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Article

Applications of a Fixed Point Result for Solving Nonlinear Fractional and Integral Differential Equations

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Abstract: In this article, we apply one fixed point theorem in the setting of b-metric-like spaces to prove the existence of solutions for one type of Caputo fractional differential equation as well as the existence of solutions for one integral equation created in mechanical engineering.

Keywords: b-metric-like; 0-d_{bl}-complete; d_{bl}-continuous; fixed point; metric; like-space; Caputo fractional differential equation; Green function

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1. Introduction and Preliminaries

It is well known that fixed point theory has applications in various fields of research. These fields include engineering, economics, natural sciences, game and graph theory, etc. The first known and perhaps most applicable result comes from Stefan Banach in 1922. It is known as the Banach contraction mapping principle. This principle claims that every contraction in a complete metric space has a unique fixed point. It is useful to say that this fixed point is also a unique fixed point for all iterations of the given contractive mapping.

After 1922, a large number of authors generalized Banach's famous result. Hundreds of papers have been written on the subject. The generalizations went in two important directions:

(1) New conditions were introduced in the given contractive relation using new relations (Kannan, Chatterje, Reich, Hardy-Rogers, Ćirić, ...).
(2) The axioms of metric space have been changed.

Thus, many classes of new spaces are obtained. For more details see papers [1–10].

One of the mentioned generalizations of Banach's result from 1922 was introduced by the Polish mathematician D. Wardowski. In 2012, he defined the $\overline{\mathbb{F}}$-contraction as follows.

The mapping $\overline{T}$ of the metric space $(\overline{X}, d)$ into itself, is an $\overline{\mathbb{F}}$ -contraction if there is a positive number τ such that for all $\overline{x}, \overline{y} \in X$

$$d(\overline{T}\overline{x}, \overline{T}\overline{y}) > 0 \text{ yields } \tau + \overline{\mathbb{F}}(d(\overline{T}\overline{x}, \overline{T}\overline{y})) \leq \overline{\mathbb{F}}(d(\overline{x}, \overline{y})), \tag{1}$$

where $\overline{\mathbb{F}}$ is a mapping of the interval $(0, +\infty)$ into the set $\overline{\mathbb{R}} = (-\infty, +\infty)$ of real numbers, which satisfies the following three properties:

(F1) $\overline{\mathbb{F}}(\overline{r}) < \overline{\mathbb{F}}(\overline{p})$ whenever $0 < \overline{r} < \overline{p}$;
(F2) If $\{\overline{\alpha}_n\} \subset (0, +\infty)$ then $\overline{\alpha}_n \to 0$ if and only if $\overline{\mathbb{F}}(\overline{\alpha}_n) \to -\infty$;
(F3) $\overline{\alpha}^k \overline{\mathbb{F}}(\overline{\alpha}) \to 0$ as $\overline{\alpha} \to 0^+$ for some $k \in (0, 1)$.

The set of all functions satisfying the above definition of D. Wardowski is denoted with $\mathcal{F}$.

The following functions $\overline{\mathbb{F}} : (0, +\infty) \to (-\infty, +\infty)$ are in $\mathcal{F}$.

1. $\overline{\mathbb{F}}(\alpha) = \ln \alpha$;
2. $\overline{\mathbb{F}}(\alpha) = \alpha + \ln \alpha$;
3. $\overline{\mathbb{F}}(\alpha) = -\alpha^{-\frac{1}{2}}$;
4. $\overline{\mathbb{F}}(\alpha) = \ln(\alpha + \alpha^2)$.

By using $\overline{\mathbb{F}}$-contraction, Wardowski [11] proved the following fixed point theorem that generalizes Banach's [3] contraction principle.

Theorem 1. *Ref. [11] Let* $(\overline{X}, d)$ *be a complete metric space and* $\overline{T} : \overline{X} \to \overline{X}$ *an* $\overline{\mathbb{F}}$*-contraction. Then* $\overline{T}$ *has a unique fixed point* $\overline{x}^* \in \overline{X}$ *and for every* $\overline{x} \in X$ *the sequence* $\left\{\overline{T}^n \overline{x}\right\}_{n \in \mathbb{N}}$ *converges to* $\overline{x}^*$.

To prove his main result in [11] D. Wardovski used all three properties **(F1)**, **(F2)** and **(F3)** of the mapping $\overline{\mathbb{F}}$. They were also used in the works [12–19]. However in the works [20–22] instead of all three properties, the authors used only property **(F1)**.

Since Wardowski's main result is true if the function $\overline{\mathbb{F}}$ satisfies only **(F1)** (see [20–22]), it is natural to ask whether it is also true for the other five classes of generalized metric spaces: b-metric spaces, partial metric spaces, metric like spaces, partial b-metric spaces, and b-metric like spaces. Clearly, it is sufficient to check it for b-metric-like spaces.

Let us recall the definitions of the b-metric like space as well as of the generalized $(s, q)-$ Jaggi-$\overline{\mathbb{F}}$-contraction type mapping.

Definition 1. *A b-metric-like on a nonempty set* $\overline{X}$ *is a function* $d_{bl} : \overline{X} \times \overline{X} \to [0, +\infty)$ *such that for all* $\overline{x}, \overline{y}, \overline{z} \in \overline{X}$ *and a constant* $\mathfrak{s} \geq 1$*, the following three conditions are satisfied:*
$(d_{bl}1)$ $d_{bl}(\overline{x}, \overline{y}) = 0$ *yields* $\overline{x} = \overline{y}$;
$(d_{bl}2)$ $d_{bl}(\overline{x}, \overline{y}) = d_{bl}(\overline{y}, \overline{x})$;
$(d_{bl}3)$ $d_{bl}(\overline{x}, z) \leq \mathfrak{s}(d_{bl}(\overline{x}, \overline{y}) + d_{bl}(\overline{y}, z))$.

In this case, the triple $(\overline{X}, d_{bl}, \mathfrak{s} \geq 1)$ is called b-metric-like space with constant $\mathfrak{s}$ or b-dislocated metric space by some author. It should be noted that the class of b-metric-like spaces is larger that the class of metric-like spaces, since a b-metric-like is a metric like with $\mathfrak{s} = 1$. For some examples of metric-like and b-metric-like spaces (see [13,15,23,24]).

The definitions of convergent and Cauchy sequences are formally the same in partial metric, metric-like, partial b-metric and b-metric-like spaces. Therefore we give only the definition of convergence and Cauchyness of the sequences in b-metric-like space.

Definition 2. *Ref. [1] Let* $\{\overline{x}_n\}$ *be a sequence in a b-metric-like space* $(\overline{X}, d_{bl}, \mathfrak{s} \geq 1)$.

