fractal and fractional

Article

Nonexistence of Global Solutions to Time-Fractional Damped Wave Inequalities in Bounded Domains with a Singular Potential on the Boundary

Areej Bin Sultan †, Mohamed Jleli † and Bessem Samet *,†

Department of Mathematics, College of Science, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia; 437203645@student.ksu.edu.sa (A.B.S.); jleli@ksu.edu.sa (M.J.)
* Correspondence: bsamet@ksu.edu.sa
† These authors contributed equally to this work.

Abstract: We first consider the damped wave inequality $\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial t} \geq x^\sigma |u|^p$, $\quad t > 0$, $x \in$ (0,L), where $L > 0$, $\sigma \in \mathbb{R}$, and $p > 1$, under the Dirichlet boundary conditions $(u(t,0), u(t,L))= (f(t), g(t))$, $\quad t > 0$. We establish sufficient conditions depending on σ, p, the initial conditions, and the boundary conditions, under which the considered problem admits no global solution. Two cases of boundary conditions are investigated: $g \equiv 0$ and $g(t) = t^\gamma$, $\gamma > -1$. Next, we extend our study to the time-fractional analogue of the above problem, namely, the time-fractional damped wave inequality $\frac{\partial^\alpha u}{\partial t^\alpha} - \frac{\partial^2 u}{\partial x^2} + \frac{\partial^\beta u}{\partial t^\beta} \geq x^\sigma |u|^p$, $\quad t > 0$, $x \in (0, L)$, where $\alpha \in (1, 2)$, $\beta \in (0, 1)$, and $\frac{\partial^\tau}{\partial t^\tau}$ is the time-Caputo fractional derivative of order τ, $\tau \in \{\alpha, \beta\}$. Our approach is based on the test function method. Namely, a judicious choice of test functions is made, taking in consideration the boundedness of the domain and the boundary conditions. Comparing with previous existing results in the literature, our results hold without assuming that the initial values are large with respect to a certain norm.

Keywords: time-fractional damped wave inequalities; bounded domain; singularity; nonexistence

MSC: 35B44; 35B33; 26A33

check for updates

Citation: Bin Sultan, A.; Jleli, M.; Samet, B. Nonexistence of Global Solutions to Time-Fractional Damped Wave Inequalities in Bounded Domains with a Singular Potential on the Boundary. *Fractal Fract.* **2021**, *5*, 258. https://doi.org/10.3390/fractalfract5040258

Academic Editor: Rodica Luca

Received: 1 November 2021
Accepted: 29 November 2021
Published: 6 December 2021

1. Introduction

In this paper, we first consider the damped wave inequality

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial u}{\partial t} \geq x^\sigma |u|^p, & t > 0,\, x \in (0, L), \\ (u(t,0), u(t,L)) = (f(t), g(t)), & t > 0, \\ \left(u(0,x), \dfrac{\partial u}{\partial t}(0,x)\right) = (u_0(x), u_1(x)), & x \in (0, L), \end{cases} \tag{1}$$

where $L > 0$, $\sigma \in \mathbb{R}$, and $p > 1$. It is supposed that $u_0, u_1 \in L^1([0, L])$, $f \in L^1_{loc}([0, \infty))$, and $g(t) = C_g t^\gamma$, where $C_g \geq 0$ and $\gamma > -1$, are constants. Namely, we establish sufficient conditions depending on the initial values, the boundary conditions, p, and σ, under which (1) admits no global weak solution, in a sense that will be specified later.

Next, we study the time-fractional analogue of (1), namely the time-fractional damped wave inequality

$$\begin{cases} \frac{\partial^\alpha u}{\partial t^\alpha} - \frac{\partial^2 u}{\partial x^2} + \frac{\partial^\beta u}{\partial t^\beta} \geq x^\sigma |u|^p, \quad t > 0,\, x \in (0, L), \\ (u(t,0), u(t,L)) = (f(t), g(t)), \quad t > 0, \\ \left(u(0,x), \frac{\partial u}{\partial t}(0,x)\right) = (u_0(x), u_1(x)), \quad x \in (0, L), \end{cases} \tag{2}$$

where $\alpha \in (1,2)$, $\beta \in (0,1)$, and $\frac{\partial^\tau}{\partial t^\tau}$, $\tau \in \{\alpha, \beta\}$, is the time-Caputo fractional derivative of order τ.

The investigation of the question of blow-up of solutions to initial boundary value problems for semilinear wave equations started in the 1970s. For example, Tsutsumi [1] considered the nonlinear damped wave equation

$$\frac{\partial^2 u}{\partial t^2} - \Delta u + b \frac{\partial u}{\partial t} = F(u),$$

under homogeneous Dirichlet boundary conditions, where $b \geq 0$ and

$$F(s)s - 2(2\kappa + 1) \int_0^s F(\tau)\, d\tau \geq d_0 |s|^{\rho+2}, \quad s \in \mathbb{R},$$

for some $\kappa > 0$ and $\rho > 0$. By means of the energy method, the author established sufficient conditions for the blow-up of solutions. In [2], using a concavity argument, Levine established sufficient conditions for the blow-up of solutions to an abstract Cauchy problem in a Hilbert space, of the form

$$P \frac{\partial^2 u}{\partial t^2} + Au + Q \frac{\partial u}{\partial t} = F(u),$$

where P and A are positive symmetric operators and F is a nonlinear operator satisfying certain conditions. Later, the concavity method was used and developed by many authors in order to study more general problems. For further blow-up results for nonlinear wave equations, obtained by means of the energy/concavity method, see e.g., [3–11] and the references therein.

Fractional operators arise in various applications, such as chemistry, biology, continuum mechanics, anomalous diffusion, and materials science, see for instance [12–16]. Consequently, many mathematicians dealt with the study of fractional differential equations in both theoretical and numerical aspects, see e.g., [17–21].

In [22], Kirane and Tatar considered the time-fractional damped wave equation

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - \Delta u + \frac{\partial^{1+\alpha} u}{\partial t^{1+\alpha}} = a|u|^{p-1}u, \quad t > 0,\, x \in \Omega, \\ u(t,x) = 0, \quad t > 0,\, x \in \partial\Omega, \\ \left(u(0,x), \frac{\partial u}{\partial t}(0,x)\right) = (u_0(x), u_1(x)), \quad x \in \Omega, \end{cases} \tag{3}$$

where $p > 1$, $\alpha \in (-1,1)$, and Ω is a bounded domain of $\mathbb{R}^N$. Using some arguments based on Fourier transforms and the Hardy–Littlewood inequality, it was shown that the energy grows exponentially for sufficiently large initial data.

By combining an argument due to Georgiev and Todorova [23] with the techniques used in [22], Tatar [24] proved that the solutions to (3) blow up in finite-time for sufficiently large initial data.

In all the above cited references, the blow-up results were obtained for sufficiently large initial data. In this paper, we use a different approach than those used in the above mentioned references. Namely, our approach is based on the test function method introduced by Mitidieri and Pohozaev [25]. Taking into consideration the boundedness of the domain as well as the boundary conditions, adequate test functions are used to obtain sufficient conditions for the nonexistence of global weak solutions to problems (1) and (2). Notice that our results hold without assuming that the initial values are large with respect to a certain norm.

Let us mention also that recently, methods for the numerical diagnostics of the solution's blow-up have been actively developing (see e.g., [26–28]), which make it possible to refine the theoretical estimates.

The rest of the paper is organized as follows: In Section 2, we provide some preliminaries on fractional calculus, and some useful lemmas. We state our main results in Section 3. The proofs are presented in Section 4.

2. Preliminaries on Fractional Calculus

For the reader's convenience, we recall below some notions from fractional calculus, see e.g., [17,20].

Let $T>0$ be fixed. Given $\rho>0$ and $v\in L^1([0,T])$, the left-sided and right-sided Riemann–Liouville fractional integrals of order ρ of v, are defined, respectively, by

$$(I_0^\rho v)(t)=\frac{1}{\Gamma(\rho)}\int_0^t (t-s)^{\rho-1}v(s)\,ds \quad \text{and} \quad (I_T^\rho v)(t)=\frac{1}{\Gamma(\rho)}\int_t^T (s-t)^{\rho-1}v(s)\,ds,$$

for almost everywhere $t\in[0,T]$, where Γ denotes the Gamma function. It can be easily seen that, if $v\in C([0,T])$, then

$$\lim_{t\to 0^+}(I_0^\rho v)(t)=\lim_{t\to T^-}(I_T^\rho v)(t)=0.$$

In this case, we may consider $I_0^\rho v$ and $I_T^\rho v$ as continuous functions in $[0,T]$, by taking

$$(I_0^\rho v)(0)=(I_T^\rho v)(T)=0.$$

Given a positive integer n, $\tau\in(n-1,n)$, and $v\in C^n([0,T])$, the (left-sided) Caputo fractional derivative of order τ of v, is defined by

$$\frac{d^\tau v}{dt^\tau}(t)=\left(I_0^{n-\tau}\frac{d^n v}{dt^n}\right)(t)=\frac{1}{\Gamma(n-\tau)}\int_0^t (t-s)^{n-\tau-1}\frac{d^n v}{dt^n}(s)\,ds,$$

for all $t\in[0,L]$.

