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A Mixed Element Algorithm Based on the Modified $L1$ Crank–Nicolson Scheme for a Nonlinear Fourth-Order Fractional Diffusion-Wave Model

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Abstract: In this article, a new mixed finite element (MFE) algorithm is presented and developed to find the numerical solution of a two-dimensional nonlinear fourth-order Riemann–Liouville fractional diffusion-wave equation. By introducing two auxiliary variables and using a particular technique, a new coupled system with three equations is constructed. Compared to the previous space–time high-order model, the derived system is a lower coupled equation with lower time derivatives and second-order space derivatives, which can be approximated by using many time discrete schemes. Here, the second-order Crank–Nicolson scheme with the modified $L1$-formula is used to approximate the time direction, while the space direction is approximated by the new MFE method. Analyses of the stability and optimal L^2 error estimates are performed and the feasibility is validated by the calculated data.

Keywords: fourth-order fractional diffusion-wave equation; modified $L1$-formula; mixed element method; a priori error estimates

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1. Introduction

Fourth-order fractional partial differential equations (PDEs) including fourth-order fractional subdiffusion models [1–3] and fourth-order fractional diffusion-wave models [2,4,5] can be founded in many fields of science and engineering. Thus far, there have been many efficient numerical algorithms for solving linear or nonlinear fourth-order fractional subdiffusion and diffusion-wave models. Liu et al. [6], Liu et al. [7], and Liu et al. [8] considered different mixed element methods to solve fourth-order nonlinear fractional subdiffusion models with the first-order time derivative and developed numerical theories including stability and convergence. Liu et al. [3] introduced a mixed element algorithm with a new approximation of the fractional derivative. Ji et al. [9], Ran et al. [10], Nandal and Pandey [11], Sun et al. [12], and Huang et al. [13] considered some difference schemes for linear or nonlinear fourth-order fractional diffusion or diffusion-wave models. Abbaszadeh and Dehghan [14] studied the direct meshless local Petrov–Galerkin method for solving fourth-order reaction-diffusion problems with a time-fractional derivative. Yang et al. [15] and Zhang et al. [16] found the numerical solutions for a fourth-order fractional model by using the orthogonal spline collocation method. Tariq and Akram [17] considered a quintic spline technique to solve a fourth-order time-fractional subdiffusion model. Guo et al. [18] and Du et al. [19] studied the LDG methods for solving some time-fractional subdiffusion models with fourth-order spatial derivative terms, respectively. In [1], Nikan et al. developed a local radial basis function generated by the finite difference scheme for a time-fractional fourth-order reaction-diffusion model. In [5], Jafari et al. solved a fourth-order fractional diffusion-wave equation by the decomposition method. Hu and Zhang [2] implemented numerical calculations via finite difference methods for fourth-order time fractional subdiffusion and diffusion-wave models. Li and Wong [20] developed an efficient numerical algorithm for a fourth-order time-fractional diffusion-wave model.

Here, we propose a new mixed element algorithm to solve the following nonlinear fourth-order time-fractional diffusion-wave model:

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} + \frac{\partial_{RL}^{\beta} u}{\partial t^{\beta}} + \frac{\partial u}{\partial t} + \triangle^2 u - \triangle f(u) = g(\mathbf{x}, t), (\mathbf{x}, t) \in \Omega \times J, \\ u(\mathbf{x}, t) = \Delta u(\mathbf{x}, t) = 0, t \in \bar{J}, \\ u(\mathbf{x}, 0) = 0, \frac{\partial u}{\partial t}(\mathbf{x}, 0) = u_1(\mathbf{x}), \mathbf{x} \in \bar{\Omega}, \end{cases} \tag{1}$$

where $\Omega \subset R^d (d \leq 2)$ and $J = (0, T]$ with $0 < T < \infty$ are the spatial domain and time interval, respectively. $u_1(\mathbf{x})$ is an initial value function, $g(\mathbf{x}, t)$ is a given source term, $f(u)$ is a polynomial function or bounded function on u satisfying $f \in C^2(R)$, and the Riemann–Liouville fractional derivative is defined by

$$\frac{\partial_{RL}^{\beta} u}{\partial t^{\beta}} = \frac{1}{\Gamma(2-\beta)} \frac{\partial^2}{\partial t^2} \int_0^t \frac{u(s) ds}{(t-s)^{\beta-1}}, 1 < \beta < 2, \tag{2}$$

where the nonlinear fourth-order fractional diffusion-wave model (1) can be generated by the classical fourth-order hyperbolic wave equation. When $\beta \to 1$ or 2 and $f(u) = u^3 - u$, the model (1) can be reduced to an important Cahn–Hilliard equation model [21].

Recently, Zeng and Li [22] developed a new Crank–Nicolson scheme based on a modified $L1$-formula, whose coefficients are different from the famous $L1$-formula (see [23,24] for the fractional parameter $\alpha \in (0, 1)$). One should note that this modified $L1$-formula can only approximate the Caputo or Riemann–Liouville fractional derivative with parameter $\alpha \in (0, 1)$, and it cannot approximate the case $\beta \in (1, 2)$. Here, we will develop the modified $L1$-formula for the case of $\beta \in (1, 2)$ by using some techniques.

In this article, by introducing two auxiliary functions and using some techniques, we propose a new mixed element algorithm. Here, our major contributions are as follows: (1) by the introduction of two auxiliary functions, we reduce the nonlinear fourth-order time-fractional diffusion-wave model to a low-order coupled system; (2) we turn order $\beta \in (1, 2)$ into order $\alpha \in (0, 1)$ for the Riemann–Liouville fractional derivative; (3) we approximate the derived coupled system with a fractional derivative with order $\alpha \in (0, 1)$ by the modified $L1$ Crank–Nicolson scheme with the developed new mixed element method; (4) we derive the stability of the new mixed element scheme and optimal error estimates in the L^2-norm for three functions.

The structure of this article is as follows: in Section 2, we provide some numerical approximation formulas, propose a new mixed element scheme, and prove the stability of the derived scheme; in Section 3, we derive optimal error estimates for three variables; in Section 4, some numerical data are computed and discussed; Finally, in Section 5, we give some concluding remarks.

2. Numerical Approximation and Stability

Based on the relation between the Riemann–Liouville fractional derivative and Caputo fractional derivative, we take $\alpha = \beta - 1$ and $v = \frac{\partial u}{\partial t}$ to obtain

$$\begin{aligned} \frac{\partial_{RL}^{\beta} u}{\partial t^{\beta}} &= \frac{1}{\Gamma(2-\beta)} \int_0^t \frac{\frac{\partial^2 u(s)}{\partial s^2} ds}{(t-s)^{\beta-1}} + \frac{u(0)}{\Gamma(1-\beta)} t^{-\beta} + \frac{\frac{\partial u(0)}{\partial t}}{\Gamma(2-\beta)} t^{1-\beta} \\ &= \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\frac{\partial v}{\partial s} ds}{(t-s)^{\alpha}} + \frac{v(0)}{\Gamma(1-\alpha)} t^{-\alpha} \\ &= \frac{\partial_{RL}^{\alpha} v}{\partial t^{\alpha}}, 0 < \alpha < 1. \end{aligned} \tag{3}$$

Let $\sigma = \triangle u - f(u)$; (1) can be rewritten as the following coupled system:

$$\begin{cases} v = \dfrac{\partial u}{\partial t}, (\mathbf{x}, t) \in \Omega \times J, \\ \dfrac{\partial \sigma}{\partial t} = \triangle v - f_u(u) v, (\mathbf{x}, t) \in \Omega \times J, \\ \dfrac{\partial v}{\partial t} + \dfrac{\partial^\alpha_{RL} v}{\partial t^\alpha} + v + \triangle \sigma = g(\mathbf{x}, t), (\mathbf{x}, t) \in \Omega \times J. \end{cases} \tag{4}$$

For formulating the fully discrete scheme, we insert the nodes $t_n = n\Delta t (n = 0, 1, 2, \cdots, N)$ in time interval $[0, T]$, where t_n satisfy $0 = t_0 < t_1 < t_2 < \cdots < t_N = T$ and the time step length size $\Delta t = T/N$, for some positive integer N. For a smooth function ϕ defined on the time interval $[0, T]$, we denote $\phi^n = \phi(t_n)$.