(i) *The sequence* $\{\overline{x}_n\}$ *is said to be convergent to* $\overline{x}$ *if* $\lim_{n \to +\infty} d_{bl}(\overline{x}_n, \overline{x}) = d_{bl}(\overline{x}, \overline{x})$;

(ii) *The sequence* $\{\overline{x}_n\}$ *is said to be* $d_{bl}-$*Cauchy in* $(\overline{X}, d_{bl}, \mathfrak{s} \geq 1)$ *if* $\lim_{n,m \to +\infty} d_{bl}(\overline{x}_n, \overline{x}_m)$ *exists and is finite. If* $\lim_{n,m \to +\infty} d_{bl}(\overline{x}_n, \overline{x}_m) = 0$*, then* $\{\overline{x}_n\}$ *is called* $0 - d_{bl}-$*Cauchy sequence.*

(iii) *One says that a b-metric-like space* $(\overline{X}, d_{bl}, \mathfrak{s} \geq 1)$ *is* $d_{bl}-$*complete (resp.* $0 - d_{bl}-$*complete) if for every* $d_{bl}-$*Cauchy (resp.* $0 - d_{bl}-$*Cauchy) sequence* $\{\overline{x}_n\}$ *in it there exists an* $\overline{x} \in \overline{X}$ *such that* $\lim_{n,m \to +\infty} d_{bl}(\overline{x}_n, \overline{x}_m) = \lim_{n \to +\infty} d_{bl}(\overline{x}_n, \overline{x}) = d_{bl}(\overline{x}, \overline{x})$.

(iv) *A mapping* $T : (\overline{X}, d_{bl}, \mathfrak{s} \geq 1) \to (\overline{X}, d_{bl}, \mathfrak{s} \geq 1)$ *is called* $d_{bl}-$*continuous if the sequence* $\{\overline{T}\overline{x}_n\}$ *tends to* $\overline{T}\overline{x}$ *whenever the sequence* $\{\overline{x}_n\} \subseteq \overline{X}$ *tends to* $\overline{x}$ *as* $n \to +\infty$, *that is, if* $\lim_{n \to +\infty} d_{bl}(\overline{x}_n, \overline{x}) = d_{bl}(\overline{x}, \overline{x})$ *yields* $\lim_{n \to +\infty} d_{bl}(\overline{T}\overline{x}_n, \overline{T}\overline{x}) = d_{bl}(\overline{T}\overline{x}, \overline{T}\overline{x})$.

Herein, we discuss first some fixed points considerations for the case of b-metric-like spaces. Then we give a $(\mathfrak{s}, q)-$Jaggi-$\overline{\mathbb{F}}$- contraction fixed point theorem in $0 - d_{bl}-$complete b-metric-like space without conditions **(F2)** and **(F3)** using the property of strictly increasing function defined on $(0, +\infty)$. Moreover, using this fixed point result we prove the existence of solutions for one type of Caputo fractional differential equation as well as existence of solutions for one integral equation created in mechanical engineering.

2. Fixed Point Remarks

Let us start this section with an important remark for the case of b-metric-like spaces.

Remark 1. *In a b-metric-like space the limit of a sequence does not need to be unique and a convergent sequence does not need to be a* $d_{bl}-$*Cauchy one. However, if the sequence* $\{\overline{x}_n\}$ *is a* $0 - d_{bl}-$*Cauchy sequence in the* $d_{bl}-$*complete b-metric-like space* $(\overline{X}, d_{bl}, \mathfrak{s} \geq 1)$, *then the limit of such sequence is unique. Indeed, in such case if* $\overline{x}_n \to \overline{x}$ *as* $n \to +\infty$ *we get that* $d_{bl}(\overline{x}, \overline{x}) = 0$. *Now, if* $\overline{x}_n \to \overline{x}$ *and* $\overline{x}_n \to \overline{y}$ *where* $\overline{x} \neq \overline{y}$, *we obtain that:*

$$\frac{1}{s} d_{bl}(\overline{x}, \overline{y}) \leq d_{bl}(\overline{x}, \overline{x}_n) + d_{bl}(\overline{x}_n, \overline{x}) \to d_{bl}(\overline{x}, \overline{x}) + d_{bl}(\overline{y}, \overline{y}) = 0 + 0 = 0. \tag{2}$$

From $(d_{bl}1)$ *follows that* $\overline{x} = \overline{y}$, *which is a contradiction.*

We shall use the following result, the proof is similar to that in the paper [25] (see also [26,27]).

Lemma 1. *Let* $\{\overline{x}_n\}$ *be a sequence in b-metric-like space* $(\overline{X}, d_{bl}, \mathfrak{s} \geq 1)$ *such that*

$$d_{bl}(\overline{x}_n, \overline{x}_{n+1}) \leq \lambda \cdot d_{bl}(\overline{x}_{n-1}, \overline{x}_n) \tag{3}$$

for some $\lambda \in [0, \frac{1}{\mathfrak{s}})$ *and for each* $n \in \mathbb{N}$. *Then* $\{\overline{x}_n\}$ *is a* $0 - d_{bl}-$*Cauchy sequence.*

Remark 2. *It is worth noting that the previous Lemma holds in the setting of b-metric-like spaces for each* $\lambda \in [0, 1)$. *For more details see [26,28].*

Definition 3. *Let* $\overline{T}$ *be a self-mapping on a b-metric-like space* $(\overline{X}, d_{bl}, \mathfrak{s} \geq 1)$. *Then the mapping* $\overline{T}$ *is said to be generalized* $(\mathfrak{s}, q)-$*Jaggi* $\overline{\mathbb{F}}-$*contraction-type if there is strictly increasing* $\overline{\mathbb{F}} : (0, +\infty) \to (-\infty, +\infty)$ *and* $\tau > 0$ *such that for all* $\overline{x}, \overline{y} \in X$:

$$\left(d_{bl}(\overline{T}\overline{x}, \overline{T}\overline{y}) > 0 \text{ and } d_{bl}(\overline{x}, \overline{y}) > 0\right) \text{ yields } \tau + \overline{\mathbb{F}}\left(\mathfrak{s}^q d_{bl}(\overline{T}\overline{x}, \overline{T}\overline{y})\right) \leq \overline{\mathbb{F}}\left(N_{bl}^{A,B,C}(\overline{x}, \overline{y})\right), \tag{4}$$

for all $\overline{x}, \overline{y} \in X$, *where* $N_{bl}^{A,B,C}(\overline{x}, \overline{y}) = A \cdot \frac{d_{bl}(\overline{x}, \overline{T}\overline{x}) \cdot d_{bl}(\overline{y}, \overline{T}\overline{y})}{d_{bl}(\overline{x}, \overline{y})} + B \cdot d_{bl}(\overline{x}, \overline{y}) + C \cdot d_{bl}(\overline{y}, \overline{T}\overline{y})$, $A, B, C \geq 0$ *with* $A + B + 2C\mathfrak{s} < 1$ *and* $q > 1$.