We have the following integration by parts rule.

Lemma 1 (see the Corollary in [17], p. 67)**.** *Let $\rho>0$, $q,r\geq 1$, and $\frac{1}{q}+\frac{1}{r}\leq 1+\rho$ ($q\neq 1$, $r\neq 1$, in the case $\frac{1}{q}+\frac{1}{r}=1+\rho$). If $(v,w)\in L^q([0,T])\times L^r([0,T])$, then*

$$\int_0^T (I_0^\rho v)(t)w(t)\,dt=\int_0^T v(t)(I_T^\rho w)(t)\,dt.$$

Lemma 2. *For sufficiently large λ, let*

$$\eta(t)=T^{-\lambda}(T-t)^\lambda,\quad 0\leq t\leq T. \tag{4}$$

Let $\rho \in (0,1)$. *Then*

$$\begin{aligned} (I_T^\rho \eta)(t) &= \frac{\Gamma(\lambda+1)}{\Gamma(\rho+\lambda+1)} T^{-\lambda}(T-t)^{\rho+\lambda}, \quad (5)\\ (I_T^\rho \eta)'(t) &= -\frac{\Gamma(\lambda+1)}{\Gamma(\rho+\lambda)} T^{-\lambda}(T-t)^{\rho+\lambda-1}, \quad (6)\\ (I_T^\rho \eta)''(t) &= \frac{\Gamma(\lambda+1)}{\Gamma(\rho+\lambda-1)} T^{-\lambda}(T-t)^{\rho+\lambda-2}. \quad (7) \end{aligned}$$

Proof. We have

$$\begin{aligned} (I_T^\rho \eta)(t) &= \frac{1}{\Gamma(\rho)} \int_t^T (s-t)^{\rho-1} \eta(s)\, ds \\ &= \frac{T^{-\lambda}}{\Gamma(\rho)} \int_t^T (s-t)^{\rho-1} (T-s)^\lambda\, ds \\ &= \frac{T^{-\lambda}}{\Gamma(\rho)} \int_t^T (s-t)^{\rho-1} ((T-t)-(s-t))^\lambda\, ds \\ &= \frac{T^{-\lambda}(T-t)^\lambda}{\Gamma(\rho)} \int_t^T (s-t)^{\rho-1} \left(1-\frac{s-t}{T-t}\right)^\lambda ds. \end{aligned}$$

Using the change of variable $z = \frac{s-t}{T-t}$, we obtain

$$\begin{aligned} (I_T^\rho \eta)(t) &= \frac{T^{-\lambda}(T-t)^{\lambda+\rho}}{\Gamma(\rho)} \int_0^1 z^{\rho-1}(1-z)^\lambda\, dz \\ &= \frac{T^{-\lambda}(T-t)^{\lambda+\rho}}{\Gamma(\rho)} B(\rho, \lambda+1), \end{aligned}$$

where B denotes the Beta function. Using the property (see e.g., [20])

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}, \quad a, b > 0,$$

we obtain

$$\begin{aligned} (I_T^\rho \eta)(t) &= \frac{T^{-\lambda}(T-t)^{\lambda+\rho}}{\Gamma(\rho)} \frac{\Gamma(\rho)\Gamma(\lambda+1)}{\Gamma(\rho+\lambda+1)} \\ &= \frac{\Gamma(\lambda+1)}{\Gamma(\rho+\lambda+1)} T^{-\lambda}(T-t)^{\rho+\lambda}, \end{aligned}$$

which proves (5).

Next, calculating the derivative of $I_T^\rho \eta$, we obtain

$$(I_T^\rho \eta)'(t) = -\frac{(\rho+\lambda)\Gamma(\lambda+1)}{\Gamma(\rho+\lambda+1)} T^{-\lambda}(T-t)^{\rho+\lambda-1}.$$

On the other hand, by the property (see e.g., [20])

$$\Gamma(a+1) = a\Gamma(a), \quad a > 0, \quad (8)$$

we obtain

$$\Gamma(\rho+\lambda+1) = (\rho+\lambda)\Gamma(\rho+\lambda).$$

Hence, we deduce that

$$(I_T^\rho \eta)'(t) = -\frac{\Gamma(\lambda+1)}{\Gamma(\rho+\lambda)} T^{-\lambda}(T-t)^{\rho+\lambda-1},$$

which proves (6).

Differentiating $(I_T^\rho \eta)'$ and using (8), we obtain

$$\begin{aligned}(I_T^\rho \eta)''(t) &= \frac{(\rho+\lambda-1)\Gamma(\lambda+1)}{\Gamma(\rho+\lambda)} T^{-\lambda}(T-t)^{\rho+\lambda-2} \\ &= \frac{(\rho+\lambda-1)\Gamma(\lambda+1)}{(\rho+\lambda-1)\Gamma(\rho+\lambda-1)} T^{-\lambda}(T-t)^{\rho+\lambda-2} \\ &= \frac{\Gamma(\lambda+1)}{\Gamma(\rho+\lambda-1)} T^{-\lambda}(T-t)^{\rho+\lambda-2},\end{aligned}$$

which proves (7). □

The following inequality will be useful later.

Lemma 3 (Young's Inequality with Epsilon, see [29], p. 36)**.** *Let $\varepsilon > 0$ and $p > 1$. Then, for all $a, b \geq 0$, there holds*

$$ab \leq \varepsilon a^p + C_{\varepsilon,p} b^{\frac{p}{p-1}},$$

where $C_{\varepsilon,p} = (p-1)p^{-1}(\varepsilon p)^{\frac{-1}{p-1}}$.

Remark 1. *For a function $u : (0,\infty) \times (0,L) \to \mathbb{R}$, the notation $\frac{\partial^\alpha u}{\partial t^\alpha}$ used in (2), where $1 < \alpha < 2$, means the following:*

$$\frac{\partial^\alpha u}{\partial t^\alpha}(t,x) = \left(I_0^{2-\alpha} \frac{\partial^2 u}{\partial t^2}(\cdot, x) \right)(t), \quad t > 0,\ 0 < x < L,$$

i.e.,

$$\frac{\partial^\alpha u}{\partial t^\alpha}(t,x) = \frac{1}{\Gamma(2-\alpha)} \int_a^t (t-s)^{1-\alpha} \frac{\partial^2 u}{\partial t^2}(s,x)\, ds.$$

Similarly, the notation $\frac{\partial^\beta u}{\partial t^\beta}$ used in (2), where $0 < \beta < 1$, means the following:

$$\frac{\partial^\beta u}{\partial t^\beta}(t,x) = \left(I_0^{1-\beta} \frac{\partial u}{\partial t}(\cdot, x) \right)(t), \quad t > 0,\ 0 < x < L,$$

i.e.,

$$\frac{\partial^\beta u}{\partial t^\beta}(t,x) = \frac{1}{\Gamma(1-\beta)} \int_a^t (t-s)^{-\beta} \frac{\partial u}{\partial t}(s,x)\, ds.$$

3. Statement of the Main Results

We first consider problem (1). Let

$$Q = [0,\infty) \times [0,L].$$

We introduce the test function space

$$\Phi = \left\{ \varphi \in C^2(Q) :\ \varphi \geq 0,\ \varphi(\cdot,0) = \varphi(\cdot,L) \equiv 0,\ \varphi(t,\cdot) \equiv 0 \text{ for sufficiently large } t \right\}.$$

Definition 1. *Let $u_0, u_1 \in L^1([0,L])$ and $f, g \in L^1_{loc}([0,\infty))$. We say that u is a global weak solution to (1), if*

(i) $x^\sigma |u|^p \in L^1_{loc}(Q)$, $u \in L^1_{loc}(Q)$;

(ii) *for every* $\varphi \in \Phi$,

$$\begin{aligned}
&\int_Q x^\sigma |u|^p \varphi \, dx\, dt + \int_0^\infty \left(f(t) \frac{\partial \varphi}{\partial x}(t,0) - g(t) \frac{\partial \varphi}{\partial x}(t,L) \right) dt \\
&+ \int_0^L \left(u_1(x)\varphi(0,x) - u_0(x) \frac{\partial \varphi}{\partial t}(0,x) + u_0(x)\varphi(0,x) \right) dx \\
&\leq -\int_Q u \frac{\partial^2 \varphi}{\partial x^2} \, dx\, dt + \int_Q u \frac{\partial^2 \varphi}{\partial t^2} \, dx\, dt - \int_Q u \frac{\partial \varphi}{\partial t} \, dx\, dt.
\end{aligned} \tag{9}$$

Remark 2. *The weak formulation* (9) *is obtained by multiplying the differential inequality in* (1) *by* φ*, integrating over* Q*, and using the initial conditions in* (1)*. So, clearly, any global solution to* (1) *is a global weak solution to* (1) *in the sense of Definition 1.*

We first consider the case $g \equiv 0$.