Now, we need to introduce some lemmas on integer and fractional derivatives.

Lemma 1. *At $t_{k+\frac{1}{2}}$, the following relation holds:*

$$\begin{aligned} \frac{\partial \phi}{\partial t}(t_{k+\frac{1}{2}}) =& \frac{\phi^{k+1} - \phi^k}{\Delta t} + O(\Delta t^2) \\ \triangleq& P_{\Delta t} \phi^{k+\frac{1}{2}} + O(\Delta t^2). \end{aligned} \tag{5}$$

Lemma 2. *At $t_{k+\frac{1}{2}}$, we have*

$$\begin{aligned} \phi(t_{k+\frac{1}{2}}) =& \frac{\phi^{k+1} + \phi^k}{2} + O(\Delta t^2) \\ \triangleq& \phi^{k+\frac{1}{2}} + O(\Delta t^2). \end{aligned} \tag{6}$$

Lemma 3 ([22])**.** *At $t_{k+\frac{1}{2}}$, the Caputo fractional derivative has the following form:*

$$\begin{aligned} \frac{\partial^\alpha_C \phi}{\partial t^\alpha}(t_{k+\frac{1}{2}}) =& \frac{1}{\Gamma(1-\alpha)} \int_0^{t_{k+\frac{1}{2}}} \frac{\frac{\partial \phi}{\partial s} ds}{(t_{k+\frac{1}{2}} - s)^\alpha} \\ =& \frac{\Delta t^{-\alpha}}{\Gamma(2-\alpha)} \Big[a_0 \phi(t_{k+\frac{1}{2}}) - \sum_{j=1}^{k} (a_{k-j} - a_{k-j+1}) \phi(t_{j-\frac{1}{2}}) \\ & - (a_k - b_k) \phi(t_{\frac{1}{2}}) - b_k \phi(t_0) \Big] + O(\Delta t^{2-\alpha}), \end{aligned} \tag{7}$$

for $k \geq 0$, we have

$$b_k = 2\Big[(k + \frac{1}{2})^{1-\alpha} - k^{1-\alpha}\Big], a_{k-j} = \big[(k-j+1)^{1-\alpha} - (k-j)^{1-\alpha}\big]. \tag{8}$$

By Lemma 3, we have

Lemma 4 ([22])**.** *At $t_{k+\frac{1}{2}}$, the Riemann–Liouville fractional derivative has the following approximation:*

$$\begin{aligned} \frac{\partial^\alpha_{RL} \phi}{\partial t^\alpha}(t_{k+\frac{1}{2}}) =& \frac{\Delta t^{-\alpha}}{\Gamma(2-\alpha)} \Big[a_0 \phi(t_{k+\frac{1}{2}}) - \sum_{j=1}^{k} (a_{k-j} - a_{k-j+1}) \phi(t_{j-\frac{1}{2}}) \\ & - (a_k - b_k) \phi(t_{\frac{1}{2}}) - \widehat{b}_k \phi(t_0) \Big] + O(\Delta t^{2-\alpha}), \end{aligned} \tag{9}$$

where $\widehat{b}_k = b_k - (1-\alpha)(k + \frac{1}{2})^{-\alpha}$.

Remark 1. *In [22], the authors provided the L1-formula above, which is different from the usual L1-formula and called the modified L1-formula.*

By the approximation scheme above, we arrive at

$$\begin{cases} (a)\ P_{\Delta t}u^{n+\frac{1}{2}} = v^{n+\frac{1}{2}} + R_1^{n+\frac{1}{2}}, \\ (b)\ P_{\Delta t}\sigma^{n+\frac{1}{2}} = \triangle v^{n+\frac{1}{2}} - \dfrac{f_u(u^{n+1})v^{n+1} + f_u(u^n)v^n}{2} + R_2^{n+\frac{1}{2}}, \\ (c)\ P_{\Delta t}v^{n+\frac{1}{2}} + \dfrac{\Delta t^{-\alpha}}{\Gamma(2-\alpha)}\Big[a_0 v^{n+\frac{1}{2}} - \sum_{j=1}^{n}(a_{n-j} - a_{n-j+1})v^{j-\frac{1}{2}} \\ \quad - (a_n - b_n)v^{\frac{1}{2}} - \widehat{b}_n v^0\Big] + v^{n+\frac{1}{2}} + \triangle\sigma^{n+\frac{1}{2}} = g(\mathbf{x}, t_{n+\frac{1}{2}}) + R_3^{n+\frac{1}{2}}, \end{cases} \tag{10}$$

where

$$\begin{aligned} R_1^{n+\frac{1}{2}} &= P_{\Delta t}u^{n+\frac{1}{2}} - \frac{\partial u}{\partial t}(t_{n+\frac{1}{2}}) + (v(t_{n+\frac{1}{2}}) - v^{n+\frac{1}{2}}) = O(\Delta t^2), \\ R_2^{n+\frac{1}{2}} &= P_{\Delta t}\sigma^{n+\frac{1}{2}} - \frac{\partial \sigma}{\partial t}(t_{n+\frac{1}{2}}) + (\triangle v(t_{n+\frac{1}{2}}) - \triangle v^{n+\frac{1}{2}}) \\ &+ f_u(u(t_{n+\frac{1}{2}}))v(t_{n+\frac{1}{2}}) - \frac{f_u(u^{n+1})v^{n+1} + f_u(u^n)v^n}{2} = O(\Delta t^2), \\ R_3^{n+\frac{1}{2}} &= P_{\Delta t}v^{n+\frac{1}{2}} - \frac{\partial v}{\partial t}(t_{n+\frac{1}{2}}) + O(\Delta t^{2-\alpha}) + (v^{n+\frac{1}{2}} - v(t_{n+\frac{1}{2}})) \\ &+ (\triangle\sigma^{n+\frac{1}{2}} - \triangle\sigma(t_{n+\frac{1}{2}})) = O(\Delta t^{2-\alpha}). \end{aligned}$$

For $(\varphi, \psi, \chi) \in L^2 \times H_0^1 \times H_0^1$, we have the following mixed weak formulation:

$$\begin{cases} (a)\ (P_{\Delta t}u^{n+\frac{1}{2}}, \varphi) = (v^{n+\frac{1}{2}}, \varphi) + (R_1^{n+\frac{1}{2}}, \varphi), \\ (b)\ (P_{\Delta t}\sigma^{n+\frac{1}{2}}, \psi) + (\nabla v^{n+\frac{1}{2}}, \nabla\psi) + \Big(\dfrac{f_u(u^{n+1})v^{n+1} + f_u(u^n)v^n}{2}, \psi\Big) = (R_2^{n+\frac{1}{2}}, \psi), \\ (c)\ (P_{\Delta t}v^{n+\frac{1}{2}}, \chi) + \Big(\dfrac{\Delta t^{-\alpha}}{\Gamma(2-\alpha)}\Big[a_0 v^{n+\frac{1}{2}} - \sum_{j=1}^{n}(a_{n-j} - a_{n-j+1})v^{j-\frac{1}{2}} \\ \quad -(a_n - b_n)v^{\frac{1}{2}} - \widehat{b}_n v^0\Big], \chi\Big) + (v^{n+\frac{1}{2}}, \chi) - (\nabla\sigma^{n+\frac{1}{2}}, \nabla\chi) = (g(\mathbf{x}, t_{n+\frac{1}{2}}), \chi) + (R_3^{n+\frac{1}{2}}, \chi). \end{cases} \tag{11}$$

