Article

Multiweighted-Type Fractional Fourier Transform: Unitarity

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Abstract: The definition of the discrete fractional Fourier transform (DFRFT) varies, and the multiweighted-type fractional Fourier transform (M-WFRFT) is its extended definition. It is not easy to prove its unitarity. We use the weighted-type fractional Fourier transform, fractional-order matrix and eigendecomposition-type fractional Fourier transform as basic functions to prove and discuss the unitarity. Thanks to the growing body of research, we found that the effective weighting term of the M-WFRFT is only four terms, none of which are extended to M terms, as described in the definition. Furthermore, the program code is analyzed, and the result shows that the previous work (Digit Signal Process 2020: 104: 18) based on MATLAB for unitary verification is inaccurate.

Keywords: fractional fourier transform; weighted-type fractional Fourier transform; multiweighted-type fractional fourier transform; unitarity

1. Introduction

The multiweighted-type fractional Fourier transform (M-WFRFT) is the extended definition of the weighted-type fractional Fourier transform (WFRFT), and its application has been described in detail in our previous research [1]. Here, we focus on summarizing and analyzing the theory of the M-WFRFT. In [2], Zhu et al. proposed the definition of the multifraction Fourier transform, i.e., the M-WFRFT. Researchers have applied this definition to image encryption but have not discussed the properties of the definition itself. Early research work [3–5] laid a solid foundation for the proposal of the M-WFRFT. In 1995, Shih proposed a new type of fractional-order Fourier transform (FRFT), which is called WFRFT because it is a linear summation [3]. This definition has period 4, so it is also called the 4-WFRFT. Subsequently, Liu et al. extended the definition of the WFRFT, and the generalized definition has period M = $4l$, where l = 1,2, ... [4,5]. Zhu's M-WFRFT is proposed on this basis, and its period is any integer M > 4 [2]. However, little is known about the properties of these definitions. Ran et al. sought to present a unified framework with the help of a generalized permutation matrix group and discussed its properties [6]. This research greatly promotes the theoretical development of WFRFTs. Unfortunately, there is no proof of unitarity, and the focus of the previous studies has been the generality of weighted coefficients. Recently, some new definitions based on the M-WFRFT have been proposed [7–11]. For example, Tao et al. proposed multiple-parameter fractional Fourier transforms (MPFRFTs) [8], Ran et al. proposed modified multiple-parameter fractional Fourier transforms (m-MPFRFTs) [9], and Zhao et al. proposed vector power multiple-parameter fractional Fourier transforms (VPMPFRFTs) [10,11]. Unfortunately, the properties of these definitions have not been discussed.

First, Santhanam et al. demonstrated the properties of the WFRFT and proved its unitarity using weighted coefficients [12]. However, this work ignores that the basis function is also a part of the definition. For the M-WFRFT, its basis function is the fractional power of the Fourier transform, so it is not easy to prove its properties. Some recent research results have also failed to prove its properties [13–18]. We proposed a new reformulation of the M-WFRFT to prove its periodicity, additivity and boundary [1]. Unfortunately,

unitarity is only discussed by means of numerical simulation. This paper is a follow-up of previous research work and mainly seeks to prove and discuss the unitarity of the M-WFRFT. However, the most recent studies have also enlightened our research [19,20].

The remainder of this paper is organized as follows. Section 2 proposes a new reformulation of the M-WFRFT. The unitarity of the M-WFRFT is proven in Section 3. The deviation caused by the numerical simulation is discussed in Section 4. Finally, the conclusions are presented in Section 5.

2. Reformulation of M-WFRFT

Shih proposed the WFRFT [3], and its definition can be expressed as

$$F^\alpha[f(t)] = \sum_{l=0}^{3} A_l^\alpha f_l(t),$$　　　(1)

with

$$A_l^\alpha = \cos\left(\frac{(\alpha - l)\pi}{4}\right) \cos\left(\frac{2(\alpha - l)\pi}{4}\right) \exp\left(\frac{3(\alpha - l)i\pi}{4}\right),$$　　　(2)

where $f_l(t) = F^l[f(t)]; l = 0, 1, 2, 3$, (F denotes Fourier transform). Shih's WFRFT with period 4 is also called the 4-weighted-type fractional Fourier transform (4-WFRFT).

We further improve the weighted coefficient A_l^α, as shown in Equation (3).

$$
\begin{aligned}
A_l^\alpha &= \cos\left(\frac{(\alpha-l)\pi}{4}\right) \cos\left(\frac{2(\alpha-l)\pi}{4}\right) \exp\left(\frac{3(\alpha-l)i\pi}{4}\right) \\
&= \frac{1}{2} \times \left[\exp\left(\frac{(\alpha-l)\pi i}{4}\right) + \exp\left(\frac{-(\alpha-l)\pi i}{4}\right)\right] \\
&\quad \times \frac{1}{2} \times \left[\exp\left(\frac{2(\alpha-l)\pi i}{4}\right) + \exp\left(\frac{-2(\alpha-l)\pi i}{4}\right)\right] \times \exp\left(\frac{3(\alpha-l)i\pi}{4}\right) \\
&= \frac{1}{4}\left(1 + \exp\left(\frac{2(\alpha-l)\pi i}{4}\right) + \exp\left(\frac{4(\alpha-l)\pi i}{4}\right) + \exp\left(\frac{6(\alpha-l)\pi i}{4}\right)\right) \\
&= \frac{1}{4}\sum_{k=0}^{3} \exp\left(\frac{2\pi ik(\alpha-l)}{4}\right).
\end{aligned}
$$　　　(3)

Then, we can obtain Equation (4) as

$$
\begin{pmatrix} A_0^\alpha \\ A_1^\alpha \\ A_2^\alpha \\ A_3^\alpha \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix} \begin{pmatrix} B_0^\alpha \\ B_1^\alpha \\ B_2^\alpha \\ B_3^\alpha \end{pmatrix},
$$　　　(4)

where $B_k^\alpha = \exp\left(\frac{2\pi ik\alpha}{4}\right)$, $k = 0, 1, 2, 3$. Equation (4) provides ideas for expanding the definition in the future.

Liu et al., generalize Shih's definition, and the generalized definition is shown to have M-periodic eigenvalues with respect to the order of Hermite–Gaussian functions ($M = 4l$, where $l = 1,2,3, ...$) [4,5].

Subsequently, Zhu et al. proposed a multifractional Fourier transform whose period can be any integer ($M > 4$), so this definition is also called the M-WFRFT [2]. Zhu et al., extended the weighting coefficient A_l^α, which is more general; the result is shown in Equation (5).

$$
\begin{pmatrix} A_0^\alpha \\ A_1^\alpha \\ \vdots \\ A_{M-1}^\alpha \end{pmatrix} = \frac{1}{M} \begin{pmatrix} u^{0\times0} & u^{0\times1} & \cdots & u^{0\times(M-1)} \\ u^{1\times0} & u^{1\times1} & \cdots & u^{1\times(M-1)} \\ \vdots & \vdots & \ddots & \vdots \\ u^{(M-1)\times0} & u^{(M-1)\times1} & \cdots & u^{(M-1)\times(M-1)} \end{pmatrix} \begin{pmatrix} B_0^\alpha \\ B_1^\alpha \\ \vdots \\ B_{M-1}^\alpha \end{pmatrix},
$$　　　(5)

where $u = \exp(-2\pi i/M)$ and $B_k^\alpha = \exp\left(\frac{2\pi i k \alpha}{M}\right)$, $k = 0, 1, \cdots, M - 1$. Then,

$$A_l^\alpha = \frac{1}{M} \sum_{k=0}^{M-1} \exp\left[\frac{2\pi i k(\alpha - l)}{M}\right];$$
$$l = 0, 1, \cdots, M - 1. \tag{6}$$

The M-WFRFT is defined as

$$F_M^\alpha[f(t)] = \sum_{l=0}^{M-1} A_l^\alpha f_l(t), \tag{7}$$

where the basic functions are $f_l(t) = F^{4l/M}[f(t)]$; $l = 0, 1, \cdots, M - 1$ (F denotes the Fourier transform).

