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Existence of Solutions to a Class of Nonlinear Arbitrary Order Differential Equations Subject to Integral Boundary Conditions

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Abstract: We investigate the existence of positive solutions for a class of fractional differential equations of arbitrary order $\delta > 2$, subject to boundary conditions that include an integral operator of the fractional type. The consideration of this type of boundary conditions allows us to consider heterogeneity on the dependence specified by the restriction added to the equation as a relevant issue for applications. An existence result is obtained for the sublinear and superlinear case by using the Guo–Krasnosel'skii fixed point theorem through the definition of adequate conical shells that allow us to localize the solution. As additional tools in our procedure, we obtain the explicit expression of Green's function associated to an auxiliary linear fractional boundary value problem, and we study some of its properties, such as the sign and some useful upper and lower estimates. Finally, an example is given to illustrate the results.

Keywords: fractional differential equations; fractional derivative of Riemann–Liouville type; integral boundary value problems; Green's functions; Guo–Krasnosel'skii fixed point theorem in cones; sublinearity and superlinearity; Arzelà-Ascoli Theorem

MSC: Primary 26A33; Secondary 34B15

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1. Introduction

Differential equations for non-integer order play an important role to describe the physical phenomena more accurately than classical integer order differential equations. The need for fractional order differential equations stems in part from the fact that many phenomena cannot be modeled by differential equations with integer derivatives. Therefore, the existence results for solutions to fractional differential equations have received considerable attention in recent years.

Some relevant monographs on fractional calculus and fractional differential equations are, for instance, [1–3]. The work [4] gives some fundamental ideas on initial value problems for fractional differential equations from the point of view of Riemann–Liouville operators, discussing local and global existence, or extremal solutions, and the monograph [5] includes different theoretical results as well as developments related to applications in the field of fractional calculus.

There are several papers dealing with the existence and uniqueness of solution to initial and boundary value problems for fractional order differential equations. For instance, in 2009, some impulsive problems for Caputo-type differential equations with $\delta \in (1, 2]$ and boundary conditions given by $x(0) + x'(0) = 0$, $x(1) + x'(1) = 0$, were studied (see [6]). Later, in 2010, initial value problems and periodic boundary value problems for linear fractional differential equations were analyzed in [7] by giving some comparison results. The authors of [8] studied the existence of positive solutions for fractional differential equations of order $\delta \in (1, 2)$, whose nonlinearity depended on a fractional derivative of the unknown function, subject to Dirichlet boundary conditions.

They completed their study by calculating the associated Green's function and by applying the compressive version of the Guo–Krasnosel'skii fixed point theorem. Green's function, Banach contraction mapping and fixed point index theory are the main tools used in [9] for the analysis of a nonlocal problem for fractional differential equations. In [10], a result that guarantees the existence of a unique fixed point for a mixed monotone operator was used to provide the existence of a unique positive solution to an initial value problem for fractional differential equations of general order $n - 1 < \delta \leq n$, with $n \geq 2$, whose nonlinearity depends on the classical derivatives of the unknown function up to order $n - 2$.

On the other hand, the development of the monotone iterative technique for periodic boundary value problems associated with impulsive fractional differential equations with Riemann–Liouville sequential derivatives was made in [11], and [12] was devoted to boundary value problems for fractional differential inclusions. We refer also to [13] for a monograph devoted to the positive solutions for differential, difference and integral equations.

Integral boundary value problems for differential equations with integer and non-integer order have been studied by several researchers [1,2,4,12,14,15]. To mention some related references, in [16], first-order problems were considered by using the method of upper and lower solutions, and, in [17], the Guo–Krasnosel'skii fixed point result was applied to study the existence of positive solutions to integral boundary value problems for classical second-order differential equations.

These kind of problems were also considered in [18], where some results were derived as a consequence of the nonlinear alternative of Leray–Schauder type. On the other hand, the monotone iterative technique was developed in [19] for integral boundary value problems relative to first-order integro-differential equations with deviating arguments. See also [20] for a similar study on analogous differential systems. Very recently, the results in [21] were devoted to the study of first-order problems with multipoint and integral boundary conditions by applying Banach or Schaefer's fixed point theorem.

In the fractional case, some sufficient conditions were established in [22] for the existence of solutions to nonlocal boundary value problems associated to Caputo-type fractional differential equations by using Banach and Schaefer's fixed point theorems. A related problem with integral boundary conditions in the context of Banach spaces was analyzed in [23] by using Green's functions and the Kuratowski measure of noncompactness.

The authors of [24] studied fractional differential equations subject to a nonlocal strip condition of integral type that, in the limit, approaches the usual integral boundary condition, and some results were derived by applying fixed point results and the Leray–Schauder degree theory. In [25], the authors considered boundary value problems for a class of fractional differential equations of order $\delta \in (1, 2]$ with three-point fractional integral boundary conditions by means of Schaefer's fixed point theorem.

In [26], the contractive mapping principle and the monotone iterative technique were the basic tools and procedures used in the study of a class of Riemann–Liouville fractional differential equations with integral boundary conditions. On the other hand, in [27], Lyapunov-type results were used to study the nonexistence, the uniqueness and the existence and uniqueness of solutions to fractional boundary value problems.

More recently, in [28], a fractional problem subject to Stieltjes and generalized fractional integral boundary conditions was analyzed by applying the Banach contraction mapping principle. An analogous method was applied in [29], where the authors studied a Cauchy problem for Caputo–Fabrizio fractional differential equations in Banach spaces, imposing an initial condition that involves an integral operator, and they deduced the existence and uniqueness of solutions by applying the Banach fixed point theorem.

Some results for Hilfer fractional differential equations subject to boundary conditions involving Riemann–Liouville fractional integral operators were given in [30], and the study was completed by applying a nonlinear alternative of Leray–Schauder type and the Nadler theorem. Classical fixed point theory was also the tool used in [31] for the analysis of sequential ψ-Hilfer fractional boundary value problems. In particular, one of the results applied was the Krasnosel'skii fixed point theorem for the addition of a contractive mapping and a compact mapping.

Several other recent papers include, for instance, [32], where the type of derivative considered was Caputo fractional derivatives with respect to a fixed function, and, under this framework the authors studied an impulsive problem subject to integral boundary conditions based on the Riemann–Stieltjes fractional integral through Leray–Schauder's nonlinear alternative; or [33], where ψ-Caputo operators were considered in the differential equation and in the integral boundary conditions, and the method of upper and lower solutions coupled with the monotone iterative technique were the main tools used.

More specifically, in 2012, Cabada and Wang [15] considered the following boundary value problem for fractional order differential equations with classical integral boundary conditions:

$$\begin{cases} {}^cD^{\delta}u(t) + f(t,u(t)) = 0,\ 0 < t < 1, \\ u(0) = u''(0) = 0,\ u(1) = \lambda \int_0^1 u(s)ds, \end{cases}$$

where $2 < \delta < 3, 0 < \lambda < 2, {}^cD^{\delta}$ is the Caputo fractional derivative and $f : [0,1] \times [0,\infty) \to [0,\infty)$ is a continuous function.

In 2014, Cabada and Hamdi [14] discussed, by defining a suitable cone on a Banach space and by applying Guo–Krasnosel'skii fixed point theorem, the existence of positive solutions for the following class of nonlinear fractional differential equations with integral boundary conditions:

$$\begin{cases} D^{\delta}u(t) + f(t,u(t)) = 0,\ 0 < t < 1, \\ u(0) = u'(0) = 0,\ u(1) = \lambda \int_0^1 u(s)ds, \end{cases}$$

where $2 < \delta \leq 3, 0 < \lambda, \lambda \neq \delta$, D^{δ} is the Riemann–Liouville fractional derivative of order δ and f is a continuous function.