Remark 3. *Due to division by* $d_{bl}(\overline{x}, \overline{y})$ *in previously it must be* $d_{bl}(\overline{x}, \overline{y}) > 0$. *Hence, we improved Definition 6 from [13].*

We give further, various results using only some conditions of the definition of $\overline{\mathbb{F}}$-contractions. Then, we prove a $(\mathfrak{s}, q)-$Jaggi-$\overline{\mathbb{F}}$- contraction fixed point theorem in $0 - d_{bl}-$complete b-metric-like space without conditions **(F2)** and **(F3)** using the property of strictly increasing function defined on $(0, +\infty)$. For all details on monotone real functions see [29].

Let us give the following main result of this section.

Theorem 2. *Let* $(\overline{X}, d_{bl}, \mathfrak{s} \geq 1)$ *be* $0 - d_{bl}-$*complete and* $\overline{T} : \overline{X} \to \overline{X}$ *be a generalized* $(\mathfrak{s}, q)$*-Jaggi-*$\overline{\mathbb{F}}$*-contraction-type mapping. Then,* $\overline{T}$ *has a unique fixed point* $\overline{x}^* \in X$*, if it is* $d_{bl}-$*continuous and* $\lim_{n\to+\infty} \overline{T}^n \overline{x} = \overline{x}^*$*, for every* $\overline{x} \in X$*.*

Proof. First of all, we will prove the uniqueness of a possible fixed point. If the mapping $\overline{T}$ has a two distinct fixed point $\overline{x}^*$ and $\overline{y}^*$ in $\overline{X}$ then since $d_{bl}\left(\overline{T}\overline{x}^*, \overline{T}\overline{y}^*\right) > 0$ and $d_{bl}(\overline{x}^*, \overline{y}^*) > 0$ we get by to (4):

$$\overline{\mathbb{F}}\left(d_{bl}\left(\overline{T}\overline{x}^*, \overline{T}\overline{y}^*\right)\right) < \tau + \overline{\mathbb{F}}\left(\mathfrak{s}^q \cdot d_{bl}\left(\overline{T}\overline{x}^*, \overline{T}\overline{y}^*\right)\right) \leq \overline{\mathbb{F}}\left(N_{bl}^{A,B,C}(\overline{x}^*, \overline{y}^*)\right), \tag{5}$$

where $N_{bl}^{A,B,C}(\overline{x}^*, \overline{y}^*) = A \cdot \frac{d_{bl}\left(\overline{x}^*, \overline{T}\overline{x}^*\right) \cdot d_{bl}\left(\overline{y}^*, \overline{T}\overline{y}^*\right)}{d_{bl}(\overline{x}^*, \overline{y}^*)} + B \cdot d_{bl}(\overline{x}^*, \overline{y}^*) + C \cdot d_{bl}\left(\overline{y}^*, \overline{T}\overline{y}^*\right)$, that is,

$$\begin{aligned}\overline{\mathbb{F}}(d_{bl}(\overline{x}^*, \overline{y}^*)) &< \overline{\mathbb{F}}\left(N_{bl}^{A,B,C}(\overline{x}^*, \overline{y}^*)\right) \\ &= \overline{\mathbb{F}}(A \cdot 0 + B \cdot d_{bl}(\overline{x}^*, \overline{y}^*) + C \cdot d_{bl}(\overline{x}^*, \overline{y}^*)),\end{aligned} \tag{6}$$

or equivalently,

$$d_{bl}(\overline{x}^*, \overline{y}^*) < (B + C) \cdot d_{bl}(\overline{x}^*, \overline{y}^*). \tag{7}$$

The last obtained relation is in fact, a contradiction. Indeed,

$$\begin{aligned} B + C &< B + Cs \\ &< B + 2Cs \\ &< A + B + 2CS < 1. \end{aligned}$$

In the previously we used that $d_{bl}(\overline{x}, \overline{x}) = 0$ if $\overline{x}$ is a fixed point in $\overline{X}$ of the mapping $\overline{T}$. Further, (4) yields

$$d_{bl}\left(\overline{T}\overline{x}, \overline{T}\overline{y}\right) \leq \mathfrak{s}^q d_{bl}\left(\overline{T}\overline{x}, \overline{T}\overline{y}\right) < A \cdot \frac{d_{bl}\left(\overline{x}, \overline{T}\overline{x}\right) \cdot d_{bl}\left(\overline{y}, \overline{T}\overline{y}\right)}{d_{bl}(\overline{x}, \overline{y})} + B \cdot d_{bl}(\overline{x}, \overline{y}) + C \cdot d_{bl}\left(\overline{y}, \overline{T}\overline{y}\right), \tag{8}$$

for all $\mathfrak{s} \geq 1, q > 1$ and $\overline{x}, \overline{y} \in X$ whenever $d_{bl}\left(\overline{T}\overline{x}, \overline{T}\overline{y}\right) > 0$ and $d_{bl}(\overline{x}, \overline{y}) > 0$.

Now, consider the following Picard sequence $\overline{x}_n = \overline{T}\overline{x}_{n-1}, n \in \mathbb{N}$ where $\overline{x}_0$ is arbitrary point in X. if $\overline{x}_k = \overline{x}_{k-1}$ for some $k \in \mathbb{N}$ then $\overline{x}_{k-1}$ is a unique fixed point of the mapping $\overline{T}$. Therefore, suppose that $\overline{x}_n \neq \overline{x}_{n-1}$ for all $n \in \mathbb{N}$. In this case we have that $d_{bl}(\overline{x}_{n-1}, \overline{x}_n) > 0$ for all $n \in \mathbb{N}$. Since, $d_{bl}\left(\overline{T}\overline{x}_{n-1}, \overline{T}\overline{x}_n\right) > 0$ and $d_{bl}(\overline{x}_{n-1}, \overline{x}_n) > 0$ then according to (4) we get

$$\begin{aligned} d_{bl}(\overline{x}_n, \overline{x}_{n+1}) &< A \cdot \frac{d_{bl}(\overline{x}_{n-1}, \overline{x}_n) \cdot d_{bl}(\overline{x}_n, \overline{x}_{n+1})}{d_{bl}(\overline{x}_{n-1}, \overline{x}_n)} + B \cdot d_{bl}(\overline{x}_{n-1}, \overline{x}_n) + C \cdot d_{bl}(\overline{x}_n, \overline{x}_n) \\ &= A \cdot d_{bl}(\overline{x}_n, \overline{x}_{n+1}) + B \cdot d_{bl}(\overline{x}_{n-1}, \overline{x}_n) + C \cdot d_{bl}(\overline{x}_n, \overline{x}_n) \\ &\leq A \cdot d_{bl}(\overline{x}_n, \overline{x}_{n+1}) + B \cdot d_{bl}(\overline{x}_{n-1}, \overline{x}_n) + 2\mathfrak{s}C \cdot d_{bl}(\overline{x}_{n-1}, \overline{x}_n). \end{aligned} \tag{9}$$

