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A Study on Controllability of a Class of Impulsive Fractional Nonlinear Evolution Equations with Delay in Banach Spaces

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Abstract: Under a new generalized definition of exact controllability we introduced and with a appropriately constructed time delay term in a special complete space to overcome the delay-induced-difficulty, we establish the sufficient conditions of the exact controllability for a class of impulsive fractional nonlinear evolution equations with delay by using the resolvent operator theory and the theory of nonlinear functional analysis. Nonlinearity in the system is only supposed to be continuous rather than Lipschitz continuous by contrast. The results obtained in the present work are generalizations and continuations of the recent results on this issue. Further, an example is presented to show the effectiveness of the new results.

Keywords: controllability; impulsive fractional evolution equations; delay; measure of noncompactness; mild solution; fixed point theorem

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1. Introduction

This paper's primary objective is to investigate the exact controllability of the following impulsive fractional nonlinear evolution equations with delay in Banach spaces:

$$\begin{cases} D^\gamma x(t) = Ax(t) + f(t, x(t), x_t) + Bu(t), \quad a.e.\ t \in I := [0, a], \\ \Delta x(t_i) = x(t_i^+) - x(t_i^-) = I_i(x(t_i)), \quad i = 1, 2, \cdots, m, \\ x(t) = \phi(t), \quad t \in [-b, 0], \end{cases} \tag{1}$$

where D^γ represents the Caputo derivative of order $\gamma \in (0,1)$. The state $x(\cdot)$ takes values in X, control function u is given in $L^2(I, U)$, and $B : U \to X$ is a bounded linear operator where X and U are Banach spaces. $0 = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = a < +\infty$. $A : \mathcal{D} \subset X \to X$ is a closed linear unbounded operator on X with dense domain $\mathcal{D}$. x_t represents the history of the state function that will be specified in (2). $\phi(t) \in C([-b, 0], X)$. The given functions f and I_i $(i = 1, 2, \cdots, m)$ are supposed to satisfy some appropriate assumptions that will be specified later.

In the last few decades, the topic of fractional calculus has received considerable and extensive attention. The modelling of many mathematical and biological problems by fractional differential equations has more superiority and accuracy than classical integral-order ones. In view of its extensive applications in the area of physics, chemistry, mathematics, medicine and economics, a growing number of researchers have devoted generous energy to the study of various types of fractional differential equations. For further details of the recent works, we refer readers to [1–5].

It is well known that impulse and delay embody lots of rich and varied dynamic behaviors. The investigation of various dynamical systems with impulsive interference and time delay effects has obtained more and more attention due to their important and potential applications in signal and image processing, weather predicting, artificial intelligence and some other optimization problems. For more details, one can see [6–8].

It is noted that the research on the controllability of fractional differential equations is becoming more and more active, since controllability is a quite important concept in mathematics and control theory. As one of the most mainstream research direction, exact controllability of many kinds of integral-order and fractional-order control systems have been well investigated by taking advantage of diverse tools and methods in some recent literatures. For example, S. Ji et al. [9] studied the exact controllability of a class of integral-order impulsive differential equations by using the measure of noncompactness and fixed point theorem under a compact condition imposed on the nonlocal item. J. Wang and Y. Zhou [10] investigated a class of fractional differential systems without assuming the compactness of the semigroup. They discussed the exact controllability of the considered control systems under the assumption that the nonlinearity satisfied Lipschitz continuity. In [11], J. Du et al. obtained a result of exact controllability for some fractional neutral integro-differential evolution systems with delay and nonlocal conditions. The Lipschitz condition and some other growth conditions on nonlocal item and nonlinearity are still necessary. Z. Tai [12] proved the exact controllability results for fractional impulsive neutral integro-differential systems in Banach spaces. The results are obtained by utilizing Banach contraction mapping theorem due to the Lipschitz conditions of the systems. In addition, some excellent results of exact controllability for various fractional differential equations have also been established recently [7,10–23], but the limitation is also that the functions in the systems are either Lipschitz continuous, compact or satisfy some special growth suppositions. Although the exact controllability studied in [13] does not require the nonlinear term to satify Lipschitz condition, the considered evolution system in [13] have no effects of time delay and impulse. At present, it seems that the exact controllability results of fractional evolution equations with both impulses and delays are rare [12,21,22]. We point out that nonlinearities and impulsive items in these papers satisfy special growth assumptions [21], Lipschitz condition [12,22], and semigroups together with the resolvent operators of some systems possess compactness, which still show the limitation to a certain extent in practical problems. Therefore, it seems interesting whether the exact controllability of the impulsive fractional evolution equations with delay can be established via noncompact resolvent operators together with the nonlinearity satisfying continuity rather than Lipschitz continuity.