The large collection of research works existing on the topic shows the increasing interest that the study of integral boundary value problems for fractional differential equations has received in the recent times, due to their applicability to the modeling of various processes for which hereditary or memory properties leave a footprint in the performance of the phenomena, and because, in many occasions, the restrictions on the real problem make it adequate to consider boundary conditions that consider the influence that the state on a certain interval has on the evolution of the system.

It is worthwhile to devote efforts to study the existence of positive solutions, since controlling the sign of the solutions is a relevant issue in many fields of application for which negative values are not admissible (populations, amount of substances etc.). In this sense, in comparison with the above mentioned works, we are interested in the consequences, in terms of the properties of the solutions, that the application of the Guo–Krasnosel'skii fixed point theorem may present for a fractional problem with a boundary condition including a fractional operator.

Motivated by the above-mentioned work [14] and its approach, this paper deals with the existence of positive solutions for the following fractional differential equation of general order $\delta > 2$ with fractional integral boundary conditions:

$$\begin{cases} D^{\delta}_{0+}w(t) + f(t,w(t)) = 0,\ 0 < t < 1, \\ w(0) = w'(0) = w''(0) = w'''(0) = \cdots = w^{(n-2)}(0) = 0, \\ w(1) = \lambda I^{\gamma}_{0+}w(\zeta),\ 0 < \zeta < 1,\ n-1 < \delta \leq n, \end{cases} \tag{1}$$

where $n \in \mathbb{N}, n \geq 3$, $\lambda > 0$ and D^{δ}_{0+} denotes the Riemann–Liouville fractional derivative of order δ, I^{γ} is the Riemann–Liouville fractional integral operator of order $\gamma > 0$ and $f : [0, 1] \times [0, \infty) \to [0, \infty)$ is a continuous function.

As original contributions of the paper, we mention the consideration of a boundary value problem that involves an integral operator of fractional type, which allows us to consider heterogeneity on the dependence specified by the restriction added to the equation and also the subsequent explicit calculation of the Green's function for this general problem, which is not easy to handle due to the high order of the equation and the introduction of fractional operators in the boundary conditions considered.

These novelties in the problem considered add more complexity to the study of the particular properties of the Green's function that are essential to build the mathematical constructs required for the application of the fixed point result, namely, the establishment of estimates, which allow us to define an appropriate cone that is mapped into itself through the integral operator corresponding to the boundary value problem.

To prove the existence of positive solutions to (1), we apply the Guo–Krasnosel'skii fixed point theorem in cones, used in [14] in the context of fractional problems with boundary conditions involving a classical integral term but different from the techniques followed in the discussed works dealing with boundary conditions involving integral operators of a fractional type. The main reason to use this fixed point result is its potential to provide a localization of the solution by handling conical shells whose boundary is defined by the boundaries of two sets, which can be, in this case, more general than open balls [34,35].

Then, it is not only possible to deduce the existence of a positive solution but also we can give an upper bound for its maximum value and establish a certain positive number that is exceeded by the values of the solution at some points. Having, at our disposal, a contractive and an expansive version of the hypotheses, it is possible to deduce the existence of a positive solution under different types of restrictions on the function defining the equation—namely, the sublinear and the superlinear case.

The organization of the paper is as follows. In Section 2, we recall some basic notations and concepts concerning fractional calculus as well as the fixed point result that we apply as a fundamental tool in our procedure. In Section 3, we explicitly obtain the Green's function for a modified linear fractional boundary value problem, and we deduce some estimates for its expression.

The study of the sign of the Green's function is relevant too, as well as the comparison between its value at different points, which is also useful to our reasoning. Then, in Section 4, we present our main result, which allows us to derive the existence of a positive solution for the nonlinear problem (1) in the sublinear and superlinear cases. The proof of the main result provides details regarding the conical shells to which the mentioned solution belongs in each case. In Section 5, an example is included, and, finally, Section 6 shows our conclusions.

2. Materials and Methods

In this section, we recall some notations, definitions and results that are essential to prove our main result.

Definition 1. *The fractional derivative of Riemann–Liouville type and fractional order $\delta > 0$ is defined for a function f as*

$$D^{\delta}_{0+} f(t) = \frac{1}{\Gamma(n-\delta)} \left(\frac{d}{dt}\right)^{n} \int_0^t (t-s)^{n-\delta-1} f(s) ds,$$

where $n = [\delta] + 1$, and $[\delta]$ is the integer part of δ, provided that the integral on the right-hand side converges pointwise on $(0,\infty)$.

Definition 2. *The fractional integral of Riemann–Liouville type and fractional order $\delta > 0$ is defined for a function f as*

$$I_{0+}^{\delta} f(t) = \frac{1}{\Gamma(\delta)} \int_0^t (t-s)^{\delta-1} f(s) ds,$$

provided that the integral on the right-hand side converges pointwise on (0,∞).

Lemma 1 ([1])**.** *Let $\delta > 0$, and then the solutions to $D_{0+}^{\delta} w(t) + y(t) = 0$ are given by*

$$w(t) = -\int_0^t \frac{(t-s)^{\delta-1}}{\Gamma(\delta)} y(s) ds + c_1 t^{\delta-1} + c_2 t^{\delta-2} + \cdots + c_n t^{\delta-n}.$$

Without loss of generality, we assume in this and later results that the fractional derivatives are developed taking 0 as base point. For a discussion on other types of conditions, we refer to Kilbas et al. [1] and Samko et al. [3].

Definition 3. *Let E be a real Banach space. A nonempty closed and convex set $K \subset E$ is called a cone if it satisfies the following two conditions:*

(i) $x \in K, \lambda \geq 0$ *implies* $\lambda x \in K$;
(ii) $x \in K, \ -x \in K$ *implies* $x = 0$, *where 0 denotes the zero element of E.*

Theorem 1 ([34])**.** *Let E be a Banach space, and let $K \subset E$ be a cone. Assume that Ω_1, Ω_2 are open and bounded subsets of E with $0 \in \Omega_1 \subset \overline{\Omega_1} \subset \Omega_2$, and let $T : K \cap (\overline{\Omega_2} \setminus \Omega_1) \longrightarrow K$ be a completely continuous mapping such that one of the following conditions holds:*

(i) $\|Tu\| \geq \|u\|$, $u \in K \cap \partial\Omega_1$, *and* $\|Tu\| \leq \|u\|$, $u \in K \cap \partial\Omega_2$; *or*
(ii) $\|Tu\| \leq \|u\|$, $u \in K \cap \partial\Omega_1$, *and* $\|Tu\| \geq \|u\|$, $u \in K \cap \partial\Omega_2$.

Then, the mapping T has at least one fixed point in $K \cap (\overline{\Omega_2} \setminus \Omega_1)$.

We define the mapping $T : C[0,1] \to C[0,1]$ as $[Tu](t) = \int_0^1 G(t,s) f(s, u(s)) ds$, with G a certain Green's function whose expression is given as indicated below (see (3)). This Green's function will be built in such a way that the fixed points of the mapping T coincide with the solutions to problem (1), and, hence, by Theorem 1, we will deduce the existence of positive solutions to problem (1).

3. Some Auxiliary Results

First, we prove the following lemma, relative to the expression of the explicit solution for a linear fractional problem subject to integral boundary conditions of fractional type.