The relation (9) yields

$$d_{bl}(\overline{x}_n, \overline{x}_{n+1}) < \frac{B + 2\mathfrak{s}C}{1 - A} \cdot d_{bl}(\overline{x}_{n-1}, \overline{x}_n). \tag{10}$$

As $\frac{B+2\mathfrak{s}C}{1-A} < 1$ then, by Lemma 1 and Remark 2, we have that the sequence $\{\overline{x}_n\}_{n\in\mathbb{N}}$ is a $0 - d_{bl}-$Cauchy in 0-complete b-metric-like space $\left(\overline{X}, d_{bl}, \mathfrak{s} \geq 1\right)$. This means that exists a unique point $\overline{x}^* \in X$ such that

$$\lim_{n,m\to+\infty} d_{bl}(\overline{x}_n, \overline{x}_m) = \lim_{n\to+\infty} d_{bl}(\overline{x}_n, \overline{x}^*) = d_{bl}(\overline{x}^*, \overline{x}^*). \tag{11}$$

Now, we will prove that $\overline{x}^*$ is a fixed point of $\overline{T}$. Since, the mapping T is continuous, then we get

$$d_{bl}\left(\overline{T}\overline{x}_n, \overline{T}\overline{x}^*\right) \to d_{bl}\left(\overline{T}\overline{x}^*, \overline{T}\overline{x}^*\right), \text{ i.e., } d_{bl}\left(\overline{x}_{n+1}, \overline{T}\overline{x}^*\right) \to d_{bl}\left(\overline{T}\overline{x}^*, \overline{T}\overline{x}^*\right), \tag{12}$$

as $n \to +\infty$. The conditions (11) and (12) show that $\overline{T}\overline{x}^* = \overline{x}^*$,, i.e., $\overline{x}^*$ is a fixed point of $\overline{T}$. This completes the proof of Theorem 2. □

Now we give some corollaries of Theorem 2.

Corollary 1. *Putting in (4) $A = C = 0$ we get that result of D. Wardowski holds true for all five classes of generalized metric spaces (partial metric, metric-like, b-metric, partial b-metric and b-metric-like) for continuous mapping T. Indeed, it this case, (4) yields*

$$\tau + \overline{\mathbb{F}}\left(\mathfrak{s}^q d_{bl}\left(\overline{T}\overline{x}, \overline{T}\overline{y}\right)\right) \le \overline{\mathbb{F}}\left(B \cdot d_{bl}(\overline{x}, \overline{y})\right), \tag{13}$$

for all $\overline{x}, \overline{y} \in \overline{X}$, with $d_{bl}\left(\overline{T}\overline{x}, \overline{T}\overline{y}\right) > 0$ and $d_{bl}(\overline{x}, \overline{y}) > 0$. Further, from (13) follows

$$\tau + \overline{\mathbb{F}}\left(d_{bl}\left(\overline{T}\overline{x}, \overline{T}\overline{y}\right)\right) \le \overline{\mathbb{F}}\left(d_{bl}(\overline{x}, \overline{y})\right), \tag{14}$$

that is., D. Wardowski $\overline{\mathbb{F}}$-contractive condition. This means that continuous mapping $\overline{T}$ has a unique fixed point $\overline{x}^*$ in $\overline{X}$ and $d_{bl}\left(\overline{T}^n\overline{x}, \overline{T}\overline{x}^*\right) = d_{bl}\left(\overline{T}^n\overline{x}, \overline{x}^*\right) \to d_{bl}(\overline{x}, \overline{x}^*)$ as $n \to +\infty$, for all $\overline{x} \in X$.

Corollary 2. *Putting in (4) $A = 0$, $B + 2\mathfrak{s}C < 1$ we get the following $\overline{\mathbb{F}}$-contractive condition:*

$$\tau + \overline{\mathbb{F}}\left(\mathfrak{s}^q d_{bl}\left(\overline{T}\overline{x}, \overline{T}\overline{y}\right)\right) \le \overline{\mathbb{F}}\left(B \cdot d_{bl}(\overline{x}, \overline{y}) + C \cdot d_{bl}\left(\overline{y}, \overline{T}\overline{y}\right)\right), \tag{15}$$

that is

$$\tau + \overline{\mathbb{F}}\left(d_{bl}\left(\overline{T}\overline{x}, \overline{T}\overline{y}\right)\right) < \overline{\mathbb{F}}\left(B \cdot d_{bl}(\overline{x}, \overline{y}) + C \cdot d_{bl}\left(\overline{y}, \overline{T}\overline{y}\right)\right), \tag{16}$$

or equivalently,

$$d_{bl}\left(\overline{T}\overline{x}, \overline{T}\overline{y}\right) < B \cdot d_{bl}(\overline{x}, \overline{y}) + C \cdot d_{bl}\left(\overline{y}, \overline{T}\overline{y}\right). \tag{17}$$

Then, continuous mapping $\overline{T} : \overline{X} \to \overline{X}$ has a unique fixed point $\overline{x}^* \in \overline{X}$ and $d_{bl}\left(\overline{T}^n\overline{x}, \overline{T}\overline{x}^*\right) = d_{bl}\left(\overline{T}^n\overline{x}, \overline{x}^*\right) \to d_{bl}(\overline{x}, \overline{x}^*)$ as $n \to +\infty$, for all $\overline{x} \in X$.

The immediately corollaries of Theorem 2 have new contraction conditions that generalize and complement results from [30,31].

Corollary 3. *Let $\left(\overline{X}, d_{bl}, \mathfrak{s} \ge 1\right)$ be a $0 - d_{bl}-$complete b-metric-like space and $\overline{T}$ be a self mapping satisfying a generalized $(\mathfrak{s}, q)-$Jaggi $\overline{\mathbb{F}}$-contraction-type (4) where $C_i > 0, i = \overline{1,3}$ such that for all $\overline{x}, \overline{y} \in \overline{X}$ with $d_{bl}\left(\overline{T}\overline{x}, \overline{T}\overline{y}\right) > 0$ and $d_{bl}(\overline{x}, \overline{y}) > 0$ the following inequalities hold true.*

$$C_1 + \exp\left(\mathfrak{s}^q \cdot d_{bl}\left(\overline{T}\overline{x}, \overline{T}\overline{y}\right)\right) \le \exp\left(N_{bl}^{A,B,C}(\overline{x}, \overline{y})\right), \tag{18}$$

$$C_2 - \frac{1}{\mathfrak{s}^q \cdot d_{bl}\left(\overline{T}\overline{x}, \overline{T}\overline{y}\right)} \le -\frac{1}{N_{bl}^{A,B,C}(\overline{x}, \overline{y})}, \tag{19}$$

$$C_3 + \exp\left(\mathfrak{s}^q \cdot d_{bl}\left(\overline{T}\overline{x}, \overline{T}\overline{y}\right)\right) \cdot \ln\left(\mathfrak{s}^q \cdot d_{bl}\left(\overline{T}\overline{x}, \overline{T}\overline{y}\right)\right) \le \exp\left(N_{bl}^{A,B,C}(\overline{x}, \overline{y})\right) \cdot \ln\left(N_{bl}^{A,B,C}(\overline{x}, \overline{y})\right), \tag{20}$$

where $N_{bl}^{A,B,C}(\overline{x}, \overline{y}) = A \cdot \frac{d_{bl}\left(\overline{x}^*, \overline{T}\overline{x}^*\right) \cdot d_{bl}\left(\overline{y}^*, \overline{T}\overline{y}^*\right)}{d_{bl}(\overline{x}^*, \overline{y}^*)} + B \cdot d_{bl}(\overline{x}^*, \overline{y}^*) + C \cdot d_{bl}\left(\overline{y}^*, \overline{T}\overline{y}^*\right), A, B, C \ge 0$ with $A + B + 2\mathfrak{s}C < 1$ and $q > 1$.

