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Existence and Uniqueness Results of Coupled Fractional-Order Differential Systems Involving Riemann–Liouville Derivative in the Space $W_{a^+}^{\gamma_1,1}(a,b) \times W_{a^+}^{\gamma_2,1}(a,b)$ with Perov's Fixed Point Theorem

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Abstract: This paper is devoted to studying the existence and uniqueness of a system of coupled fractional differential equations involving a Riemann–Liouville derivative in the Cartesian product of fractional Sobolev spaces $\mathcal{E} = W_{a^+}^{\gamma_1,1}(a,b) \times W_{a^+}^{\gamma_2,1}(a,b)$. Our strategy is to endow the space $\mathcal{E}$ with a vector-valued norm and apply the Perov fixed point theorem. An example is given to show the usefulness of our main results.

Keywords: coupled system; fractional differential equations; Riemann–Liouville derivative; generalized Banach space; fixed point theorems; convergent to zero matrix

1. Introduction

The beauty of fractional calculus lies in finding the derivative and integration of an operator for any order. Therefore, fractional derivatives became helpful in studying the anomalous behavior of dynamical systems in biology, viscoelasticity, bioengineering, electrochemistry, etc. [1–4]. So, the subject of fractional calculus has become the center of attractive research.

The origins of fractional calculus go back to the end of the 17th century, starting from the discussion between Leibniz and de l'Hôpital regarding the meaning of $\frac{d^{\frac{1}{2}}f}{dt^{\frac{1}{2}}}$ [5]. Moreover, several investigations have been introduced to develop and study this important mathematical field, including Liouville, Riemann, Abel, Riesz, Weyl, Hadamard, and Caputo.

To ensure a solution of some nonlinear problems, researchers utilize some suitable fixed point theorems. One of these theorems is the Banach contraction principle. Perov, in 1965, extended the Banach contraction principle to the vector-valued metric spaces by replacing the contraction factor with a matrix that converges to zero [6]. Perov's fixed point theorem is one of the crucial methods to prove an existence solution of systems of differential equations, fractional differential equations, and integral equations in N variables; see [7–10], and the references cited therein.

Recently, a number of interesting papers [11–13] on the solvability of mathematical problems in Sobolev spaces $W^{n,p}(\mathbb{R}^+)$ [14] with the help of fixed point theory have been presented. In [15], the authors utilized the Riemann–Liouville derivative to introduce the left fractional Sobolev spaces $W^{\gamma,p}_{a^+}(a,b)$, where $-\infty < a < b < +\infty$, $1 \le p < +\infty$, and $n-1 < \gamma \le n, n \in \mathbb{N}$.

Boucenna et al. [16] proved the existence of the solution for the following initial value problem:

$$\begin{cases} {}^{RL}\mathcal{D}^{\alpha}_{0^+}\varrho(t) = f\left(t, \varrho(t), {}^{RL}\mathcal{D}^{\alpha-1}_{0^+}\varrho(t)\right), t \in (0,1) \\ {}^{RL}\mathcal{D}^{\alpha-1}_{0^+}\varrho(0) = \varrho_0, \quad {}^{RL}\mathcal{I}^{2-\alpha}_{0^+}\varrho(0) = \varrho_1 \end{cases}$$

in the fractional Sobolev space $W^{\alpha-1,1}_{0^+}(0,1)$ where ${}^{RL}\mathcal{D}^{\alpha}_{0^+}$ is the Riemann–Liouville fractional derivative of order α, $1 < \alpha \le 2$, ${}^{RL}\mathcal{I}^{2-\alpha}_{0^+}$ is the Riemann–Liouville fractional integral, and f satisfies some certain conditions.

Our work is devoted to studying the existence and the uniqueness of a coupled system of fractional differential equations of the form:

$$\begin{cases} {}^{RL}\mathcal{D}^{\alpha_1}_{0^+}\varrho_1(t) = f_1(t, \varrho_1(t), \varrho_2(t), {}^{RL}\mathcal{D}^{\alpha_1-1}_{0^+}\varrho_1(t)), \\ {}^{RL}\mathcal{D}^{\alpha_2}_{0^+}\varrho_2(t) = f_2(t, \varrho_1(t), \varrho_2(t), {}^{RL}\mathcal{D}^{\alpha_2-1}_{0^+}\varrho_2(t)), \\ {}^{RL}\mathcal{D}^{\alpha_1-1}_{0^+}\varrho_1(0) = \varrho^0_1, \quad {}^{RL}\mathcal{I}^{2-\alpha_1}_{0^+}\varrho_1(0) = \varrho^1_1 \\ {}^{RL}\mathcal{D}^{\alpha_2-1}_{0^+}\varrho_2(0) = \varrho^0_2, \quad {}^{RL}\mathcal{I}^{2-\alpha_2}_{0^+}\varrho_2(0) = \varrho^1_2, \end{cases} \tag{1}$$

in the generalized Banach space $W^{\alpha_1-1,1}_{0^+}(0,b) \times W^{\alpha_2-1,1}_{0^+}(0,b)$ where $\varrho^0, \varrho^1 \in \mathbb{R}^2$, $1 < \alpha_1, \alpha_2 \le 2$, and for each $i = 1,2$, ${}^{RL}\mathcal{D}^{\alpha_i}_{0^+}$ is the Riemann–Liouville fractional derivative of order α_i and $f_i : (0,b) \times \mathbb{R}^3 \longrightarrow \mathbb{R}$.

We organize our paper as follows. In Section 2, we present the essential definitions and background that will be used in the rest of our paper. In Section 3, we use the Perov fixed point theorem to establish the existence and uniqueness of a solution of problem (1). In the last section, we give an example to show the applicability of our main result.

2. Preliminaries

In this section, we introduce some notations, definitions, and auxiliary results that will be used later. We begin with the following basic definitions of fractional calculus.