For $(\varphi_h, \psi_h, \chi_h) \in L_h \times V_h \times V_h \subset L^2 \times H_0^1 \times H_0^1$, based on the mixed weak formulation above, we formulate the following new mixed element system:

$$\begin{cases} (a)\ (P_{\Delta t}u_h^{n+\frac{1}{2}}, \varphi_h) = (v_h^{n+\frac{1}{2}}, \varphi_h), \\ (b)\ (P_{\Delta t}\sigma_h^{n+\frac{1}{2}}, \psi_h) + (\nabla v_h^{n+\frac{1}{2}}, \nabla\psi_h) + \Big(\dfrac{f_u(u_h^{n+1})v_h^{n+1} + f_u(u_h^n)v_h^n}{2}, \psi_h\Big) = 0, \\ (c)\ (P_{\Delta t}v_h^{n+\frac{1}{2}}, \chi_h) + \Big(\dfrac{\Delta t^{-\alpha}}{\Gamma(2-\alpha)}\Big[a_0 v_h^{n+\frac{1}{2}} - \sum_{j=1}^{n}(a_{n-j} - a_{n-j+1})v_h^{j-\frac{1}{2}} \\ \quad - (a_n - b_n)v_h^{\frac{1}{2}} - \widehat{b}_n v_h^0\Big], \chi_h\Big) + (v_h^{n+\frac{1}{2}}, \chi_h) - (\nabla\sigma_h^{n+\frac{1}{2}}, \nabla\chi_h) = (g(\mathbf{x}, t_{n+\frac{1}{2}}), \chi_h). \end{cases} \tag{12}$$

Lemma 5 (See [22])**.** *For $\widehat{b}_{n-j}$, the following important inequality holds:*

$$\frac{\Delta t^{1-\alpha}}{\Gamma(2-\alpha)} \sum_{j=0}^{n-1} \widehat{b}_{n-j} \le \frac{CT^{1-\alpha}}{\Gamma(2-\alpha)}, \tag{13}$$

where C is a positive constant that is independent of space–time step length sizes h and Δt.

Proof. Applying the Taylor formula, we have

$$\begin{aligned}
&\frac{\Delta t^{1-\alpha}}{\Gamma(2-\alpha)}\sum_{j=0}^{n-1}\widehat{b}_{n-j}\\
=&\frac{\Delta t^{1-\alpha}}{\Gamma(2-\alpha)}\sum_{j=0}^{n-1}\Big[2(n-j+\frac{1}{2})^{1-\alpha}-2(n-j)^{1-\alpha}-(1-\alpha)(n-j+\frac{1}{2})^{-\alpha}\Big]\\
=&\frac{2\Delta t^{1-\alpha}}{\Gamma(2-\alpha)}\sum_{j=0}^{n-1}(n-j)^{1-\alpha}\Big[(1+\frac{1}{2(n-j)})^{1-\alpha}-1-\frac{1-\alpha}{2(n-j+\frac{1}{2})}(1+\frac{1}{2(n-j)})^{1-\alpha}\Big]\\
=&\frac{\Delta t^{1-\alpha}}{\Gamma(2-\alpha)}\sum_{j=0}^{n-1}(n-j)^{1-\alpha}\Big[\frac{1-\alpha}{2(n-j)}+\frac{(1-\alpha)\alpha}{2!}\Big(1+\kappa\frac{1}{2(n-j)}\Big)^{1-\alpha}\frac{1}{4(n-j)^2}\\
&-\frac{1-\alpha}{2(n-j+\frac{1}{2})}\Big(1+\frac{1-\alpha}{2(n-j)}+\frac{(1-\alpha)\alpha}{2!}\Big(1+\kappa\frac{1}{2(n-j)}\Big)^{1-\alpha}\frac{1}{4(n-j)^2}\Big)\Big]\\
=&\frac{\Delta t^{1-\alpha}}{\Gamma(2-\alpha)}\sum_{j=0}^{n-1}(n-j)^{1-\alpha}\Big[\frac{1-\alpha}{2(n-j)}-\frac{1-\alpha}{2(n-j+\frac{1}{2})}+O\Big(\frac{1}{(n-j)^2}\Big)\Big]\\
=&\frac{\Delta t^{1-\alpha}}{\Gamma(2-\alpha)}\sum_{j=0}^{n-1}(n-j)^{1-\alpha}\Big[\frac{1-\alpha}{2(n-j+\frac{1}{2})(n-j)}+O\Big(\frac{1}{(n-j)^2}\Big)\Big].
\end{aligned}\tag{14}$$

Noting that $\Delta t=\frac{T}{N}$ and $n-j\leq N$, we have

$$\begin{aligned}
&\frac{\Delta t^{1-\alpha}}{\Gamma(2-\alpha)}\sum_{j=0}^{n-1}(n-j)^{1-\alpha}\Big[\frac{1-\alpha}{2(n-j+\frac{1}{2})(n-j)}+O\Big(\frac{1}{(n-j)^2}\Big)\Big]\\
=&\frac{1}{\Gamma(2-\alpha)}\sum_{j=0}^{n-1}T^{1-\alpha}(\frac{n-j}{N})^{1-\alpha}\Big[\frac{1-\alpha}{2(n-j+\frac{1}{2})(n-j)}+O\Big(\frac{1}{(n-j)^2}\Big)\Big]\\
\leq&\frac{T^{1-\alpha}}{\Gamma(2-\alpha)}\sum_{j=0}^{n-1}\Big[\frac{1}{(n-j)^2}+O\Big(\frac{1}{(n-j)^2}\Big)\Big]\leq\frac{CT^{1-\alpha}}{\Gamma(2-\alpha)}\sum_{n=1}^{+\infty}\frac{1}{n^2}\leq\frac{CT^{1-\alpha}}{\Gamma(2-\alpha)}.
\end{aligned}\tag{15}$$

Substitute (15) into (14) to obtain the conclusion. □

Next, we will prove the stability.

Theorem 1. *For $n\geq 0$, the stability for the fully discrete system (12) holds:*

$$\begin{aligned}
&(a).\ \|v_h^{n+1}\|+\|\sigma_h^{n+1}\|+\Big(\frac{\Delta t^{1-\alpha}}{\Gamma(2-\alpha)}\sum_{j=1}^{n+1}a_{n-j+1}\|v_h^{j-\frac{1}{2}}\|^2\Big)^{\frac{1}{2}}\\
\leq&C\Big(\|v_h^0\|+\|\sigma_h^0\|+\max_{0\leq j\leq n}\{\|g(\boldsymbol{x},t_{j+\frac{1}{2}})\|\}\Big),\\
&(b).\ \|u_h^{n+1}\|\leq C\Big(\|u_h^0\|+\|v_h^0\|+\|\sigma_h^0\|+\max_{0\leq j\leq n}\{\|g(\boldsymbol{x},t_{j+\frac{1}{2}})\|\}\Big).
\end{aligned}\tag{16}$$

Proof. In (12) (a), we take $\varphi_h=u_h^{n+\frac{1}{2}}$, and use Cauchy–Schwarz inequality as well as Young inequality to obtain

$$\begin{aligned}
\frac{1}{2}(\|v_h^{n+\frac{1}{2}}\|^2+\|u_h^{n+\frac{1}{2}}\|^2)\geq(v_h^{n+\frac{1}{2}},u_h^{n+\frac{1}{2}})=&(P_{\Delta t}u_h^{n+\frac{1}{2}},u_h^{n+\frac{1}{2}})\\
\geq&\frac{\|u_h^{n+1}\|^2-\|u_h^n\|^2}{2\Delta t}.
\end{aligned}\tag{17}$$

In (12) (b), set $\psi_h = \sigma_h^{n+\frac{1}{2}}$ and make use of Cauchy–Schwarz inequality to arrive at

$$\begin{aligned}
&(\nabla v_h^{n+\frac{1}{2}}, \nabla \sigma_h^{n+\frac{1}{2}}) \\
&= -(P_{\Delta t}\sigma_h^{n+\frac{1}{2}}, \sigma_h^{n+\frac{1}{2}}) - \Big(\frac{f_u(u_h^{n+1})v_h^{n+1} + f_u(u_h^n)v_h^n}{2}, \sigma_h^{n+\frac{1}{2}}\Big) \\
&\leq -\frac{\|\sigma_h^{n+1}\|^2 - \|\sigma_h^n\|^2}{2\Delta t} + \frac{1}{2}(\|f_u(u_h^{n+1})\|_\infty \|v_h^{n+1}\| + \|f_u(u_h^n)\|_\infty \|v_h^n\|)\|\sigma_h^{n+\frac{1}{2}}\| \\
&\leq -\frac{\|\sigma_h^{n+1}\|^2 - \|\sigma_h^{n+1}\|^2}{2\Delta t} + C(\|v_h^{n+1}\|^2 + \|v_h^n\|^2 + \|\sigma_h^{n+1}\|^2 + \|\sigma_h^n\|^2).
\end{aligned} \tag{18}$$