Then $\overline{T}$ has a unique fixed point $\overline{x}^* \in \overline{X}$ if it is continuous and then for every $\overline{x} \in \overline{X}$ the sequence $\left\{\overline{T}^n\overline{x}\right\}_{n \in \mathbb{N}}$ converges to $\overline{x}^*$.

Proof. First of all, put in Theorem 2. $\overline{\mathbb{F}}(r) = \exp(r)$, $\overline{\mathbb{F}}(r) = -\frac{1}{r}$, $\overline{\mathbb{F}}(r) = \exp(r) \cdot \ln(r)$, respectively. Since every of the functions $r \mapsto \overline{\mathbb{F}}(r)$ is strictly increasing on $(0, +\infty)$ the result follows by Theorem 2. □

3. Main Results

Fixed point theory is an important tool for developing studies and calculations of solutions to differential and integral equations, dynamical systems, models in economy, game theory, physics, computer science, engineering, neural networks and many others. In this section, let us give two applications of our fixed point theorems previously discussed in fractional differential equations and in an initial value problem from mechanical engineering.

Let $\overline{p} : [0, +\infty) \to \overline{\mathbb{R}}$ be a continuous function. Next, we recall the definition of Caputo derivative of function $\overline{p}$ order $\overline{\beta} > 0$ (see [32,33]):

$$^{C}\mathcal{D}^{\overline{\beta}}(\overline{p}(t)) := \frac{1}{\Gamma(n-\overline{\beta})} \int_0^t (t-s)^{n-\overline{\beta}-1} \overline{p}^{(n)}(s) ds \, \left(n - 1 < \overline{\beta} < n, n = \left[\overline{\beta}\right] + 1\right),$$

where $\left[\overline{\beta}\right]$ denotes the integer part of the positive real number $\overline{\beta}$ and Γ is a gamma function.

Further, we will provide an application of the Theorem 2 for proving the existence of a solution of the following nonlinear fractional differential equation

$$^{C}\mathcal{D}^{\overline{\beta}}(\overline{x}(t)) + f(t, \overline{x}(t)) = 0 \, \left(0 \le t \le 1, \overline{\beta} < 1\right) \tag{21}$$

with the boundary conditions $\overline{x}(0) = 0 = \overline{x}(1)$, with $\overline{x} \in \overline{\mathcal{C}}\left([0,1], \overline{\mathbb{R}}\right)$, $\overline{\mathcal{C}}\left([0,1], \overline{\mathbb{R}}\right)$ denotes the set of all continuous functions with real values from $[0,1]$ and $f : [0,1] \times \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ is a continuous function (see [34–37]). The Green function connected with the problem (21) is

$$\mathfrak{G}(t,s) = \begin{cases} (t(1-s))^{\alpha-1} - (t-s)^{\alpha-1} \text{ if } 0 \le s \le t \le 1 \\ \frac{(t(1-s))^{\alpha-1}}{\Gamma(\alpha)} \text{ if } 0 \le t \le s \le 1. \end{cases}$$

Let $\overline{X} = \overline{\mathcal{C}}\left([0,1], \overline{\mathbb{R}}\right)$ endowed with the b-metric-like

$$d_{bl}(\overline{x}, \overline{y}) = \sup_{t \in [0,1]} |\overline{x}(t) + \overline{y}(t)|^q, \text{ for all } \overline{x}, \overline{y} \in \overline{X}.$$

We can prove easily that $\left(\overline{X}, d_{bl}, \mathfrak{s} \ge 1\right)$ is a $0 - d_{bl}-$complete b-metric-like space with parameter $\mathfrak{s} = 2^{q-1}$. For simplicity let us denote the triple $\left(\overline{X}, d_{bl}, \mathfrak{s} \ge 1\right)$ by $\overline{X}$. Obviously $\overline{x}^* \in \overline{X}$ is a solution of (21) if and only if $\overline{x}^* \in \overline{X}$ is a solution of the equation

$$\overline{x}(t) = \int_0^1 \mathfrak{G}(t,s) f(s, \overline{x}(s)) ds \text{ for all } t \in [0,1].$$

Let us give our first main result of this section.

Theorem 3. *Consider the nonlinear fractional order differential Equation* (21)*. Let* ϑ*:* $\overline{\mathbb{R}} \times \mathbb{R} \to \overline{\mathbb{R}}$ *be a given mapping and* $f : [0;1] \times \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ *be a continuous function. Suppose that the following assertions are true:*

(i) *there exists* $\overline{x}_0 \in \overline{X}$ *such that* $\vartheta\left(\overline{x}_0(t), \int_0^t \overline{T}\overline{x}_0(t)\right) \geq 0$ *for all* $t \in [0,1]$, *where* $\overline{T}: \overline{C}\left([0,1],\overline{\mathbb{R}}\right) \to \overline{C}\left([0,1],\overline{\mathbb{R}}\right)$ *is defined by*

$$\overline{T}\overline{x}(t) = \int_0^t \mathfrak{G}(t,s)f(s,\overline{x}(s))ds; \tag{22}$$

(ii) *there exists* $\tau > 0$ *such that for all* $\overline{x}, \overline{y} \in \overline{X}$:

$$d_{bl}\left(\overline{T}\overline{x}, \overline{T}\overline{y}\right) > 0 \text{ and } d_{bl}(\overline{x}, \overline{y}) > 0 \text{ yields}$$

$$|f(t,\overline{x}(t)) + f(t,\overline{y}(t))| \leq \frac{1}{\mathfrak{s}}\sqrt[q]{N_{bl}^{A,B,C}(\overline{x},\overline{y})e^{-\tau}}, \tag{23}$$

for all $t \in [0,1]$, *where*

$$N_{bl}^{A,B,C}(\overline{x},\overline{y}) = A \cdot \frac{d_{bl}(\overline{x}, T\overline{x}) \cdot d_{bl}\left(\overline{y}, \overline{T}\overline{y}\right)}{d_{bl}(\overline{x},\overline{y})} + B \cdot d_{bl}(\overline{x},\overline{y}) + C \cdot d_{bl}\left(\overline{y}, \overline{T}\overline{x}\right), \tag{24}$$