Theorem 1. *Let* $u_0, u_1 \in L^1([0,L])$*,* $f \in L^1_{loc}([0,\infty))$*, and* $g \equiv 0$*. Suppose that*

$$\int_0^L (u_0(x) + u_1(x))(L-x)\, dx > 0. \tag{10}$$

If

$$\sigma < -(p+1), \tag{11}$$

then (1) *admits no global weak solution.*

Remark 3. *Comparing with the existing results in the literature, in Theorem 1, it is not required that the initial data are sufficiently large with respect to a certain norm. The same remark holds for the next theorems.*

Example 1. *Consider problem* (1) *with*

$$f(t) = \frac{1}{\sqrt{t}}, \; t>0, \quad g \equiv 0, \quad u_0(x) = -(L-x), \quad u_1(x) = 2(L-x), \quad \sigma = -4, \quad p = 2.$$

Then, all the assumptions of Theorem 1 are satisfied. Consequently, we deduce that (1) *admits no global weak solution.*

Next, we consider the case when

$$g(t) = C_g t^\gamma, \quad \gamma > -1, \quad t > 0, \tag{12}$$

where $C_g > 0$ is a constant.

Theorem 2. *Let* $u_0, u_1 \in L^1([0,L])$*,* $f \in L^1_{loc}([0,\infty))$*, and* g *be the function defined by* (12)*. If one of the following conditions is satisfied:*

(i) $\sigma < -(p+1)$;
(ii) $\sigma \geq -(p+1)$, $\gamma > 0$,

then (1) *admits no global weak solution.*

Example 2. *Consider problem* (1) *with*

$$f(t) = \frac{e^t}{\sqrt{t}}, \; t>0, \quad u_0(x) = x, \quad u_1(x) = x^2, \quad g(t) = \sqrt{t}, \; t>0, \quad \sigma = -2, \quad p = 2.$$

Then, by the statement (ii) of Theorem 2, we deduce that (1) *admits no global weak solution.*

Consider now problem (2). For all $T > 0$, let

$$Q_T = [0, T] \times [0, L].$$

We introduce the test function space

$$\Phi_T = \left\{ \varphi \in C^2(Q_T) : \varphi \geq 0,\, \varphi(\cdot, 0) = \varphi(\cdot, L) \equiv 0,\, \frac{\partial (I_T^{2-\alpha}\varphi)}{\partial t}(T, \cdot) \equiv 0 \right\}.$$

Definition 2. *Let $u_0, u_1 \in L^1([0, L])$ and $f, g \in L^1_{loc}([0, \infty))$. We say that u is a global weak solution to* (2)*, if*

(i) $x^\sigma |u|^p \in L^1_{loc}(Q)$, $u \in L^1_{loc}(Q)$;
(ii) *for all $T > 0$ and $\varphi \in \Phi_T$,*

$$\begin{aligned} &\int_{Q_T} x^\sigma |u|^p \varphi \, dx\, dt + \int_0^T \left(f(t) \frac{\partial \varphi}{\partial x}(t, 0) - g(t) \frac{\partial \varphi}{\partial x}(t, L) \right) dt \\ &+ \int_0^L \left(u_1(x)(I_T^{2-\alpha}\varphi)(0, x) - u_0(x) \frac{\partial (I_T^{2-\alpha}\varphi)}{\partial t}(0, x) + u_0(x)(I_T^{1-\beta}\varphi)(0, x) \right) dx \\ &\leq - \int_{Q_T} u \frac{\partial^2 \varphi}{\partial x^2}\, dx\, dt + \int_{Q_T} u \frac{\partial^2 (I_T^{2-\alpha}\varphi)}{\partial t^2}\, dx\, dt - \int_{Q_T} u \frac{\partial (I_T^{1-\beta}\varphi)}{\partial t}\, dx\, dt. \end{aligned} \tag{13}$$

Remark 4. *The weak formulation* (13) *is obtained by multiplying the differential inequality in* (2) *by φ, integrating over Q_T, using the initial conditions in* (2)*, and using the fractional integration by parts rule provided by Lemma* 1*. So, clearly, any global solution to* (2) *is a global weak solution to* (2) *in the sense of Definition* 2*.*

As for problem (1), we first consider the case $g \equiv 0$.

Theorem 3. *Let $u_0, u_1 \in L^1([0, L])$, $f \in L^1_{loc}([0, \infty))$, and $g \equiv 0$. If*

$$\sigma < -(p + 1),$$

and one of the following conditions is satisfied:

$$\alpha < \beta + 1, \quad \int_0^L u_1(x)(L - x)\, dx > 0; \tag{14}$$

$$\alpha = \beta + 1, \quad \int_0^L (u_0(x) + u_1(x))(L - x)\, dx > 0; \tag{15}$$

$$\alpha > \beta + 1, \quad \int_0^L u_0(x)(L - x)\, dx > 0, \tag{16}$$

then (2) *admits no global weak solution.*

Example 3. *Consider problem* (2) *with*

$$f(t) = \frac{1}{\sqrt{t}},\ t > 0, \quad u_0 \equiv 0, \quad u_1(x) = 2(L - x), \quad \alpha = \frac{3}{2}, \quad \beta = \frac{3}{4}, \quad \sigma = -4, \quad p = 2.$$

Since (14) *is satisfied and $\sigma < -(p + 1)$, by Theorem* 3*, we deduce that* (2) *admits no global weak solution.*

Next, we consider the inhomogeneous case, where the function g is given by (12).

Theorem 4. *Let $u_0, u_1 \in L^1([0,L])$, $f \in L^1_{loc}([0,\infty))$, and g be the function defined by (12). If*

$$\alpha > \max\{1-\gamma, 1\}, \quad \beta > \max\{-\gamma, 0\}, \tag{17}$$

and one of the following conditions is satisfied:

(i) $\sigma < -(p+1)$;

(ii) $\sigma \geq -(p+1)$, $\gamma > 0$,

then (2) *admits no global weak solution.*

Example 4. *Consider problem* (2) *with*

$$f(t) = \frac{1}{\sqrt{t}},\ t>0, \quad u_0(x) = -x, \quad u_1(x) = x^2, \quad g(t) = t^{\frac{2}{3}},\ t>0, \quad \alpha = \frac{3}{2}, \quad \beta = \frac{1}{2},$$

and

$$\sigma = -3, \quad p = 3.$$

Then (17) *is satisfied, $\sigma \geq -(p+1)$, and $\gamma > 0$. Then, by Theorem 4, we deduce that* (2) *admits no global weak solution.*

4. Proof of the Main Results

Throughout this section, any positive constant independent on T and R, is denoted by C. Namely, in the proofs, we use several asymptotic estimates as $T \to \infty$ and $R \to \infty$; therefore, the value of any positive constant independent of T and R has no influence in our analysis.

4.1. Proof of Theorem 1

Proof. Suppose that u is a global weak solution to (1). Then, by (9), for every $\varphi \in \Phi$, there holds

$$\begin{aligned} &\int_Q x^\sigma |u|^p \varphi\,dx\,dt + \int_0^\infty \left(f(t)\frac{\partial \varphi}{\partial x}(t,0) - g(t)\frac{\partial \varphi}{\partial x}(t,L)\right) dt \\ &+ \int_0^L \left(u_1(x)\varphi(0,x) - u_0(x)\frac{\partial \varphi}{\partial t}(0,x) + u_0(x)\varphi(0,x)\right) dx \\ &\leq \int_Q |u| \left|\frac{\partial^2 \varphi}{\partial x^2}\right| dx\,dt + \int_Q |u| \left|\frac{\partial^2 \varphi}{\partial t^2}\right| dx\,dt + \int_Q |u| \left|\frac{\partial \varphi}{\partial t}\right| dx\,dt. \end{aligned} \tag{18}$$

On the other hand, using Lemma 3 with $\varepsilon = \frac{1}{3}$ and adequate choices of a and b, we obtain

$$\int_Q |u| \left|\frac{\partial^2 \varphi}{\partial x^2}\right| dx\,dt \leq \frac{1}{3}\int_Q x^\sigma |u|^p \varphi\,dx\,dt + C\int_Q x^{\frac{-\sigma}{p-1}} \varphi^{\frac{-1}{p-1}} \left|\frac{\partial^2 \varphi}{\partial x^2}\right|^{\frac{p}{p-1}} dx\,dt, \tag{19}$$

$$\int_Q |u| \left|\frac{\partial^2 \varphi}{\partial t^2}\right| dx\,dt \leq \frac{1}{3}\int_Q x^\sigma |u|^p \varphi\,dx\,dt + C\int_Q x^{\frac{-\sigma}{p-1}} \varphi^{\frac{-1}{p-1}} \left|\frac{\partial^2 \varphi}{\partial t^2}\right|^{\frac{p}{p-1}} dx\,dt, \tag{20}$$

$$\int_Q |u| \left|\frac{\partial \varphi}{\partial t}\right| dx\,dt \leq \frac{1}{3}\int_Q x^\sigma |u|^p \varphi\,dx\,dt + C\int_Q x^{\frac{-\sigma}{p-1}} \varphi^{\frac{-1}{p-1}} \left|\frac{\partial \varphi}{\partial t}\right|^{\frac{p}{p-1}} dx\,dt. \tag{21}$$