At present, the M-WFRFT is widely used in image encryption and signal processing [7–11,21–25]. Unfortunately, few researchers have discussed its properties, and the proponents of the definition have not explained the properties. We find that it is not easy to prove the properties of the M-WFRFT (Equation (7)). Some researchers have discussed the properties using the weighted coefficient A_l^α but ignore that the basis function is also a part of the definition [6,12]. Therefore, we present a new reformulation of the M-WFRFT. As such, Equation (7) can be expressed as

$$
\begin{aligned}
F_M^\alpha[f(t)] &= A_0^\alpha f_0(t) + A_1^\alpha f_1(t) + \cdots + A_{M-1}^\alpha f_{M-1}(t) \\
&= A_0^\alpha F^{\frac{0}{M}}[f(t)] + A_1^\alpha F^{\frac{4}{M}}[f(t)] + \cdots + A_{M-1}^\alpha F^{\frac{4(M-1)}{M}}[f(t)] \\
&= \left(A_0^\alpha I + A_1^\alpha F^{\frac{4}{M}} + \cdots + A_{M-1}^\alpha F^{\frac{4(M-1)}{M}} \right) f(t) \\
&= \left(I, F^{\frac{4}{M}}, \cdots, F^{\frac{4(M-1)}{M}} \right)
\begin{pmatrix}
A_0^\alpha \\
A_1^\alpha \\
\vdots \\
A_{M-1}^\alpha
\end{pmatrix} f(t).
\end{aligned} \tag{8}
$$

By Equations (5) and (8), Equation (9) is obtained as

$$F_M^\alpha[f(t)] = \frac{1}{M}\left(I, F^{\frac{4}{M}}, \cdots, F^{\frac{4(M-1)}{M}} \right)
\begin{pmatrix}
u^{0\times 0} & u^{0\times 1} & \cdots & u^{0\times(M-1)} \\
u^{1\times 0} & u^{1\times 1} & \cdots & u^{1\times(M-1)} \\
\vdots & \vdots & \ddots & \vdots \\
u^{(M-1)\times 0} & u^{(M-1)\times 1} & \cdots & u^{(M-1)\times(M-1)}
\end{pmatrix}
\begin{pmatrix}
B_0^\alpha \\
B_1^\alpha \\
\vdots \\
B_{M-1}^\alpha
\end{pmatrix} f(t), \tag{9}$$

where $u = \exp(-2\pi i/M)$, $B_k^\alpha = \exp\left(\frac{2\pi i k \alpha}{M}\right)$, $k = 0, 1, \ldots, M - 1$ and F denotes the Fourier transform. Here, let

$$
\begin{cases}
Y_0 = u^{0\times 0}I + u^{1\times 0}F^{\frac{4}{M}} + \cdots + u^{(M-1)\times 0}F^{\frac{4(M-1)}{M}}, \\
Y_1 = u^{0\times 1}I + u^{1\times 1}F^{\frac{4}{M}} + \cdots + u^{(M-1)\times 1}F^{\frac{4(M-1)}{M}}, \\
Y_2 = u^{0\times 2}I + u^{1\times 2}F^{\frac{4}{M}} + \cdots + u^{(M-1)\times 2}F^{\frac{4(M-1)}{M}}, \\
\quad\vdots \\
Y_{M-1} = u^{0\times(M-1)}I + u^{1\times(M-1)}F^{\frac{4}{M}} + \cdots + u^{(M-1)\times(M-1)}F^{\frac{4(M-1)}{M}}.
\end{cases} \tag{10}
$$

Definition 1. A new reformulation of the M-WFRFT as

$$
\begin{aligned}
T_{MW}^{\alpha}[f(t)] \quad &= \tfrac{1}{M}(Y_0, Y_1, \cdots, Y_{M-1})
\begin{pmatrix}
B_0^{\alpha} \\
B_1^{\alpha} \\
\vdots \\
B_{M-1}^{\alpha}
\end{pmatrix} f(t) \\
&= \tfrac{1}{M} \sum_{k=0}^{M-1} Y_k B_k^{\alpha} f(t),
\end{aligned}
\tag{11}
$$

where $B_k^{\alpha} = \exp\left(\frac{2\pi i k \alpha}{M}\right)$; $k = 0, 1, \cdots, M - 1$.

Remark 1. *Our previous work [1] discussed that the new reformulation helps to prove the properties. Unitarity is often used in signal processing. Unfortunately, previous research work only presents simulation verification. This paper will seek to prove and discuss the unitarity.*

3. Unitarity

A complex matrix U satisfies

$$
UU^H = U^H U = I,
\tag{12}
$$

where H denotes the conjugate transpose and I is the identity matrix. Then, U is called a unitary matrix. The greatest difficulty in proving the unitarity of the M-WFRFT is considering the basis function $F^{4l/M}$, $l = 0, 1, \cdots, M - 1$. The basis function is related to the discrete fractional Fourier transform (DFRFT), and the definition of the DFRFT varies. Therefore, we seek to use different types of DFRFT as the basis function to verify the unitarity of M-WFRFT.

3.1. 4-WFRFT as the Basis Function

Proposition 1. *4-WFRFT is used as the basis function, so the M-WFRFT has unitarity.*

Proof. The definition of the 4-WFRFT is shown in Equation (1), and Equation (13) can be obtained as

$$
\begin{aligned}
F^{\alpha}[f(t)] \quad &= \left(A_0^{\alpha} \cdot I + A_1^{\alpha} \cdot F + A_2^{\alpha} \cdot F^2 + A_3^{\alpha} \cdot F^3\right) f(t) \\
&= (I, F, F^2, F^3)
\begin{pmatrix}
A_0^{\alpha} \\
A_1^{\alpha} \\
A_2^{\alpha} \\
A_3^{\alpha}
\end{pmatrix} f(t).
\end{aligned}
\tag{13}
$$

From Equations (4) and (13), we obtain

$$
F^{\alpha}[f(t)] = \frac{1}{4}\left(I, F, F^2, F^3\right)
\begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & -i & -1 & i \\
1 & -1 & 1 & -1 \\
1 & i & -1 & -i
\end{pmatrix}
\begin{pmatrix}
B_0^{\alpha} \\
B_1^{\alpha} \\
B_2^{\alpha} \\
B_3^{\alpha}
\end{pmatrix} f(t),
\tag{14}
$$

where $B_k^{\alpha} = \exp\left(\frac{2\pi i k \alpha}{4}\right)$, $k = 0, 1, 2, 3$. Here, let

$$
\begin{cases}
P_0 = I + F + F^2 + F^3 \\
P_1 = I - F * i - F^2 + F^3 * i \\
P_2 = I - F + F^2 - F^3 \\
P_3 = I + F * i - F^2 - F^3 * i.
\end{cases}
\tag{15}
$$

Then, the 4-WFRFT can be re-expressed as

$$
T_{4W}^{\alpha}[f(t)] = \frac{1}{4}(P_0, P_1, P_2, P_3)
\begin{pmatrix} B_0^{\alpha} \\ B_1^{\alpha} \\ B_2^{\alpha} \\ B_3^{\alpha} \end{pmatrix} f(t). \tag{16}
$$

Thus, the discrete 4-WFRFT can be expressed as

$$
T_{4W}^{\alpha} = \frac{1}{4}(P_0, P_1, P_2, P_3)
\begin{pmatrix} B_0^{\alpha} \\ B_1^{\alpha} \\ B_2^{\alpha} \\ B_3^{\alpha} \end{pmatrix}. \tag{17}
$$

From Equation (10), Y_k can be expressed as

$$
Y_k = u^{0 \times k} \times I + u^{1 \times k} \times F^{\frac{4}{M}} + \cdots + u^{(M-1) \times k} \times F^{\frac{4(M-1)}{M}}; \\
k = 0, 1, \cdots, M-1, \tag{18}
$$