Lemma 2. *Let $\delta > 0$, $n-1 < \delta \leq n$, $0 < \zeta < 1$, $y \in C[0,1]$, and suppose that $P := 1 - \frac{\lambda\Gamma(\delta)}{\Gamma(\delta+\gamma)} \zeta^{\delta+\gamma-1} \neq 0$. Then, the problem*

$$\begin{cases} D_{0+}^{\delta} w(t) + y(t) = 0,\ 0 < t < 1, \\ w(0) = w'(0) = w''(0) = w'''(0) = \cdots = w^{(n-2)}(0) = 0, \\ w(1) = \lambda I_{0+}^{\gamma} w(\zeta),\ 0 < \zeta < 1,\ n-1 < \delta \leq n, \end{cases} \tag{2}$$

has a unique solution $w \in C^1[0,1]$, given by $w(t) = \int_0^1 G(t,s) y(s) ds$, where

$$G(t,s) = \begin{cases} \frac{-P\Gamma(\delta+\gamma)(t-s)^{\delta-1} + \Gamma(\delta+\gamma)(1-s)^{\delta-1} t^{\delta-1} - \Gamma(\delta)\lambda(\zeta-s)^{\delta+\gamma-1} t^{\delta-1}}{P\Gamma(\delta)\Gamma(\delta+\gamma)}, & 0 \leq s \leq t \leq 1, s \leq \zeta, \\ \frac{\Gamma(\delta+\gamma)(1-s)^{\delta-1} t^{\delta-1} - \Gamma(\delta)\lambda(\zeta-s)^{\delta+\gamma-1} t^{\delta-1}}{P\Gamma(\delta)\Gamma(\delta+\gamma)}, & 0 \leq t \leq s \leq \zeta \leq 1, \\ \frac{-P\Gamma(\delta+\gamma)(t-s)^{\delta-1} + \Gamma(\delta+\gamma)(1-s)^{\delta-1} t^{\delta-1}}{P\Gamma(\delta)\Gamma(\delta+\gamma)}, & 0 \leq \zeta \leq s \leq t \leq 1, \\ \frac{\Gamma(\delta+\gamma)(1-s)^{\delta-1} t^{\delta-1}}{P\Gamma(\delta)\Gamma(\delta+\gamma)}, & 0 \leq t \leq s \leq 1, s \geq \zeta. \end{cases} \tag{3}$$

Here, $G(t,s)$ is called the Green's function associated to the boundary value problem (1). *Note that $G(t,s)$ is a continuous function on* $[0,1] \times [0,1]$.

Proof. The first equation in problem (2) is equivalent to the following integral equation:

$$w(t) = -I_{0+}^{\delta} y(t) + c_1 t^{\delta-1} + c_2 t^{\delta-2} + \cdots + c_n t^{\delta-n}.$$

By using

$$w(0) = w'(0) = \cdots = w^{(n-2)}(0) = 0,$$

we obtain that

$$w(t) = -I_{0+}^{\delta} y(t) + c_1 t^{\delta-1}.$$

It follows from

$$w(1) = \lambda I_{0+}^{\gamma} w(\zeta),$$

combined with

$$w(1) = -I_{0+}^{\delta} y(1) + c_1$$

and

$$\lambda I_{0+}^{\gamma} w(\zeta) = -\lambda I_{0+}^{\delta+\gamma} y(\zeta) + \lambda c_1 \frac{\Gamma(\delta)}{\Gamma(\delta+\gamma)} \zeta^{\delta+\gamma-1},$$

that

$$\begin{aligned} w(t) = &-\frac{1}{\Gamma(\delta)} \int_0^t (t-s)^{\delta-1} y(s) ds + \frac{t^{\delta-1}}{P\Gamma(\delta)} \int_0^1 (1-s)^{\delta-1} y(s) ds \\ &- \frac{\lambda t^{\delta-1}}{P\Gamma(\delta+\gamma)} \int_0^{\zeta} (\zeta-s)^{\delta+\gamma-1} y(s) ds. \end{aligned}$$

For $t \leq \zeta$, we have

$$\begin{aligned} w(t) = &\frac{-1}{\Gamma(\delta)} \int_0^t (t-s)^{\delta-1} y(s) ds + \frac{t^{\delta-1}}{P\Gamma(\delta)} \left\{ \int_0^t + \int_t^{\zeta} + \int_{\zeta}^1 \right\} (1-s)^{\delta-1} y(s) ds \\ &- \frac{\lambda t^{\delta-1}}{P\Gamma(\delta+\gamma)} \left\{ \int_0^t + \int_t^{\zeta} \right\} (\zeta-s)^{\delta+\gamma-1} y(s) ds \\ = &\int_0^t \frac{-P\Gamma(\delta+\gamma)(t-s)^{\delta-1} + \Gamma(\delta+\gamma)(1-s)^{\delta-1} t^{\delta-1} - \Gamma(\delta)\lambda(\zeta-s)^{\delta+\gamma-1} t^{\delta-1}}{P\Gamma(\delta)\Gamma(\delta+\gamma)} y(s) ds \\ &+ \int_t^{\zeta} \frac{\Gamma(\delta+\gamma)(1-s)^{\delta-1} t^{\delta-1} - \Gamma(\delta)\lambda(\zeta-s)^{\delta+\gamma-1} t^{\delta-1}}{P\Gamma(\delta)\Gamma(\delta+\gamma)} y(s) ds \\ &+ \int_{\zeta}^1 \frac{\Gamma(\delta+\gamma)(1-s)^{\delta-1} t^{\delta-1}}{P\Gamma(\delta)\Gamma(\delta+\gamma)} y(s) ds \\ = &\int_0^1 G(t,s) y(s) ds. \end{aligned}$$

For $t \geq \zeta$, we deduce that

$$w(t) = -\frac{1}{\Gamma(\delta)}\left\{\int_0^\zeta + \int_\zeta^t\right\}(t-s)^{\delta-1}y(s)ds + \frac{t^{\delta-1}}{P\Gamma(\delta)}\left\{\int_0^\zeta + \int_\zeta^t + \int_t^1\right\}(1-s)^{\delta-1}y(s)ds$$
$$-\frac{\lambda t^{\delta-1}}{P\Gamma(\delta+\gamma)}\int_0^\zeta(\zeta-s)^{\delta+\gamma-1}y(s)ds$$
$$= \int_0^\zeta \frac{-P\Gamma(\delta+\gamma)(t-s)^{\delta-1} + \Gamma(\delta+\gamma)(1-s)^{\delta-1}t^{\delta-1} - \Gamma(\delta)\lambda(\zeta-s)^{\delta+\gamma-1}t^{\delta-1}}{P\Gamma(\delta)\Gamma(\delta+\gamma)}y(s)ds$$
$$+ \int_\zeta^t \frac{-P\Gamma(\delta+\gamma)(t-s)^{\delta-1} + \Gamma(\delta+\gamma)(1-s)^{\delta-1}t^{\delta-1}}{P\Gamma(\delta)\Gamma(\delta+\gamma)}y(s)ds$$
$$+ \int_t^1 \frac{\Gamma(\delta+\gamma)(1-s)^{\delta-1}t^{\delta-1}}{P\Gamma(\delta)\Gamma(\delta+\gamma)}y(s)ds$$
$$= \int_0^1 G(t,s)y(s)ds.$$

□

A careful analysis of the Green's function G allows us to prove some of its properties that will be useful to our procedure, such as the nonnegativity or the establishment of upper and lower estimates.

Lemma 3. *Let G be the Green's funtion corresponding to the problem (2), which is given in Lemma 2. Then, for all $\delta \in (n-1, n]$, and $\lambda > 0$ with $P := 1 - \frac{\lambda\Gamma(\delta)}{\Gamma(\delta+\gamma)}\zeta^{\delta+\gamma-1} > 0$, the following properties hold:*

(I) $G(t,s) \geq \frac{\lambda t^{\delta-1}\zeta^{\delta+\gamma-1}}{P\Gamma(\delta+\gamma)}[(1-s)^{\delta-1} - (1-s)^{\delta+\gamma-1}]$ *for all* $t, s \in (0,1)$.
(II) $G(t,s) \leq \frac{(1-s)^{\delta-1}t^{\delta-1}}{P\Gamma(\delta)}$ *for all* $t, s \in (0,1)$.
(III) $G(t,s) > 0$ *for all* $t, s \in (0,1)$.
(IV) $G(1,s) > 0$ *for all* $s \in (0,1)$.
(V) $G(t,s)$ *is a continuous function for all* $t, s \in (0,1)$.