Definition 1. *The Riemann–Liouville fractional integral [17] of order γ for a function $f \in L^1(0,b)$ is defined as*

$${}^{RL}\mathcal{I}^{\gamma}_{0^+}f(t) = \frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1}f(s)ds,$$

where

$$\Gamma(\gamma) = \int_0^{\infty} t^{\gamma-1}e^{-t}dt \qquad \mathcal{R}(\gamma) > 0.$$

Definition 2. *The Riemann–Liouville fractional derivative of order γ of a function f is given by*

$${}^{RL}\mathcal{D}^{\gamma}_{0^+}f(t) = \frac{1}{\Gamma(n-\gamma)}\left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\gamma-1}f(s)ds,$$

where $n = In(\gamma) + 1$, and $In(\gamma)$ denotes the integer part of γ [17,18].

Definition 3. *The Mittag-Leffler function is defined by [17,19]*

$$E_{\gamma,\lambda}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\gamma k + \lambda)}, \quad (z \in \mathbb{C}, \mathcal{R}(\lambda), \mathcal{R}(\gamma) > 0).$$

The following assertion shows that fractional differentiation is an operation inverse to the fractional integration from the left.

Lemma 1. *If $f \in L^1[0,b]$, then the following equality*

$$^{RL}\mathcal{D}^{\gamma}_{0^+}\,^{RL}\mathcal{I}^{\gamma}_{0^+} f(t) = f(t), \quad \mathcal{R}(\gamma) > 0$$

holds almost everywhere on $[0,b]$ [18,19].

Lemma 2. *Let $z, t \in \mathbb{R}$ [20]. Then,*

$$E_{\gamma,\lambda}(z) \le \frac{1}{\gamma} z^{\frac{1-\lambda}{\gamma}} e^{z^{\frac{1}{\gamma}}}, \quad 0 < \gamma < 2, \lambda > 1.$$

On $\mathcal{M}_{m\times n}(\mathbb{R}_+)$, we define a partial order relation as follows: Let $M, N \in \mathcal{M}_{m\times n}(\mathbb{R}_+)$, $m \ge 1$, and $n \ge 1$. Put $M = (M_{i,j})_{\substack{1\le j\le m\\ 1\le i\le n}}$ and $N = (N_{i,j})_{\substack{1\le j\le m\\ 1\le i\le n}}$. Then,

$$M \preccurlyeq N \text{ if } N_{i,j} \ge M_{i,j} \quad \text{for all } j = 1,\cdots,m,\ i = 1,\cdots,n.$$
$$M \prec N \text{ if } N_{i,j} > M_{i,j} \quad \text{for all } j = 1,\cdots,m,\ i = 1,\cdots,n.$$

Definition 4. *Let $\mathcal{E}$ be a vector space on $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. By a generalized norm on $\mathcal{E}$, we mean a map*

$$\|\cdot\|_G : \mathcal{E} \longrightarrow \mathbb{R}^n_+$$
$$\varrho \mapsto \|\varrho\|_G = \begin{pmatrix} \|\varrho\|_1 \\ \vdots \\ \|\varrho\|_n \end{pmatrix}$$

satisfying the following properties:

(i) For all $\varrho \in \mathcal{E}$; if $\|\varrho\|_G = 0_{\mathbb{R}^n_+}$, then $\varrho = 0$,

(ii) $\|\lambda\varrho\|_G = |\lambda|\|\varrho\|_G$ for all $\varrho \in \mathcal{E}$ and $\lambda \in \mathbb{K}$, and

(iii) $\|\varrho + v\|_G \preccurlyeq \|\varrho\|_G + \|v\|_G$ for all $\varrho, v \in \mathcal{E}$.

The pair $(\mathcal{E}, \|\cdot\|_G)$ is called a vector (generalized) normed space. Furthermore, $(\mathcal{E}, \|\cdot\|_G)$ is called a generalized Banach space (in short, GBS), if the vector metric space generated by its vector metric is complete.

Remark 1. *In the sense of Perov, the definitions of convergence sequence, continuity, and open and closed subsets in a GBS are similar to those for usual Banach spaces [21].*

Let $(\mathcal{E}, \|\cdot\|_G)$ be a generalized Banach space. For $r = (r_1,\cdots,r_n) \in \mathbb{R}^n_+, \varrho_0 \in \mathcal{E}$, and $i = 1,\cdots,n$, we define the open ball $B(\varrho_0, r)$ and the closed ball $\overline{B}(\varrho_0, r)$ centered at ϱ_0 as follows:

$$B(\varrho_0, r) = \{\varrho \in \mathcal{E} : \|\varrho_0 - \varrho\|_G \prec r\} \ \ (\text{resp. } B_i(\varrho_0, r_i) = \{\varrho \in \mathcal{E} : \|\varrho_0 - \varrho\|_i < r_i\}),$$

and

$$\overline{B}(\varrho_0, r) = \{\varrho \in \mathcal{E} : \|\varrho_0 - \varrho\|_G \preccurlyeq r\} \ \ (\text{resp. } \overline{B_i}(\varrho_0, r_i) = \{\varrho \in \mathcal{E} : \|\varrho_0 - \varrho\|_i \le r_i\}),$$

If $\varrho_0 = 0$, then we simply denote $B_r = B(0, r)$ and $\overline{B_r} = \overline{B}(0, r)$.

Definition 5. *Let $(\mathcal{E}, \|\cdot\|_G)$ be a GBS and let $\mathcal{K}$ be a subset of $\mathcal{E}$. Then, $\mathcal{K}$ is said to be G-bounded, if there exists a vector $V \in \mathbb{R}^n_+$ such that*

$$\text{for all } \varrho \in \mathcal{K}, \qquad \|\varrho\|_G \preccurlyeq V.$$

Definition 6. *A matrix $M \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$ is said to be convergent to zero if*

$$M^m \longrightarrow 0, \quad \text{as} \quad m \longrightarrow \infty.$$

Lemma 3. *Let $M \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$. Then, the following affirmations are equivalent [22] :*

(i) *The matrix M converges to zero.*
(ii) *The matrix $I - M$ is invertible, and $(I - M)^{-1} \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$.*
(iii) *The spectral radius $\rho(M)$ is strictly less than 1.*

Definition 7. *Let $(\mathcal{E}, d_G)$ be a complete generalized metric space and let N be an operator from $\mathcal{E}$ into itself. N is called G-Lipschitzian if there exists a square matrix M of non-negative numbers such that*

$$d_G(N(\varrho), N(v)) \preccurlyeq M d_G(\varrho, v), \quad \text{for all } \varrho,\ v \in \mathcal{E}.$$

If the matrix M converges to zero, then N is called an M-contraction.