Inspired by the abovementioned papers and the ideas adopted in [13], in this work, we present a new depiction of the exact controllability of the system (1) by using the theory of resolvent operator and the theory of nonlinear functional analysis. The main contributions of this article are as follows. (i) The Lipschitz and other restrictive conditions on nonlinear and impulsive items have been removed. (ii) The application of C_0-semigroup based on probability density function [24] is replaced by resolvent operators without compact conditions, which is different from most of the existing literatures such as [7,10–12,17,21,22,25,26]. (iii) With the properly defined delay item in a corresponding complete space we introduced, we have solved the delay-induced-difficulty during the investigation of exact controllability by measures of noncompactness.

The organization of this work is as follows. Some necessary notations, definitions and lemmas are introduced in next section. In the third section, sufficient conditions ensuring exact controllability of the addressed systems are provided. An example is worked out in the last section to illustrate our theory of the main results.

2. Preliminaries

We denote by X a Banach space with the norm $\|\cdot\|$. By $C(I,X)$ and $C([-b,a],X)$ we denote the spaces of continuous functions from I into X, $[-b,a]$ into X with suprema norms $\|\cdot\|_{C(I,X)}$ and $\|\cdot\|_{C([-b,a],X)}$, respectively. For the case of $a=0$, norm $\|\cdot\|_{C([-b,a],X)}$ is abbreviated as $\|\cdot\|_b$. Also consider the usual Banach space $PC(I,X)=\{x: I\to X \mid x\in C((t_k,t_{k+1}],X), x(t_k^-)\ \mathit{and}\ x(t_k^+)\ \mathit{exist\ with}\ x(t_k^-)=x(t_k),\ k=1,2,\cdots,m\}$, with the norm $\|x\|_{PC}=\sup_{t\in I}\{\|x(t)\|\}$. $\mathcal{D}$ stands for the domain of the operator A in (1) with the graph norm $\|x\|_{\mathcal{D}}=\|x\|+\|Ax\|$. Denote by U a Banach space with the norm $\|\cdot\|_U$. By $C^\gamma(I,X), \gamma\in(0,1)$, we denote the space of all the γ-Hölder continuous functions from I into X with the norm $\|x\|_{C^\gamma(I,X)} = \|x\|_{C(I,X)} + [|x|]_{C^\gamma(I,X)}$, where $[|x|]_{C^\gamma(I,X)} = \sup\limits_{t,s\in I, t\neq s} \frac{\|x(t)-x(s)\|}{(t-s)^\gamma}$. For any measurable function $x : I \to R$, define the norm

$$\|x\|_{L^p(I)} = \begin{cases} \left(\int_I |x(t)|^p dt\right)^{\frac{1}{p}}, & 1 \leq p < \infty, \\ \inf\limits_{\mu(\bar{I})=0} \{\sup\limits_{t\in I-\bar{I}} |x(t)|\}, & p = \infty, \end{cases}$$

where $\mu(\bar{I})$ is the Lebesgue measure on $\bar{I}$. Let $\mathcal{L}(X,Y)$ be the space of all bounded linear operators from X into Banach space Y equipped with operator norm $\|\cdot\|_{\mathcal{L}(X,Y)}$.

Introduce a complete and integrable space $L([-b,0],X)$ which contains all the integrable functions from $[-b,0]$ into X. For $x \in PC(I,X)$ and $t \in I$, define a piecewise function as follows:

$$x_t(\theta) = \begin{cases} x(t+\theta), & t+\theta \geq 0, \\ \phi(t+\theta), & t+\theta \leq 0, \end{cases} \quad (2)$$

for every $\theta \in [-b,0]$, where ϕ is the same as in (1). It is not hard to verify that $x_t \in L([-b,0],X)$.

Remark 1. *Based on (2) and Lemma 4 together with Lemma 5 which we will present in the following discussions, it is much more convenient to study the exact controllability of system (1) by using the theory of noncompact measures.*

Next we list the well known definitions as follows.