Using (18)–(21), we obtain

$$\begin{aligned}
&\int_0^\infty \left(f(t)\frac{\partial \varphi}{\partial x}(t,0) - g(t)\frac{\partial \varphi}{\partial x}(t,L) \right) dt \\
&+ \int_0^L \left(u_1(x)\varphi(0,x) - u_0(x)\frac{\partial \varphi}{\partial t}(0,x) + u_0(x)\varphi(0,x) \right) dx \\
&\le C\sum_{j=1}^{3} I_j(\varphi),
\end{aligned} \tag{22}$$

where

$$\begin{aligned}
I_1(\varphi) &= \int_Q x^{\frac{-\sigma}{p-1}} \varphi^{\frac{-1}{p-1}} \left| \frac{\partial^2 \varphi}{\partial x^2} \right|^{\frac{p}{p-1}}, \\
I_2(\varphi) &= \int_Q x^{\frac{-\sigma}{p-1}} \varphi^{\frac{-1}{p-1}} \left| \frac{\partial^2 \varphi}{\partial t^2} \right|^{\frac{p}{p-1}}, \\
I_3(\varphi) &= \int_Q x^{\frac{-\sigma}{p-1}} \varphi^{\frac{-1}{p-1}} \left| \frac{\partial \varphi}{\partial t} \right|^{\frac{p}{p-1}}.
\end{aligned}$$

Consider now two cut-off functions $\xi, \mu \in C^\infty([0,\infty))$ satisfying the following properties:

$$0 \le \xi, \mu \le 1, \quad \xi(s) = \begin{cases} 1 & \text{if} \quad 0 \le s \le \frac{1}{2} \\ 0 & \text{if} \quad s \ge 1 \end{cases}, \quad \mu(s) = \begin{cases} 0 & \text{if} \quad 0 \le s \le \frac{1}{2} \\ 1 & \text{if} \quad s \ge 1 \end{cases}.$$

For sufficiently large ℓ and R, let

$$\varphi_1(t) = \xi^\ell(R^{-\theta} t), \quad \varphi_2(x) = (L-x)\mu^\ell(Rx), \quad t \ge 0,\ x \in [0,L], \tag{23}$$

where $\theta > 0$ is a constant that will be determined later. Consider the function

$$\varphi(t,x) = \varphi_1(t)\varphi_2(x), \quad t \ge 0,\ x \in [0,L]. \tag{24}$$

By the properties of the cut-off functions ξ and μ, it can be easily seen that the function φ defined by (24), belongs to Φ. Thus, the estimate (22) holds for this function.

Now, let us estimate the terms $I_j(\varphi)$, $j = 1,2,3$. For $j = 1$, by (24), we obtain

$$I_1(\varphi) = \left(\int_0^\infty \varphi_1(t)\, dt \right) \left(\int_0^L x^{\frac{-\sigma}{p-1}} \varphi_2^{\frac{-1}{p-1}}(x) |\varphi_2''(x)|^{\frac{p}{p-1}}\, dx \right) := I_1^{(1)}(\varphi_1) I_1^{(2)}(\varphi_2). \tag{25}$$

On the other hand, by the definitions of the function φ_1 and the cut-off function ξ, there holds

$$\begin{aligned}
I_1^{(1)}(\varphi_1) &= \int_0^\infty \xi^\ell \left(R^{-\theta} t \right) dt \\
&= \int_0^{R^\theta} \xi^\ell \left(R^{-\theta} t \right) dt \\
&\le R^\theta.
\end{aligned} \tag{26}$$

By the definitions of the function φ_2 and the cut-off function μ, we obtain

$$\begin{aligned}
\varphi_2''(x) &= \ell R^2 \mu^{\ell-2}(Rx) \times \\
&\quad \left[(L-x)\left((\ell-1)\mu'^2(Rx) + \mu(Rx)\mu''(Rx) \right) - 2R^{-1}\mu(Rx)\mu'(Rx) \right] \chi_{[\frac{1}{2}R^{-1}, R^{-1}]}(x),
\end{aligned}$$

which yields

$$|\varphi_2''(x)| \le CR^2 \mu^{\ell-2}(Rx) \chi_{[\frac{1}{2}R^{-1}, R^{-1}]}(x),$$

where $\chi_{\left[\frac{1}{2}R^{-1},R^{-1}\right]}$ is the indicator function of the interval $\left[\frac{1}{2}R^{-1},R^{-1}\right]$. Then, there holds

$$\begin{aligned} I_1^{(2)}(\varphi_2) &\leq CR^{\frac{2p}{p-1}}\int_{\frac{1}{2}R^{-1}}^{R^{-1}} x^{\frac{-\sigma}{p-1}}(L-x)^{\frac{-1}{p-1}}\mu^{\ell-\frac{2p}{p-1}}(Rx)\,dx \\ &\leq CR^{\frac{2p}{p-1}}\int_{\frac{1}{2}R^{-1}}^{R^{-1}} x^{\frac{-\sigma}{p-1}}\,dx \\ &\leq CR^{\frac{\sigma}{p-1}+\frac{2p}{p-1}-1}. \end{aligned} \tag{27}$$

Thus, it follows from (25)–(27) that

$$I_1(\varphi) \leq CR^{\theta+\frac{p+1+\sigma}{p-1}}. \tag{28}$$

For $j=2$, $I_j(\varphi)$ can be written as

$$I_2(\varphi) = \left(\int_0^\infty \varphi_1^{\frac{-1}{p-1}}(t)|\varphi_1''(t)|^{\frac{p}{p-1}}\,dt\right)\left(\int_0^L x^{\frac{-\sigma}{p-1}}\varphi_2(x)\,dx\right) := I_2^{(1)}(\varphi_1)I_2^{(2)}(\varphi_2). \tag{29}$$

By the definitions of the function φ_1 and the cut-off function ξ, we obtain

$$\varphi_1''(t) = \ell R^{-2\theta}\xi^{\ell-2}(R^{-\theta}t)\left[(\ell-1)\xi'^2(R^{-\theta}t)+\xi^{\ell-1}(R^{-\theta}t)\xi''(R^{-\theta}t)\right]\chi_{\left[\frac{1}{2}R^{\theta},R^{\theta}\right]}(t),$$

which yields

$$|\varphi_1''(t)| \leq CR^{-2\theta}\xi^{\ell-2}(R^{-\theta}t)\chi_{\left[\frac{1}{2}R^{\theta},R^{\theta}\right]}(t).$$

Thus, there holds

$$\begin{aligned} I_2^{(1)}(\varphi_1) &\leq CR^{\frac{-2\theta p}{p-1}}\int_{\frac{1}{2}R^{\theta}}^{R^{\theta}} \xi^{\ell-\frac{2p}{p-1}}(R^{-\theta}t)\,dt \\ &\leq CR^{\theta\left(1-\frac{2p}{p-1}\right)}. \end{aligned} \tag{30}$$

Moreover, we have

$$\begin{aligned} I_2^{(2)}(\varphi_2) &= \int_0^L x^{\frac{-\sigma}{p-1}}\varphi_2(x)\,dx \\ &= \int_{\frac{1}{2}R^{-1}}^{L} x^{\frac{-\sigma}{p-1}}(L-x)\mu^{\ell}(Rx)\,dx \\ &\leq C\int_{\frac{1}{2}R^{-1}}^{L} x^{\frac{-\sigma}{p-1}}\,dx. \end{aligned}$$

On the other hand, by (11), we have $\sigma < p-1$, thus we deduce that

$$I_2^{(2)}(\varphi_2) \leq C. \tag{31}$$

Combining (29)–(31), there holds

$$I_2(\varphi) \leq CR^{\theta\left(1-\frac{2p}{p-1}\right)}. \tag{32}$$

Now, let us estimate $I_3(\varphi)$. This term can be written as

$$I_3(\varphi) = \left(\int_0^\infty \varphi_1^{\frac{-1}{p-1}}(t)|\varphi_1'(t)|^{\frac{p}{p-1}}\,dt\right)\left(\int_0^L x^{\frac{-\sigma}{p-1}}\varphi_2(x)\,dx\right) := I_3^{(1)}(\varphi_1)I_3^{(2)}(\varphi_2). \tag{33}$$

A similar calculation as above yields

$$I_3^{(1)}(\varphi_1) \leq C R^{\theta\left(1-\frac{p}{p-1}\right)}. \tag{34}$$

Observe that $I_3^{(2)}(\varphi_2) = I_2^{(2)}(\varphi_2)$. Thus, by (31), (33), and (34), we obtain

$$I_3(\varphi) \leq C R^{\theta\left(1-\frac{p}{p-1}\right)}. \tag{35}$$

Next, combining (28), (32), and (35), we obtain

$$\sum_{j=1}^{3} I_j(\varphi) \leq C\left(R^{\theta+\frac{p+1+\sigma}{p-1}} + R^{\theta\left(1-\frac{p}{p-1}\right)}\right). \tag{36}$$