In (12) (c), set $\chi_h = v_h^{n+\frac{1}{2}}$ and use Cauchy–Schwarz inequality to obtain

$$\begin{aligned}
&-(\nabla \sigma_h^{n+\frac{1}{2}}, \nabla v_h^{n+\frac{1}{2}}) \\
&= -(P_{\Delta t}v_h^{n+\frac{1}{2}}, v_h^{n+\frac{1}{2}}) - \Big(\frac{\Delta t^{-\alpha}}{\Gamma(2-\alpha)}\Big[a_0 v_h^{n+\frac{1}{2}} - \sum_{j=1}^{n}(a_{n-j} - a_{n-j+1})v_h^{j-\frac{1}{2}} \\
&\quad - (a_n - b_n)v_h^{\frac{1}{2}} - b_n v_h^0\Big], v_h^{n+\frac{1}{2}}\Big) - \|v_h^{n+\frac{1}{2}}\|^2 + (g(\mathbf{x}, t_{n+\frac{1}{2}}), v_h^{n+\frac{1}{2}}) \\
&\leq -\frac{\|v_h^{n+1}\|^2 - \|v_h^n\|^2}{2\Delta t} - \Big(\frac{\Delta t^{-\alpha}}{\Gamma(2-\alpha)}\Big[a_0 v_h^{n+\frac{1}{2}} - \sum_{j=1}^{n}(a_{n-j} - a_{n-j+1})v_h^{j-\frac{1}{2}} \\
&\quad - (a_n - b_n)v_h^{\frac{1}{2}} - \widehat{b}_n v_h^0\Big], v_h^{n+\frac{1}{2}}\Big) - \frac{1}{2}\|v_h^{n+\frac{1}{2}}\|^2 + \frac{1}{2}\|g(\mathbf{x}, t_{n+\frac{1}{2}})\|^2.
\end{aligned} \tag{19}$$

Add (18) and (19) to obtain

$$\begin{aligned}
&\frac{\|v_h^{n+1}\|^2 - \|v_h^n\|^2}{2\Delta t} + \frac{\|\sigma_h^{n+1}\|^2 - \|\sigma_h^n\|^2}{2\Delta t} + \frac{1}{2}\|v_h^{n+\frac{1}{2}}\|^2 \\
&\leq -\Big(\frac{\Delta t^{-\alpha}}{\Gamma(2-\alpha)}\Big[a_0 v_h^{n+\frac{1}{2}} - \sum_{j=1}^{n}(a_{n-j} - a_{n-j+1})v_h^{j-\frac{1}{2}} - (a_n - b_n)v_h^{\frac{1}{2}} - \widehat{b}_n v_h^0\Big], v_h^{n+\frac{1}{2}}\Big) \\
&\quad + C(\|v_h^{n+1}\|^2 + \|v_h^n\|^2 + \|\sigma_h^{n+1}\|^2 + \|\sigma_h^n\|^2 + \|g(\mathbf{x}, t_{n+\frac{1}{2}})\|^2).
\end{aligned} \tag{20}$$

Refer to Lemma 4.2 in [22] to easily obtain

$$\begin{aligned}
&-\Big(\frac{\Delta t^{-\alpha}}{\Gamma(2-\alpha)}\Big[a_0 v_h^{n+\frac{1}{2}} - \sum_{j=1}^{n}(a_{n-j} - a_{n-j+1})v_h^{j-\frac{1}{2}} - (a_n - b_n)v_h^{\frac{1}{2}} - b_n v_h^0\Big], v_h^{n+\frac{1}{2}}\Big) \\
&\leq \frac{\Delta t^{-\alpha}}{2\Gamma(2-\alpha)}\Big(\sum_{j=1}^{n} a_{n-j}\|v_h^{j-\frac{1}{2}}\|^2 - \sum_{j=1}^{n+1} a_{n-j+1}\|v_h^{j-\frac{1}{2}}\|^2 + \widehat{b}_n\|v_h^0\|^2\Big).
\end{aligned} \tag{21}$$

Combine (20) with (21) to obtain

$$\begin{aligned}
&\|v_h^{n+1}\|^2 + \|\sigma_h^{n+1}\|^2 + \Delta t\|v_h^{n+\frac{1}{2}}\|^2 + \frac{\Delta t^{1-\alpha}}{\Gamma(2-\alpha)}\sum_{j=1}^{n+1} a_{n-j+1}\|v_h^{j-\frac{1}{2}}\|^2 \\
&\leq \|v_h^n\|^2 + \|\sigma_h^n\|^2 + \frac{\Delta t^{1-\alpha}}{\Gamma(2-\alpha)}\sum_{j=1}^{n} a_{n-j}\|v_h^{j-\frac{1}{2}}\|^2 + \frac{\Delta t^{1-\alpha}}{\Gamma(2-\alpha)}\widehat{b}_n\|v_h^0\|^2 \\
&\quad + C\Delta t(\|v_h^{n+1}\|^2 + \|v_h^n\|^2 + \|\sigma_h^{n+1}\|^2 + \|\sigma_h^n\|^2 + \|g(\mathbf{x}, t_{n+\frac{1}{2}})\|^2).
\end{aligned} \tag{22}$$

We denote

$$\Xi(v_h^{n+1}, \sigma_h^{n+1}) = \|v_h^{n+1}\|^2 + \|\sigma_h^{n+1}\|^2 + \frac{\Delta t^{1-\alpha}}{\Gamma(2-\alpha)}\sum_{j=1}^{n+1} a_{n-j+1}\|v_h^{j-\frac{1}{2}}\|^2. \tag{23}$$

Remove the non-negative term to obtain

$$\begin{aligned}\Xi(v_h^{n+1},\sigma_h^{n+1}) \leq& \Big(\frac{1+\Delta t}{1-\Delta t}\Big)\Xi(v_h^n,\sigma_h^n) + \frac{1}{1-\Delta t}\frac{\Delta t^{1-\alpha}}{\Gamma(2-\alpha)}\widehat{b}_n\|v_h^0\|^2 + \frac{C\Delta t}{1-\Delta t}\|g(\mathbf{x},t_{n+\frac{1}{2}})\|^2\\ \leq& \Big(\frac{1+\Delta t}{1-\Delta t}\Big)^2\Xi(v_h^{n-1},\sigma_h^{n-1}) + \frac{1}{1-\Delta t}\frac{\Delta t^{1-\alpha}}{\Gamma(2-\alpha)}\|v_h^0\|^2\sum_{j=0}^{1}\widehat{b}_{n-j}\Big(\frac{1+\Delta t}{1-\Delta t}\Big)^j\\ &+\frac{C\Delta t}{1-\Delta t}\sum_{j=0}^{1}\Big(\frac{1+\Delta t}{1-\Delta t}\Big)^j\|g(\mathbf{x},t_{n-j+\frac{1}{2}})\|^2\\ \leq&\cdots\cdots\\ \leq& \Big(\frac{1+\Delta t}{1-\Delta t}\Big)^n\Xi(v_h^1,\sigma_h^1) + \frac{1}{1-\Delta t}\frac{\Delta t^{1-\alpha}}{\Gamma(2-\alpha)}\|v_h^0\|^2\sum_{j=0}^{n-1}\widehat{b}_{n-j}\Big(\frac{1+\Delta t}{1-\Delta t}\Big)^j\\ &+\frac{C\Delta t}{1-\Delta t}\sum_{j=0}^{n-1}\Big(\frac{1+\Delta t}{1-\Delta t}\Big)^j\|g(\mathbf{x},t_{n-j+\frac{1}{2}})\|^2.\end{aligned} \tag{24}$$

Noting that $\left(\frac{1+\Delta t}{1-\Delta t}\right) > 1$, $\Delta t = T/N \leq T/n$, we have

$$\begin{aligned}\Big(\frac{1+\Delta t}{1-\Delta t}\Big)^n \leq& \Big(\frac{1+\Delta t}{1-\Delta t}\Big)^{n+1} \leq \cdots \leq \Big(1+\frac{2\Delta t}{1-\Delta t}\Big)^{\frac{T}{\Delta t}}\\ \leq& \lim_{\Delta t\to 0}\Big(1+\frac{2\Delta t}{1-\Delta t}\Big)^{\frac{T(1-\Delta t)}{2\Delta t}\frac{2}{1-\Delta t}} = e^2.\end{aligned} \tag{25}$$