$A, B, C \geq 0$ *with* $A + B + 2C\mathfrak{s} < 1$ *and* $q > 1$;

(iii) *for all* $t \in [0,1]$ *and* $\mu, \nu \in \overline{C}\left([0,1],\overline{\mathbb{R}}\right)$, $\vartheta(\overline{x}(t), \overline{y}(t)) \geq 0$ *yields* $\vartheta\left(\overline{T}\overline{x}(t), T\overline{y}(t)\right) \geq 0$;

(iv) *for all* $t \in [0,1]$, *if* $\{\overline{x}_n\}$ *is a sequence in* $\overline{C}\left([0,1],\overline{\mathbb{R}}\right)$ *such that* $\overline{x}_n \to \overline{x}$ *in* $\overline{C}\left([0,1],\overline{\mathbb{R}}\right)$ *and* $\vartheta(\overline{x}_n(t), \overline{x}_{n+1}(t)) \geq 0$ *for all* $n \in \mathbb{N}$, *then* $\vartheta(\overline{x}_n(t), \overline{x}(t)) \geq 0$ *for all* $n \in \mathbb{N}$.

Then problem (21) *has a solution.*

Proof. It is obvious that the problem (21) can be reduced to find an element $\overline{x}^* \in \overline{X}$, which is a fixed point for the mapping $\overline{T}$.

Let $\overline{x}, \overline{y} \in \overline{X}$ such that $\vartheta(\overline{x}(t), \overline{y}(t)) \geq 0$ for all $t \in [0,1]$. By (iii) we have $\vartheta\left(\overline{T}\overline{x}, \overline{T}\overline{y}\right) \geq 0$. Then by hypothesis (i) and (ii) we have the following inequalities

$$\begin{aligned} \left|\overline{T}\overline{x}(t) + \overline{T}\overline{y}(t)\right| &= \left|\int_0^1 \mathfrak{G}(t,s)f(s,\overline{x}(s))ds + \int_0^1 \mathfrak{G}(t,s)f(s,\overline{y}(s))ds\right| \\ &= \left|\int_0^1 \mathfrak{G}(t,s)[f(s,\overline{x}(s)) + f(s,\overline{y}(s))]ds\right| \\ &\leq \int_0^1 |f(s,\overline{x}(s)) + f(s,\overline{y}(s))|ds \int_0^1 \mathfrak{G}(t,s)ds \\ &\leq \tfrac{1}{\mathfrak{s}}\sqrt[q]{N_{bl}^{A,B,C}(\overline{x},\overline{y})e^{-\tau}} \int_0^1 \mathcal{G}(t,s)ds \end{aligned} \tag{25}$$

Since $\int_0^1 \mathfrak{G}(t,s)ds \leq 1$ and taking supremum in both sides we get

$$\begin{aligned} \sup_{t\in[0,1]}\left|\overline{T}\overline{x}(t) + \overline{T}\overline{y}(t)\right| &\leq \tfrac{1}{\mathfrak{s}} \sup_{t\in[0,1]} \sqrt[q]{N_{bl}^{A,B,C}(\overline{x},\overline{y})e^{-\tau}} \cdot \sup_{t\in[0,1]} \int_0^1 \mathfrak{G}(t,s)ds \\ &\leq \tfrac{1}{\mathfrak{s}} \sup_{t\in[0,1]} \sqrt[q]{N_{bl}^{A,B,C}(\overline{x},\overline{y})e^{-\tau}}. \end{aligned} \tag{26}$$

This means

$$\sup_{t\in[0,1]}\left|\overline{T}\overline{x}(t) + \overline{T}\overline{y}(t)\right|^q \leq \frac{1}{\mathfrak{s}^q} N_{bl}^{A,B,C}(\overline{x},\overline{y})e^{-\tau}. \tag{27}$$

Then we get

$$\mathfrak{s}^q d_{bl}\left(\overline{T}\overline{x}, \overline{T}\overline{y}\right) \leq N_{bl}^{A,B,C}(\overline{x},\overline{y})e^{-\tau}. \tag{28}$$

If we take $\overline{\mathbb{F}}(\omega) = \ln(\omega)$ for all $\omega > 0$ and since $\overline{\mathbb{F}} \in \mathcal{F}$ we get

$$\ln\left(\mathfrak{s}^q d_{bl}\left(\overline{T}\overline{x}, \overline{T}\overline{y}\right)\right) \leq \ln\left(N_{bl}^{A,B,C}(\overline{x}, \overline{y})e^{-\tau}\right) \text{ or}$$

$$\tau + \ln\left(\mathfrak{s}^q d_{bl}\left(\overline{T}\overline{x}, \overline{T}\overline{y}\right)\right) \leq \ln\left(N_{bl}^{A,B,C}(\overline{x}, \overline{y})\right). \tag{29}$$

Equivalently

$$\tau + \overline{\mathbb{F}}\left(\mathfrak{s}^q d_{bl}\left(\overline{T}\overline{x}, \overline{T}\overline{y}\right)\right) \leq \overline{\mathbb{F}}\left(A \cdot \frac{d_{bl}\left(\overline{x}, \overline{T}\overline{x}\right) \cdot d_{bl}\left(\overline{y}, \overline{T}\overline{y}\right)}{d_{bl}(\overline{x}, \overline{y})} + B \cdot d_{bl}(\overline{x}, \overline{y}) + C \cdot d_{bl}\left(\overline{y}, \overline{T}\overline{x}\right)\right) \tag{30}$$

where $A, B, C \geq 0$ with $A + B + 2C\mathfrak{s} < 1$ and $q > 1$.

Applying Theorem 2, we deduce that $\overline{T}$ has a fixed point, which yields that Equation (21) has at least one solution. □

Next, we will give a new application of Theorem 2, in mechanical engineering. Then, an automobile suspension system is the practicable application for the spring-mass system in engineering matters. Let us study the motion of an automobile spring when motion of it is upon a rugged and pitted road, where the forcing term is the rugged road and shock absorbers provide the damping. The possible external forces acting on the system are the gravity, the tension force, the earthquake, etc. We denote by m be the spring mass and by Φ the external force acting on it. Then, the next initial value problem express the critical damped motion of the spring-mass system under the action of an external force Φ.

$$\begin{cases} m\frac{d^2\overline{x}}{dt^2} + \pi\frac{d\overline{x}}{dt} - \Phi(t, \overline{x}(t)) = 0; \\ \overline{x}(0) = 0; \\ \overline{x}'(0) = 0; \end{cases} \tag{31}$$

where $\pi > 0$ denote the dumping constant and $\Phi : [0, \theta] \times \overline{\mathbb{R}^+} \to \overline{\mathbb{R}}$ is a continuous map.