Let θ be such that

$$\theta + \frac{p+1+\sigma}{p-1} = \theta\left(1 - \frac{p}{p-1}\right),$$

that is,

$$\theta = \frac{-(p+1)-\sigma}{p}.$$

Notice that by (11), we have $\theta > 0$. Then, (36) reduces to

$$\sum_{j=1}^{3} I_j(\varphi) \leq C R^{\theta\left(1-\frac{p}{p-1}\right)}. \tag{37}$$

Next, let us estimate the terms from the right side of (22). Observe that by the definition of the function φ, and the properties of the cut-off function μ, we have

$$\frac{\partial \varphi}{\partial x}(t,0) = 0, \quad t > 0.$$

Moreover, since $g \equiv 0$, there holds

$$\int_0^\infty \left(f(t)\frac{\partial \varphi}{\partial x}(t,0) - g(t)\frac{\partial \varphi}{\partial x}(t,L)\right) dt = 0. \tag{38}$$

By the properties of the cut-off function ξ, we have

$$\varphi(0,x) = \varphi_2(x), \quad \frac{\partial \varphi}{\partial t}(0,x) = 0, \quad x \in (0,L).$$

Thus, we obtain

$$\begin{aligned}
&\int_0^L \left(u_1(x)\varphi(0,x) - u_0(x)\frac{\partial \varphi}{\partial t}(0,x) + u_0(x)\varphi(0,x)\right) dx \\
&= \int_0^L (u_0(x) + u_1(x))\varphi(0,x)\, dx \\
&= \int_0^L (u_0(x) + u_1(x))\varphi_2(x)\, dx \\
&= \int_0^L (u_0(x) + u_1(x))(L-x)\mu^\ell(Rx)\, dx.
\end{aligned}$$

Then, taking into consideration that $u_0, u_1 \in L^1([0,L])$, by the dominated convergence theorem, we obtain

$$\begin{aligned} &\lim_{R\to\infty} \int_0^L \left(u_1(x)\varphi(0,x) - u_0(x)\frac{\partial\varphi}{\partial t}(0,x) + u_0(x)\varphi(0,x) \right) dx \\ &= \int_0^L (u_0(x)+u_1(x))(L-x)\,dx. \end{aligned} \tag{39}$$

Hence, by (10), for sufficiently large R, there holds

$$\int_0^L \left(u_1(x)\varphi(0,x) - u_0(x)\tfrac{\partial\varphi}{\partial t}(0,x) + u_0(x)\varphi(0,x) \right) dx \geq \tfrac{1}{2}\int_0^L (u_0(x)+u_1(x))(L-x)\,dx. \tag{40}$$

Next, combining (22), (37), (38), and (40), we obtain

$$\frac{1}{2}\int_0^L (u_0(x)+u_1(x))(L-x)\,dx \leq CR^{\theta\left(1-\frac{p}{p-1}\right)}.$$

Passing to the limit as $R \to \infty$ in the above inequality, we obtain

$$\frac{1}{2}\int_0^L (u_0(x)+u_1(x))(L-x)\,dx \leq 0,$$

which contradicts (10). Consequently, (1) admits no global weak solution. The proof is completed. □

4.2. *Proof of Theorem 2*

Proof. As was performed previously, suppose that u is a global weak solution to (1). From the proof of Theorem 1, for sufficiently large R, there holds

$$\begin{aligned} &-\int_0^\infty g(t)\frac{\partial\varphi}{\partial x}(t,L)\,dt \\ &+\int_0^L \left(u_1(x)\varphi(0,x) - u_0(x)\frac{\partial\varphi}{\partial t}(0,x) + u_0(x)\varphi(0,x) \right) dx \\ &\leq C\left(R^{\theta+\frac{p+1+\sigma}{p-1}} + R^{\theta\left(1-\frac{p}{p-1}\right)} \int_{\frac{1}{2}R^{-1}}^L x^{\frac{-\sigma}{p-1}}\,dx \right), \end{aligned} \tag{41}$$

where $\theta > 0$ and φ is the function defined by (24). On the other hand, by the definition of the function φ, for sufficiently large R, there holds

$$\frac{\partial\varphi}{\partial x}(t,L) = -\varphi_1(t), \quad t > 0,$$

which yields

$$\begin{aligned} -\int_0^\infty g(t)\frac{\partial\varphi}{\partial x}(t,L)\,dt &= \int_0^\infty g(t)\varphi_1(t)\,dt \\ &= C\int_0^\infty t^\gamma \xi^\ell(R^{-\theta}t)\,dt \\ &\geq C\int_0^{\frac{1}{2}R^\theta} t^\gamma\,dt \\ &= CR^{\theta(\gamma+1)}. \end{aligned}$$

Then, by (41), we deduce that

$$\begin{aligned} &C+R^{-\theta(\gamma+1)}\int_0^L\left(u_1(x)\varphi(0,x)-u_0(x)\frac{\partial\varphi}{\partial t}(0,x)+u_0(x)\varphi(0,x)\right)dx \\ &\leq C\left(R^{-\theta\gamma+\frac{p+1+\sigma}{p-1}}+R^{-\theta\left(\gamma+\frac{p}{p-1}\right)}\int_{\frac{1}{2}R^{-1}}^L x^{\frac{-\sigma}{p-1}}\,dx\right). \end{aligned} \tag{42}$$

Let $\sigma<-(p+1)$. In this case, (42) reduces to

$$\begin{aligned} &C+R^{-\theta(\gamma+1)}\int_0^L\left(u_1(x)\varphi(0,x)-u_0(x)\frac{\partial\varphi}{\partial t}(0,x)+u_0(x)\varphi(0,x)\right)dx \\ &\leq C\left(R^{-\theta\gamma+\frac{p+1+\sigma}{p-1}}+R^{-\theta\left(\gamma+\frac{p}{p-1}\right)}\right). \end{aligned} \tag{43}$$

Taking $\theta>0$ so that

$$\theta\gamma>\frac{p+1+\sigma}{p-1}, \tag{44}$$

passing to the limit as $R\to\infty$ in (43), and using (39), we obtain a contradiction with $C>0$. This proves part (i) of Theorem 2.

Let $\sigma\geq-(p+1)$ and $\gamma>0$.

If $-(p+1)\leq\sigma<p-1$, then (43) holds. Since $\gamma>0$, there exists $\theta>0$ such that (44) holds. Thus, passing to the limit as $R\to\infty$ in (43), we obtain a contradiction.
If $\sigma=p-1$, then (42) yields

$$\begin{aligned} &C+R^{-\theta(\gamma+1)}\int_0^L\left(u_1(x)\varphi(0,x)-u_0(x)\frac{\partial\varphi}{\partial t}(0,x)+u_0(x)\varphi(0,x)\right)dx \\ &\leq C\left(R^{-\theta\gamma+\frac{p+1+\sigma}{p-1}}+R^{-\theta\left(\gamma+\frac{p}{p-1}\right)}\ln R\right). \end{aligned}$$

As in the previous case, since $\gamma>0$, there exists $\theta>0$ such that (44) holds. Thus, passing to the limit as $R\to\infty$ in the above inequality, we obtain a contradiction.
If $\sigma>p-1$, then (42) yields

$$\begin{aligned} &C+R^{-\theta(\gamma+1)}\int_0^L\left(u_1(x)\varphi(0,x)-u_0(x)\frac{\partial\varphi}{\partial t}(0,x)+u_0(x)\varphi(0,x)\right)dx \\ &\leq C\left(R^{-\theta\gamma+\frac{p+1+\sigma}{p-1}}+R^{-\theta\left(\gamma+\frac{p}{p-1}\right)+\frac{\sigma}{p-1}-1}\right). \end{aligned}$$

Taking θ such that (44) is satisfied, and passing to the limit as $R\to\infty$ in the above inequality, a contradiction follows. Thus, part (ii) of Theorem 2 is proved. □

4.3. Proof of Theorem 3

Proof. Suppose that u is a global weak solution to (2). Then, by (13), for every $T>0$ and $\varphi\in\Phi_T$, there holds

$$\begin{aligned} &\int_{Q_T}x^\sigma|u|^p\varphi\,dx\,dt+\int_0^T\left(f(t)\frac{\partial\varphi}{\partial x}(t,0)-g(t)\frac{\partial\varphi}{\partial x}(t,L)\right)dt \\ &+\int_0^L\left(u_1(x)(I_T^{2-\alpha}\varphi)(0,x)-u_0(x)\frac{\partial(I_T^{2-\alpha}\varphi)}{\partial t}(0,x)+u_0(x)(I_T^{1-\beta}\varphi)(0,x)\right)dx \\ &\leq\int_{Q_T}|u|\left|\frac{\partial^2\varphi}{\partial x^2}\right|dx\,dt+\int_{Q_T}|u|\left|\frac{\partial^2(I_T^{2-\alpha}\varphi)}{\partial t^2}\right|dx\,dt+\int_{Q_T}|u|\left|\frac{\partial(I_T^{1-\beta}\varphi)}{\partial t}\right|dx\,dt. \end{aligned} \tag{45}$$