The following result is due to Perov, which is a generalization of the Banach contraction principle.

Theorem 1. *Let $\mathcal{E}$ be a complete generalized metric space and let $N : \mathcal{E} \longrightarrow \mathcal{E}$ be an M-contraction operator [6]. Then, N has a unique fixed point in $\mathcal{E}$.*

From [15], the left fractional Sobolev space of order γ is the set $W^{\gamma,1}_{0^+}(0,b)$ defined as follows:

$$W^{\gamma,1}_{0^+}(0,b) = \left\{ f : f \in L^1(0,b), {}^{RL}\mathcal{D}^{\gamma}_{0^+} f \in L^1(0,b) \right\}, \quad 0 < \gamma < 1,$$

endowed with the norm

$$\|f\|_{W^{\gamma,1}_{0^+}(0,b)} = \|f\|_{L^1(0,b)} + \left\| {}^{RL}\mathcal{D}^{\gamma}_{0^+} f \right\|_{L^1(0,b)}.$$

Lemma 4. *$W^{\gamma_1,1}_{0^+}(0,b) \times W^{\gamma_2,1}_{0^+}(0,b)$ is a generalized Banach space endowed with the generalized norm*

$$\|x\|_G = \begin{pmatrix} \|\varrho_1\|_{W^{\gamma_1,1}_{0^+}(0,b)} \\ \|\varrho_2\|_{W^{\gamma_2,1}_{0^+}(0,b)} \end{pmatrix} = \begin{pmatrix} \|\varrho_1\|_{L^1(0,b)} + \|{}^{RL}\mathcal{D}^{\gamma_1}_{0^+} \varrho_1\|_{L^1(0,b)} \\ \|\varrho_2\|_{L^1(0,b)} + \|{}^{RL}\mathcal{D}^{\gamma_2}_{0^+} \varrho_2\|_{L^1(0,b)} \end{pmatrix}.$$

3. Main Results

In this section, we study the existence and the uniqueness of a solution for a coupled system of fractional differential equations (1).

Lemma 5. *ϱ is a solution of System (1) if and only if it is a solution of the following problem:*

$$\begin{cases} {}^{RL}\mathcal{D}^{\beta_1}_{0^+} \varrho_1(t) = e^{\theta t} \int_0^t F_1\left(s, \varrho_1(s), \varrho_2(s), {}^{RL}\mathcal{D}^{\beta_1}_{0^+} \varrho_1(s)\right) e^{-\theta s} ds + \varrho_1^0 e^{\theta t}, \\ {}^{RL}\mathcal{D}^{\beta_2}_{0^+} \varrho_2(t) = e^{\theta t} \int_0^t F_2\left(s, \varrho_1(s), \varrho_2(s), {}^{RL}\mathcal{D}^{\beta_2}_{0^+} \varrho_2(s)\right) e^{-\theta s} ds + \varrho_2^0 e^{\theta t}, \\ {}^{RL}\mathcal{I}^{1-\beta_1}_{0^+} \varrho_1(0) = \varrho_1^1, \\ {}^{RL}\mathcal{I}^{1-\beta_2}_{0^+} \varrho_2(0) = \varrho_2^1 \end{cases} \tag{2}$$

where $\beta_i = \alpha_i - 1$, $\theta > 0$, *and*

$$F_i\Big(s, \varrho_1(s), \varrho_2(s), {}^{RL}\mathcal{D}_{0^+}^{\beta_i}\varrho_i(s)\Big) = f_i\Big(s, \varrho_1(s), \varrho_2(s), {}^{RL}\mathcal{D}_{0^+}^{\beta_i}\varrho_i(s)\Big) - \theta\, {}^{RL}\mathcal{D}_{0^+}^{\beta_i}\varrho_i(s),$$

for each $i = 1, 2$ *.*

Proof. For each $i = 1, 2$, we have

$$\int_0^t {}^{RL}\mathcal{D}_{0^+}^{\alpha_i}\varrho_i(s)e^{-\theta s}ds = \int_0^t f_i\Big(s, \varrho_1(s), \varrho_2(s), {}^{RL}\mathcal{D}_{0^+}^{\alpha_i-1}\varrho_i(s)\Big)e^{-\theta s}ds.$$

Thus

$${}^{RL}\mathcal{D}_{0^+}^{\alpha_i-1}\varrho_i(t)e^{-\theta t} = \int_0^t f_i\Big(s, \varrho_1(s), \varrho_2(s), {}^{RL}\mathcal{D}_{0^+}^{\alpha_i-1}\varrho_i(s)\Big)e^{-\theta s} - \theta\, {}^{RL}\mathcal{D}_{0^+}^{\beta_i}\varrho_i(s)e^{-\theta s}ds + \varrho_i^0.$$

Hence,

$${}^{RL}\mathcal{D}_{0^+}^{\beta_i}\varrho_i(t) = e^{\theta t}\int_0^t F_i\Big(s, \varrho_1(s), \varrho_2(s), {}^{RL}\mathcal{D}_{0^+}^{\beta_i}\varrho_i(s)\Big)e^{-\theta s}ds + \varrho_i^0 e^{\theta t}.$$

Reciprocally, by returning to (2), for each $i = 1, 2$, ${}^{RL}\mathcal{D}_{0^+}^{\beta_i}\varrho_i(0) = \varrho_i^0$, we find

$$\begin{aligned}{}^{RL}\mathcal{D}_{0^+}^{\beta_i}\varrho_i(t)e^{\theta t} - {}^{RL}\mathcal{D}_{0^+}^{\beta_i}\varrho_i(0)e^{\theta t} = &\int_0^t f_i\Big(s, \varrho_1(s), \varrho_2(s), {}^{RL}\mathcal{D}_{0^+}^{\beta_i}\varrho_i(s)\Big)e^{-\theta s}ds \\ &- \int_0^t \theta\, {}^{RL}\mathcal{D}_{0^+}^{\beta_i}\varrho_i(s)e^{-\theta s}ds.\end{aligned}$$

