *1*, 87–92.