The 4-WFRFT as the basis function is

$$
Y_k = u^{0 \times k} \times T_{4M}^{0} + u^{1 \times k} \times T_{4M}^{\frac{4}{M}} + \cdots + u^{(M-1) \times k} \times T_{4M}^{\frac{4(M-1)}{M}}. \tag{19}
$$

From Equations (17) and (19), we can obtain

$$
\begin{aligned}
Y_k &= \tfrac{1}{4}(P_0, P_1, P_2, P_3) \left(u^{0 \times k} \times \begin{pmatrix} B_0^0 \\ B_1^0 \\ B_2^0 \\ B_3^0 \end{pmatrix} + u^{1 \times k} \times \begin{pmatrix} B_0^{\frac{4}{M}} \\ B_1^{\frac{4}{M}} \\ B_2^{\frac{4}{M}} \\ B_3^{\frac{4}{M}} \end{pmatrix} + \cdots + u^{(M-1) \times k} \times \begin{pmatrix} B_0^{\frac{4(M-1)}{M}} \\ B_1^{\frac{4(M-1)}{M}} \\ B_2^{\frac{4(M-1)}{M}} \\ B_3^{\frac{4(M-1)}{M}} \end{pmatrix} \right) \\
&= \tfrac{1}{4}(P_0, P_1, P_2, P_3) \begin{pmatrix} u^{0 \times k} \times B_0^0 + u^{1 \times k} \times B_0^{\frac{4}{M}} + \cdots + u^{(M-1) \times k} \times B_0^{\frac{4(M-1)}{M}} \\ u^{0 \times k} \times B_1^0 + u^{1 \times k} \times B_1^{\frac{4}{M}} + \cdots + u^{(M-1) \times k} \times B_1^{\frac{4(M-1)}{M}} \\ u^{0 \times k} \times B_2^0 + u^{1 \times k} \times B_2^{\frac{4}{M}} + \cdots + u^{(M-1) \times k} \times B_2^{\frac{4(M-1)}{M}} \\ u^{0 \times k} \times B_3^0 + u^{1 \times k} \times B_3^{\frac{4}{M}} + \cdots + u^{(M-1) \times k} \times B_3^{\frac{4(M-1)}{M}} \end{pmatrix},
\end{aligned} \tag{20}
$$

where $k = 0, 1, \cdots, M-1$ and $u = \exp(-2\pi i/M)$. Therefore, we obtain

$$
\begin{aligned}
Y_k &= \tfrac{1}{4}(P_0, P_1, P_2, P_3) \begin{pmatrix} 1 + \exp\left(\frac{-2\pi i 1 k}{M}\right) + \exp\left(\frac{-2\pi i 2 k}{M}\right) + \cdots + \exp\left(\frac{-2\pi i (M-1) k}{M}\right) \\ 1 + \exp\left(\frac{-2\pi i 1 (k-1)}{M}\right) + \exp\left(\frac{-2\pi i 2 (k-1)}{M}\right) + \cdots + \exp\left(\frac{-2\pi i (M-1)(k-1)}{M}\right) \\ 1 + \exp\left(\frac{-2\pi i 1 (k-2)}{M}\right) + \exp\left(\frac{-2\pi i 2 (k-2)}{M}\right) + \cdots + \exp\left(\frac{-2\pi i (M-1)(k-2)}{M}\right) \\ 1 + \exp\left(\frac{-2\pi i 1 (k-3)}{M}\right) + \exp\left(\frac{-2\pi i 2 (k-3)}{M}\right) + \cdots + \exp\left(\frac{-2\pi i (M-1)(k-3)}{M}\right) \end{pmatrix} \\
&= \tfrac{1}{4}(P_0, P_1, P_2, P_3) \begin{pmatrix} S_0(k) \\ S_1(k) \\ S_2(k) \\ S_3(k) \end{pmatrix}.
\end{aligned} \tag{21}
$$

For sequence $S_0(k)$, it can be expressed as

$$
S_0(k) = \frac{a_1 (1 - q^M)}{1 - q} = \frac{1 - \exp\left(\frac{-2\pi i k}{M}\right)^M}{1 - \exp\left(\frac{-2\pi i k}{M}\right)}. \tag{22}
$$

where $a_1 = 1$. Then, we obtain

$$S_0(k) = \begin{cases} M, & k \equiv 0 \bmod M \\ 0, & k \not\equiv 0 \bmod M. \end{cases} \tag{23}$$

For sequence $S_1(k)$,

$$S_1(k) = \frac{1 - \left(e^{-2\pi i(k-1)/M}\right)^M}{1 - e^{-2\pi i(k-1)/M}}, \tag{24}$$

we obtain

$$S_1(k) = \begin{cases} M, & k \equiv 1 \bmod M \\ 0, & k \not\equiv \ \bmod M. \end{cases} \tag{25}$$

For sequence $S_2(k)$,

$$S_2(k) = \frac{1 - \left(e^{-2\pi i(k-2)/M}\right)^M}{1 - e^{-2\pi i(k-2)/M}}, \tag{26}$$

we obtain

$$S_2(k) = \begin{cases} M, & k \equiv 2 \bmod M \\ 0, & k \not\equiv 2 \bmod M. \end{cases} \tag{27}$$

For sequence $S_3(k)$,

$$S_3(k) = \frac{1 - \left(e^{-2\pi i(k-3)/M}\right)^M}{1 - e^{-2\pi i(k-3)/M}}, \tag{28}$$

we obtain

$$S_3(k) = \begin{cases} M, & k \equiv 3 \bmod M \\ 0, & k \not\equiv 3 \bmod M. \end{cases} \tag{29}$$

Then, Equation (21) can be expressed as

$$Y_k = \begin{cases} \frac{M}{4} P_k, & k = 0, 1, 2, 3 \\ 0, & k = 4, 5, \cdots, M-1. \end{cases} \tag{30}$$

Therefore, the M-WFRFT Equation (11) is written as

$$\begin{aligned} T_{MW}^{\alpha} &= \frac{1}{M}(Y_0, Y_1, \cdots, Y_{M-1}) \begin{pmatrix} B_0^{\alpha} \\ B_1^{\alpha} \\ \vdots \\ B_{M-1}^{\alpha} \end{pmatrix} \\ &= \frac{1}{4}(P_0, P_1, P_2, P_3, 0, \cdots, 0) \begin{pmatrix} B_0^{\alpha} \\ B_1^{\alpha} \\ \vdots \\ B_{M-1}^{\alpha} \end{pmatrix} \\ &= \frac{1}{4}(P_0, P_1, P_2, P_3) \begin{pmatrix} B_0^{\alpha} \\ B_1^{\alpha} \\ B_2^{\alpha} \\ B_3^{\alpha} \end{pmatrix}. \end{aligned} \tag{31}$$

From the expressions, we notice that Equations (17) and (31) are the same, but in fact they are different. The difference is that for Equation (31), $B_k^{\alpha} = \exp\left(\frac{2\pi i k \alpha}{M}\right)$; $k = 0, 1, \cdots, M - 1$. However, this does not affect the proof of unitarity. $\square$

Remark 2. *In our previous work [1], we proved the unitarity of Equation (17). When the 4-WFRFT is selected as the basis function, the M-WFRFT has unitarity. From Equation (31), we notice that the weighted sum of the M-WFRFT is only four terms.*

3.2. Fractional-Order Matrix as the Basis Function

In our previous numerical simulation, a fractional-order matrix was used to verify the unitarity of the M-WFRFT [1]. In this section, we present the theoretical analysis to improve the previous work.

Proposition 2. *Fractional-order matrix is used as the basis function, so the M-WFRFT has unitarity.*

Proof. The calculation of the fractional power of the matrix is applied to the eigenvalues, so eigenvalue decomposition of the matrix is required. Therefore, the eigendecomposition of the matrix can be expressed as

$$F = VDV^H, \tag{32}$$

where F is the DFT matrix, V is the eigenvector, and D is the eigenvalue matrix.