Proof. We start by proving (I) and (II) simultaneously. First, assume that $0 \leq s \leq t \leq 1$, $s \leq \zeta$. Since $0 < \frac{\lambda\Gamma(\delta)\zeta^{\delta+\gamma-1}}{\Gamma(\delta+\gamma)} < 1$, then we obtain

$$P\Gamma(\delta)\Gamma(\delta+\gamma)G(t,s)$$
$$= -P\Gamma(\delta+\gamma)(t-s)^{\delta-1} + \Gamma(\delta+\gamma)(1-s)^{\delta-1}t^{\delta-1} - \Gamma(\delta)\lambda(\zeta-s)^{\delta+\gamma-1}t^{\delta-1}$$
$$= \lambda\Gamma(\delta)\zeta^{\delta+\gamma-1}(t-s)^{\delta-1} + [-\Gamma(\delta+\gamma)(t-s)^{\delta-1} + \Gamma(\delta+\gamma)(1-s)^{\delta-1}t^{\delta-1}]$$
$$- \Gamma(\delta)\lambda(\zeta-s)^{\delta+\gamma-1}t^{\delta-1}$$
$$\geq \lambda\Gamma(\delta)\zeta^{\delta+\gamma-1}(t-s)^{\delta-1} - \lambda\Gamma(\delta)\zeta^{\delta+\gamma-1}(t-s)^{\delta-1}$$
$$+ \lambda\Gamma(\delta)\zeta^{\delta+\gamma-1}(1-s)^{\delta-1}t^{\delta-1} - \Gamma(\delta)\lambda(\zeta-s)^{\delta+\gamma-1}t^{\delta-1}$$
$$= \lambda\Gamma(\delta)\zeta^{\delta+\gamma-1}(1-s)^{\delta-1}t^{\delta-1} - \Gamma(\delta)\lambda(\zeta-s)^{\delta+\gamma-1}t^{\delta-1}$$
$$\geq \lambda\Gamma(\delta)\zeta^{\delta+\gamma-1}t^{\delta-1}[(1-s)^{\delta-1} - (1-s)^{\delta+\gamma-1}],$$

and

$$
\begin{aligned}
&P\Gamma(\delta)\Gamma(\delta+\gamma)G(t,s)\\
&=-P\Gamma(\delta+\gamma)(t-s)^{\delta-1}+\Gamma(\delta+\gamma)(1-s)^{\delta-1}t^{\delta-1}-\Gamma(\delta)\lambda(\zeta-s)^{\delta+\gamma-1}t^{\delta-1}\\
&=\lambda\Gamma(\delta)\zeta^{\delta+\gamma-1}(t-s)^{\delta-1}-\Gamma(\delta+\gamma)(t-s)^{\delta-1}\\
&+\Gamma(\delta+\gamma)(1-s)^{\delta-1}t^{\delta-1}-\Gamma(\delta)\lambda(\zeta-s)^{\delta+\gamma-1}t^{\delta-1}\\
&\leq \Gamma(\delta+\gamma)(1-s)^{\delta-1}t^{\delta-1}-\Gamma(\delta)\lambda(\zeta-s)^{\delta+\gamma-1}t^{\delta-1}\\
&\leq \Gamma(\delta+\gamma)(1-s)^{\delta-1}t^{\delta-1}.
\end{aligned}
$$

For $0\leq t\leq s\leq\zeta\leq 1$, we have

$$
\begin{aligned}
&P\Gamma(\delta)\Gamma(\delta+\gamma)G(t,s)\\
&=\Gamma(\delta+\gamma)(1-s)^{\delta-1}t^{\delta-1}-\Gamma(\delta)\lambda(\zeta-s)^{\delta+\gamma-1}t^{\delta-1}\\
&\geq \lambda\Gamma(\delta)\zeta^{\delta+\gamma-1}(1-s)^{\delta-1}t^{\delta-1}-\Gamma(\delta)\lambda(\zeta-s)^{\delta+\gamma-1}t^{\delta-1}\\
&\geq \lambda\Gamma(\delta)\zeta^{\delta+\gamma-1}t^{\delta-1}[(1-s)^{\delta-1}-(1-s)^{\delta+\gamma-1}],
\end{aligned}
$$

and

$$
\begin{aligned}
&P\Gamma(\delta)\Gamma(\delta+\gamma)G(t,s)\\
&=\Gamma(\delta+\gamma)(1-s)^{\delta-1}t^{\delta-1}-\Gamma(\delta)\lambda(\zeta-s)^{\delta+\gamma-1}t^{\delta-1}\\
&\leq \Gamma(\delta+\gamma)(1-s)^{\delta-1}t^{\delta-1}.
\end{aligned}
$$

For $0\leq\zeta\leq s\leq t\leq 1$, we find

$$
\begin{aligned}
&P\Gamma(\delta)\Gamma(\delta+\gamma)G(t,s)\\
&=-P\Gamma(\delta+\gamma)(t-s)^{\delta-1}+\Gamma(\delta+\gamma)(1-s)^{\delta-1}t^{\delta-1}\\
&=\lambda\Gamma(\delta)\zeta^{\delta+\gamma-1}(t-s)^{\delta-1}-\Gamma(\delta+\gamma)(t-s)^{\delta-1}+\Gamma(\delta+\gamma)(1-s)^{\delta-1}t^{\delta-1}\\
&\geq \lambda\Gamma(\delta)\zeta^{\delta+\gamma-1}(t-s)^{\delta-1}-\lambda\Gamma(\delta)\zeta^{\delta+\gamma-1}(t-s)^{\delta-1}+\lambda\Gamma(\delta)\zeta^{\delta+\gamma-1}(1-s)^{\delta-1}t^{\delta-1}\\
&\geq \lambda\Gamma(\delta)\zeta^{\delta+\gamma-1}t^{\delta-1}[(1-s)^{\delta-1}-(1-s)^{\delta+\gamma-1}],
\end{aligned}
$$

and

$$
\begin{aligned}
&P\Gamma(\delta)\Gamma(\delta+\gamma)G(t,s)\\
&=-P\Gamma(\delta+\gamma)(t-s)^{\delta-1}+\Gamma(\delta+\gamma)(1-s)^{\delta-1}t^{\delta-1}\\
&=\lambda\Gamma(\delta)\zeta^{\delta+\gamma-1}(t-s)^{\delta-1}-\Gamma(\delta+\gamma)(t-s)^{\delta-1}+\Gamma(\delta+\gamma)(1-s)^{\delta-1}t^{\delta-1}\\
&\leq \Gamma(\delta+\gamma)(1-s)^{\delta-1}t^{\delta-1}.
\end{aligned}
$$

For $0\leq t\leq s\leq 1$ $s\geq\zeta$, we have

$$
\begin{aligned}
&P\Gamma(\delta)\Gamma(\delta+\gamma)G(t,s)\\
&=\Gamma(\delta+\gamma)(1-s)^{\delta-1}t^{\delta-1}\\
&\geq \lambda\Gamma(\delta)\zeta^{\delta+\gamma-1}t^{\delta-1}[(1-s)^{\delta-1}-(1-s)^{\delta+\gamma-1}].
\end{aligned}
$$

Property (III) is derived from (I). On the other hand, for the validity of (IV), we observe that

$$
G(1,s)=\begin{cases}\frac{(1-P)\Gamma(\delta+\gamma)(1-s)^{\delta-1}-\Gamma(\delta)\lambda(\zeta-s)^{\delta+\gamma-1}}{P\Gamma(\delta)\Gamma(\delta+\gamma)}=\frac{\lambda[\zeta^{\delta+\gamma-1}(1-s)^{\delta-1}-(\zeta-s)^{\delta+\gamma-1}]}{P\Gamma(\delta+\gamma)}, & s\leq\zeta,\\ \frac{(1-P)\Gamma(\delta+\gamma)(1-s)^{\delta-1}}{P\Gamma(\delta)\Gamma(\delta+\gamma)}=\frac{\lambda\zeta^{\delta+\gamma-1}(1-s)^{\delta-1}}{P\Gamma(\delta+\gamma)}, & \zeta\leq s,\end{cases}
$$

which is obviously positive for $s \in (0,1)$. Finally, (V) is trivially derived. □

The previous result is consistent with those obtained in [14] for the problem with $2 < \delta \leq 3$. In fact, for $\gamma = 1$, we have $P = 1 - \frac{\lambda}{\delta}\zeta^{\delta}$, and thus the assumption $\lambda \in (0,\delta)$ (as considered in [14]) guarantees that $P > 0$.