Obviously, the problem (31) with the following integral equation are equivalent.

$$\overline{x}(t) = \int_0^\theta \mathfrak{G}(t, s)\Phi(s, \overline{x}(s))ds, \text{ with } t, s \in [0, \theta]. \tag{32}$$

where $\mathfrak{G}(t, s)$ is the corresponding Green's function, defined as follows

$$\mathfrak{G}(t, s) = \begin{cases} \frac{1 - e^{\xi(t-s)}}{\xi}, \text{ for } 0 \leq s \leq t \leq \theta; \\ 0, \text{ for } 0 \leq t \leq s \leq \theta; \end{cases}$$

where $\xi = \frac{\pi}{m}$ is a constant ratio.

Let us consider $\overline{X} = \overline{C}\left([0, \theta], \overline{\mathbb{R}}\right)$ be the set of real continuous functions defined on $[0, \theta]$. Then, for $q \geq 1$ we consider the following b-metric-like

$$d_{bl}(\overline{x}, \overline{y}) = (\|\overline{x}\|_\infty + \|\overline{y}\|_\infty)^2, \text{ for all } \overline{x}, \overline{y} \in \overline{X}. \tag{33}$$

where $\|\overline{x}\|_\infty = \sup\limits_{t \in [0, \theta]} |\overline{x}(t)|e^{-\tau t}$, with $\tau > 1$ and $t \in [0, \theta]$.

Then, it is easy to check that $\left(\overline{X}, d_{bl}, \mathfrak{s} \geq 1\right)$ form a $0 - d_{bl}-$complete b-metric-like space with the coefficient $\mathfrak{s} = 2$. Let us denote again the triple $\left(\overline{X}, d_{bl}, \mathfrak{s} \geq 1\right)$ by $\overline{X}$.

Then, we have the prove that the problem (31) admits a solution if and only if there exists $\overline{x}^* \in \overline{X}$, a solution of the equation

$$\overline{x}(t) = \int_0^\theta \mathfrak{G}(t, s)\Phi(s, \overline{x}(s))ds, \text{ with } t, s \in [0, \theta].$$

Further, let us give the following second main theorem of this section.

Theorem 4. *Consider the problem* (31) *and the operator* $\overline{T} : \overline{C}\Big([0,\theta], \overline{\mathbb{R}}\Big) \to \overline{C}\Big([0,\theta], \overline{\mathbb{R}}\Big)$

$$\overline{T}\overline{x}(t) = \int_0^\theta \mathfrak{G}(t,s)\Phi(s,\overline{x}(s))ds, \text{ with } t,s \in [0,\theta].$$

Suppose that:

(i) *the function* $\Phi : [0,\theta] \times \overline{\mathbb{R}^+} \to \overline{\mathbb{R}}$ *is a continuous function;*

(ii) *there exists* $\tau > 0$ *such that, for all* $\overline{x}, \overline{y} \in \overline{X}$, *we have:*

$$d_{bl}\big(\overline{T}\overline{x}, \overline{T}\overline{y}\big) > 0 \text{ and } d_{bl}(\overline{x}, \overline{y}) > 0 \text{ yields}$$

$$|\Phi(t,\overline{x}(t))| + |\Phi(t,\overline{y}(t))| \leq \sqrt{\frac{e^{-\tau(1-t)}}{\mathfrak{s}^2} N_{bl}^{A,B,C}(\overline{x},\overline{y})}, \tag{34}$$

for all $t \in [0,\theta]$ *and* $\tau > 1$, *where*

$$N_{bl}^{A,B,C}(\overline{x},\overline{y}) = A \cdot \frac{d_{bl}\big(\overline{x}, \overline{T}\overline{x}\big) \cdot d_{bl}\big(\overline{y}, \overline{T}\overline{y}\big)}{d_{bl}(\overline{x},\overline{y})} + B \cdot d_{bl}(\overline{x},\overline{y}) + C \cdot d_{bl}\big(\overline{y}, \overline{T}\overline{x}\big), \tag{35}$$

$A, B, C \geq 0$ *with* $A + B + 2C\mathfrak{s} < 1$ *and* $q > 1$.

(iii) *for all* $t \in [0,\theta]$ *and* $\mu, \nu \in \overline{C}\Big([0,\theta], \overline{\mathbb{R}}\Big)$,

$$\vartheta(\overline{x}(t), \overline{y}(t)) \geq 0 \text{ yields } \vartheta\big(\overline{T}\overline{x}(t), T\overline{y}(t)\big) \geq 0.$$

Then, the integral Equation (31) *has a unique solution.*

Proof. Then problem (31) can be considered to find an element $\overline{x}^* \in \overline{X}$, which is a fixed point for the operator $\overline{T}$.

Let $\overline{x}, \overline{y} \in \overline{X}$ such that $d_{bl}(\overline{x}(t), \overline{y}(t)) > 0$ for all $t \in [0,\theta]$. By hypothesis (iii) we have $d_{bl}(\overline{T}\overline{x}, \overline{T}\overline{y}) > 0$. According with the hypothesis (i) and (ii) of the theorem, we have the following inequalities

$$\begin{aligned}
\mathfrak{s}^2\big(\big|\overline{T}\overline{x}(t)\big| + \big|\overline{T}\overline{x}(t)\big|\big)^2 &= \mathfrak{s}^2\left(\left|\int_0^\theta \mathfrak{G}(t,s)\Phi(s,\overline{x}(s))ds\right| + \left|\int_0^\theta \mathfrak{G}(t,s)\Phi(s,\overline{y}(s))ds\right|\right)^2 \\
&\leq \mathfrak{s}^2\left(\int_0^\theta |\mathfrak{G}(t,s)\Phi(s,\overline{x}(s))|ds + \int_0^\theta |\mathfrak{G}(t,s)\Phi(s,\overline{y}(s))|ds\right)^2 \\
&\leq \mathfrak{s}^2\left(\int_0^\theta \mathfrak{G}(t,s)(|\Phi(s,\overline{x}(s))| + |\Phi(s,\overline{y}(s))|)ds\right)^2 \\
&\leq \mathfrak{s}^2\left(\int_0^\theta \mathfrak{G}(t,s)\sqrt{\tfrac{e^{-\tau(1-t)}}{\mathfrak{s}^2} N_{bl}^{A,B,C}(\overline{x},\overline{y})}ds\right)^2 \\
&\leq \mathfrak{s}^2 \tfrac{e^{-\tau(1-t)}}{\mathfrak{s}^2} N_{bl}^{A,B,C}(\overline{x},\overline{y})\left(\int_0^\theta \mathfrak{G}(t,s)ds\right)^2 \\
&\leq \tfrac{e^{-\tau}}{e^{-2\tau t}} N_{bl}^{A,B,C}(\overline{x},\overline{y})e^{-\tau t}\left(\int_0^\theta \mathfrak{G}(t,s)ds\right)^2.
\end{aligned} \tag{36}$$