On the other hand, using Lemma 3 with $\varepsilon = \frac{1}{3}$ and adequate choices of a and b, we obtain

$$\begin{aligned} &\int_{Q_T} |u| \left| \frac{\partial^2 \varphi}{\partial x^2} \right| dx\,dt \\ &\leq \frac{1}{3} \int_{Q_T} x^{\sigma} |u|^p \varphi\, dx\,dt + C \int_{Q_T} x^{\frac{-\sigma}{p-1}} \varphi^{\frac{-1}{p-1}} \left| \frac{\partial^2 \varphi}{\partial x^2} \right|^{\frac{p}{p-1}} dx\,dt, \end{aligned} \tag{46}$$

$$\begin{aligned} &\int_{Q_T} |u| \left| \frac{\partial^2 (I_T^{2-\alpha}\varphi)}{\partial t^2} \right| dx\,dt \\ &\leq \frac{1}{3} \int_{Q_T} x^{\sigma} |u|^p \varphi\, dx\,dt + C \int_{Q_T} x^{\frac{-\sigma}{p-1}} \varphi^{\frac{-1}{p-1}} \left| \frac{\partial^2 (I_T^{2-\alpha}\varphi)}{\partial t^2} \right|^{\frac{p}{p-1}} dx\,dt, \end{aligned} \tag{47}$$

and

$$\begin{aligned} &\int_{Q_T} |u| \left| \frac{\partial (I_T^{1-\beta}\varphi)}{\partial t} \right| dx\,dt \\ &\leq \frac{1}{3} \int_{Q_T} x^{\sigma} |u|^p \varphi\, dx\,dt + C \int_{Q_T} x^{\frac{-\sigma}{p-1}} \varphi^{\frac{-1}{p-1}} \left| \frac{\partial (I_T^{1-\beta}\varphi)}{\partial t} \right|^{\frac{p}{p-1}} dx\,dt. \end{aligned} \tag{48}$$

Using (45)–(48), we obtain

$$\begin{aligned} &\int_0^T \left(f(t) \frac{\partial \varphi}{\partial x}(t,0) - g(t) \frac{\partial \varphi}{\partial x}(t,L) \right) dt \\ &+ \int_0^L \left(u_1(x) (I_T^{2-\alpha}\varphi)(0,x) - u_0(x) \frac{\partial (I_T^{2-\alpha}\varphi)}{\partial t}(0,x) + u_0(x) (I_T^{1-\beta}\varphi)(0,x) \right) dx \\ &\leq \sum_{j=1}^{3} J_j(\varphi), \end{aligned} \tag{49}$$

where

$$\begin{aligned} J_1(\varphi) &= \int_{Q_T} x^{\frac{-\sigma}{p-1}} \varphi^{\frac{-1}{p-1}} \left| \frac{\partial^2 \varphi}{\partial x^2} \right|^{\frac{p}{p-1}} dx\,dt, \\ J_2(\varphi) &= \int_{Q_T} x^{\frac{-\sigma}{p-1}} \varphi^{\frac{-1}{p-1}} \left| \frac{\partial^2 (I_T^{2-\alpha}\varphi)}{\partial t^2} \right|^{\frac{p}{p-1}} dx\,dt, \\ J_3(\varphi) &= \int_{Q_T} x^{\frac{-\sigma}{p-1}} \varphi^{\frac{-1}{p-1}} \left| \frac{\partial (I_T^{1-\beta}\varphi)}{\partial t} \right|^{\frac{p}{p-1}} dx\,dt. \end{aligned}$$

For sufficiently large T, λ, ℓ, and R, let

$$\varphi(t,x) = \eta(t)\varphi_2(x), \quad t \geq 0,\ x \in [0,L], \tag{50}$$

where η is the function defined by (4), and φ_2 is the function given by (23). Using Lemma 2 and the properties of the cut-off function μ, it can be easily seen that the function φ defined by (50), belongs to Φ_T. Thus, (49) holds for this function.

Let us estimate the terms $J_j(\varphi)$, $j = 1,2,3$. For $j = 1$, by (50), we have

$$J_1(\varphi) = \left(\int_0^T \eta(t)\, dt \right) \left(\int_0^L x^{\frac{-\sigma}{p-1}} \varphi_2^{\frac{-1}{p-1}}(x) |\varphi_2''(x)|^{\frac{p}{p-1}}\, dx \right). \tag{51}$$

An elementary calculation shows that

$$\int_0^T \eta(t)\,dt = \frac{T}{\lambda+1}. \tag{52}$$

Hence, using (27), (51), and (52), we obtain

$$J_1(\varphi) \leq CTR^{\frac{\sigma+2p}{p-1}-1}. \tag{53}$$

For $j=2$, we have

$$J_2(\varphi) = \left(\int_0^T \eta^{\frac{-1}{p-1}}(t)|(I_T^{2-\alpha}\eta)''(t)|^{\frac{p}{p-1}}\,dt\right)\left(\int_0^L x^{\frac{-\sigma}{p-1}}\varphi_2(x)\,dx\right). \tag{54}$$

Moreover, by Lemma 2, we obtain

$$\eta^{\frac{-1}{p-1}}(t)|(I_T^{2-\alpha}\eta)''(t)|^{\frac{p}{p-1}} = \left[\frac{\Gamma(\lambda+1)}{\Gamma(1-\alpha+\lambda)}\right]^{\frac{p}{p-1}} T^{-\lambda}(T-t)^{\lambda-\frac{\alpha p}{p-1}}.$$

Integrating over $(0,T)$, there holds

$$\int_0^T \eta^{\frac{-1}{p-1}}(t)|(I_T^{2-\alpha}\eta)''(t)|^{\frac{p}{p-1}}\,dt = CT^{\frac{-\alpha p}{p-1}+1}. \tag{55}$$

Next, taking into consideration that $\sigma < -(p+1)$ (so $\sigma < p-1$), it follows from (31), (54), and (55) that

$$J_2(\varphi) \leq CT^{1-\frac{\alpha p}{p-1}}. \tag{56}$$

Proceeding as above, we obtain

$$J_3(\varphi) \leq CT^{1-\frac{\beta p}{p-1}}. \tag{57}$$

Hence, by (53), (56), and (57), we obtain

$$\sum_{j=1}^{3} J_j(\varphi) \leq C\left(TR^{\frac{\sigma+2p}{p-1}-1} + T^{1-\frac{\beta p}{p-1}}\right). \tag{58}$$

Consider now the terms from the right side of (49). By (50) and the properties of the cut-off function μ, since $g \equiv 0$, there holds

$$\int_0^T \left(f(t)\frac{\partial\varphi}{\partial x}(t,0) - g(t)\frac{\partial\varphi}{\partial x}(t,L)\right)dt = 0. \tag{59}$$

On the other hand, using (50) and Lemma 2, for all $x \in [0,L]$, we obtain

$$\begin{array}{rclcl}
(I_T^{2-\alpha}\varphi)(0,x) & = & \frac{\Gamma(\lambda+1)}{\Gamma(3-\alpha+\lambda)}T^{2-\alpha}\varphi_2(x) & := & C_1T^{2-\alpha}\varphi_2(x), \\
\frac{\partial(I_T^{2-\alpha}\varphi)}{\partial t}(0,x) & = & -\frac{\Gamma(\lambda+1)}{\Gamma(2-\alpha+\lambda)}T^{1-\alpha}\varphi_2(x) & := & -C_2T^{1-\alpha}\varphi_2(x), \\
(I_T^{1-\beta}\varphi)(0,x) & = & \frac{\Gamma(\lambda+1)}{\Gamma(2-\beta+\lambda)}T^{1-\beta}\varphi_2(x) & := & C_3T^{1-\beta}\varphi_2(x).
\end{array}$$

Consequently, we obtain

$$\begin{aligned}
&\int_0^L \left(u_1(x)(I_T^{2-\alpha}\varphi)(0,x) - u_0(x)\frac{\partial(I_T^{2-\alpha}\varphi)}{\partial t}(0,x) + u_0(x)(I_T^{1-\beta}\varphi)(0,x) \right) dx \\
&= \int_0^L \Big(C_1 T^{2-\alpha} u_1(x) + C_2 T^{1-\alpha} u_0(x) + C_3 T^{1-\beta} u_0(x) \Big) \varphi_2(x)\, dx \\
&= \int_0^L \Big(C_1 T^{2-\alpha} u_1(x) + C_2 T^{1-\alpha} u_0(x) + C_3 T^{1-\beta} u_0(x) \Big) (L-x)\mu^\ell(Rx)\, dx.
\end{aligned} \tag{60}$$

Thus, combining (49), (58)–(60), we obtain

$$\begin{aligned}
&\int_0^L \Big(C_1 T^{2-\alpha} u_1(x) + C_2 T^{1-\alpha} u_0(x) + C_3 T^{1-\beta} u_0(x) \Big) (L-x)\mu^\ell(Rx)\, dx \\
&\le C \left(T R^{\frac{\sigma+2p}{p-1}-1} + T^{1-\frac{\beta p}{p-1}} \right).
\end{aligned}$$