In [26,27], the eigenvalues of the DFT can be expressed as $\lambda_n = e^{n\pi i/2}$. Then, the possible values of the eigenvalue are $\lambda_r = \{1, -1, i, -i\}; r = 1, 2, \cdots, n$. In this way, the eigenvalue matrix D can be expressed as

$$D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}. \tag{33}$$

Then, the fractional power operation of matrix F can be expressed as

$$F^{4l/M} = VD^{4l/M}V^H. \tag{34}$$

where $l = 0, 1, \cdots, M-1$. For Equation (10), Y_k can be expressed as

$$
\begin{aligned}
Y_k &= u^{0 \times k}I + u^{1 \times k} \times F^{\frac{4}{M}} + \cdots + u^{(M-1) \times k} \times F^{\frac{4(M-1)}{M}} \\
&= u^{0 \times k}VD^0V^H + u^{1 \times k}VD^{4/M}V^H + \cdots + u^{(M-1) \times k}VD^{4(M-1)/M}V^H.
\end{aligned} \tag{35}
$$

where $k = 0, 1, \cdots, M-1$. Therefore, we can obtain

$$
\begin{aligned}
Y_k &= V\left(u^{0 \times k} \times D^0 + u^{1 \times k} \times D^{4/M} + \cdots + u^{(M-1) \times k} \times D^{4(M-1)/M}\right)V^H \\
&= V\begin{pmatrix} u^{0 \times k}\lambda_1^0 + u^{1 \times k}\lambda_1^{4/M} + \cdots + u^{(M-1) \times k}\lambda_1^{4(M-1)/M} & 0 & \cdots & 0 \\ 0 & u^{0 \times k}\lambda_2^0 + u^{1 \times k}\lambda_2^{4/M} + \cdots + u^{(M-1) \times k}\lambda_2^{4(M-1)/M} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & u^{0 \times k}\lambda_n^0 + u^{1 \times k}\lambda_n^{4/M} + \cdots + u^{(M-1) \times k}\lambda_n^{4(M-1)/M} \end{pmatrix}V^H \\
&= V\begin{pmatrix} Q_1(k) & 0 & \cdots & 0 \\ 0 & Q_2(k) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & Q_n(k) \end{pmatrix}V^H.
\end{aligned} \tag{36}
$$

Here, let

$$
\begin{cases}
Q_1(k) = u^{0 \times k}\lambda_1^0 + u^{1 \times k}\lambda_1^{4/M} + \cdots + u^{(M-1) \times k}\lambda_1^{4(M-1)/M} \\
Q_2(k) = u^{0 \times k}\lambda_2^0 + u^{1 \times k}\lambda_2^{4/M} + \cdots + u^{(M-1) \times k}\lambda_2^{4(M-1)/M} \\
\vdots \\
Q_n(k) = u^{0 \times k}\lambda_n^0 + u^{1 \times k}\lambda_n^{4/M} + \cdots + u^{(M-1) \times k}\lambda_n^{4(M-1)/M}.
\end{cases} \tag{37}
$$

The multiplicities of the DFT eigenvalues [26,27] are shown in Table 1. Therefore, there is

$$
\begin{aligned}
\lambda_r &= \{1, i, -1, -i\} \\
&= \left\{ e^{4n\pi i/2}, e^{(4n+1)\pi i/2}, e^{(4n+2)\pi i/2}, e^{(4n+3)\pi i/2} \right\} \\
&= \left\{ e^{2n\pi i}e^{0\pi i/2}, e^{2n\pi i}e^{\pi i/2}, e^{2n\pi i}e^{2\pi i/2}, e^{2n\pi i}e^{3\pi i/2} \right\} \\
&= \left\{ e^{0\pi i/2}, e^{\pi i/2}, e^{2\pi i/2}, e^{3\pi i/2} \right\}.
\end{aligned}
\tag{38}
$$

Table 1. Multiplicities of the DFT eigenvalues.

N	1	-1	$-i$	i
$4n$	$n+1$	n	n	$n-1$
$4n+1$	$n+1$	n	n	n
$4n+2$	$n+1$	$n+1$	n	n
$4n+3$	$n+1$	$n+1$	$n+1$	n

For Equation (37), $Q_r(k), r = 1, 2, \cdots, n$ can be expressed as

$$
Q_r(k) = u^{0 \times k}\lambda_r^0 + u^{1 \times k}\lambda_r^{4/M} + \cdots + u^{(M-1) \times k}\lambda_r^{4(M-1)/M}.
\tag{39}
$$

When the eigenvalues $\lambda_r = e^{0\pi i/2} = 1$ and $u = e^{-2\pi i/M}$, $Q_r^{(1)}(k)$ can be expressed using Equation (39), as

$$
\begin{aligned}
Q_r^{(1)}(k) &= u^{0 \times k}\lambda_r^0 + u^{1 \times k}\lambda_r^{4/M} + \cdots + u^{(M-1) \times k}\lambda_r^{4(M-1)/M} \\
&= 1 + e^{-2\pi i 1(k-0)/M} + e^{-2\pi i 2(k-0)/M} + \cdots + e^{-2\pi i(M-1)(k-0)/M} \\
&= \frac{1 - \left(e^{-2\pi i(k-0)/M}\right)^M}{1 - e^{-2\pi i(k-0)/M}}.
\end{aligned}
\tag{40}
$$

Therefore, we obtain

$$
Q_r^{(1)}(k) = \begin{cases} 0, & k \equiv 0 \bmod M \\ M, & k \not\equiv 0 \bmod M. \end{cases}
\tag{41}
$$

When the eigenvalue $\lambda_r = e^{\pi i/2} = i$, $Q_r^{(i)}(k)$ can be expressed using Equation (39), as

$$
\begin{aligned}
Q_r^{(i)}(k) &= u^{0 \times k}\lambda_r^0 + u^{1 \times k}\lambda_r^{4/M} + \cdots + u^{(M-1) \times k}\lambda_r^{4(M-1)/M} \\
&= 1 + e^{-2\pi i 1(k-1)/M} + e^{-2\pi i 2(k-1)/M} + \cdots + e^{-2\pi i(M-1)(k-1)/M} \\
&= \frac{1 - \left(e^{-2\pi i(k-1)/M}\right)^M}{1 - e^{-2\pi i(k-1)/M}}.
\end{aligned}
\tag{42}
$$

Therefore, there is

$$
Q_r^{(i)}(k) = \begin{cases} 0, & k \equiv 1 \bmod M \\ M, & k \not\equiv 1 \bmod M. \end{cases}
\tag{43}
$$

When the eigenvalue $\lambda_r = e^{2\pi i/2} = -1$, $Q_r^{(-1)}(k)$ can be expressed using Equation (39), as

$$
\begin{aligned}
Q_r^{(-1)}(k) &= u^{0 \times k}\lambda_r^0 + u^{1 \times k}\lambda_r^{4/M} + \cdots + u^{(M-1) \times k}\lambda_r^{4(M-1)/M} \\
&= 1 + e^{-2\pi i 1(k-2)/M} + e^{-2\pi i 2(k-2)/M} + \cdots + e^{-2\pi i(M-1)(k-2)/M} \\
&= \frac{1 - \left(e^{-2\pi i(k-2)/M}\right)^M}{1 - e^{-2\pi i(k-2)/M}}.
\end{aligned}
\tag{44}
$$

Then, we can obtain

$$Q_r^{(-1)}(k) = \begin{cases} 0, & k \equiv 2 \bmod M \\ M, & k \not\equiv 2 \bmod M. \end{cases} \tag{45}$$