Then, we have

$$\mathfrak{s}^2\left(\left|\overline{T}\overline{x}(t)\right| + \left|\overline{T}\overline{x}(t)\right|\right)^2 e^{-2\tau t} \leq e^{-\tau} N_{bl}^{A,B,C}(\overline{x},\overline{y}) e^{-\tau t}\left(\int_0^{\theta} \mathfrak{G}(t,s)ds\right)^2. \tag{37}$$

Since $\int_0^{\theta} \mathfrak{G}(t,s)ds \leq 1$ and taking supremum on both sides, results

$$\mathfrak{s}^2\left(\|\overline{T}\overline{x}(t)\|_{\infty} + \|\overline{T}\overline{y}(t)\|_{\infty}\right)^2 \leq e^{-\tau}\left\|N_{bl}^{A,B,C}(\overline{x},\overline{y})\right\|_{\infty}. \tag{38}$$

Then $\mathfrak{s}^2 d_{bl}(\overline{T}\overline{x},\overline{T}\overline{y}) \leq e^{-\tau}\left\|N_{bl}^{A,B,C}(\overline{x},\overline{y})\right\|_{\infty}$.
For $\overline{\mathbb{F}}(w) = \ln w$, for all $w > 0$ and $\overline{\mathbb{F}}(w) \in \mathcal{F}$ we obtain

$$\ln\left(\mathfrak{s}^2 d_{bl}(\overline{T}\overline{x},\overline{T}\overline{y})\right) \leq \ln\left(e^{-\tau}\left\|N_{bl}^{A,B,C}(\overline{x},\overline{y})\right\|_{\infty}\right). \tag{39}$$

Equivalently

$$\tau + \overline{\mathbb{F}}(\mathfrak{s}^q d_{bl}(\overline{T}\overline{x},\overline{T}\overline{y})) \leq \overline{\mathbb{F}}\left(A \cdot \frac{d_{bl}(\overline{x},\overline{T}\overline{x}) \cdot d_{bl}(\overline{y},\overline{T}\overline{y})}{d_{bl}(\overline{x},\overline{y})} + B \cdot d_{bl}(\overline{x},\overline{y}) + C \cdot d_{bl}(\overline{y},\overline{T}\overline{x})\right). \tag{40}$$

By Theorem 2 with the coefficient $q = 2$, we get that $\overline{T}$ has a fixed point, which is the unique solution of the problem 31. □

4. Numerical Example

In this section, we provide a numerical example to sustain our applications. For the case of the first application of the previous section, Theorem 3, let us consider the following nonlinear differential equation

$$\overline{x}(t) = \int_0^t \left[(t(1-s))^{\alpha-1} - (t-s)^{\alpha-1}\right] cos(\overline{x}(s))ds, \text{ with } 0 \leq s \leq t \leq 1. \tag{41}$$

Then, we consider the operator $\overline{T} : C\left([0,\theta],\overline{\mathbb{R}}\right) \to C\left([0,\theta],\overline{\mathbb{R}}\right)$ defined as

$$\overline{T}\overline{x}(t) = \int_0^t \left[(t(1-s))^{\alpha-1} - (t-s)^{\alpha-1}\right] cos(\overline{x}(s))ds.$$

It is easy to check that, for $q = 1$ and $\mathfrak{s} = 1$, under the assumptions of Theorem 3, the integral Equation (41) has a unique solution, such that $\overline{x}(t) = \overline{T}\overline{x}(t) = \frac{t}{3}$.
Further, we shall use the iteration method to underline the validity of our approaches

$$\overline{x}_{n+1}(t) = \int_0^1 \left[(t(1-s))^{\alpha-1} - (t-s)^{\alpha-1}\right] cos(\overline{x}_n(s))ds.$$

Let $\alpha \in (1,2)$. Then, we consider $\alpha = 1.5$ and $\overline{x}_0(t) = 0$ as starting point. Table 1 show that for $t = 0.1$ the sequence $\overline{x}_{n+1}(t) = \int_0^1 \left[(t(1-s))^{\alpha-1} - (t-s)^{\alpha-1}\right] cos(\overline{x}_n(s))ds$ converge to the exact solution $\overline{x}(0.1) = \overline{T}(\overline{x}(0.1)) = 0.033$.

Table 1. For $t = 0.1$ exact solution is $\overline{x}(0.1) = 0.033$.

n	$\overline{x}_{n+1}(0.1)$	Approximate Solution	Absolute Error
0	$\overline{x}_1(0.1)$	0.0308	$2.5 * 10^{-3}$
1	$\overline{x}_2(0.1)$	0.0307	$2.6 * 10^{-3}$
2	$\overline{x}_3(0.1)$	0.0307	$2.6 * 10^{-3}$

Using Python, a well known scientific computer program, in order to obtain the interpolated graphs of nonlinear integral equation for two cases, $t = 0.1$ and $t = 0.9$, we get the following interpolated graphs, Figure 1 respectively, Figure 2.

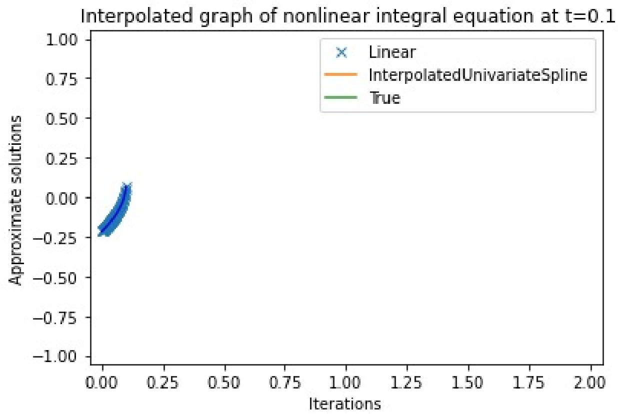

Figure 1. Interpolated graph for $t = 0.1$.

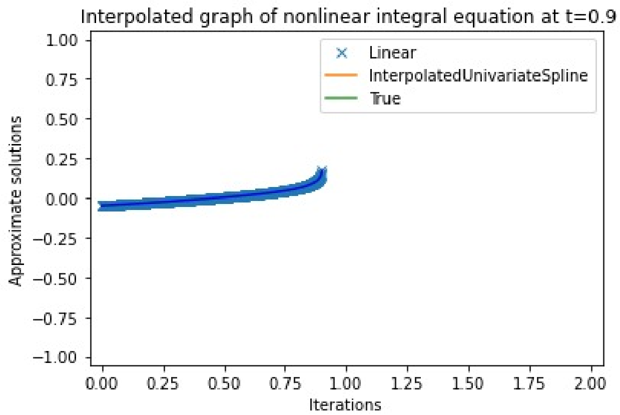

Figure 2. Interpolated graph for $t = 0.9$.

5. Conclusions

In this manuscript, among other things, using one theorem from the fixed point theory, we prove the following:

- One type of Caputo fractional differential equation has at least one solution.
- A special integral equation created in mechanical engineering has a solution.

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