Hence, by replacing β_i with $\alpha_i - 1$, we get

$${}^{RL}\mathcal{D}_{0^+}^{\alpha_i}\varrho_i(t) = f_i\Big(s, \varrho_1(s), \varrho_2(s), {}^{RL}\mathcal{D}_{0^+}^{\alpha_i-1}\varrho_i(s)\Big).$$

□

Lemma 6. *ϱ is a solution of System* (2) *if and only if it is a solution of the following fractional integral equation system:*

$$\begin{cases}\varrho_1(t) = \displaystyle\int_0^t (t-s)^{\beta_1}E_{1,\beta_1+1}(\theta(t-s))F_1\Big(s, \varrho_1(s), \varrho_2(s), {}^{RL}\mathcal{D}_{0^+}^{\beta_1}\varrho_1(s)\Big)ds \\ \qquad\quad + \varrho_1^0 t^{\beta_1}E_{1,\beta_1+1}(\theta t) + \dfrac{\varrho_1^1}{\Gamma(\beta)}t^{\beta_1-1} \\ \varrho_2(t) = \displaystyle\int_0^t (t-s)^{\beta_2}E_{1,\beta_2+1}(\theta(t-s))F_2\Big(s, \varrho_1(s), \varrho_2(s), {}^{RL}\mathcal{D}_{0^+}^{\beta_2}\varrho_2(s)\Big)ds \\ \qquad\quad + \varrho_2^0 t^{\beta_2}E_{1,\beta_2+1}(\theta t) + \dfrac{\varrho_2^1}{\Gamma(\beta)}t^{\beta_2-1}.\end{cases} \tag{3}$$

Proof. The proof of the above lemma can be found in [16]. □

In the rest of the paper, we assume the following hypotheses:

$(\mathcal{H}_1)$ The functions $F_i : (0, b) \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are integrable and there exists the matrix $M := \begin{pmatrix}\lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22}\end{pmatrix}$ such that

$$\begin{pmatrix}\operatorname{ess\,sup}_{t\in(0,b)}|\lambda_{11}(t)e^{-\theta t}| & \operatorname{ess\,sup}_{t\in(0,b)}|\lambda_{12}(t)e^{-\theta t}| \\ \operatorname{ess\,sup}_{t\in(0,b)}|\lambda_{21}(t)e^{-\theta t}| & \operatorname{ess\,sup}_{t\in(0,b)}|\lambda_{22}(t)e^{-\theta t}|\end{pmatrix} \preccurlyeq \begin{pmatrix}\Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22}\end{pmatrix} \in \mathcal{M}_{2\times 2}(\mathbb{R}^+),$$

and for each $(t, \varrho_1, \varrho_2, \varrho_3), (t, x_1, x_2, x_3) \in (0, b) \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ and for $i \in \{1, 2\}$, we have:

$$|F_i(t, \varrho_1, \varrho_2, \varrho_3) - F_i(t, x_1, x_2, x_3)| \leq \lambda_{i1}(t)|\varrho_1 - x_1| + \lambda_{i2}(t)|\varrho_2 - x_2|. \tag{4}$$

$(\mathcal{H}_2)$ For each $i = 1, 2$, there is a positive number ξ_i such that

$$\int_0^b |F_i(t, 0, 0, 0)e^{-\theta t}|dt \leq \xi_i.$$

Theorem 2. *Suppose that the assumptions $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$ are satisfied. Then System (3) has a unique solution in $\mathcal{E}$ if the matrix*

$$M_* = \begin{pmatrix} \Lambda_{1,1} & \Lambda_{1,2} \\ \Lambda_{2,1} & \Lambda_{2,2} \end{pmatrix}$$

converges to zero, and there is $r \in \mathbb{R}^2_+$ that fulfills

$$\begin{aligned} r \succcurlyeq & \begin{pmatrix} 1 - \Lambda_{1,1}\dfrac{e^b(\theta^{\beta_i} + 1)}{\theta^{\beta_i+1}} & \Lambda_{1,2}\dfrac{e^b(\theta^{\beta_i} + 1)}{\theta^{\beta_i+1}} \\ \Lambda_{2,1}\dfrac{e^b(\theta^{\beta_i} + 1)}{\theta^{\beta_i+1}} & 1 - \Lambda_{2,2}\dfrac{e^b(\theta^{\beta_i} + 1)}{\theta^{\beta_i+1}} \end{pmatrix}^{-1} \\ & \times \begin{pmatrix} \dfrac{e^b(\theta^{\beta_i} + 1)}{\theta^{\beta_i+1}}\xi_1 + |\varrho_1^0|\theta^{-\beta_1}(e^{\theta b} + \theta^{\beta_1 - 1}e^b) + \frac{|\varrho_1^1|}{\Gamma(\beta_i)\beta_1}b^{\beta_i} \\ \dfrac{e^b(\theta^{\beta_i} + 1)}{\theta^{\beta_i+1}}\xi_2 + |\varrho_2^0|\theta^{-\beta_2}(e^{\theta b} + \theta^{\beta_1 - 1}e^b) + \frac{|\varrho_2^1|}{\Gamma(\beta_2)\beta_i}b^{\beta_2} \end{pmatrix}. \end{aligned} \tag{5}$$