Corollary 1. *For all $\delta \in (n-1,n]$, and $\lambda > 0$ with $P := 1 - \frac{\lambda\Gamma(\delta)}{\Gamma(\delta+\gamma)}\zeta^{\delta+\gamma-1} > 0$, the Green's function $G(t,s)$ satisfies*

$$t^{\delta-1}w_1(s) \leq G(t,s) \leq t^{\delta-1}w_2(s), \quad \forall\, t,\, s \in (0,1), \tag{4}$$

where

$$w_1(s) = \frac{\lambda\zeta^{\delta+\gamma-1}}{P\Gamma(\delta+\gamma)}[(1-s)^{\delta-1} - (1-s)^{\delta+\gamma-1}],$$
$$w_2(s) = \frac{(1-s)^{\delta-1}}{P\Gamma(\delta)}.$$

Similarly to [14], we derive the following Lemma, which expresses a correspondence between the values $G(t,s)$ and $G(1,s)$. This relation will be essential in the proof of the main result.

Lemma 4. *For all $\delta \in (n-1,n]$, and $\lambda > 0$ with $P := 1 - \frac{\lambda\Gamma(\delta)}{\Gamma(\delta+\gamma)}\zeta^{\delta+\gamma-1} > 0$, the Green's function $G(t,s)$ also satisfies*

$$t^{\delta-1}G(1,s) \leq G(t,s) \leq \frac{1}{1-P}G(1,s) = \frac{\Gamma(\delta+\gamma)}{\lambda\Gamma(\delta)\zeta^{\delta+\gamma-1}}G(1,s), \quad \forall\, t,\, s \in (0,1). \tag{5}$$

Proof. By Lemma 3 (IV), the sought inequality is equivalent to prove that

$$t^{\delta-1} \leq \frac{G(t,s)}{G(1,s)} \leq \frac{1}{1-P} = \frac{\Gamma(\delta+\gamma)}{\lambda\Gamma(\delta)\zeta^{\delta+\gamma-1}}, \quad \forall\, t,\, s \in (0,1). \tag{6}$$

Note also that, under the hypotheses imposed, $G(t,s) > 0$ for all t, $s \in (0,1)$.

First, we consider the case $0 < s \leq t < 1$, with $s \leq \zeta$, and then

$$\begin{aligned}
\varphi(t,s) &:= \frac{G(t,s)}{G(1,s)} \\
&= \frac{-P\Gamma(\delta+\gamma)(t-s)^{\delta-1} + \Gamma(\delta+\gamma)(1-s)^{\delta-1}t^{\delta-1} - \Gamma(\delta)\lambda(\zeta-s)^{\delta+\gamma-1}t^{\delta-1}}{-P\Gamma(\delta+\gamma)(1-s)^{\delta-1} + \Gamma(\delta+\gamma)(1-s)^{\delta-1} - \Gamma(\delta)\lambda(\zeta-s)^{\delta+\gamma-1}} \\
&= t^{\delta-1}\frac{-P\Gamma(\delta+\gamma)(1-\frac{s}{t})^{\delta-1} + \Gamma(\delta+\gamma)(1-s)^{\delta-1} - \Gamma(\delta)\lambda(\zeta-s)^{\delta+\gamma-1}}{-P\Gamma(\delta+\gamma)(1-s)^{\delta-1} + \Gamma(\delta+\gamma)(1-s)^{\delta-1} - \Gamma(\delta)\lambda(\zeta-s)^{\delta+\gamma-1}} \\
&= t^{\delta-1}\frac{-P\frac{(1-\frac{s}{t})^{\delta-1}}{(1-s)^{\delta-1}} + 1 - \frac{\Gamma(\delta)\lambda(\zeta-s)^{\delta+\gamma-1}}{\Gamma(\delta+\gamma)(1-s)^{\delta-1}}}{-P + 1 - \frac{\Gamma(\delta)\lambda(\zeta-s)^{\delta+\gamma-1}}{\Gamma(\delta+\gamma)(1-s)^{\delta-1}}} \\
&\in \left[t^{\delta-1}, \frac{1 - \frac{\Gamma(\delta)\lambda(\zeta-s)^{\delta+\gamma-1}}{\Gamma(\delta+\gamma)(1-s)^{\delta-1}}}{1 - P - \frac{\Gamma(\delta)\lambda(\zeta-s)^{\delta+\gamma-1}}{\Gamma(\delta+\gamma)(1-s)^{\delta-1}}}\right] \subseteq \left[t^{\delta-1}, \frac{1}{1-P}\right] = \left[t^{\delta-1}, \frac{\Gamma(\delta+\gamma)}{\lambda\Gamma(\delta)\zeta^{\delta+\gamma-1}}\right].
\end{aligned}$$

For $0 < t \le s \le \zeta < 1$, we have

$$\begin{aligned}\varphi(t,s) &:= \frac{G(t,s)}{G(1,s)}\\ &= t^{\delta-1}\frac{\Gamma(\delta+\gamma)(1-s)^{\delta-1}-\Gamma(\delta)\lambda(\zeta-s)^{\delta+\gamma-1}}{-P\Gamma(\delta+\gamma)(1-s)^{\delta-1}+\Gamma(\delta+\gamma)(1-s)^{\delta-1}-\Gamma(\delta)\lambda(\zeta-s)^{\delta+\gamma-1}}\\ &\ge t^{\delta-1}.\end{aligned}$$

Next, we prove that $\varphi(t,s) \le \frac{1}{1-P}$, for $0 < t \le s \le \zeta < 1$. We study the behavior of the auxiliary one-variable function

$$\psi(s) := \frac{\Gamma(\delta+\gamma)(1-s)^{\delta-1}-\Gamma(\delta)\lambda(\zeta-s)^{\delta+\gamma-1}}{(1-P)\Gamma(\delta+\gamma)(1-s)^{\delta-1}-\Gamma(\delta)\lambda(\zeta-s)^{\delta+\gamma-1}}$$

in the interval $[t,\zeta]$, with $t \in (0,\zeta]$ fixed. The sign of $\psi'(s)$ coincides with the sign of

$$\begin{aligned}\phi(s) :=&\Big(-\Gamma(\delta+\gamma)(\delta-1)(1-s)^{\delta-2}+\Gamma(\delta)\lambda(\delta+\gamma-1)(\zeta-s)^{\delta+\gamma-2}\Big)\\ &\times\Big((1-P)\Gamma(\delta+\gamma)(1-s)^{\delta-1}-\Gamma(\delta)\lambda(\zeta-s)^{\delta+\gamma-1}\Big)\\ &-\Big(\Gamma(\delta+\gamma)(1-s)^{\delta-1}-\Gamma(\delta)\lambda(\zeta-s)^{\delta+\gamma-1}\Big)\\ &\times\Big(-(1-P)\Gamma(\delta+\gamma)(\delta-1)(1-s)^{\delta-2}+\Gamma(\delta)\lambda(\delta+\gamma-1)(\zeta-s)^{\delta+\gamma-2}\Big)\\ =&\Gamma(\delta+\gamma)\Gamma(\delta)(1-s)^{\delta-2}\lambda(\zeta-s)^{\delta+\gamma-2}P\{(\delta-1)(\zeta-1)-(1-s)\gamma\},\end{aligned}$$

which is, clearly, nonpositive for $s \in [t,\zeta]$. Hence, $\psi(s) \le \psi(t)$, for $s \in [t,\zeta]$. Since $\varphi(t,s) = t^{\delta-1}\psi(s)$, this proves that, in the case $0 < t \le s \le \zeta < 1$, we have

$$\varphi(t,s) \le t^{\delta-1}\psi(t) = \frac{\Gamma(\delta+\gamma)t^{\delta-1}(1-t)^{\delta-1}-\Gamma(\delta)\lambda t^{\delta-1}(\zeta-t)^{\delta+\gamma-1}}{(1-P)\Gamma(\delta+\gamma)(1-t)^{\delta-1}-\Gamma(\delta)\lambda(\zeta-t)^{\delta+\gamma-1}} =: \mathcal{M}(t).$$