Further, noting that $\widehat{b}_{n-j} > 0$ and using Lemma 5, we have

$$\begin{aligned}&\frac{1}{1-\Delta t}\frac{\Delta t^{1-\alpha}}{\Gamma(2-\alpha)}\|v_h^0\|^2\sum_{j=0}^{n-1}b_{n-j}\Big(\frac{1+\Delta t}{1-\Delta t}\Big)^j + \frac{C\Delta t}{1-\Delta t}\sum_{j=0}^{n-1}\Big(\frac{1+\Delta t}{1-\Delta t}\Big)^j\|g(\mathbf{x},t_{n-j+\frac{1}{2}})\|^2\\ \leq&\frac{e^2}{1-\Delta t}\frac{\Delta t^{1-\alpha}}{\Gamma(2-\alpha)}\|v_h^0\|^2\sum_{j=0}^{n-1}\widehat{b}_{n-j} + \frac{C\Delta t}{1-\Delta t}\sum_{j=0}^{n-1}\|g(\mathbf{x},t_{n-j+\frac{1}{2}})\|^2\\ \leq& C\Big(\frac{T^{1-\alpha}}{\Gamma(2-\alpha)}\|v_h^0\|^2 + \max_{1\leq j\leq n}\{\|g(\mathbf{x},t_{j+\frac{1}{2}})\|^2\}\Big).\end{aligned} \tag{26}$$

Substitute (25) and (26) into (24) to arrive at

$$\Xi(v_h^{n+1},\sigma_h^{n+1}) \leq C\Big(\Xi(v_h^1,\sigma_h^1) + \frac{T^{1-\alpha}}{\Gamma(2-\alpha)}\|v_h^0\|^2 + \max_{1\leq j\leq n}\{\|g(\mathbf{x},t_{j+\frac{1}{2}})\|^2\}\Big). \tag{27}$$

Now, we estimate $\Xi(v_h^1,\sigma_h^1)$. Using (12) (c), taking $\chi_h = v_h^{\frac{1}{2}}$, and using Cauchy–Schwarz inequality, we have

$$\begin{aligned}&-(\nabla\sigma_h^{\frac{1}{2}},\nabla v_h^{\frac{1}{2}})\\ =&-(P_{\Delta t}v_h^{\frac{1}{2}},v_h^{\frac{1}{2}}) - \Big(\frac{\Delta t^{-\alpha}}{\Gamma(2-\alpha)}\Big[(\frac{1}{2})^{-\alpha}v_h^{\frac{1}{2}} - \alpha(\frac{1}{2})^{-\alpha}v_h^0\Big],v_h^{\frac{1}{2}}\Big) - \|v_h^{\frac{1}{2}}\|^2 + (g(\mathbf{x},t_{\frac{1}{2}}),v_h^{\frac{1}{2}})\\ \leq&-\frac{\|v_h^1\|^2-\|v_h^0\|^2}{2\Delta t} - (\frac{1}{2})^{-\alpha}\frac{\Delta t^{-\alpha}}{\Gamma(2-\alpha)}\|v_h^{\frac{1}{2}}\|^2\\ &+\frac{\Delta t^{-\alpha}}{\Gamma(2-\alpha)}\alpha(\frac{1}{2})^{-\alpha}(\|v_h^0\|^2+\|v_h^{\frac{1}{2}}\|^2) - \frac{1}{2}\|v_h^{\frac{1}{2}}\|^2 + \frac{1}{2}\|g(\mathbf{x},t_{\frac{1}{2}})\|^2.\end{aligned} \tag{28}$$

For $n = 0$, we sum for (18) and (28) to obtain

$$\frac{\|v_h^1\|^2 - \|v_h^0\|^2}{2\Delta t} + \frac{\|\sigma_h^1\|^2 - \|\sigma_h^0\|^2}{2\Delta t} + \left(\frac{1}{2} + \frac{2^\alpha \Delta t^{-\alpha}}{\Gamma(1-\alpha)}\right)\|v_h^{\frac{1}{2}}\|^2 \\ \leq \frac{\alpha 2^\alpha \Delta t^{-\alpha}}{\Gamma(2-\alpha)}\|v_h^0\|^2 + C(\|v_h^1\|^2 + \|v_h^0\|^2 + \|\sigma_h^1\|^2 + \|\sigma_h^0\|^2 + \|g(\mathbf{x}, t_{\frac{1}{2}})\|^2). \tag{29}$$

Noting that $1 - \alpha \leq 2^\alpha$, $(0 < \alpha < 1)$ and (23), we have, for sufficiently small Δt,

$$\Xi(v_h^1, \sigma_h^1) = \|v_h^1\|^2 + \|\sigma_h^1\|^2 + \frac{\Delta t^{1-\alpha}}{\Gamma(2-\alpha)} a_0 \|v_h^{\frac{1}{2}}\|^2 \\ \leq \|v_h^1\|^2 + \|\sigma_h^1\|^2 + \left(1 + \frac{2\Delta t^{1-\alpha}}{\Gamma(1-\alpha)}\right)\|v_h^{\frac{1}{2}}\|^2 \leq C(\|v_h^0\|^2 + \|\sigma_h^0\|^2 + \|g(\mathbf{x}, t_{\frac{1}{2}})\|^2). \tag{30}$$

Substitute (30) into (27) to obtain

$$\Xi(v_h^{n+1}, \sigma_h^{n+1}) \leq C\left(\|v_h^0\|^2 + \|\sigma_h^0\|^2 + \max_{0\leq j\leq n}\{\|g(\mathbf{x}, t_{j+\frac{1}{2}})\|^2\}\right), n \geq 0. \tag{31}$$

Combine (31) with (17) and use the Gronwall lemma to obtain

$$\|u_h^{n+1}\|^2 \leq C\left(\|u_h^0\|^2 + \|v_h^0\|^2 + \|\sigma_h^0\|^2 + \max_{0\leq j\leq n}\{\|g(\mathbf{x}, t_{j+\frac{1}{2}})\|^2\}\right), n \geq 0. \tag{32}$$

Using (31) and (32), we obtain the conclusion. □

3. A Priori Error Estimate

Now, we provide two projection operators [25] to derive a priori error estimates of our mixed finite element method.

Lemma 6. *Define the L^2 projection $\mathcal{P}_h : L^2(\Omega) \to L_h$ as*

$$(u - \mathcal{P}_h u, \varphi_h) = 0, \ \forall \varphi_h \in L_h, \tag{33}$$

with the estimate inequality

$$\|u - \mathcal{P}_h u\| + \|u_t - \mathcal{P}_h u_t\| \leq Ch^{m+1}\|u\|_{m+1}, \ \forall u \in L^2(\Omega). \tag{34}$$

Lemma 7. *Define the elliptic projection $\mathcal{Q}_h : H_0^1(\Omega) \to V_h$ as*

$$(\nabla(v - \mathcal{Q}_h v), \nabla \phi_h) = 0, \ \forall \phi_h \in V_h, \tag{35}$$

with the following inequality:

$$\|v - \mathcal{Q}_h v\| + \|v_t - \mathcal{Q}_h v_t\| + h\|v - \mathcal{Q}_h v\|_1 \leq Ch^{k+1}(\|v\|_{k+1} + \|v_t\|_{k+1}), \\ \forall v \in H_0^1(\Omega) \cap H^{k+1}(\Omega). \tag{36}$$

In what follows, we derive the proof of error estimates in L^2-norm in detail.

Theorem 2. *For $\mathcal{P}_h u(0) = u_h^0$, $\mathcal{Q}_h v(0) = v_h^0$ and $\mathcal{Q}_h \sigma(0) = \sigma_h^0$, there exists a positive constant C that is independent of space–time step length sizes $(h, \Delta t)$ and we have for $n \geq 0$*

$$\|u(t_{n+1}) - u_h^{n+1}\| + \|v(t_{n+1}) - v_h^{n+1}\| + \|\sigma(t_{n+1}) - \sigma_h^{n+1}\| \\ \leq C\left[(1 + \mu t_{n+\frac{1}{2}}^{1-\beta})h^{k+1} + \Delta t^{3-\beta} + h^{m+1}\right], \tag{37}$$

where, for the Caputo fractional derivative, we take μ as 0; for the Riemann–Liouville fractional derivative, we take μ as 1.