Next, taking $T = R^\theta$, where $\theta > 0$ is a constant that will be determined later, the above inequality reduces to

$$\begin{aligned}
&\int_0^L \Big(C_1 R^{\theta(2-\alpha)} u_1(x) + C_2 R^{\theta(1-\alpha)} u_0(x) + C_3 R^{\theta(1-\beta)} u_0(x) \Big) (L-x)\mu^\ell(Rx)\, dx \\
&\le C \left(R^{\theta+\frac{\sigma+2p}{p-1}-1} + R^{\theta\left(1-\frac{\beta p}{p-1}\right)} \right).
\end{aligned} \tag{61}$$

Suppose that (14) holds. In this case, we obtain

$$\begin{aligned}
&\lim_{R\to\infty} R^{-\theta(2-\alpha)} \int_0^L \Big(C_1 R^{\theta(2-\alpha)} u_1(x) + C_2 R^{\theta(1-\alpha)} u_0(x) + C_3 R^{\theta(1-\beta)} u_0(x) \Big) (L-x)\mu^\ell(Rx)\, dx \\
&= C_1 \int_0^L u_1(x)(L-x)\, dx \\
&> 0.
\end{aligned}$$

Hence, for sufficiently large R,

$$\int_0^L \Big(C_1 R^{\theta(2-\alpha)} u_1(x) + C_2 R^{\theta(1-\alpha)} u_0(x) + C_3 R^{\theta(1-\beta)} u_0(x) \Big) (L-x)\mu^\ell(Rx)\, dx \ge C R^{\theta(2-\alpha)}. \tag{62}$$

Combining (61) with (62), we obtain

$$C \le R^{\theta(\alpha-1)+\frac{\sigma+2p}{p-1}-1} + R^{\theta\left(\alpha-\frac{\beta p}{p-1}-1\right)}. \tag{63}$$

Observe that, since $\alpha < \beta + 1$, we have

$$\alpha - \frac{\beta p}{p-1} - 1 < 0.$$

Hence, taking into consideration that $\sigma < -(p+1)$, picking $\theta > 0$ so that

$$\theta < \frac{-(p+1)-\sigma}{(p-1)(\alpha-1)},$$

and passing to the limit as $R \to \infty$ in (63), we obtain a contradiction with $C > 0$.

Suppose that (15) holds. Then,

$$(I_T^{2-\alpha}\varphi)(0,x) = (I_T^{1-\beta}\varphi)(0,x).$$

Thus, (61) reduces to

$$\begin{aligned}
&\int_0^L \left(C_1 R^{\theta(2-\alpha)}(u_0(x)+u_1(x)) + C_2 R^{\theta(1-\alpha)} u_0(x)\right)(L-x)\mu^\ell(Rx)\,dx \\
&\leq C\left(R^{\theta+\frac{\sigma+2p}{p-1}-1} + R^{\theta\left(1-\frac{\beta p}{p-1}\right)}\right).
\end{aligned} \tag{64}$$

Moreover, we have

$$\begin{aligned}
&\lim_{R\to\infty} R^{-\theta(2-\alpha)} \int_0^L \left(C_1 R^{\theta(2-\alpha)}(u_0(x)+u_1(x)) + C_2 R^{\theta(1-\alpha)} u_0(x)\right)(L-x)\mu^\ell(Rx)\,dx \\
&= C_1 \int_0^L (u_0(x)+u_1(x))(L-x)\,dx \\
&> 0,
\end{aligned}$$

which yields

$$\int_0^L \left(C_1 R^{\theta(2-\alpha)}(u_0(x)+u_1(x)) + C_2 R^{\theta(1-\alpha)} u_0(x)\right)(L-x)\mu^\ell(Rx)\,dx \geq C R^{\theta(2-\alpha)},$$

for sufficiently large R. Hence, using (64), and following the same argument as above, a contradiction follows.

Finally, suppose that (16) holds. In this case, we obtain

$$\begin{aligned}
&\lim_{R\to\infty} R^{-\theta(1-\beta)} \int_0^L \left(C_1 R^{\theta(2-\alpha)} u_1(x) + C_2 R^{\theta(1-\alpha)} u_0(x) + C_3 R^{\theta(1-\beta)} u_0(x)\right)(L-x)\mu^\ell(Rx)\,dx \\
&= C_3 \int_0^L u_0(x)(L-x)\,dx \\
&> 0.
\end{aligned}$$

Hence, for sufficiently large R,

$$\int_0^L \left(C_1 R^{\theta(2-\alpha)} u_1(x) + C_2 R^{\theta(1-\alpha)} u_0(x) + C_3 R^{\theta(1-\beta)} u_0(x)\right)(L-x)\mu^\ell(Rx)\,dx \geq C R^{\theta(1-\beta)}. \tag{65}$$

Combining (61) with (65), we obtain

$$C \leq R^{\theta\beta+\frac{\sigma+2p}{p-1}-1} + R^{\frac{-\theta\beta}{p-1}}. \tag{66}$$

Taking $\theta > 0$ such that

$$\theta < \frac{-\sigma-(p+1)}{\beta(p-1)},$$

and passing to the limit as $R \to \infty$ in (66), a contradiction follows. This completes the proof of Theorem 3. □

4.4. Proof of Theorem 4

Proof. Suppose that u is a global weak solution to (2). From the proof of Theorem 3, for sufficiently large T and R, there holds

$$\begin{aligned}
&-\int_0^T g(t)\frac{\partial\varphi}{\partial x}(t,L)\,dt \\
&+\int_0^L \left(C_1 T^{2-\alpha} u_1(x) + C_2 T^{1-\alpha} u_0(x) + C_3 T^{1-\beta} u_0(x)\right)(L-x)\mu^\ell(Rx)\,dx \\
&\leq C\left(TR^{\frac{\sigma+2p}{p-1}-1} + T^{1-\frac{\beta p}{p-1}} \int_{\frac{1}{2}R^{-1}}^L x^{\frac{-\sigma}{p-1}}\,dx\right),
\end{aligned} \tag{67}$$

where φ is the function defined by (50). On the other hand, by (50) and the properties of the cut-off function μ, we have

$$\begin{aligned} -\int_0^T g(t)\frac{\partial\varphi}{\partial x}(t,L)\,dt &= \int_0^T g(t)\eta(t)\,dt \\ &= T^{-\lambda}\int_0^T t^{\gamma}(T-t)^{\lambda}\,dt \\ &= B(\gamma+1,\lambda+1)T^{\gamma+1} \\ &:= CT^{\gamma+1}, \end{aligned}$$

where B denotes the Beta function. Thus, by (67), we obtain

$$\begin{aligned} &C+\int_0^L \Big(C_1T^{1-\alpha-\gamma}u_1(x)+C_2T^{-\gamma-\alpha}u_0(x)+C_3T^{-\beta-\gamma}u_0(x)\Big)(L-x)\mu^{\ell}(Rx)\,dx \\ &\leq C\left(T^{-\gamma}R^{\frac{\sigma+2p}{p-1}-1}+T^{-\frac{\beta p}{p-1}-\gamma}\int_{\frac{1}{2}R^{-1}}^{L} x^{\frac{-\sigma}{p-1}}\,dx\right). \end{aligned}$$

Taking $T=R^{\theta}$, where $\theta>0$ is a constant that will be determined later, the above inequality reduces to

$$\begin{aligned} &C+\int_0^L \Big(C_1R^{\theta(1-\alpha-\gamma)}u_1(x)+C_2R^{-\theta(\gamma+\alpha)}u_0(x)+C_3R^{-\theta(\beta+\gamma)}u_0(x)\Big)(L-x)\mu^{\ell}(Rx)\,dx \\ &\leq C\left(R^{-\theta\gamma+\frac{\sigma+2p}{p-1}-1}+R^{-\theta\left(\frac{\beta p}{p-1}+\gamma\right)}\int_{\frac{1}{2}R^{-1}}^{L} x^{\frac{-\sigma}{p-1}}\,dx\right). \end{aligned} \tag{68}$$

Let $\sigma<-(p+1)$. In this case, for sufficiently large R, there holds

$$\int_{\frac{1}{2}R^{-1}}^{L} x^{\frac{-\sigma}{p-1}}\,dx \leq C.$$

Hence, (68) yields

$$\begin{aligned} &C+\int_0^L \Big(C_1R^{\theta(1-\alpha-\gamma)}u_1(x)+C_2R^{-\theta(\gamma+\alpha)}u_0(x)+C_3R^{-\theta(\beta+\gamma)}u_0(x)\Big)(L-x)\mu^{\ell}(Rx)\,dx \\ &\leq C\left(R^{-\theta\gamma+\frac{\sigma+2p}{p-1}-1}+R^{-\theta\left(\frac{\beta p}{p-1}+\gamma\right)}\right). \end{aligned} \tag{69}$$

Since by (17), $\beta+\gamma>0$, there holds

$$\frac{\beta p}{p-1}+\gamma>0.$$

Thus, taking $\theta>0$ so that

$$\theta\gamma>\frac{\sigma+p+1}{p-1}, \tag{70}$$

using (17), and passing to the limit as $R\to\infty$ in (69), we obtain a contradiction with $C>0$. This proves part (i) of Theorem 4.