We now check that $\mathcal{M}(t) \le \frac{1}{1-P}$, for $t \in (0,\zeta]$, which is equivalent to

$$(1-P)\Gamma(\delta+\gamma)(1-t)^{\delta-1}(1-t^{\delta-1}) \ge \Gamma(\delta)\lambda(\zeta-t)^{\delta+\gamma-1}(1-(1-P)t^{\delta-1}),\ t \in (0,\zeta].$$

By substituting the value of P, the previous condition is equivalent to the nonnegativity on the interval $(0,\zeta]$ of the function

$$R(t) := \zeta^{\delta+\gamma-1}(1-t)^{\delta-1}(1-t^{\delta-1}) - (\zeta-t)^{\delta+\gamma-1}\left(1-\frac{\Gamma(\delta)\lambda\zeta^{\delta+\gamma-1}}{\Gamma(\delta+\gamma)}t^{\delta-1}\right).$$

Indeed, $R(0) = \zeta^{\delta+\gamma-1} - \zeta^{\delta+\gamma-1} = 0$, $R(\zeta) = \zeta^{\delta+\gamma-1}(1-\zeta)^{\delta-1}(1-\zeta^{\delta-1}) > 0$, and

$$\begin{aligned}R'(t) =&\zeta^{\delta+\gamma-1}(\delta-1)(1-t)^{\delta-2}\Big(1-t^{\delta-1}-(1-t)t^{\delta-2}\Big)\\ &+(\zeta-t)^{\delta+\gamma-1}\left\{(\delta+\gamma-1)\left(1-\frac{\Gamma(\delta)\lambda\zeta^{\delta+\gamma-1}}{\Gamma(\delta+\gamma)}t^{\delta-1}\right)+\frac{\Gamma(\delta)\lambda\zeta^{\delta+\gamma-1}}{\Gamma(\delta+\gamma)}(\delta-1)t^{\delta-2}\right\},\end{aligned}$$

which is clearly positive on $(0,\zeta]$, since

$$\frac{\Gamma(\delta)\lambda\zeta^{\delta+\gamma-1}}{\Gamma(\delta+\gamma)}t^{\delta-1} < \frac{\Gamma(\delta)\lambda\zeta^{\delta+\gamma-1}}{\Gamma(\delta+\gamma)} < 1,$$

and $S(t) := 1 - t^{\delta-1} - (1-t)t^{\delta-2}$ satisfies $S(0) = 1$, $S(1) = 0$, and $S'(t) = t^{\delta-3}(2-\delta) < 0$ for $t \in (0,1)$; thus, $S > 0$ on $(0,\zeta]$. This proves that $R > 0$ on $(0,\zeta]$.

For $0 < \zeta \leq s \leq t < 1$,

$$\begin{aligned}
\varphi(t,s) &:= \frac{G(t,s)}{G(1,s)} \\
&= \frac{-P\Gamma(\delta+\gamma)(t-s)^{\delta-1} + \Gamma(\delta+\gamma)(1-s)^{\delta-1}t^{\delta-1}}{-P\Gamma(\delta+\gamma)(1-s)^{\delta-1} + \Gamma(\delta+\gamma)(1-s)^{\delta-1}} \\
&= t^{\delta-1}\frac{-P\Gamma(\delta+\gamma)(1-\frac{s}{t})^{\delta-1} + \Gamma(\delta+\gamma)(1-s)^{\delta-1}}{-P\Gamma(\delta+\gamma)(1-s)^{\delta-1} + \Gamma(\delta+\gamma)(1-s)^{\delta-1}} \\
&= t^{\delta-1}\frac{-P\frac{(1-\frac{s}{t})^{\delta-1}}{(1-s)^{\delta-1}} + 1}{-P+1} \\
&\in \left[t^{\delta-1}, \frac{1}{1-P}\right] = \left[t^{\delta-1}, \frac{\Gamma(\delta+\gamma)}{\lambda\Gamma(\delta)\zeta^{\delta+\gamma-1}}\right].
\end{aligned}$$

Finally, for $0 < t \leq s < 1, s \geq \zeta$,

$$\begin{aligned}
\varphi(t,s) &:= \frac{G(t,s)}{G(1,s)} \\
&= t^{\delta-1}\frac{\Gamma(\delta+\gamma)(1-s)^{\delta-1}}{-P\Gamma(\delta+\gamma)(1-s)^{\delta-1} + \Gamma(\delta+\gamma)(1-s)^{\delta-1}} \\
&= t^{\delta-1}\frac{1}{-P+1} \in \left[t^{\delta-1}, \frac{1}{1-P}\right] = \left[t^{\delta-1}, \frac{\Gamma(\delta+\gamma)}{\lambda\Gamma(\delta)\zeta^{\delta+\gamma-1}}\right].
\end{aligned}$$

□

4. Main Results

This section of the paper is focused on the study of the existence of at least one positive solution to the nonlinear boundary value problem specified in expression (1). The main tool used is the fixed point result by Guo and Krasnosel'skii [34], i.e., Theorem 1.

The base space of interest is $E = C[0,1]$, which is a Banach space if we consider the usual supremum norm $\|\cdot\|$.

Next, similarly to [14], we consider the cone $K \subset E$ defined in the following way:

$$K := \left\{u \in E \,:\, u(t) \geq 0 \text{ for all } t \in [0,1],\ u(t) \geq t^{\delta-1}(1-P)\|u\|, \text{ for all } t \in \left[\frac{1}{2}, 1\right]\right\}, \quad (7)$$

and develop, in the rest of the section, a procedure similar to that in the mentioned reference [14]. Hence, one of the assumptions that will be used is specified below:

(a) The function $f : [0,1] \times [0,\infty) \to [0,\infty)$ is continuous.

We take the following finite or infinite values:

$$f_0 := \lim_{h\to 0^+}\left\{\min_{t\in[\frac{1}{2},1]}\frac{f(t,h)}{h}\right\}, \quad f_\infty := \lim_{h\to\infty}\left\{\min_{t\in[\frac{1}{2},1]}\frac{f(t,h)}{h}\right\},$$

$$f^0 := \lim_{h\to 0^+}\left\{\max_{t\in[0,1]}\frac{f(t,h)}{h}\right\}, \text{ and } f^\infty := \lim_{h\to\infty}\left\{\max_{t\in[0,1]}\frac{f(t,h)}{h}\right\}.$$

Then, it is possible to extend Theorem 3.2 [14] to the context of the general-order problem (1). This fact is the main conclusion of this paper.

Theorem 2. *Suppose that the hypothesis* (a) *is satisfied, and that one of the following assumptions also holds:*

(i) $f_0 = \infty$ *and* $f^\infty = 0$ *(that is, the sublinear case).*

(ii) $f^0 = 0$ *and* $f_\infty = \infty$ *(that is, the superlinear case).*

Then, for all $\delta \in (n-1, n]$*, and* $\lambda > 0$ *with* $P := 1 - \frac{\lambda \Gamma(\delta)}{\Gamma(\delta+\gamma)} \zeta^{\delta+\gamma-1} > 0$*, the problem* (1) *has a positive solution that belongs to the cone* K *given by* (7)*.*

Proof. We consider the mapping T defined by $[Tu](t) := \int_0^1 G(t,s) f(s,u(s))\, ds$, where G is the Green's function given in expression (3). In the first place, we check that the mapping $T : K \to K$ is a self-mapping and that T is also completely continuous. Indeed, using the continuity and the nonnegative character of the functions G and f on $[0,1] \times [0,1]$ and $[0,1] \times [0, \infty)$, respectively, it is clear that, if $u \in K$, then Tu is continuous and nonnegative on $[0,1]$.