Definition 1. *([27]) The fractional integral with order $\gamma > 0$ for a function $u : (0,+\infty) \to R$ can be defined as*

$$I^\gamma_{0+}u(t) = \frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1}u(s)ds,$$

provided that the right side integral is pointwise defined on $[0,+\infty)$.

Definition 2. *([27]) The Caputo fractional derivative with order $\gamma > 0$ for a function $u : (0,\infty) \to R$ is written as*

$$D^\gamma_{0+}u(t) = \frac{1}{\Gamma(n-\gamma)}\int_0^t \frac{u^{(n)}(s)}{(t-s)^{\gamma-n+1}}ds,$$

where $n = [\gamma] + 1$, provided the right side integral is pointwise defined on $[0,\infty)$.

Definition 3. *([28]) A family of bounded linear operators $\{\wp(t)\}_{t\geq 0} \subset \mathcal{L}(X)$ on X is called a resolvent operator of integral equation*

$$x(t) = \int_0^t \frac{(t-s)^{\gamma-1}}{\Gamma(\gamma)} Ax(s)ds,\ t \geq 0, \quad (3)$$

if the following assumptions are satisfied:

(i) $\wp(t)$ is strongly continuous on R^+ and $\wp(0) = I$;

(ii) $\wp(t)\mathcal{D} \subset \mathcal{D}$, $A\wp(t)x = \wp(t)Ax$ for every $t \geq 0$ and $x \in \mathcal{D}$;

(iii) the resolvent equation holds

$$\wp(t)x = x + \int_0^t \frac{(t-s)^{\gamma-1}}{\Gamma(\gamma)} A\wp(s)xds.$$

Definition 4. *([28]) A resolvent operator $\wp(t)$ of (3) is called differentiable, if there is a function $\varphi \in L^1_{loc}(R^+)$ such that the following inequality holds:*

$$\|\dot{\wp}(t)x\| \leq \varphi(t)\|x\|_{\mathcal{D}} \ \text{a.e. on} \ R^+, \ \forall x \in \mathcal{D},$$

where $\wp(\cdot)x \in W^{1,1}_{loc}(R^+, X)$ for each $x \in \mathcal{D}$.

Consider the equation

$$x(t) = g(t) + \int_0^t \frac{(t-s)^{\gamma-1}}{\Gamma(\gamma)} Ax(s)ds, \ \ t \in I, \tag{4}$$

where $g \in L^1(I, X)$. According to [28], the mild solution of (4) can be defined as follows.

Definition 5. *We call $x \in C(I, X)$ a mild solution for (4) if $\int_0^t \frac{(t-s)^{\gamma-1}}{\Gamma(\gamma)} x(s)ds \in \mathcal{D}$, and satisfies*

$$x(t) = g(t) + A\int_0^t \frac{(t-s)^{\gamma-1}}{\Gamma(\gamma)} x(s)ds,$$

for each $t \in I$.

Now, let us give a useful lemma about differentiable resolvent operator from which one can get the equivalent definition of mild solution for Equation (4).

Lemma 1. *([28]) Assume that $\wp(t)$ is a differentiable resolvent operator for (4) and $g \in C(I, \mathcal{D})$. Then*

$$x(t) = \int_0^t \dot{\wp}(t-s)g(s)ds + g(t), \ \ t \in I,$$

is a mild solution of (4).

We now recall some useful properties of Kuratowski measures of noncompactness. For more details, please refer [29].

Lemma 2. *Let X be a Banach space and $\beta(\cdot)$ be the Kuratowski measures of noncompactness which is given by $\beta(\Omega) = \inf\{\delta > 0 : \Omega = \bigcup_{i=1}^k \Omega_i$ with $diam(\Omega_i) \leq \delta, \ i = 1, 2, \cdots, k\}$ for a bounded subset Ω in X.*
(I) Let D_1, D_2 be bounded sets of X and $\lambda \in R$. Then
(i) $\beta(D_1) = 0 \Leftrightarrow D_1$ is relatively compact;
(ii) $\beta(\lambda D_1) = |\lambda|\beta(D_1)$;
(iii) $\beta(D_1 + D_2) \leq \beta(D_1) + \beta(D_2)$;
(II) Assume that $D = \{u_n\}$ is a countable set of strongly measurable functions from I into Banach space X, and there has a function $\psi \in L^1(I)$ such that $\|u_n(t)\| \leq \psi(t)$ a.e.$t \in I, \ n = 1, 2, \cdots,$ then $\beta(D(t))$ is integrable on I, and satisfies

$$\beta\left(\left\{\int_I u_n(t)dt : n \in N\right\}\right) \leq 2\int_I \beta(D(t))dt.$$

For convenience, the Kuratowski measures of noncompactness of a bounded subset in spaces X, $PC(I, X)$ and $L([-b, 0], X)$ are all denoted by $\beta(\cdot)$, on the premise of no confusion.