Let $\sigma\geq-(p+1)$ and $\gamma>0$.

If $-(p+1)\leq\sigma<p-1$, then (69) holds. Since $\gamma>0$, there exists $\theta>0$ satisfying (70). Thus, passing to the limit as $R\to\infty$ in (69), a contradiction follows.

If $\sigma=p-1$, then (68) yields

$$\begin{aligned} &C+\int_0^L \Big(C_1R^{\theta(1-\alpha-\gamma)}u_1(x)+C_2R^{-\theta(\gamma+\alpha)}u_0(x)+C_3R^{-\theta(\beta+\gamma)}u_0(x)\Big)(L-x)\mu^{\ell}(Rx)\,dx \\ &\leq C\left(R^{-\theta\gamma+\frac{\sigma+2p}{p-1}-1}+R^{-\theta\left(\frac{\beta p}{p-1}+\gamma\right)}\ln R\right). \end{aligned} \tag{71}$$

As in the previous case, since $\gamma > 0$, there exists $\theta > 0$ satisfying (70). Thus, passing to the limit as $R \to \infty$ in (71), a contradiction follows.
If $\sigma > p-1$, then (68) yields

$$
\begin{aligned}
&C + \int_0^L \left(C_1 R^{\theta(1-\alpha-\gamma)} u_1(x) + C_2 R^{-\theta(\gamma+\alpha)} u_0(x) + C_3 R^{-\theta(\beta+\gamma)} u_0(x) \right) (L-x) \mu^{\ell}(Rx)\, dx \\
&\leq C \left(R^{-\theta\gamma + \frac{\sigma+2p}{p-1} - 1} + R^{-\theta\left(\frac{\beta p}{p-1} + \gamma\right) + \frac{\sigma}{p-1} - 1} \right).
\end{aligned} \tag{72}
$$

So, taking $\theta > 0$ satisfying (70) and

$$
\theta \left(\frac{\beta p}{p-1} + \gamma \right) > \frac{\sigma}{p-1} - 1,
$$

and passing to the limit as $R \to \infty$ in (72), a contradiction follows. This proves part (ii) of Theorem 4. □

5. Conclusions

Using the test function method, sufficient conditions for the nonexistence of global weak solutions to problems (1) and (2) are obtained. For each problem, an adequate choice of a test function is made, taking into consideration the boundedness of the domain and the boundary conditions. Comparing with previous existing results in the literature, our results hold without assuming that the initial values are large with respect to a certain norm.

In this paper, we treated only the one dimensional case. It will be interesting to study problems (1) and (2) in a bounded domain $\Omega \subset \mathbb{R}^N$ under different types of boundary conditions, such as Dirichlet boundary conditions, Neumann boundary conditions, and Robin boundary conditions.

Author Contributions: Investigation, A.B.S.; Supervision M.J. and B.S. All authors have read and agreed to the published version of the manuscript.

Funding: The second author is supported by the Researchers Supporting Project number (RSP-2021/57), King Saud University, Riyadh, Saudi Arabia.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Tsutsumi, M. On solutions of semilinear differential equations in a Hilbert space. *Math. Japon.* **1972**, *17*, 173–193.
2. Levine, H.A. Some additional remarks on the nonexistence of global solutions to nonlinear wave equations. *SIAM J. Math. Anal.* **1974**, *5*, 138–146. [CrossRef]
3. Galaktionov, V.A.; Pohozaev, S.I. Blow-up and critical exponents for nonlinear hyperbolic equations. *Nonlinear Anal.* **2003**, *53*, 453–466. [CrossRef]
4. Erbay, H.A.; Erbay, S.; Erkip, A. Thresholds for global existence and blow-up in a general class of doubly dispersive nonlocal wave equations. *Nonlinear Anal.* **2014**, *95*, 313–322. [CrossRef]
5. Kalantarov, V.K.; Ladyzhenskaya, O.A. The occurrence of collapse for quasilinear equations of parabolic and hyperbolic type. *J. Soviet Math.* **1978**, *10*, 53–70. [CrossRef]
6. Li, F. Global existence and blow-up of solutions for a higher-order Kirchhoff-type equation with nonlinear dissipation. *Appl. Math. Lett.* **2004**, *17*, 1409–1414. [CrossRef]
7. Wang, X.; Chen, Y.; Yang, Y.; Li, J.; Xu, R. Kirchhoff-type system with linear weak damping and logarithmic nonlinearities. *Nonlinear Anal.* **2019**, *188*, 475–499. [CrossRef]
8. Guedda, M.; Labani, H. Nonexistence of global solutions to a class of nonlinear wave equations with dynamic boundary conditions. *Bull. Belg. Math. Soc. Simon Stevin.* **2002**, *9*, 39–46. [CrossRef]
9. Messaoudi, S.A. Blow-up of positive-initial-energy solutions of a nonlinear viscoelastic hyperbolic equation. *J. Math. Anal. Appl.* **2006**, *320*, 902–915. [CrossRef]

10. Kafini, M.; Messaoudi, S.A. blow-up result for a viscoelastic system in $\mathbb{R}^N$. *Electron. J. Differ. Equ.* **2007**, *113*, 1–7.
11. Kafini, M.; Messaoudi, S.A. On the decay and global nonexistence of solutions to a damped wave equation with variable-exponent nonlinearity and delay. *Ann. Pol. Math*. **2019**, *122*, 49–70. [CrossRef]
12. Freeborn, T.J. A survey of fractional-order circuit models for biology and biomedicine. *IEEE J. Emerg. Sel. Top. Circuits Syst*. **2013**, *3*, 416–423. [CrossRef]
13. Bagley, R.L.; Torvik, P.J. A theoretical basis for the application of fractional calculus to viscoelasticity. *J. Rheol*. **1983**, *27*, 201–210. [CrossRef]
14. Povstenko, Y.Z. Fractional heat conduction equation and associated thermal Stresses. *J. Therm. Stress*. **2005**, *28*, 83–102. [CrossRef]
15. Chen, W.; Sun, H.G.; Zhang, X.; Korosak, D. Anomalous diffusion modeling by fractal and fractional derivatives. *Comput. Math. Appl.* **2010**, *59*, 1754–1758. [CrossRef]
16. Zhao, Y.; Hou, Z. Two viscoelastic constitutive models of rubber materials using fractional derivations. *J. Tsinghua Univ*. **2013**, *53*, 378–383.
17. Samko, S.G.; Kilbas, A.A.; Marichev, O.I. *Fractional Integrals and Derivatives: Theory and Applications*; Gordon and Breach: Yverdon, Switzerland, 1993.
18. Agarwal, R.P.; Benchohra, M.; Hamani, S. A survey on existence results for boundary value problems of nonlinear fractional differential equations and inclusions. *Acta Appl. Math.* **2010**, *109*, 973–1033. [CrossRef]
19. Podlubny, I. *Fractional Differential Equations*; Academic Press: New York, NY, USA, 1999.
20. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*; Elsevier Science Limited: Amsterdam, The Netherlands, 2006.
21. Li C.; Zeng F.H. *Numerical Methods for Fractional Calculus*; Chapman and Hall/CRC: Boca Raton, FL, USA, 2015.
22. Kirane, M.; Tatar, N.-E. Exponential growth for a fractionally damped wave equation. *Z. Anal. Anwend.* **2003**, *22*, 167–177. [CrossRef]
23. Georgiev, V.; Todorova, G. Existence of a solution of the wave equation with nonlinear damping and source terms. *J. Differ. Equ.* **1994**, *109*, 295–308. [CrossRef]
24. Tatar, N.-E. A blow up result for a fractionally damped wave equation. *Nonlinear Differ. Equ. Appl*. **2005**, *12*, 215–226. [CrossRef]
25. Mitidieri, E.; Pohozaev, S. A priori estimates and blow-up of solutions to nonlinear partial differential equations and inequalities. *Proc. Steklov Inst. Math*. **2001**, *234*, 1–383.
26. Korpusov, M.O.; Lukyanenko, D.V.; Panin, A.A.; Shlyapugin, G.I. On the blow-up phenomena for a 1-dimensional equation of ion sound waves in a plasma: Analytical and numerical investigation. *Math. Methods Appl. Sci*. **2018**, *41*, 2906–2929. [CrossRef]
27. Pelinovsky, D.; Xu, C. On numerical modelling and the blow-up behavior of contact lines with a 180 degrees contact angle. *J. Eng. Math*. **2015**, *92*, 31–44. [CrossRef]
28. Cangiani, A.; Georgoulis, E.H.; Kyza, I.; Metcalfe, S. Adaptivity and blow-up detection for nonlinear evolution problems. *SIAM J. Sci. Comput.* **2016**, *38*, A3833–A3856. [CrossRef]
29. Carl, S.; Le, V.K.; Motreanu, D. *Nonsmooth Variational Problems and Their Inequalities*; Springer: New York, NY, USA, 2007.