To prove that T is self-mapping, let $u \in K$, and then, by Lemma 4, we have

$$
\begin{aligned}
[Tu](t) &= \int_0^1 G(t,s) f(s,u(s))\, ds \\
&\geq t^{\delta-1} \int_0^1 G(1,s) f(s,u(s))\, ds \\
&\geq t^{\delta-1}(1-P) \int_0^1 \max_{t\in[0,1]} G(t,s) f(s,u(s))\, ds \\
&\geq t^{\delta-1}(1-P) \max_{t\in[0,1]} \left\{ \int_0^1 G(t,s) f(s,u(s))\, ds \right\} \\
&= t^{\delta-1}(1-P) \|Tu\|.
\end{aligned}
$$

It is clear that the mapping $T : K \to K$ is continuous, since G and f are both continuous.

Next, to check that T is completely continuous, let $\mathcal{B} \subset K$ be a bounded set, i.e., such that there exists a positive constant $N > 0$ with $\|u\| \leq N$ for all $u \in \mathcal{B}$. Consider the compact set $[0,1] \times [0,N]$, and take $L := \max\limits_{(t,u)\in[0,1]\times[0,N]} |f(t,u)| + 1 > 0$.

Now we check that $T(\mathcal{B})$ is a bounded set. Indeed, for an arbitrary $u \in \mathcal{B}$, we have, by Corollary 1, that

$$
\|[Tu](t)\| \leq \max_{t\in[0,1]} \int_0^1 G(t,s) |f(s,u(s))|\, ds \leq L \max_{t\in[0,1]} \int_0^1 t^{\delta-1} \frac{(1-s)^{\delta-1}}{P\Gamma(\delta)}\, ds \leq \frac{L}{P\Gamma(\delta)},
$$

for every $t \in [0,1]$, so that $T(\mathcal{B})$ is a bounded subset of E.

On the other hand, we seek an estimate for the derivative of the functions in $T(\mathcal{B})$. Given an arbitrary $u \in \mathcal{B}$, we have, from the calculations in Lemma 2, that

$$
\begin{aligned}
[Tu](t) &= \int_0^1 G(t,s) f(s,u(s))\, ds \\
&= -\frac{1}{\Gamma(\delta)} \int_0^t (t-s)^{\delta-1} f(s,u(s))\, ds + \frac{t^{\delta-1}}{P\Gamma(\delta)} \int_0^1 (1-s)^{\delta-1} f(s,u(s))\, ds \\
&\quad - \frac{\lambda t^{\delta-1}}{P\Gamma(\delta+\gamma)} \int_0^\zeta (\zeta-s)^{\delta+\gamma-1} f(s,u(s))\, ds,
\end{aligned}
$$

so that

$$
\begin{aligned}
&\big|(Tu)'(t)\big| = \Bigg| -\frac{1}{\Gamma(\delta-1)}\int_0^t (t-s)^{\delta-2} f(s,u(s))ds \\
&+\frac{t^{\delta-2}}{P\Gamma(\delta-1)}\int_0^1 (1-s)^{\delta-1} f(s,u(s))ds - \frac{(\delta-1)\lambda t^{\delta-2}}{P\Gamma(\delta+\gamma)}\int_0^\zeta (\zeta-s)^{\delta+\gamma-1} f(s,u(s))ds\Bigg| \\
&\le \frac{1}{\Gamma(\delta-1)}\int_0^t (t-s)^{\delta-2}|f(s,u(s))|ds \\
&+\frac{t^{\delta-2}}{|P|\Gamma(\delta-1)}\int_0^1 (1-s)^{\delta-1}|f(s,u(s))|ds + \frac{(\delta-1)\lambda t^{\delta-2}}{|P|\Gamma(\delta+\gamma)}\int_0^\zeta (\zeta-s)^{\delta+\gamma-1}|f(s,u(s))|ds \\
&\le \frac{Lt^{\delta-1}}{\Gamma(\delta)} + \frac{Lt^{\delta-2}}{|P|\Gamma(\delta-1)\delta} + \frac{(\delta-1)\lambda L t^{\delta-2}\zeta^{\delta+\gamma}}{|P|\Gamma(\delta+\gamma)(\delta+\gamma)} \\
&\le \frac{Lt^{\delta-1}}{\Gamma(\delta)} + \frac{t^{\delta-2}L}{|P|\Gamma(\delta)} + \frac{(\delta-1)Lt^{\delta-2}\zeta^{\delta+\gamma}\lambda}{|P|\Gamma(\delta+\gamma+1)} \le \frac{L}{\Gamma(\delta)} + \frac{L}{|P|\Gamma(\delta)} + \frac{(\delta-1)L\zeta^{\delta+\gamma}\lambda}{|P|\Gamma(\delta+\gamma+1)} =: M.
\end{aligned}
$$

Therefore, for every $t_1, t_2 \in [0,1]$ with $t_1 < t_2$, we obtain

$$|[Tu](t_2) - [Tu](t_1)| \le M(t_2 - t_1),$$

and we deduce that $T(\mathcal{B})$ is an equicontinuous set in E.

With these ingredients, the application of the Arzelà–Ascoli Theorem proves that $T(\mathcal{B})$ is relatively compact. As a consequence, $T : K \to K$ is completely continuous.

Once we have proven some relevant properties of the mapping T, we distinguish two cases and complete the proof following the ideas in [14]. We include the explanations and adaptations here for completeness.

Case (i): ($f_0 = \infty$ and $f^\infty = 0$).

We choose $\tilde{\delta} > 0$ to be sufficiently large such that

$$\tilde{\delta}(1-P)\max_{t\in[0,1]}\left\{\int_{\frac{1}{2}}^1 s^{\delta-1}G(t,s)ds\right\} \ge 1. \tag{8}$$

Since $f_0 = \infty$, we can affirm the existence of a constant $\tilde{\rho} > 0$ such that $f(t,h) \ge \tilde{\delta}h$ for every $t \in [\frac{1}{2}, 1]$ and every $0 < h \le \tilde{\rho}$.

Then, for an arbitrary $u \in K$ with $\|u\| = \tilde{\rho}$, we have that $u(t) > 0$ for $t \in [\frac{1}{2}, 1]$ and, using the selection for $\tilde{\delta}$, we obtain that

$$
\begin{aligned}
\|Tu\| &= \max_{t\in[0,1]}\left\{\int_0^1 G(t,s)f(s,u(s))ds\right\} \\
&\ge \tilde{\delta}\max_{t\in[0,1]}\left\{\int_{\frac{1}{2}}^1 G(t,s)u(s)ds\right\} \\
&\ge \tilde{\delta}\|u\|(1-P)\max_{t\in[0,1]}\left\{\int_{\frac{1}{2}}^1 s^{\delta-1}G(t,s)ds\right\} \\
&\ge \|u\|.
\end{aligned}
$$

By the continuity of $f(t,\cdot)$ on the interval $[0,\infty)$, we can consider the function:

$$\tilde{f}(t,h) = \max_{z\in[0,h]} f(t,z),$$

which is clearly a nondecreasing function on $[0,\infty)$. By the hypothesis $f^\infty = 0$, it is deduced that

$$\lim_{h\to\infty}\left\{\max_{t\in[0,1]}\frac{\tilde{f}(t,h)}{h}\right\} = 0.$$

Next, we select $\delta^* > 0$ small enough such that $\frac{\delta^*}{P\Gamma(\delta)} \leq 1$.

By virtue of the previous limit, we can prove the existence of a constant $\rho^* > \tilde{\rho} > 0$ such that $\tilde{f}(t,h) \leq \delta^* h$ for every $t \in [0,1]$ and all $h \geq \rho^*$.