Proof. We define the operator $N : W_{0^+}^{\beta_1,1}(0, b) \times W_{0^+}^{\beta_2,1}(0, b) \to W_{0^+}^{\beta_1,1}(0, b) \times W_{0^+}^{\beta_2,1}(0, b)$ by

$$\begin{aligned} N(\varrho)(t) &= \begin{pmatrix} N_1(\varrho)(t) \\ N_2(\varrho)(t) \end{pmatrix} \\ &= \begin{pmatrix} \int_0^t (t-s)^{\beta_1} E_{1,\beta_1+1}(\theta(t-s))F_1\left(s, \varrho_1(s), \varrho_2(s), {}^{RL}\mathcal{D}_{0^+}^{\beta_1}\varrho_1(s)\right)ds + \varrho_1^0 t^{\beta_1} E_{1,\beta_1+1}(\theta t) + \frac{\varrho_1^1}{\Gamma(\beta)}t^{\beta_1-1} \\ \int_0^t (t-s)^{\beta_2} E_{1,\beta_2+1}(\theta(t-s))F_2\left(s, \varrho_1(s), \varrho_2(s), {}^{RL}\mathcal{D}_{0^+}^{\beta_2}\varrho_2(s)\right)ds + \varrho_2^0 t^{\beta_2} E_{1,\beta_2+1}(\theta t) + \frac{\varrho_2^1}{\Gamma(\beta)}t^{\beta_2-1} \end{pmatrix}, \end{aligned}$$

to see if each $i = 1, 2$ $N_i(\varrho)$ and ${}^{RL}\mathcal{D}_{0^+}^{\beta_i}N_i(\varrho)$ are measurable for any
$\varrho \in W_{0^+}^{\beta_1,1}(0, b) \times W_{0^+}^{\beta_2,1}(0, b)$.
Step 1: First, we shall show that the mapping

$$N : W_{0^+}^{\beta_1,1}(0, b) \times W_{0^+}^{\beta_2,1}(0, b) \to W_{0^+}^{\beta_1,1}(0, b) \times W_{0^+}^{\beta_2,1}(0, b)$$

is well defined. Using our hypotheses, for arbitrarily fixed $t \in (0, b)$, $\varrho \in W_{0^+}^{\beta_1,1}(0, b) \times W_{0^+}^{\beta_2,1}(0, b)$ and $i = 1, 2$, we obtain

$$
\begin{aligned}
|N_i\varrho(t)| \leq & \int_0^t (t-s)^{\beta_i} E_{1,\beta_i+1}(\theta(t-s)) \left| F_i\left(s, \varrho_1(s), \varrho_2(s), {}^{RL}\mathcal{D}_{0^+}^{\beta_i} \varrho_i(s)\right) \right| ds \\
& + \left|\varrho_i^0\right| t^{\beta_i} E_{1,\beta_i+1}(\theta t) + \frac{|\varrho_i^1|}{\Gamma(\beta_i)} t^{\beta_i - 1} \\
\leq & \int_0^t (t-s)^{\beta_i} E_{1,\beta_i+1}(\theta(t-s)) \left| F_i\left(s, \varrho_1(s), \varrho_2(s), {}^{RL}\mathcal{D}_{0^+}^{\beta_i} \varrho(s)\right) - F_i(s,0,0,0) \right| ds \\
& + \int_0^t (t-s)^{\beta_i} E_{1,\beta_i+1}(\theta(t-s)) |F_i(s,0,0,0)| ds + \left|\varrho_i^0\right| t^{\beta_i} E_{1,\beta_i+1}(\theta t) + \frac{|\varrho_i^1|}{\Gamma(\beta_i)} t^{\beta_i - 1} \\
\leq & \int_0^t \theta^{-\beta_i} e^{\theta(t-s)} \left| F_i\left(s, \varrho_1(s), \varrho_2(s), {}^{RL}\mathcal{D}_{0^+}^{\beta_i} \varrho_i(s)\right) - F_i(s,0,0,0) \right| ds \\
& + \int_0^t \theta^{-\beta_i} e^{\theta(t-s)} |F_i(s,0,0,0)| ds + \left|\varrho_i^0\right| \theta^{-\beta_i} e^{\theta t} + \frac{|\varrho_i^1|}{\Gamma(\beta_i)} t^{\beta_i - 1}.
\end{aligned}
$$

Lemma 2 implies that

$$
\begin{aligned}
\int_0^b |N_i\varrho(t)| dt \leq & \int_0^b \int_0^t \theta^{-\beta_i} e^{\theta(t-s)} \left| F_i\left(s, \varrho_1(s), \varrho_2(s), {}^{RL}\mathcal{D}_{0^+}^{\beta_i} \varrho_i(s)\right) - F_i(s,0,0,0) \right| ds dt \\
& + \int_0^b \int_0^t \theta^{-\beta_i} e^{\theta(t-s)} |F_i(s,0,0,0)| ds dt + \int_0^b \left|\varrho_i^0\right| \theta^{-\beta_i} e^{\theta t} + \frac{|\varrho_i^1|}{\Gamma(\beta_i)} t^{\beta_i - 1} dt \\
\leq & \theta^{-\beta_i} \int_0^b e^{\theta t} \int_0^t e^{-\theta s} (\lambda_{i,1}(s)|\varrho_1(s)| + \lambda_{i,2}(s)|\varrho_2(s)| + |F_i(s,0,0,0)|) ds dt \\
& + \left|\varrho_i^0\right| \theta^{-\beta_i} (e^{\theta b} - 1) + \frac{|\varrho_i^1|}{\Gamma(\beta_i)\beta_i} b^{\beta_i} \\
\leq & \frac{(e^b - 1)}{\theta^{1+\beta_i}} \int_0^b e^{-\theta s} (\lambda_{i,1}(s)|\varrho_1(s)| + \lambda_{i,2}(s)|\varrho_2(s)| + |F_i(s,0,0,0)|) ds \\
& + \left|\varrho_i^0\right| \theta^{-\beta_i} (e^{\theta b} - 1) + \frac{|\varrho_i^1|}{\Gamma(\beta_i)\beta_i} b^{\beta_i} \\
\leq & \frac{(e^b - 1)}{\theta^{1+\beta_i}} \left(\Lambda_{i,1} \|\varrho_1\|_{W_{0^+}^{\beta_1,1}(0,b)} + \Lambda_{i,2} \|\varrho_2\|_{W_{0^+}^{\beta_2,1}(0,b)} + \xi_i \right) \\
& + \left|\varrho_i^0\right| \theta^{-\beta_i} (e^{\theta b} - 1) + \frac{|\varrho_i^1|}{\Gamma(\beta_i)\beta_i} b^{\beta_i}.
\end{aligned}
$$