Lemma 3. *(Mönch) Suppose X to be a Banach space and $D \subset X$ is a closed and convex set, $x_0 \in D$. If $A : D \to D$ is continuous and satisfies: $C \subset D$ countable, $C \subset \overline{co}(\{x_0\} \bigcup A(C)) \Rightarrow C$ is relatively compact. Then A has a fixed point in D.*

At last of this section, we present two important lemmas as follows.

Lemma 4. *Suppose that x_n converges to x_0 in $PC(I,X)$ as $n \to +\infty$. Then $(x_n)_t$ converges to $(x_0)_t$ in $L([-b,0],X)$ for each $t \in I$ as $n \to +\infty$.*

Proof. By (2), we can obtain
(i) if $t \leq b$, then

$$\begin{aligned} \|(x_n)_t - (x_0)_t\|_{L[-b,0]} &= \int_{-b}^{0} \|(x_n)_t(\theta) - (x_0)_t(\theta)\| d\theta \\ &= \int_{t-b}^{t} \|(x_n)_t(\theta) - (x_0)_t(\theta)\| d(t+\theta) \\ &= \int_{0}^{t} \|x_n(t+\theta) - x_0(t+\theta)\| d(t+\theta) \\ &= \int_{0}^{t} \|x_n(s) - x_0(s)\| ds. \end{aligned}$$

(ii) if $t \geq b$, then

$$\begin{aligned} \|(x_n)_t - (x_0)_t\|_{L[-b,0]} &= \int_{-b}^{0} \|(x_n)_t(\theta) - (x_0)_t(\theta)\| d\theta \\ &= \int_{t-b}^{t} \|(x_n)_t(\theta) - (x_0)_t(\theta)\| d(t+\theta) \\ &= \int_{t-b}^{t} \|x_n(t+\theta) - x_0(t+\theta)\| d(t+\theta) \\ &= \int_{t-b}^{t} \|x_n(s) - x_0(s)\| ds. \end{aligned}$$

Obviously, (i) and (ii) imply that

$$\|(x_n)_t - (x_0)_t\|_{L[-b,0]} \leq b\|x_n - x_0\|_{PC(I,X)}, \ \forall t \in I.$$

This completes the proof. □

Lemma 5. *Let $D = \{x_n\}_{n=1}^{\infty}$ be a bounded countable sequence in $PC(I,X)$. Then for each $t \in I$, one has*

$$\beta(D_t) \leq b\beta(D),$$

where $D_t = \{(x_n)_t\}_{n=1}^{\infty}$.

Proof. From the definition of Kuratowski measures of noncompactness in Lemma 2, we can infer that for any $\varepsilon > 0$, there is a partition $D = \bigcup_{i=1}^{k} D_i$ such that

$$diam(D_i) < \beta(D) + \varepsilon, \ i = 1, 2, \cdots, k. \tag{5}$$

As already done in Lemma 4, we also deduce

$$\|(x_n)_t - (x_m)_t\|_{L[-b,0]} \leq b\|x_n - x_m\|_{PC(I,X)}, \ t \in I. \tag{6}$$

Hence, from (5) and (6) one derives

$$diam(D_{i_t}) \leq b diam(D_i) < b(\beta(D) + \varepsilon), \ i = 1, 2, \cdots, k,$$

which means

$$\beta(D_t) < b\beta(D) + b\varepsilon.$$

The arbitrariness of ε implies that the conclusion is true. □

3. Main Results

In this section, we always suppose the resolvent operator $\{\wp(t)\}_{t\geq 0}$ for (4) to be differentiable. Based on [30], Definition 5 and the Riemann-Liouville standard fractional integral, the mild solution of system (1) can be defined as below.