If we take $u \in K$ such that $\|u\| = \rho^*$, then, using the nondecreasing character of $\tilde{f}$ and Lemma 3 (II) (or Corollary 1), the next inequalities are satisfied:

$$\|Tu\| = \max_{t\in[0,1]} \left\{ \int_0^1 G(t,s) f(s,u(s))\, ds \right\} \leq \max_{t\in[0,1]} \left\{ \int_0^1 G(t,s) \tilde{f}(s, \|u\|)\, ds \right\}$$
$$\leq \delta^* \|u\| \max_{t\in[0,1]} \left\{ \int_0^1 G(t,s)\, ds \right\} \leq \frac{\delta^*}{P\Gamma(\delta)} \|u\| \leq \|u\|.$$

Therefore, by part (i) in Theorem 1, we can affirm that problem (1) has at least one positive solution u with $\tilde{\rho} \leq \|u\| \leq \rho^*$.

Case (ii): $f^0 = 0$ and $f_\infty = \infty$.

We take $\delta^* > 0$ with $\frac{\delta^*}{P\Gamma(\delta)} \leq 1$.

Using $f^0 = 0$, it is possible to find a constant $r^* > 0$ such that $f(t,h) \leq \delta^* h$ for every $t \in [0,1]$ and $0 < h \leq r^*$. From $f^0 = 0$, it is clear that $\lim_{h\to 0^+} \frac{f(t,h)}{h} = 0$ for every $t \in [0,1]$; hence, $\lim_{h\to 0^+} f(t,h) = 0$, and thus, by the continuity of f, $f(t,0) = 0$, for every $t \in [0,1]$. This, together with the previous inequality, implies that $f(t,h) \leq \delta^* h$ for every $t \in [0,1]$ and $0 \leq h \leq r^*$.

Then, for every $u \in K$ with $\|u\| = r^*$, we deduce that

$$\|Tu\| = \max_{t\in[0,1]} \left\{ \int_0^1 G(t,s) f(s,u(s))\, ds \right\} \leq \delta^* \|u\| \max_{t\in[0,1]} \left\{ \int_0^1 G(t,s)\, ds \right\}$$
$$\leq \frac{\delta^*}{P\Gamma(\delta)} \|u\| \leq \|u\|.$$

Finally, we select $\hat{\delta} > 0$ large enough such that

$$\frac{\hat{\delta}}{2^{\delta-1}} (1-P) \max_{t\in[0,1]} \left\{ \int_{\frac{1}{2}}^1 G(t,s)\, ds \right\} \geq 1.$$

Since $f_\infty = \infty$, we can affirm the existence of $\hat{r} > r^* > 0$, which can be taken satisfying the additional condition $\hat{r} 2^{\delta-1} > r^*(1-P)$, such that $f(t,h) \geq \hat{\delta} h$ for all $t \in [\frac{1}{2}, 1]$ and all $h \geq \hat{r}$.

Next, we choose a convenient shell, in particular, we take an arbitrary $u \in K$ with $\|u\| = \frac{\hat{r}}{1-P} 2^{\delta-1}$. The definition of the cone K implies that $u(t) \geq \hat{r}$ for every $t \in [\frac{1}{2}, 1]$.

In summary, in this case, we obtain that

$$\begin{aligned}
\|Tu\| &= \max_{t\in[0,1]} \left\{ \int_0^1 G(t,s) f(s,u(s))\, ds \right\} \\
&\geq \max_{t\in[0,1]} \left\{ \int_{\frac{1}{2}}^1 G(t,s) f(s,u(s))\, ds \right\} \\
&\geq \hat{\delta} \max_{t\in[0,1]} \left\{ \int_{\frac{1}{2}}^1 G(t,s) u(s)\, ds \right\} \\
&\geq \frac{\hat{\delta}}{2^{\delta-1}} (1-P) \|u\| \max_{t\in[0,1]} \left\{ \int_{\frac{1}{2}}^1 G(t,s)\, ds \right\} \\
&\geq \|u\|.
\end{aligned}$$

In consequence, by case (ii) in Theorem 1, we deduce that problem (1) has at least one positive solution such that $r^* \leq \|u\| \leq \frac{\hat{r}}{1-P} 2^{\delta-1}$. □

5. Example

In this section, we discuss an example to show the applicability of our result.

Example 1. *Consider the following fractional integral boundary value problem on the interval* $[0,1]$*:*

$$\begin{cases} D_{0+}^{\frac{5}{2}} u(t) + f(t,u(t)) = 0 \\ u(0) = u'(0) = 0,\ u(1) = 2I_{0+}^{\frac{1}{2}} u(\zeta), \end{cases} \tag{9}$$

where $f(t,u(t)) = u^{\frac{1}{3}}(t) + \log(1+u^2(t)) + \sin^2(e^{u(t)})$, $D_{0+}^{\frac{5}{2}}$ *denotes the Riemann–Liouville fractional derivative operator of order* $\delta = \frac{5}{2}$, $I_{0+}^{\frac{1}{2}}$ *is the Riemann–Liouville fractional integral operator of order* $\gamma = \frac{1}{2}$ *and* $0 < \zeta < 1$. *Here,* $f : [0,1] \times [0,\infty) \to [0,\infty)$ *is a continuous function. It is clear that* $f_0 = \infty$, $f^\infty = 0$, *and thus the function* f *is sublinear. Note that, since* $\frac{2\Gamma(\frac{5}{2})}{\Gamma(3)} > 1$, $P := 1 - \frac{2\Gamma(\frac{5}{2})}{\Gamma(3)}\zeta^2$ *vanishes at a certain* $\zeta \in (0,1)$, *exactly at* $\zeta_* := \sqrt{\frac{\Gamma(3)}{2\Gamma(\frac{5}{2})}}$. *Therefore, we must impose that* $\zeta \in (0, \zeta_*)$ *in order to guarantee* $P > 0$. *Under this restriction, from case (i) in Theorem 2, the particular problem (9) has, at least, a positive solution.*

6. Conclusions

In this paper, we extended the results in [14] to general fractional problems of order greater than 2, dealing with the existence of positive solutions for differential equations of arbitrary order with fractional integral boundary conditions of the type (1). The introduction of a boundary condition that involves an integral operator of fractional type is interesting from the point of view of applications, since it allows for the mathematical expression of heterogeneity that may affect the dependence specified by the restriction added to the equation—a fact that is consistent with many physical problems.

The main tool used in the paper was Guo–Krasnosel'skii fixed point theorem in cones. In particular, in Lemma 2, we obtained, by imposing some adequate restrictions on the parameters, the integral expression of the solution to a modified linear fractional boundary value problem, which provides the Green's function of interest. Then, in Lemma 3, we studied some properties of the Green's function, including its positivity on $(0,1) \times (0,1)$ under some restrictions on the parameters, as well as some upper and lower estimates for its expression.

Another useful result is Lemma 4, which establishes the relation between the value of the Green's function at an arbitrary point and the value at the point with the same ordinate and abscise 1. The explicit calculations for this general problem were developed in detail due to the high order of the equation and the difficulty generated by the introduction of fractional operators in the boundary conditions.

Theorem 2 provides the existence of a positive solution to (1) by assuming that the nonlinearity f is sublinear or superlinear. The proof, based on the Guo–Krasnosel'skii fixed point theorem, makes a selection of the conical shells that allow localization of the solution in each case. Then, we have not only deduced the existence of a positive solution but the details of the proof also provide the procedure to obtain an estimate for its maximum value and to determine positive numbers that are not upper bounds for the solution.

Since the fixed point theorem used has two contexts of application (a contractive and expansive case), it is possible to consider the problem under two types of hypotheses; that is, two types of restrictions on the function defining the equation. The consideration of other types of restrictions on the function f can be one of the possible future lines of research.

Finally, an example was presented.

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