Additionally,

$$
\begin{aligned}
\left| {}^{RL}\mathcal{D}_{0^+}^{\beta_i} (N_i\varrho)(t) \right| \leq & e^{\theta t} \int_0^t \left| F_i\left(s, \varrho_1(s), \varrho_2(s), {}^{RL}\mathcal{D}_{0^+}^{\beta_i} \varrho_i(s)\right) - F_i(s,0,0,0) \right| e^{-\theta s} + |F_i(s,0,0,0)| e^{-\theta s} ds \\
& + |\varrho_i^0| e^{\theta t}.
\end{aligned}
$$

Then,

$$
\begin{aligned}
\int_0^b \left| {}^{RL}\mathcal{D}_{0^+}^{\beta_i} (N_i\varrho)(t) \right| dt \leq & \int_0^b e^{\theta t} \int_0^t \left| F_i\left(s, \varrho_1(s), \varrho_2(s), {}^{RL}\mathcal{D}_{0^+}^{\beta_i} \varrho_i(s)\right) - F_i(s,0,0,0) \right| e^{-\theta s} ds dt \\
& + \int_0^b e^{\theta t} \int_0^t |F_i(s,0,0,0)| e^{-\theta s} ds dt + |\varrho_i^0| \int_0^b e^{\theta t} dt \\
\leq & \int_0^b \frac{(e^b - 1)}{\theta} (\lambda_{i,1}(s)|\varrho_1(s)| + \lambda_{i,2}(s)|\varrho_2(s)| + |F_i(s,0,0,0)|) e^{-\theta s} ds + \frac{1}{\theta} |\varrho_i^0| (e^b - 1) \\
\leq & \frac{(e^b - 1)}{\theta} \left(\Lambda_{i,1} \|\varrho_1\|_{W_{0^+}^{\beta_1,1}(0,b)} + \Lambda_{i,2} \|\varrho_2\|_{W_{0^+}^{\beta_2,1}(0,b)} + \xi_i + |\varrho_i^0| \right).
\end{aligned}
$$

Thus,

$$\|N_i(\varrho)\|_{W^{\beta_i,1}_{0^+}(0,b)} \leq \frac{e^b(\theta^{\beta_i}+1)}{\theta^{\beta_i+1}}\Big(\Lambda_{i,1}\|\varrho_1\|_{W^{\beta_1,1}_{0^+}(0,b)} + \Lambda_{i,2}\|\varrho_2\|_{W^{\beta_2,1}_{0^+}(0,b)} + \xi_i\Big) + \left|\varrho_i^0\right|\theta^{-\beta_i}(e^{\theta b}+\theta^{\beta_1-1}e^b) + \frac{|\varrho_i^1|}{\Gamma(\beta_i)\beta_i}b^{\beta_i} < \infty.$$

This means that the operator N maps $W^{\beta_1,1}_{0^+}(0,b)\times W^{\beta_2,1}_{0^+}(0,b)$ into itself. Keeping in mind that the vector r fulfills (5), we find that for all $\varrho \in \bar{B}_r$ and $i=1,2$,

$$\begin{aligned}\|N(\varrho)\|_G &\preccurlyeq \frac{e^b(\theta^{\beta_i}+1)}{\theta^{\beta_i+1}}\begin{pmatrix}\Lambda_{1,1} & \Lambda_{1,2}\\ \Lambda_{2,1} & \Lambda_{2,2}\end{pmatrix}\begin{pmatrix}\|\varrho_1\|_{W^{\beta_1,1}_{0^+}(0,b)}\\ \|\varrho_2\|_{W^{\beta_2,1}_{0^+}(0,b)}\end{pmatrix}\\ &+\begin{pmatrix}\frac{e^b(\theta^{\beta_i}+1)}{\theta^{\beta_i+1}}\xi_i + |\varrho_1^0|\theta^{-\beta_1}(e^{\theta b}+\theta^{\beta_1-1}e^b) + \frac{|\varrho_1^1|}{\Gamma(\beta_i)\beta_1}b^{\beta_i}\\ \frac{e^b(\theta^{\beta_i}+1)}{\theta^{\beta_i+1}}\xi_i + |\varrho_2^0|\theta^{-\beta_2}(e^{\theta b}+\theta^{\beta_1-1}e^b) + \frac{|\varrho_2^1|}{\Gamma(\beta_2)\beta_i}b^{\beta_2}\end{pmatrix}\\ &\preccurlyeq \begin{pmatrix}r_1\\ r_2\end{pmatrix}.\end{aligned} \tag{6}$$

Due to (6), we derive that N is a mapping from $\bar{B}_r$ into $\bar{B}_r$.

Step 2: Our claim here is to prove that the operator N is G-contractive. To this end, let $\varrho := (\varrho_1,\varrho_2)$, $v := (v_1,v_2) \in W^{\beta_1,1}_{0^+}(0,b)\times W^{\beta_2,1}_{0^+}(0,b)$. Then, for each $i=1,2$,

$$\begin{aligned}\int_0^b |N_i(\varrho)(t)-N_i(v)(t)|dt =&\Big|\int_0^t (t-s)^{\beta_i}E_{1,\beta_i+1}(\theta(t-s))F_i\Big(s,\varrho_1(s),\varrho_2(s),{}^{RL}\mathcal{D}^{\beta_i}_{0^+}\varrho_i(s)\Big)ds\\ &-\int_0^t (t-s)^{\beta_i}E_{1,\beta_i+1}(\theta(t-s))F_i\Big(s,v_1(s),v_2(s),{}^{RL}\mathcal{D}^{\beta_i}_{0^+}v_i(s)\Big)ds\Big|\\ &\leq \int_0^b\int_0^t (t-s)^{\beta_i}E_{1,\beta_i+1}(\theta(t-s))\Big|F_i\Big(s,\varrho_1(s),\varrho_2(s),{}^{RL}\mathcal{D}^{\beta_i}_{0^+}\varrho_i(s)\Big)\\ &-F_i\Big(s,v_1(s),v_2(s),{}^{RL}\mathcal{D}^{\beta_i}_{0^+}v_i(s)\Big)\Big|dsdt,\end{aligned}$$