When the eigenvalue $\lambda_r = e^{3\pi i/2} = -i$, $Q_r^{(-i)}(k)$ can be expressed using Equation (39), as

$$\begin{aligned} Q_r^{(-i)}(k) &= u^{0 \times k} \lambda_r^0 + u^{1 \times k} \lambda_r^{4/M} + \cdots + u^{(M-1) \times k} \lambda_r^{4(M-1)/M} \\ &= 1 + e^{-2\pi i 1(k-3)/M} + e^{-2\pi i 2(k-3)/M} + \cdots + e^{-2\pi i (M-1)(k-3)/M} \\ &= \frac{1 - \left(e^{-2\pi i(k-3)/M}\right)^M}{1 - e^{-2\pi i(k-3)/M}}. \end{aligned} \tag{46}$$

Therefore, there is

$$Q_r^{(-i)}(k) = \begin{cases} 0, & k \equiv 3 \bmod M \\ M, & k \not\equiv 3 \bmod M. \end{cases} \tag{47}$$

Using Equations (41), (43), (45) and (47), we can formulate Equation (36) as

$$Y_k = \begin{cases} Y_k, & k = 0, 1, 2, 3 \\ 0, & k = 4, 5, \cdots, M-1. \end{cases} \tag{48}$$

In this way, the M-WFRFT of Equation (11) can also be expressed as Equation (49).

$$\begin{aligned} T_{MW}^\alpha &= \frac{1}{M}(Y_0, Y_1, \cdots, Y_{M-1}) \begin{pmatrix} B_0^\alpha \\ B_1^\alpha \\ \vdots \\ B_{M-1}^\alpha \end{pmatrix} \\ &= \frac{1}{M}(Y_0, Y_1, Y_2, Y_3, 0, \cdots, 0) \begin{pmatrix} B_0^\alpha \\ B_1^\alpha \\ \vdots \\ B_{M-1}^\alpha \end{pmatrix} \\ &= \frac{1}{M}(Y_0, Y_1, Y_2, Y_3) \begin{pmatrix} B_0^\alpha \\ B_1^\alpha \\ B_2^\alpha \\ B_3^\alpha \end{pmatrix}, \end{aligned} \tag{49}$$

where $B_k^\alpha = \exp\left(\frac{2\pi i k \alpha}{M}\right); k = 0, 1, \cdots, M-1$.

The effective weighted sum of the M-WFRFT based on the fractional-order matrix is also four terms. In order to prove its unitarity, we denote

$$(T_{MW}^\alpha)^H = \frac{1}{M}\left(Y_0^H B_0^{-\alpha} + Y_1^H B_1^{-\alpha} + Y_2^H B_2^{-\alpha} + Y_3^H B_3^{-\alpha}\right), \tag{50}$$

Therefore, there is

$$T_{MW}^\alpha (T_{MW}^\alpha)^H = \frac{1}{M^2}\left(\sum_{k=0}^{3}\sum_{l=0}^{3} Y_k Y_l^H B_k^\alpha B_l^{-\alpha}\right). \tag{51}$$

From Equation (36), we can obtain

$$Y_k Y_l^H = V \begin{pmatrix} Q_1(k) & 0 & \cdots & 0 \\ 0 & Q_2(k) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & Q_n(k) \end{pmatrix} V^H \left[V \begin{pmatrix} Q_1(l) & 0 & \cdots & 0 \\ 0 & Q_2(l) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & Q_n(l) \end{pmatrix} V^H \right]^H. \tag{52}$$

The eigenvector V of the DFT can be defined as a real symmetric matrix [27–29]; and through Equations (41), (43), (45) and (47), we know that the value of $Q_r(k)$ is 0 or M (M is an integer greater than 4). Therefore, $Y_l^H = Y_l$. Then, Equation (52) can be expressed as

$$Y_k Y_l^H = Y_k Y_l = V \begin{pmatrix} Q_1(k)Q_1(l) & 0 & \cdots & 0 \\ 0 & Q_2(k)Q_2(l) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & Q_n(k)Q_n(l) \end{pmatrix} V^H, \tag{53}$$

and

$$Q_r(k)Q_r(l) = \begin{cases} M^2, & k = l \\ 0, & k \neq l. \end{cases} \tag{54}$$

Therefore, we can obtain

$$Y_k Y_l^H = \begin{cases} MY_k, & k = l \\ 0, & k \neq l. \end{cases} \tag{55}$$

Then, the result of Equation (51) is

$$\begin{aligned} T_{MW}^\alpha (T_{MW}^\alpha)^H &= \frac{1}{M^2} \left(\sum_{k=0}^{3} \sum_{l=0}^{3} Y_k Y_l^H B_k^\alpha B_l^{-\alpha} \right) \\ &= \frac{1}{M} (Y_0 + Y_1 + Y_2 + Y_3) \\ &= \frac{1}{M} \left(V \begin{pmatrix} M & 0 & \cdots & 0 \\ 0 & M & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & M \end{pmatrix} V^H \right) \\ &= I. \end{aligned} \tag{56}$$

$\square$

Remark 3. *With the help of theoretical analysis, we can confirm that the M-WFRFT based on the fractional-order matrix has unitarity. However, we find that the theoretical analysis deviates from the previous numerical simulation [1], which we will discuss further in Section 4.*

3.3. Eigendecomposition-Type FRFT as the Basis Function

Proposition 3. *Eigendecomposition-type FRFT is used as the basis function, so the M-WFRFT has unitarity.*

Proof. In [2], Zhu et al. proposed the M-WFRFT and stated that the basis function is the FRFT, as shown in Equation (57).

$$F^\alpha[f(t)] = \int_{-\infty}^{\infty} K_\alpha(u,t) f(t) dt, \tag{57}$$

where the transform kernel is given by

$$K_\alpha(u,t) = \begin{cases} A_\alpha e^{i\frac{u^2+t^2}{2}\cot\phi - iut\csc\phi} & \alpha \neq k\pi \\ \delta(u-t) & \alpha = 2k\pi \\ \delta(u+t) & \alpha = (2k+1)\pi \end{cases}, \tag{58}$$

where $\phi = \alpha\pi/2$ is interpreted as a rotation angle in the phase plane and $A_\alpha = \sqrt{(1 - i\cot\alpha)/2\pi}$.

As we know, Equation (57) is a continuous FRFT, and a discrete FRFT is used for numerical simulation. At present, the discrete definition [29] closest to the continuous FRFT is

$$F^\alpha(m,n) = \sum_{k=0}^{N-1} v_k(m)e^{-i\frac{\pi}{2}k\alpha}v_k(n), \tag{59}$$

where $v_k(n)$ is an arbitrary orthonormal eigenvector set of the $N \times N$ DFT matrix. Equation (59) can be written as

$$F^\alpha = VD^\alpha V^H, \tag{60}$$

where $V = (v_0, v_1, \cdots, v_{N-1})$, v_k is the kth-order DFT Hermite eigenvector, and D^α is a diagonal matrix, defined as

$$D^\alpha = \mathrm{diag}\left(1, e^{-i\frac{\pi}{2}\alpha}, \cdots, e^{-i\frac{\pi}{2}(N-2)\alpha}, e^{-i\frac{\pi}{2}(N-1)\alpha}\right), \text{ when } N \text{ is odd,} \tag{61}$$

and

$$D^\alpha = \mathrm{diag}\left(1, e^{-i\frac{\pi}{2}\alpha}, \cdots, e^{-i\frac{\pi}{2}(N-2)\alpha}, e^{-i\frac{\pi}{2}(N)\alpha}\right), \text{ when } N \text{ is even.} \tag{62}$$

We only prove that N is odd (when N is even, the proof process is the same). Therefore, there is

$$D^\alpha = \mathrm{diag}\left((1)^\alpha, (-i)^\alpha, (-1)^\alpha, (i)^\alpha, (1)^\alpha, (-i)^\alpha, (-1)^\alpha, (i)^\alpha, \cdots\cdots, (1 \text{ or } -1)^\alpha\right). \tag{63}$$