Proof. For convenience, we write

$$\begin{aligned} u(t_n) - u_h^n &= (u(t_n) - \mathcal{P}_h u^n) + (\mathcal{P}_h u^n - u_h^n) = \mathcal{E}^n + \mathfrak{E}^n, \\ v(t_n) - v_h^n &= (v(t_n) - \mathcal{Q}_h v^n) + (\mathcal{Q}_h v^n - v_h^n) = \mathcal{F}^n + \mathfrak{F}^n, \\ \sigma(t_n) - \sigma_h^n &= (\sigma(t_n) - \mathcal{Q}_h \sigma^n) + (\mathcal{Q}_h \sigma^n - \sigma_h^n) = \mathcal{H}^n + \mathfrak{H}^n. \end{aligned}$$

Applying triangle inequality, we have

$$\begin{aligned} \|u(t_n) - u_h^n\| &\le \|\mathcal{E}^n\| + \|\mathfrak{E}^n\|, \\ \|v(t_n) - v_h^n\| &\le \|\mathcal{F}^n\| + \|\mathfrak{F}^n\|, \\ \|\sigma(t_n) - \sigma_h^n\| &\le \|\mathcal{H}^n\| + \|\mathfrak{H}^n\|. \end{aligned} \tag{38}$$

Using Lemmas 6 and 7, we arrive at the estimates of $\|\mathcal{E}^n\|$, $\|\mathcal{F}^n\|$, and $\|\mathcal{H}^n\|$. Consequently, in the discussion below, we only need to derive the estimates of $\|\mathfrak{E}^n\|$, $\|\mathfrak{F}^n\|$, and $\|\mathfrak{H}^n\|$. Using projections (33) and (35), we have error equations as follows:

$$\begin{cases} (a)\ (P_{\Delta t}\mathfrak{E}^{n+\frac{1}{2}}, \varphi_h) = -(P_{\Delta t}\mathcal{E}^{n+\frac{1}{2}}, \varphi_h) + (\mathcal{F}^{n+\frac{1}{2}} + \mathfrak{F}^{n+\frac{1}{2}}, \varphi_h) + (R_1^{n+\frac{1}{2}}, \varphi_h), \\ (b)\ (P_{\Delta t}\mathfrak{H}^{n+\frac{1}{2}}, \psi_h) + (\nabla\mathfrak{F}^{n+\frac{1}{2}}, \nabla\psi_h) \\ \quad = -\Big(\dfrac{f_u(u^{n+1})v^{n+1} + f_u(u^n)v^n}{2} - \dfrac{f_u(u_h^{n+1})v_h^{n+1} + f_u(u_h^n)v_h^n}{2}, \psi_h\Big) \\ \qquad - (P_{\Delta t}\mathcal{H}^{n+\frac{1}{2}}, \psi_h) + (R_2^{n+\frac{1}{2}}, \psi_h), \\ (c)\ (P_{\Delta t}\mathfrak{F}^{n+\frac{1}{2}}, \chi_h) + \Big(\dfrac{\Delta t^{-\alpha}}{\Gamma(2-\alpha)}\Big[a_0\mathfrak{F}^{n+\frac{1}{2}} - \displaystyle\sum_{j=1}^{n}(a_{n-j} - a_{n-j+1})\mathfrak{F}^{j-\frac{1}{2}} \\ \qquad - (a_n - b_n)\mathfrak{F}^{\frac{1}{2}} - \widehat{b}_n\mathfrak{F}^0\Big], \chi_h\Big) + (\mathcal{F}^{n+\frac{1}{2}} + \mathfrak{F}^{n+\frac{1}{2}}, \psi_h) - (\nabla\mathfrak{H}^{n+\frac{1}{2}}, \nabla\chi_h) \\ \quad = -(P_{\Delta t}\mathcal{F}^{n+\frac{1}{2}}, \chi_h) - \Big(\dfrac{\Delta t^{-\alpha}}{\Gamma(2-\alpha)}\Big[a_0\mathcal{F}^{n+\frac{1}{2}} - \displaystyle\sum_{j=1}^{n}(a_{n-j} - a_{n-j+1})\mathcal{F}^{j-\frac{1}{2}} \\ \qquad - (a_n - b_n)\mathcal{F}^{\frac{1}{2}} - \widehat{b}_n\mathcal{F}^0\Big], \chi_h\Big) + (R_3^{n+\frac{1}{2}}, \chi_h). \end{cases} \tag{39}$$

In (39), we set $\varphi_h = \mathfrak{E}^{n+\frac{1}{2}}$, $\chi_h = \mathfrak{F}^{n+\frac{1}{2}}$, and $\psi_h = \mathfrak{H}^{n+\frac{1}{2}}$, and add the resulting equations to obtain

$$\begin{aligned} &(P_{\Delta t}\mathfrak{E}^{n+\frac{1}{2}}, \mathfrak{E}^{n+\frac{1}{2}}) + (P_{\Delta t}\mathfrak{F}^{n+\frac{1}{2}}, \mathfrak{F}^{n+\frac{1}{2}}) + (P_{\Delta t}\mathfrak{H}^{n+\frac{1}{2}}, \mathfrak{H}^{n+\frac{1}{2}}) \\ &\quad + \Big(\frac{\Delta t^{-\alpha}}{\Gamma(2-\alpha)}\Big[a_0\mathfrak{F}^{n+\frac{1}{2}} - \sum_{j=1}^{n}(a_{n-j} - a_{n-j+1})\mathfrak{F}^{j-\frac{1}{2}} - (a_n - b_n)\mathfrak{F}^{\frac{1}{2}} - \widehat{b}_n\mathfrak{F}^0\Big], \mathfrak{F}^{n+\frac{1}{2}}\Big) \\ &= -(P_{\Delta t}\mathcal{E}^{n+\frac{1}{2}}, \mathfrak{E}^{n+\frac{1}{2}}) - (P_{\Delta t}\mathcal{F}^{n+\frac{1}{2}}, \mathfrak{F}^{n+\frac{1}{2}}) - (P_{\Delta t}\mathcal{H}^{n+\frac{1}{2}}, \mathfrak{H}^{n+\frac{1}{2}}) \\ &\quad + (\mathcal{F}^{n+\frac{1}{2}} + \mathfrak{F}^{n+\frac{1}{2}}, \mathfrak{E}^{n+\frac{1}{2}}) + (\mathcal{F}^{n+\frac{1}{2}} + \mathfrak{F}^{n+\frac{1}{2}}, \mathfrak{F}^{n+\frac{1}{2}}) \\ &\quad - \Big(\frac{f_u(u^{n+1})v^{n+1} + f_u(u^n)v^n}{2} - \frac{f_u(u_h^{n+1})v_h^{n+1} + f_u(u_h^n)v_h^n}{2}, \mathfrak{H}^{n+\frac{1}{2}}\Big) \\ &\quad - \Big(\frac{\Delta t^{-\alpha}}{\Gamma(2-\alpha)}\Big[a_0\mathcal{F}^{n+\frac{1}{2}} - \sum_{j=1}^{n}(a_{n-j} - a_{n-j+1})\mathcal{F}^{j-\frac{1}{2}} - (a_n - b_n)\mathcal{F}^{\frac{1}{2}} - \widehat{b}_n\mathcal{F}^0\Big], \mathfrak{F}^{n+\frac{1}{2}}\Big) \\ &\quad + (R_1^{n+\frac{1}{2}}, \mathfrak{E}^{n+\frac{1}{2}}) + (R_2^{n+\frac{1}{2}}, \mathfrak{H}^{n+\frac{1}{2}}) + (R_3^{n+\frac{1}{2}}, \mathfrak{F}^{n+\frac{1}{2}}). \end{aligned} \tag{40}$$