so

$$\begin{aligned}\int_0^b |N_i(\varrho)(t)-N_i(v)(t)|dt \leq& \int_0^b\int_0^t \theta^{-\beta_i}e^{\theta(t-s)}\Big|F_i\Big(s,\varrho_1(s),\varrho_2(s),{}^{RL}\mathcal{D}^{\beta_i}_{0^+}\varrho_i(s)\Big)\\ &-F_i\Big(s,v_1(s),v_2(s),{}^{RL}\mathcal{D}^{\beta_i}_{0^+}v_i(s)\Big)\Big|dsdt\\ &\leq \int_0^b\int_0^t \theta^{-\beta_i}e^{\theta(t-s)}(\lambda_{i,1}(s)|\varrho_1(s)-v_1(s)|+\lambda_{i,2}(s)|\varrho_2(s)-v_2(s)|)dsdt\\ &\leq \frac{(e^b-1)}{\theta^{1+\beta_i}}\int_0^b e^{-\theta s}(\lambda_{i,1}(s)|\varrho_1(s)-v_1(s)|+\lambda_{i,2}(s)|\varrho_2(s)-v_2(s)|)ds\\ &\leq \frac{(e^b-1)}{\theta^{1+\beta_i}}\Big(\Lambda_{i,1}\|\varrho_1-v_1\|_{W^{\beta_1,1}_{0^+}(0,b)}+\Lambda_{i,2}\|\varrho_2-v_2\|_{W^{\beta_2,1}_{0^+}(0,b)}\Big).\end{aligned}$$

On the other hand, for $i=1,2$, we have

$$\begin{aligned}\int_0^b\Big|{}^{RL}\mathcal{D}^{\beta_i}_{0^+}(N_i\varrho)(t)-{}^{RL}\mathcal{D}^{\beta_i}_{0^+}(N_iv)(t)\Big|dt \leq& \int_0^b e^{\theta t}\int_0^t\Big|F_i\Big(s,\varrho_1(s),\varrho_2(s),{}^{RL}\mathcal{D}^{\beta_i}_{0^+}\varrho_i(s)\Big)\\ &-F_i\Big(s,,\varrho_1(s),\varrho_2(s),{}^{RL}\mathcal{D}^{\beta_i}_{0^+}\varrho_i(s)\Big)\Big|e^{-\theta s}dsdt\\ &\leq \int_0^b\frac{(e^b-1)}{\theta}(\lambda_{i,1}(s)|\varrho_1(s)-v_1(s)|+\lambda_{i,2}(s)|\varrho_2(s)-v_2(s)|)e^{-\theta s}ds\\ &\leq \frac{(e^b-1)}{\theta}(\Lambda_{i,1}\|\varrho_1-v_1\|_{W^{\beta_1,1}_{0^+}(0,b)}+\Lambda_{i,2}\|\varrho_2-v_2\|_{W^{\beta_2,1}_{0^+}(0,b)}).\end{aligned}$$

Then,

$$\|N(\varrho) - N(v)\|_G \preccurlyeq \frac{e^b(\theta^{\beta_i}+1)}{\theta^{\beta_i+1}} M_* \begin{pmatrix} \|\varrho_1 - v_1\|_{W^{\beta_1,1}_{0^+}(0,b)} \\ \|\varrho_2 - v_2\|_{W^{\beta_2,1}_{0^+}(0,b)} \end{pmatrix}.$$

This means that the operator N is G-contractive, and thus Perov fixed point Theorem 1 ensures that System (1) has a unique solution. □

Example 1. *Consider the following initial value problem of a nonlinear coupled fractional derivative system defined on* $\mathcal{E} = W^{\frac{1}{2},1}_{0^+}(0,1) \times W^{\frac{1}{4},1}_{0^+}(0,1)$*:*

$$\begin{cases} {}^{RL}\mathcal{D}^{\frac{3}{2}}_{0^+}\varrho_1(t) = \dfrac{1}{2}\dfrac{\left({}^{RL}\mathcal{D}^{\frac{1}{2}}_{0^+}\varrho_1^2(t)\right)}{1+\left({}^{RL}\mathcal{D}^{\frac{1}{2}}_{0^+}\varrho_1^2(t)\right)}\cos\left(e^t(\varrho_1(t)+\varrho_2(t))\right) + 5\,{}^{RL}\mathcal{D}^{\frac{1}{2}}_{0^+}\varrho_1(t), \\ {}^{RL}\mathcal{D}^{\frac{5}{4}}_{0^+}\varrho_2(t) = \dfrac{1}{2}\dfrac{\left({}^{RL}\mathcal{D}^{\frac{1}{4}}_{0^+}\varrho_1^2(t)\right)}{1+\left({}^{RL}\mathcal{D}^{\frac{1}{4}}_{0^+}\varrho_2^2(t)\right)}\sin(\arctan(t)(\varrho_1(t)+\varrho_2(t))) + 5\,{}^{RL}\mathcal{D}^{\frac{1}{4}}_{0^+}\varrho_1(t), \\ {}^{RL}\mathcal{D}^{\alpha_1-1}_{0^+}\varrho_1(0) = \varrho_1^0, \quad {}^{RL}\mathcal{I}^{2-\alpha_1}_{0^+}\varrho_1(0) = \varrho_1^1 \\ {}^{RL}\mathcal{D}^{\alpha_2-1}_{0^+}\varrho_2(0) = \varrho_2^0, \quad {}^{RL}\mathcal{I}^{2-\alpha_2}_{0^+}\varrho_2(0) = \varrho_2^1. \end{cases} \tag{7}$$