Then, Equation (10) can be written as

$$\begin{aligned}
Y_k &= u^{0\times k} \times F^0 + u^{1\times k} \times F^{\frac{4}{M}} + \cdots + u^{(M-1)\times k} \times F^{\frac{4(M-1)}{M}} \\
&= u^{0\times k}V\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & (-i)^0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & (1 \text{ or } -1)^0 \end{pmatrix}V^H + u^{1\times k}V\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & (-i)^{\frac{4}{M}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & (1 \text{ or } -1)^{\frac{4}{M}} \end{pmatrix}V^H + \cdots + u^{(M-1)\times k}V\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & (-i)^{\frac{4(M-1)}{M}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & (1 \text{ or } -1)^{\frac{4(M-1)}{M}} \end{pmatrix}V^H.
\end{aligned} \tag{64}$$

We can further obtain Equation (65) as

$$Y_k = V\begin{pmatrix} Q^{(1)}(k) & 0 & \cdots & 0 \\ 0 & Q^{(-i)}(k) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & Q^{(1 \text{ or}-1)}(k) \end{pmatrix}V^H. \tag{65}$$

The diagonal matrix of Equation (65) can be expressed as

$$\mathrm{diag}\left(Q^{(1)}(k), Q^{(-i)}(k), Q^{(-1)}(k), Q^{(i)}(k), Q^{(1)}(k), Q^{(-i)}(k), \cdots\cdots, Q^{(1 \text{ or}-1)}(k)\right). \tag{66}$$

Then, $Q^{(1)}(k)$ is the same as Equation (40), $Q^{(-i)}(k)$ is the same as Equation (46), $Q^{(-1)}(k)$ is the same as Equation (44), and $Q^{(i)}(k)$ is the same as Equation (42). Thus, Y_k can be obtained as

$$Y_k = \begin{cases} Y_k, & k = 0, 1, 2, 3 \\ 0, & k = 4, 5, \cdots, M-1. \end{cases} \tag{67}$$

All the following proofs are the same as Section 3.2. In other words, the M-WFRFT has unitarity. $\qquad\square$

Remark 4. *From Equation (67), it is not difficult to find that there are only four weighted terms of the M-WFRFT based on the eigendecomposition-type FRFT.*

3.4. Other Types of FRFTs

There are three types of discrete definitions of the FRFT. In Section 3.1, the linear WFRFT is used. The fractional-order matrix is used in Section 3.2. The discrete FRFT, which is called the eigendecomposition type, is used in Section 3.3. Then, there is a sampling-type FRFT.

In [30], a sampling-type FRFT is proposed, and its process can be written as follows:

(a)　Chirp multiplication

$$g(x_0) = \exp\left[-ipx_0^2 \tan(f/2)\right] f(x_0); \tag{68}$$

(b)　Chirp convolution

$$g'(x) = A_\phi \int_{-\infty}^{\infty} \exp[i\pi \csc(\phi)(x - x_0)^2] g(x_0) dx_0; \tag{69}$$

(c)　Chirp multiplication

$$f_\alpha(x) = \exp\left[-i\pi x^2 \tan(\phi/2)\right] g'(x). \tag{70}$$

The definition of the sampling type is the numerical simulation of a continuous FRFT.

The discretization of the FRFT has been extensively studied [12], and the three main types of DFRFTs are compared, as shown in Table 2. We noticed that the sampling-type FRFT did not satisfy additivity and unitarity.

Table 2. Comparison of the three main types of DFRFT.

	Linear Weighted Type	Eigendecomposition Type	Sampling Type
Unitarity	√	√	×
Additivity	√	√	×
Approximation	×	√	√
Closed-form	√	×	√
Complexity	$O(N\log N)$	$O(N^2)$	$O(N\log N)$

Remark 5. *The M-WFRFT is an extended definition, and its basis function can be expressed as shown in Figure 1. The sampling type FRFT does not satisfy the additivity and unitarity, so it cannot be used as a basis function.*
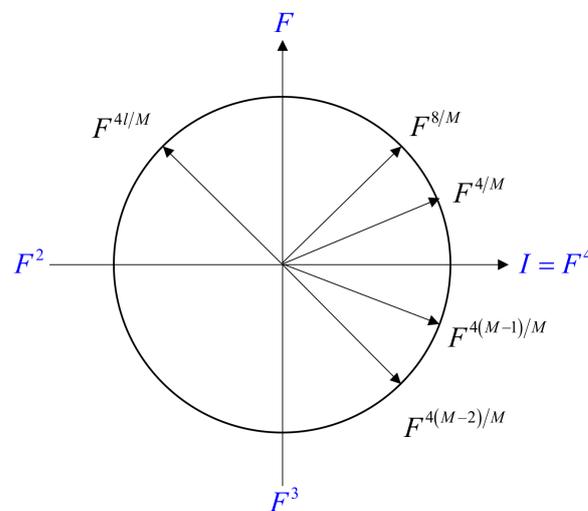
Figure 1. Time-frequency denotation of the M-WFRFT operator.

4. Discussion

Our previous research only verified the unitarity of the M-WFRFT via numerical simulation [1], but the simulation results are different from the theoretical proof in Section 3.2. Next, we will analyze and discuss this issue. Equation (10) can be verified using MATLAB, and its program is shown in Code 1.

Code 1. The program of Equation (10).

```
1.    function Yk = Yk(N,M)
2.    % M is the resulting weighting term, for example: M = 4(4-WFRFT); M = 5(5-WFRFT)
3.    % N is the length of the signal;
4.    F = zeros(N);
5.    for k = 1:N
6.        for h = 1:N
7.            F(h,k) = exp(2*pi*i*(h−1)*(k−1)/N)/sqrt(N); % IDFT
8.        end
9.    end
10.   F = fftshift(F);
11.   for k = 0:M−1
12.       yy = F^(4*k/M);      % Fractional power of Fourier transform
13.       y{k + 1} = yy;
14.   end
15.   % celldisp(y);
16.   u = zeros(M);
17.   for k = 1:M
18.       for h = 1:M
19.           u(h,k) = exp(−2*pi*i*(h−1)*(k−1)/M); %DFT
20.       end
21.   end
22.   for k = 1:M
23.       YY = zeros(N);
24.       for h = 1:M
25.           YY = YY + u(h,k)*y{h};
26.       end
27.       Y{k} = YY;     % Yk in the paper is obtained
28.   end
29.   Celldisp(y)
```

We tested from 2 to 1000 dimension and found that Y_k was a real matrix only when the dimensions were

$$N = 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 16, 18, 21, 28, 29, 32, 33, 44. \tag{71}$$

Therefore, the unitarity of the M-WFRFT is only available in the aforementioned cases. Y_k has the following rules:

$$\begin{cases} 5\text{-}WFRFT \Rightarrow & Y_0 \quad Y_1 \quad Y_2 \quad Y_3 \quad Y_4 \\ 6\text{-}WFRFT \Rightarrow & Y_0 \quad Y_1 \quad Y_2 \quad Y_3 \quad Y_4 \quad Y_5 \\ 7\text{-}WFRFT \Rightarrow & Y_0 \quad Y_1 \quad Y_2 \quad Y_3 \quad Y_4 \quad Y_5 \quad Y_6 \\ 8\text{-}WFRFT \Rightarrow & Y_0 \quad Y_1 \quad Y_2 \quad Y_3 \quad Y_4 \quad Y_5 \quad Y_6 \quad Y_7 \\ \quad \vdots & \qquad\qquad \vdots \\ M\text{-}WFRFT \Rightarrow & Y_0 \quad Y_1 \quad Y_2 \quad Y_3 \quad Y_4 \quad \cdots \quad Y_{M-3} \quad Y_{M-2} \quad Y_{M-1} \end{cases} \tag{72}$$

where the blue Y_k indicates that the result is zero.