Now, we need to estimate all terms on the right-hand side of (40). Using Cauchy–Schwarz inequality, we have

$$\begin{aligned}
&-(P_{\Delta t}\mathcal{E}^{n+\frac{1}{2}},\mathfrak{E}^{n+\frac{1}{2}})-(P_{\Delta t}\mathcal{F}^{n+\frac{1}{2}},\mathfrak{F}^{n+\frac{1}{2}})-(P_{\Delta t}\mathcal{H}^{n+\frac{1}{2}},\mathfrak{H}^{n+\frac{1}{2}})\\
&+(\mathcal{F}^{n+\frac{1}{2}}+\mathfrak{F}^{n+\frac{1}{2}},\mathfrak{E}^{n+\frac{1}{2}})+(\mathcal{F}^{n+\frac{1}{2}}+\mathfrak{F}^{n+\frac{1}{2}},\mathfrak{F}^{n+\frac{1}{2}})\\
\le& C(\|P_{\Delta t}\mathcal{E}^{n+\frac{1}{2}}\|^2+\|P_{\Delta t}\mathcal{F}^{n+\frac{1}{2}}\|^2+\|P_{\Delta t}\mathcal{H}^{n+\frac{1}{2}}\|^2+\|\mathcal{F}^{n+\frac{1}{2}}\|^2)\\
&+C(\|\mathfrak{E}^{n+\frac{1}{2}}\|^2+\|\mathfrak{F}^{n+\frac{1}{2}}\|^2+\|\mathfrak{H}^{n+\frac{1}{2}}\|^2).
\end{aligned} \tag{41}$$

Applying the mean value theorem and Cauchy–Schwarz inequality, we have

$$\begin{aligned}
&-\Big(\frac{f_u(u^{n+1})v^{n+1}+f_u(u^n)v^n}{2}-\frac{f_u(u_h^{n+1})v_h^{n+1}+f_u(u_h^n)v_h^n}{2},\mathfrak{H}^{n+\frac{1}{2}}\Big)\\
=&-\frac{1}{2}\Big(f_u(u^{n+1})(v^{n+1}-v_h^{n+1})+(f_u(u^{n+1})-f_u(u_h^{n+1}))v_h^{n+1}\\
&+f_u(u^n)(v^n-v_h^n)+(f_u(u^n)-f_u(u_h^n))v_h^n,\mathfrak{H}^{n+\frac{1}{2}}\Big)\\
\le&\frac{1}{2}\Big(\|f_u(u^{n+1})\|_\infty\|v^{n+1}-v_h^{n+1}\|+\|f_{uu}(\overline{\theta}^{n+1})\|_\infty\|u^{n+1}-u_h^{n+1}\|\|v_h^{n+1}\|_\infty\\
&+\|f_u(u^n)\|_\infty\|v^n-v_h^n\|+\|f_{uu}(\overline{\theta}^n)\|_\infty\|u^n-u_h^n\|\|v_h^n\|_\infty\Big)\|\mathfrak{H}^{n+\frac{1}{2}}\|.\\
\le& C(\|\mathcal{E}^{n+1}\|^2+\|\mathcal{F}^{n+1}\|^2+\|\mathcal{E}^n\|^2+\|\mathcal{F}^n\|^2+\|\mathfrak{E}^{n+1}\|^2\\
&+\|\mathfrak{F}^{n+1}\|^2+\|\mathfrak{H}^{n+1}\|^2+\|\mathfrak{E}^n\|^2+\|\mathfrak{F}^n\|^2+\|\mathfrak{H}^n\|^2),
\end{aligned} \tag{42}$$

where we use the boundedness of $\|f_u(u^n)\|_\infty$ and the following bounded inequality:

$$\|f_{uu}(\overline{\theta}^n)\|_\infty+\|v_h^n\|_\infty\le C, \tag{43}$$

where one can apply inverse inequality [25], and use a similar method as the one in [7,26].

Making use of (9), (3), Cauchy–Schwarz inequality, as well as Young inequality, we have

$$\begin{aligned}
&-\Big(\frac{\Delta t^{-\alpha}}{\Gamma(2-\alpha)}\Big[a_0\mathcal{F}^{n+\frac{1}{2}}-\sum_{j=1}^{n}(a_{n-j}-a_{n-j+1})\mathcal{F}^{j-\frac{1}{2}}-(a_n-b_n)\mathcal{F}^{\frac{1}{2}}-\widehat{b}_n\mathcal{F}^0\Big],\mathfrak{F}^{n+\frac{1}{2}}\Big)\\
&+(R_1^{n+\frac{1}{2}},\mathfrak{E}^{n+\frac{1}{2}})+(R_2^{n+\frac{1}{2}},\mathfrak{H}^{n+\frac{1}{2}})+(R_3^{n+\frac{1}{2}},\mathfrak{F}^{n+\frac{1}{2}})\\
=&-\Big(\frac{1}{\Gamma(1-\alpha)}\int_0^{t_{n+\frac{1}{2}}}\frac{\frac{\partial\mathcal{F}}{\partial s}ds}{(t_{n+\frac{1}{2}}-s)^\alpha}+\frac{\mu\mathcal{F}^0}{\Gamma(1-\alpha)}t_{n+\frac{1}{2}}^{-\alpha}+O(\Delta t^{2-\alpha}),\mathfrak{F}^{n+\frac{1}{2}}\Big)\\
&+(R_1^{n+\frac{1}{2}},\mathfrak{E}^{n+\frac{1}{2}})+(R_2^{n+\frac{1}{2}},\mathfrak{H}^{n+\frac{1}{2}})+(R_3^{n+\frac{1}{2}},\mathfrak{F}^{n+\frac{1}{2}})\\
\le& C\Big[(1+\mu t_{n+\frac{1}{2}}^{-\alpha})h^{2k+2}+\Delta t^{4-2\alpha}+\|\mathfrak{E}^{n+\frac{1}{2}}\|^2+\|\mathfrak{F}^{n+\frac{1}{2}}\|^2+\|\mathfrak{H}^{n+\frac{1}{2}}\|^2\Big].
\end{aligned} \tag{44}$$

Making a combination for (41)–(44) and using (18), we have

$$\begin{aligned}
&\frac{(\|\mathfrak{E}^{n+1}\|^2+\|\mathfrak{F}^{n+1}\|^2+\|\mathfrak{H}^{n+1}\|^2)-(\|\mathfrak{E}^{n}\|^2+\|\mathfrak{F}^{n}\|^2+\|\mathfrak{H}^{n}\|^2)}{2\Delta t}\\
&+\frac{\Delta t^{-\alpha}}{2\Gamma(2-\alpha)}\sum_{j=1}^{n+1}a_{n-j+1}\|\mathfrak{F}^{j-\frac{1}{2}}\|^2\\
=&\frac{\Delta t^{-\alpha}}{2\Gamma(2-\alpha)}\sum_{j=1}^{n}a_{n-j}\|\mathfrak{F}^{j-\frac{1}{2}}\|^2+\frac{\Delta t^{-\alpha}}{2\Gamma(2-\alpha)}\widehat{b}_n\|\mathfrak{F}^0\|^2+C\Big[(1+\mu t_{n+\frac{1}{2}}^{-\alpha})h^{2k+2}+\Delta t^{4-2\alpha}\\
&+\|P_{\Delta t}\mathcal{E}^{n+\frac{1}{2}}\|^2+\|P_{\Delta t}\mathcal{F}^{n+\frac{1}{2}}\|^2+\|P_{\Delta t}\mathcal{H}^{n+\frac{1}{2}}\|^2+\|\mathcal{E}^{n+1}\|^2+\|\mathcal{F}^{n+1}\|^2\\
&+\|\mathcal{E}^{n}\|^2+\|\mathcal{F}^{n}\|^2+\|\mathfrak{E}^{n+1}\|^2+\|\mathfrak{F}^{n+1}\|^2+\|\mathfrak{H}^{n+1}\|^2+\|\mathfrak{E}^{n}\|^2+\|\mathfrak{F}^{n}\|^2+\|\mathfrak{H}^{n}\|^2\Big].
\end{aligned} \tag{45}$$