Then,

$$F_1(t,\varrho_1(t),\varrho_2(t),{}^{RL}\mathcal{D}^{\alpha_1-1}_{0^+}\varrho_1(t)) = \frac{1}{2}\frac{\left({}^{RL}\mathcal{D}^{\frac{1}{2}}_{0^+}\varrho_1^2(t)\right)}{1+\left({}^{RL}\mathcal{D}^{\frac{1}{2}}_{0^+}\varrho_1^2(t)\right)}\cos\left(e^t(\varrho_1(t)+\varrho_2(t))\right),$$

and

$$F_2(t,\varrho_1(t),\varrho_2(t),{}^{RL}\mathcal{D}^{\alpha_2-1}_{0^+}\varrho_1(t)) = \frac{1}{2}\frac{\left({}^{RL}\mathcal{D}^{\frac{1}{4}}_{0^+}\varrho_2^2(t)\right)}{1+\left({}^{RL}\mathcal{D}^{\frac{1}{4}}_{0^+}\varrho_2^2(t)\right)}\sin(\arctan(t)(\varrho_1(t)+\varrho_2(t))).$$

So, we have

$$\frac{\partial F_1(t,\varrho_1(t),\varrho_2(t),{}^{RL}\mathcal{D}_{0^+}^{\frac{1}{2}}\varrho_1(t))}{\partial \varrho_1} = \frac{\left({}^{RL}\mathcal{D}_{0^+}^{\frac{1}{2}}\varrho_1^2(t)\right)}{\left(1+\left({}^{RL}\mathcal{D}_{0^+}^{\frac{1}{2}}\varrho_1^2(t)\right)\right)^2}\cos\left(e^t(\varrho_1(t)+\varrho_2(t))\right) - \frac{1}{2}\frac{\left({}^{RL}\mathcal{D}_{0^+}^{\frac{1}{2}}\varrho_1^2(t)\right)}{1+\left({}^{RL}\mathcal{D}_{0^+}^{\frac{1}{2}}\varrho_1^2(t)\right)}\sin\left(e^t(\varrho_1(t)+\varrho_2(t))\right)e^t,$$

$$\frac{\partial F_1(t,\varrho_1(t),\varrho_2(t),{}^{RL}\mathcal{D}_{0^+}^{\frac{1}{2}}\varrho_1(t))}{\partial \varrho_2} = -\frac{1}{2}\frac{\left({}^{RL}\mathcal{D}_{0^+}^{\frac{1}{2}}\varrho_1^2(t)\right)}{1+\left({}^{RL}\mathcal{D}_{0^+}^{\frac{1}{2}}\varrho_1^2(t)\right)}\sin\left(e^t(\varrho_1(t)+\varrho_2(t))\right)e^t,$$

$$\frac{\partial F_2(t,\varrho_1(t),\varrho_2(t),{}^{RL}\mathcal{D}_{0^+}^{\frac{1}{4}}\varrho_2(t))}{\partial \varrho_1} = \frac{1}{2}\frac{\left({}^{RL}\mathcal{D}_{0^+}^{\frac{1}{4}}\varrho_2^2(t)\right)}{1+\left({}^{RL}\mathcal{D}_{0^+}^{\frac{1}{4}}\varrho_2^2(t)\right)}\cos(\arctan(t)(\varrho_1(t)+\varrho_2(t)))\arctan(t),$$

$$\frac{\partial F_2(t,\varrho_1(t),\varrho_2(t),{}^{RL}\mathcal{D}_{0^+}^{\frac{1}{4}}\varrho_2(t))}{\partial \varrho_2} = \frac{\left({}^{RL}\mathcal{D}_{0^+}^{\frac{1}{2}}\varrho_2^2(t)\right)}{\left(1+\left({}^{RL}\mathcal{D}_{0^+}^{\frac{1}{2}}\varrho_2^2(t)\right)\right)^2}\sin(\arctan(t)(\varrho_1(t)+\varrho_2(t))) + \frac{1}{2}\frac{\left({}^{RL}\mathcal{D}_{0^+}^{\frac{1}{4}}\varrho_2^2(t)\right)}{1+\left({}^{RL}\mathcal{D}_{0^+}^{\frac{1}{4}}\varrho_2^2(t)\right)}\cos(\arctan(t)(\varrho_1(t)+\varrho_2(t)))\arctan(t).$$

Thus,

$$\sup_{\varrho_1,\varrho_2\in\mathbb{R}}\left|\frac{\partial F_1(t,\varrho_1,\varrho_2,{}^{RL}\mathcal{D}_{0^+}^{\frac{1}{2}}\varrho_1)}{\partial\varrho_1}\right| \le 1+\frac{e^t}{2} := \lambda_{1,1}(t)$$

$$\sup_{\varrho_1,\varrho_2\in\mathbb{R}}\left|\frac{\partial F_1(t,\varrho_1,\varrho_2,{}^{RL}\mathcal{D}_{0^+}^{\frac{1}{2}}\varrho_1)}{\partial\varrho_2}\right| \le \frac{e^t}{2} := \lambda_{1,2}(t)$$

$$\sup_{\varrho_1,\varrho_2\in\mathbb{R}}\left|\frac{\partial F_2(t,\varrho_1,\varrho_2,{}^{RL}\mathcal{D}_{0^+}^{\frac{1}{4}}\varrho_2)}{\partial\varrho_1}\right| \le \frac{\arctan(t)}{2} := \lambda_{2,1}(t),$$

$$\sup_{\varrho_1,\varrho_2\in\mathbb{R}}\left|\frac{\partial F_2(t,\varrho_1,\varrho_2,{}^{RL}\mathcal{D}_{0^+}^{\frac{1}{4}}\varrho_2)}{\partial\varrho_2}\right| \le 1+\frac{\arctan(t)}{2} := \lambda_{2,2}(t).$$

Hence, the equalities in (4) *hold. Now, we simply check that*

$$M_* = \begin{pmatrix} \frac{e+2}{2e^5} & \frac{1}{2e^4} \\ \frac{\pi}{8e^5} & \frac{\pi+8}{8e^5} \end{pmatrix}.$$

This matrix has two eigenvalues, $\mu_1 = \frac{1}{e^5} < 1$, $\mu_2 = \frac{\pi+4e+8}{8e^5} < 1$. *Therefore,* M_* *converges to zero. All the conditions in Theorem* 2 *are satisfied, so System* (7) *has a unique solution.*

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