Definition 6. *For any given $u \in L^2(I,U)$, a function $x \in PC(J,X)$ is called a mild solution of system (1) on J, provided that $\int_{t_{k-1}}^{t_k}(t_k-s)^{\gamma-1}x(s)ds$, $\int_{t_k}^{t}(t-s)^{\gamma-1}x(s)ds \in \mathcal{D}$ for all $0<t_k<t$, $t\in[0,\tau]$ and*

$$x(t)=\begin{cases}\phi(0)+\frac{1}{\Gamma(\gamma)}A\left(\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{\gamma-1}x(s)ds+\int_{t_k}^{t}(t-s)^{\gamma-1}x(s)ds\right)\\ \qquad+\frac{1}{\Gamma(\gamma)}\left(\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{\gamma-1}(f(s,x(s),x_s)+Bu(s))ds\right)\\ \qquad+\frac{1}{\Gamma(\gamma)}\int_{t_k}^{t}(t-s)^{\gamma-1}(f(s,x(s),x_s)+Bu(s))ds+\sum_{0<t_k<t}I_k(x(t_k)),\quad t\in[0,\tau],\\ \phi(t),\quad t\in[-b,0],\end{cases}$$

where $J=[-b,\tau]$, $\tau\in(0,a]$.

Based on the exact controllability considered in [13], we give the following definition

Definition 7. *System (1) is exact controllability on $I=[0,a]$ if for any initial function $\phi(t)\in C([-b,0],X)$ and $x_1\in X$, there has a control $u\in L^2(I,U)$ and a constant $\tau\in(0,a]$ such that the mild solution x of (1) on $J=[-b,\tau]$ satisfies $x(\tau)=x_1$.*

Remark 2. *In contrast with the existing definitions in [9–11,15,17], our target point x_1 taking value at $\tau\in(0,a]$ is likely to be achieved ahead of time a, which means that, from a conceptual point of view, it can be considered as an generalization of the existing notion of exact controllability.*

In order to obtain the main results, we present the hypotheses as follows:

(H1) $f\in C(I\times X\times L([-b,0],X),\mathcal{D})$ and satisfies
(i) f maps bounded sets in $I\times X\times L([-b,0],X)$ into bounded sets in $\mathcal{D}$;
(ii) There exist a constant $q\in(0,\gamma)$ and a function $l\in L^{\frac{1}{q}}(I,R^+)$ such that for any bounded subsets $D_1\subset X$, $D_2\subset L([-b,0],X)$,

$$\beta(f(t,D_1,D_2))\leq l(t)(\beta(D_1)+\beta(D_2)),\ t\in I.$$

(H2) (i) The linear operator $B: L^2(I,U)\to L^1(I,\mathcal{D})$ is bounded, and there exists a constant $M_1>0$ satisfying $\|B\|_{\mathcal{L}(U,\mathcal{D})}\leq M_1$;
(ii) Linear operators $\mathfrak{J}(t), t\in I$, denoted by $\mathfrak{J}(\cdot)$ from $L^2(I,U)$ to X defined as

$$\begin{aligned}\mathfrak{J}(t)u &= \frac{1}{\Gamma(\gamma)}\left(\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{\gamma-1}Bu(s)ds+\int_{t_k}^{t}(t-s)^{\gamma-1}Bu(s)ds\right)\\ &+\frac{1}{\Gamma(\gamma)}\int_0^t\dot{\wp}(t-s)\left(\sum_{0<t_k<s}\int_{t_{k-1}}^{t_k}(t_k-\eta)^{\gamma-1}Bu(\eta)d\eta+\int_{t_k}^{s}(s-\eta)^{\gamma-1}Bu(\eta)d\eta\right)ds, t\in I,\end{aligned}$$

have invertible operators $\mathfrak{J}^{-1}(\cdot)$ taking values in $L^2(I,U)/ker\mathfrak{J}(\cdot)$, which satisfy, for some constant $M_2 > 0$, $\sup \|\mathfrak{J}^{-1}(\cdot)\|_{\mathcal{L}(X,L^2(I,U)/ker\mathfrak{J}(\cdot))} \leq M_2$, and there is a constant $p \in (0,\gamma)$ and a function $k \in L^{\frac{1}{p}}(I,R^+)$ satisfying

$$\beta(\mathfrak{J}^{-1}(\cdot)(D)(s)) \leq k(s)\beta(D),\ s \in I,$$

for any bounded subset $D \subset X$.
(H3) $I_i : X \to \mathcal{D}\ (i = 1,2,\cdots,m)$ is continuous and satisfies
(i) There exists a constant C_0 such that