With given conditions $\mathfrak{E}^0=0$, $\mathfrak{F}^0=0$, $\mathfrak{H}^0=0$, we use (23) to arrive at

$$\begin{aligned}
&\Xi(\mathfrak{F}^{n+1},\mathfrak{H}^{n+1})+\|\mathfrak{E}^{n+1}\|^2\\
\le&\Xi(\mathfrak{F}^{n},\mathfrak{H}^{n})+\|\mathfrak{E}^{n}\|^2+C\Delta t\Big[(1+\mu t_{n+\frac{1}{2}}^{-\alpha})h^{2k+2}+\Delta t^{4-2\alpha}\\
&+h^{2m+2}+\|\mathfrak{E}^{n+1}\|^2+\|\mathfrak{F}^{n+1}\|^2+\|\mathfrak{H}^{n+1}\|^2+\|\mathfrak{E}^{n}\|^2+\|\mathfrak{F}^{n}\|^2+\|\mathfrak{H}^{n}\|^2\Big].
\end{aligned} \tag{46}$$

Sum for (46) with respect to n to arrive at

$$\begin{aligned}
&\Xi(\mathfrak{F}^{n+1},\mathfrak{H}^{n+1})+\|\mathfrak{E}^{n+1}\|^2\\
\le&\Xi(\mathfrak{F}^{1},\mathfrak{H}^{1})+\|\mathfrak{E}^{1}\|^2+C\Delta t\sum_{j=1}^{n}\Big[(1+\mu t_{j+\frac{1}{2}}^{-\alpha})h^{2k+2}+\Delta t^{4-2\alpha}+h^{2m+2}\Big]\\
&+C\Delta t\sum_{j=1}^{n+1}\Big[\|\mathfrak{E}^{j}\|^2+\|\mathfrak{F}^{j}\|^2+\|\mathfrak{H}^{j}\|^2\Big].
\end{aligned} \tag{47}$$

For $n=0$, we use a similar derivation to the one of $n\ge 1$ and apply triangle inequality to arrive at

$$\Xi(\mathfrak{F}^{1},\mathfrak{H}^{1})+\|\mathfrak{E}^{1}\|^2\le C[(1+\mu t_{\frac{1}{2}}^{-\alpha})h^{2k+2}+\Delta t^{4-2\alpha}+h^{2m+2}]. \tag{48}$$

Substitute (48) into (47) and use the Gronwall lemma to obtain

$$\Xi(\mathfrak{F}^{n+1},\mathfrak{H}^{n+1})+\|\mathfrak{E}^{n+1}\|^2\le C\Big[(1+\mu t_{n+\frac{1}{2}}^{-\alpha})h^{2k+2}+\Delta t^{4-2\alpha}+h^{2m+2}\Big],\forall n\ge 0. \tag{49}$$

Combining (49), (34), and (36) with (38) and noting that $\alpha=\beta-1$, we complete the proof of the theorem. □

Remark 2. *Compared with the classical mixed element method for fourth-order partial differential equations, our method can approximate simultaneously three variables with optimal error estimates in L^2-norm. More importantly, we can obtain directly optimal error estimates in L^2-norm for auxiliary variables in solving fourth-order PDEs, which are difficult to achieve by using classical mixed element methods [6–8].*

4. Numerical Tests

Here, we will verify the theoretical results by numerical computing. In (1), we take space domain $\bar{\Omega}=[0,1]^2$, time interval $\bar{J}=[0,1]$, nonlinear term $f(u)=u^3-u$, initial conditions with $u(x,y,0)=0$, $u_1(x,y)=0$, and exact solution $u=t^3\sin(2\pi x)\sin(2\pi y)$; we can obtain the source term $g(x,y,t)$ and two auxiliary variables $v=3t^2\sin(2\pi x)\sin(2\pi y)$ and $\sigma=-t^3\sin(2\pi x)\sin(2\pi y)(8\pi^2+t^6\sin(2\pi x)^2\sin(2\pi y)^2-1)$. In the following numerical calculations, the order of convergence in space is calculated by the following formula with a sufficiently small time step size Δt

$$\text{Order} = \log_{\frac{h_1}{h_2}} \frac{\|\phi - \phi_{h_1}\|}{\|\phi - \phi_{h_2}\|},$$

where h_k $(k = 1, 2)$ represents different space mesh step lengths.

For implementing the new mixed element algorithm, we approximate the spatial direction by the finite element method with the basis function $P(x, y) = a + bx + cy + dxy$ and discretize the time direction by using the modified $L1$ Crank–Nicolson scheme. In Table 1, by taking the fixed time mesh parameter $\Delta t = 1/200$, changed spatial step length sizes $h = \sqrt{2}/9$, $\sqrt{2}/16$ and $\sqrt{2}/25$, and different parameters $\beta = 1.1, 1.5, 1.9$, we show the L^2-norm error estimates and second-order convergence data in space. In Tables 2 and 3, we compute the convergence results v and σ, respectively. From Tables 1–3, one can see that the numerical method is effective for solving nonlinear fourth-order fractional diffusion-wave equation models with a smooth solution.

Table 1. The convergence results for u with $\Delta t = 1/200$.

β	h	$\|u - u_h\|$	Order	CPU-Time (s)
1.1	$\sqrt{2}/9$	4.1181×10^{-2}		1.13
	$\sqrt{2}/16$	1.3341×10^{-2}	1.9590	4.04
	$\sqrt{2}/25$	5.5001×10^{-3}	1.9855	19.03
1.5	$\sqrt{2}/9$	4.1175×10^{-2}		1.10
	$\sqrt{2}/16$	1.3339×10^{-2}	1.9590	4.02
	$\sqrt{2}/25$	5.4991×10^{-3}	1.9855	19.40
1.9	$\sqrt{2}/9$	4.1169×10^{-2}		1.14
	$\sqrt{2}/16$	1.3336×10^{-2}	1.9591	4.07
	$\sqrt{2}/25$	5.4977×10^{-3}	1.9857	19.33

Table 2. The convergence results for v with $\Delta t = 1/200$.

β	h	$\|v - v_h\|$	Order	CPU-Time (s)
1.1	$\sqrt{2}/9$	1.2293×10^{-1}		1.13
	$\sqrt{2}/16$	3.9815×10^{-2}	1.9594	4.04
	$\sqrt{2}/25$	1.6417×10^{-2}	1.9851	19.03
1.5	$\sqrt{2}/9$	1.2292×10^{-1}		1.10
	$\sqrt{2}/16$	3.9816×10^{-2}	1.9593	4.02
	$\sqrt{2}/25$	1.6417×10^{-2}	1.9852	19.40
1.9	$\sqrt{2}/9$	1.2293×10^{-1}		1.14
	$\sqrt{2}/16$	3.9819×10^{-2}	1.9592	4.07
	$\sqrt{2}/25$	1.6418×10^{-2}	1.9852	19.33

Table 3. The convergence results for σ with $\Delta t = 1/200$.

β	h	$\|\sigma - \sigma_h\|$	Order	CPU-Time (s)
1.1	$\sqrt{2}/9$	$1.8858 \times 10^{+0}$		1.13
	$\sqrt{2}/16$	5.9584×10^{-1}	2.0024	4.04
	$\sqrt{2}/25$	2.4398×10^{-1}	2.0007	19.03
1.5	$\sqrt{2}/9$	$1.8853 \times 10^{+0}$		1.10
	$\sqrt{2}/16$	5.9568×10^{-1}	2.0025	4.02
	$\sqrt{2}/25$	2.4391×10^{-1}	2.0007	19.40
1.9	$\sqrt{2}/9$	$1.8849 \times 10^{+0}$		1.14
	$\sqrt{2}/16$	5.9550×10^{-1}	2.0026	4.07
	$\sqrt{2}/25$	2.4381×10^{-1}	2.0010	19.33

5. Concluding Remarks

From the calculated results in Tables 1–3, one can see that our method for solving fourth-order fractional diffusion-wave equations in this article can obtain optimal error estimates in L^2-norm for three variables, which is in agreement with the derived theoretical results. These results for auxiliary variables are difficult to achieve directly by using classical mixed element methods [6–8].

In the future, we will improve our mixed element method by combining other techniques [7,27,28] with high-order time approximate schemes and develop their optimal numerical theories.

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Abbreviations

The following abbreviations are used in this manuscript:

MDPI	Multidisciplinary Digital Publishing Institute
DOAJ	Directory of Open Access Journals
TLA	Three-letter acronym
LD	Linear dichroism

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