For other dimensions, the M-WFRFT does not have unitarity, and Y_k is as follows:

$$\begin{cases} 5\text{-}WFRFT \Rightarrow & Y_0 \quad Y_1 \quad Y_2 \quad Y_3 \quad Y_4 \\ 6\text{-}WFRFT \Rightarrow & Y_0 \quad Y_1 \quad Y_2 \quad Y_3 \quad Y_4 \quad Y_5 \\ 7\text{-}WFRFT \Rightarrow & Y_0 \quad Y_1 \quad Y_2 \quad Y_3 \quad Y_4 \quad Y_5 \quad Y_6 \\ 8\text{-}WFRFT \Rightarrow & Y_0 \quad Y_1 \quad Y_2 \quad Y_3 \quad Y_4 \quad Y_5 \quad Y_6 \quad Y_7 \\ \quad\vdots & \quad\vdots \\ M\text{-}WFRFT \Rightarrow & Y_0 \quad Y_1 \quad Y_2 \quad Y_3 \quad Y_4 \quad \cdots \quad Y_{M-3} \quad Y_{M-2} \quad Y_{M-1} \end{cases} \tag{73}$$

where the blue Y_k indicates that the result is zero.

The numerical simulation results show that the M-WFRFT has unitarity only in certain dimensions. Following the theory of Section 3.2, the program is shown in Code 2. Our purpose is to compare the results of Code 2 with the results of Code 1.

Code 2. The program of Equation (36).

```
1.    function Yk = Yk1(N.M)
2.    % M is the resulting weighting term, for example: M = 4(4-WFRFT); M = 5(5-WFRFT)
3.    % N is the length of the signal;
4.    F = zeros(N);
5.    for k = 1:N
6.            for h = 1:N
7.                F(h,k) = exp(2*pi*i*(h−1)*(k−1)/N)/sqrt(N); % IDFT
8.            end
9.    end
10.   F = fftshift(F);
11.   [V,D] = eig(F);
12.   for k = 0:M−1
13.           YY = zeros(N);
14.            for l = 0:M−1
15.                YY = YY + exp(−2*pi*i*k*l/M)*D^(4*l/M);
16.            end
17.           YY = V*YY*inv(V);
18.           Y{k + 1} = YY;            % Yk in the paper is obtained
19.   end
20.   celldisp(Y)
```

After verification, we found that the results of Codes 1 and 2 are the same. Therefore, the numerical simulation shows that the unitarity of the M-WFRFT is related to signal length. However, our theoretical analysis shows that the unitarity of the M-WFRFT does not depend on signal length. Therefore, there is a problem insofar as the simulation verification is inconsistent with the theoretical analysis. In order to solve this problem, we will analyze it with a specific numerical value. For Code 2, when $M = 7$ and $N = 13$, we can obtain the eigenvalue of the DFT in *line* 11 of Code 2. Therefore, the eigenvalue matrix D is

$$D = \begin{pmatrix} i & & & & & & & & & & & & \\ & i & & & & & & & & & & & \\ & & i & & & & & \huge 0 & & & & & \\ & & & -i & & & & & & & & & \\ & & & & -i & & & & & & & & \\ & & & & & -i & & & & & & & \\ & & & & & & -1 & & & & & & \\ & & & & & & & -1 & & & & & \\ & & & & & & & & -1 & & & & \\ & & & & & & & & & 1 & & & \\ & \huge 0 & & & & & & & & & 1 & & \\ & & & & & & & & & & & 1 & \\ & & & & & & & & & & & & 1 \end{pmatrix}_{13\times13} \tag{74}$$

Then, for Equation (36), the calculated values of Y_k $(k = 0, 1, \cdots, 6)$ are

$$
Y_0 = V \begin{pmatrix} 0 & & & & & & & & & & \\ & 0 & & & & & & & & & \\ & & 0 & & & & & 0 & & & \\ & & & 0 & & & & & & & \\ & & & & 0 & & & & & & \\ & & & & & 0 & & & & & \\ & & & & & & 0 & & & & \\ & & & & & & & 7 & & & \\ & & 0 & & & & & & 7 & & \\ & & & & & & & & & 7 & \\ & & & & & & & & & & 7 \end{pmatrix} V^{-1};
\tag{75}
$$

$$
Y_1 = V \begin{pmatrix} 7 & & & & & & & & & & \\ & 7 & & & & & & & & & \\ & & 7 & & & & & 0 & & & \\ & & & 0 & & & & & & & \\ & & & & 0 & & & & & & \\ & & & & & 0 & & & & & \\ & & & & & & 0 & & & & \\ & & & & & & & 0 & & & \\ & & 0 & & & & & & 0 & & \\ & & & & & & & & & 0 & \\ & & & & & & & & & & 0 \end{pmatrix} V^{-1};
\tag{76}
$$

$$
Y_2 = V \begin{pmatrix} 0 & & & & & & & & & & \\ & 0 & & & & & & & & & \\ & & 0 & & & & & 0 & & & \\ & & & 0 & & & & & & & \\ & & & & 0 & & & & & & \\ & & & & & 7 & & & & & \\ & & & & & & 7 & & & & \\ & & & & & & & 7 & & & \\ & & 0 & & & & & & 0 & & \\ & & & & & & & & & 0 & \\ & & & & & & & & & & 0 \end{pmatrix} V^{-1};
\tag{77}
$$

$$
Y_3 = V \begin{pmatrix} 0 & & & 0 \\ & 0 & & \\ & & \ddots & \\ 0 & & & 0 \end{pmatrix} V^{-1};
\tag{78}
$$

and $Y_4 = Y_5 = Y_3$;

$$
Y_6 = V \begin{pmatrix} 0 & & & & & & & & & & \\ & 0 & & & & & & & & & \\ & & 0 & & & & & 0 & & & \\ & & & 7 & & & & & & & \\ & & & & 7 & & & & & & \\ & & & & & 7 & & & & & \\ & & & & & & 0 & & & & \\ & & & & & & & 0 & & & \\ & & 0 & & & & & & 0 & & \\ & & & & & & & & & 0 & \\ & & & & & & & & & & 0 \end{pmatrix} V^{-1}.
\tag{79}
$$

In the results obtained, the values of Y_3, Y_4 and Y_5 are zero; Equation (72) is verified.

If $M = 7$ and $N = 14$, we can obtain the eigenvalue of the DFT in *line* 11 of Code 2. Therefore, the eigenvalue matrix D is

$$D = \begin{pmatrix} 1 & & & & & & & & & & & & & \\ & 1 & & & & & & & & & & & & \\ & & 1 & & & & & & & & & \mathbf{0} & & \\ & & & 1 & & & & & & & & & & \\ & & & & i & & & & & & & & & \\ & & & & & i & & & & & & & & \\ & & & & & & -i & & & & & & & \\ & & & & & & & i & & & & & & \\ & & & & & & & & -i & & & & & \\ & & & & & & & & & -i & & & & \\ & & & & & & & & & & -1 & & & \\ & \mathbf{0} & & & & & & & & & & -1 & & \\ & & & & & & & & & & & & -1 & \\ & & & & & & & & & & & & & -1 \end{pmatrix}_{14\times14} \tag{80}$$

Then, using Equation (36), the calculated values of Y_k $(k = 0, 1, \cdots, 6)$ are

$$Y_0 = V \begin{pmatrix} 7 & & & & & & & & & & & & & \\ & 7 & & & & & & & & & & & & \\ & & 7 & & & & & & & & \mathbf{0} & & & \\ & & & 7 & & & & & & & & & & \\ & & & & 0 & & & & & & & & & \\ & & & & & 0 & & & & & & & & \\ & & & & & & 0 & & & & & & & \\ & & & & & & & 0 & & & & & & \\ & & & & & & & & 0 & & & & & \\ & & & & & & & & & 0 & & & & \\ & \mathbf{0} & & & & & & & & & 0 & & & \\ & & & & & & & & & & & 0 & & \\ & & & & & & & & & & & & 0 & \\ & & & & & & & & & & & & & 0 \end{pmatrix} V^{-1}; \tag{81}$$

$$Y_1 = V \begin{pmatrix} 0 & & & & & & & & & & & & & \\ & 0 & & & & & & & & & & & & \\ & & 0 & & & & & & & & \mathbf{0} & & & \\ & & & 0 & & & & & & & & & & \\ & & & & 7 & & & & & & & & & \\ & & & & & 7 & & & & & & & & \\ & & & & & & 0 & & & & & & & \\ & & & & & & & 7 & & & & & & \\ & & & & & & & & 0 & & & & & \\ & & & & & & & & & 0 & & & & \\ & \mathbf{0} & & & & & & & & & 0 & & & \\ & & & & & & & & & & & 0 & & \\ & & & & & & & & & & & & 0 & \\ & & & & & & & & & & & & & 0 \end{pmatrix} V^{-1}; \tag{82}$$

$$Y_2 = V \begin{pmatrix} 0 & & & & & & & & & & & & & \\ & 0 & & & & & & & & & & & & \\ & & 0 & & & & & & & & \mathbf{0} & & & \\ & & & 0 & & & & & & & & & & \\ & & & & 0 & & & & & & & & & \\ & & & & & 0 & & & & & & & & \\ & & & & & & 0 & & & & & & & \\ & & & & & & & 0 & & & & & & \\ & & & & & & & & 0 & & & & & \\ & & & & & & & & & 7 & & & & \\ & \mathbf{0} & & & & & & & & & 0 & & & \\ & & & & & & & & & & & 7 & & \\ & & & & & & & & & & & & 7 \end{pmatrix} V^{-1}; \tag{83}$$

$$Y_3 = V \begin{pmatrix} 0 & & & 0 \\ & 0 & & \\ & & \ddots & \\ 0 & & & 0 \end{pmatrix} V^{-1}; \tag{84}$$

and $Y_4 = Y_3$;

$$Y_5 = V \begin{pmatrix} 0 & & & & & & & & & & & & \\ & 0 & & & & & & & & & & & \\ & & 0 & & & & & & & & & & \\ & & & 0 & & & & & & & & & \\ & & & & 0 & & & & & & & & \\ & & & & & 0 & & & & & & & \\ & & & & & & 0 & & & & & & \\ & & & & & & & 0 & & & & & \\ & & & & & & & & 0 & & & & \\ & & & & & & & & & 0 & & & \\ & & & & & & & & & & 7 & & \\ & & & & & & & & & & & 0 & \\ & & & & & & & & & & & & 0 \end{pmatrix} V^{-1}; \tag{85}$$

$$Y_6 = V \begin{pmatrix} 0 & & & & & & & & & & & & \\ & 0 & & & & & & & & & & & \\ & & 0 & & & & & & & & & & \\ & & & 0 & & & & & & & & & \\ & & & & 0 & & & & & & & & \\ & & & & & 0 & & & & & & & \\ & & & & & & 7 & & & & & & \\ & & & & & & & 0 & & & & & \\ & & & & & & & & 7 & & & & \\ & & & & & & & & & 7 & & & \\ & & & & & & & & & & 0 & & \\ & & & & & & & & & & & 0 & \\ & & & & & & & & & & & & 0 \end{pmatrix} V^{-1}; \tag{86}$$

In the results obtained, the values of Y_3 and Y_4 are zero, and Equation (73) is verified.

When $N = 13$, Equation (72) is obtained by means of Code 1. However, in the theoretical analysis, the nonzero terms of Y_k are Y_0, Y_1, Y_2 and Y_3, which are different from the simulation results presented by Equation (72). This problem is generated by fractional power operation, based on MATLAB, mainly in *line* 15 of Code 2 (*line* 12 of Code 1), and its operation $D^{4l/M}$ ($F^{4l/M}$).

According to the deMoivre theorem, we know that

$$[r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta). \tag{87}$$

where n is a positive integer. Therefore, for Equation (87),

$$x^n = r(\cos\theta + i\sin\theta), \tag{88}$$

the results have n roots

$$x_k = \sqrt[n]{r}(\cos((\theta + 2k\pi)/n) + i\sin((\theta + 2k\pi)/n)). \tag{89}$$

where $k = 0, 1, \cdots, n-1$. However, in the numerical simulation, we only obtained one of the roots. For example, $-i = \cos(3\pi/2) + i\sin(3\pi/2)$. Using MATLAB to calculate $(-i)^{1/2}$, we obtain $0.7071 - 0.7071i$. The actual results should be that the two roots are $0.7071 - 0.7071i$ and $-0.7071 + 0.7071i$, respectively. This leads to the deviation between the simulation results (Equation (72)) and the theory (Section 3.2).

For $N = 14$, the simulation results (Equation (73)) show that the M-WFRFT does not have unitarity. However, the theoretical (Section 3.2) explanation has unitarity. This problem is caused by fractional exponentiation operation based on MATLAB. In Equation (80), we notice the position of the eigenvalue (-1), but after the fractional power operation based on MATLAB, Equations (83) and (85) appear. Therefore, the final numerical simulation results show that the M-WFRFT does not have unitarity. In fact, the correct result is the sum of Equations (83) and (85).

Using the above analysis, we explain the error of the operation based on MATLAB, which is also the clarification of our previous research work. The final conclusion is that

the M-WFRFT has unitarity. The M-WFRFT code is shown in Appendix A, and interested researchers can verify it.

5. Conclusions

In this paper, we present a new reformulation of the M-WFRFT to prove its unitarity. The M-WFRFT uses the DFRFT as the basis function, and the diversity of the DFRFT leads to different definitions of the M-WFRFT. We use the linear weighted-type, fractional-order matrix and eigendecomposition-type FRFT as the basis functions and prove the unitarity of the M-WFRFT. The results show that M-WFRFTs based on these three definitions have unitarity. However, with greater research, the results also show that the effective weighted sum of the M-WFRFT is only four terms. That is to say, as an extended definition of the WFRFT, the M-WFRFT shows no increase in its weighting term. It has great reference value for the application of the M-WFRFT. Furthermore, we note the deviation between the numerical simulation and the theoretical analysis, which reveals that the unitary verification based on MATLAB is inaccurate for the previous work. Finally, we analyze two examples and establish the reasons for the deviation. In other words, the fractional power operation directly based on MATLAB can only obtain one root at a time.

Appendix A

M-WFRFT code is written; its basis function is the WFRFT. By calling "celldisp (Y)", Y_k is verified in Section 3.1.

```
%% M-WFRFT (multi-weighted type fractional Fourier transform)
% The basis function F^(4*l/M) is WFRFT
function F = mwfrft(alpha,M,N)
% This code is written by Tieyu Zhao, E-mail: zhaotieyu@neuq.edu.cn;
% alpha is the transform order;
% M is the resulting weighting term, for example: M = 4(4-WFRFT); M = 5(5-WFRFT)
% N is the length of the signal;
for l = 0:M−1
    yy = wfrft(N,4*l/M); % WFRFT
    y{l + 1} = yy;
end
% celldisp(y);
D = zeros(M);
for k = 1:M
    for h = 1:M
        D(h,k) = exp(−2*pi*i*(h−1)*(k−1)/M); % DFT
    end
end
for k = 1:M
    YY = zeros(N);
    for h = 1:M
        YY = YY + D(h,k)*y{h};
    end
    Y{k} = YY; % Yk is obtained in Section 3.1
end
```

```
% celldisp(Y)
B = zeros(1,M);
for k = 0:M−1
      B(k + 1) = B(k + 1) + exp(2*pi*i*k*alpha/M); % B_alpha
end
F = zeros(N);
for k = 0:M−1
      F = F + B(k + 1)*Y{k + 1}/M; % M-WFRFT
end

function F = wfrft(N,beta) % WFRFT
Y = eye(N);
y1 = fftshift(fft(Y))/(sqrt(N));
y2 = y1*y1;
y3 = conj(y1);
pl = zeros(1,4);
for k = 0:3
          pl(k + 1) = pl(k + 1) + exp(i*3*pi*(beta−k)/4)*cos(pi*(beta−k)/2)*cos(pi*(beta−k)/4);
end
F = pl(1)*Y + pl(2)*y1 + pl(3)*y2 + pl(4)*y3;
```

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