$$\sup\{\|I_i(x)\|_{\mathcal{D}},\ x \in X,\ i = 1,2,\cdots,m\} \leq C_0;$$

(ii) There exist constants $k_i \geq 0$ such that

$$\beta(I_i(D)) \leq k_i\beta(D),\ i = 1,2,\cdots,m,$$

hold for each bounded subset $D \subset X$.
(H4)

$$\Big(1 + 2\|\varphi\|_{L^1(I)}\Big)(1 + M)\sum_{i=1}^{m} k_i < 1,$$

where

$$M = \frac{1 + 2\rho M_1\Big(1 + 2\|\varphi\|_{L^1(I)}\Big)}{\Gamma(\gamma)},\ \ \rho = (m+1)a^{\gamma}\left(\frac{1-p}{\gamma-p}\right)^{1-p}\|k\|_{L^{\frac{1}{p}}},$$

and φ is the function mentioned in Definition 4.

In the sequel, suppose R_0 to be a fixed constant such that $R_0 > (\|\phi(0)\|_{\mathcal{D}} + mC_0)(1+\|\varphi\|_{L^1(I)})$. By (H1), let

$$M_0 = \sup\Big\{\|f(t,x,y)\|_{\mathcal{D}} : \|x\|_{PC(I,X)} \leq R_0,\ \|y\|_{L[-b,0]} \leq b(\|\phi\|_b + R_0),\ t \in I\Big\}.$$

For simplicity, take

$$\begin{aligned}\Theta(t;x;u) \ &= \ \frac{1}{\Gamma(\gamma)}\left(\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{\gamma-1}f(s,x(s),x_s)ds + \int_{t_k}^{t}(t-s)^{\gamma-1}f(s,x(s),x_s)ds\right)\\ &\quad + \frac{1}{\Gamma(\gamma)}\left(\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{\gamma-1}Bu(s)ds + \int_{t_k}^{t}(t-s)^{\gamma-1}Bu(s)ds\right),\end{aligned}$$

set

$$I(t;x) = \sum_{0<t_k<t} I_k(x(t_k)),$$

and present two notations as follows:

$$\lambda = 2(b+1)\left(\frac{1-q}{\gamma-q}\right)^{1-q}\|l\|_{L^{\frac{1}{q}}},\ \ \mu = \frac{(m+1)a^{\gamma}}{\Gamma(\gamma+1)}.$$

In view of condition (H2) and (H4), for any $x(\cdot) \in PC(I,X)$ and any $x_1 \in X,\ t \in I$, define a control

$$u_x(t) := \mathfrak{J}^{-1}(\tau)\left(x_1 - \phi(0) - \Theta_f(\tau;x) - I(\tau;x) - \int_0^{\tau}\wp(\tau-s)(\phi(0) + \Theta_f(s;x) + I(s;x))ds\right)(t),$$

where

$$\tau=\min\left\{a,\ \left(\frac{\left(R_0-(\|\phi(0)\|_{\mathcal{D}}+mC_0)\left(1+\|\varphi\|_{L^1(I)}\right)\right)\Gamma(\gamma+1)}{(m+1)(M_0+M_1M_3)\left(1+\|\varphi\|_{L^1(I)}\right)}\right)^{\frac{1}{\gamma}},\ \left(\frac{\left(1-\left(1+2\|\varphi\|_{L^1(I)}\right)(1+M)\sum_{i=1}^{m}k_i\right)}{2\left(1+2\|\varphi\|_{L^1(I)}\right)(m+1)\lambda M}\right)^{\frac{1}{\gamma}}\right\}, \tag{7}$$

$$M_3=M_2\left(\left(1+\|\varphi\|_{L^1(I)}\right)(\|\phi(0)\|_{\mathcal{D}}+\mu M_0+mC_0)+\|x_1\|\right);$$

and

$$\Theta_f(s;x)=\frac{1}{\Gamma(\gamma)}\left(\sum_{0<t_k<s}\int_{t_{k-1}}^{t_k}(t_k-\eta)^{\gamma-1}f(\eta,x(\eta),x_\eta)d\eta+\int_{t_k}^{s}(s-\eta)^{\gamma-1}f(\eta,x(\eta),x_\eta)d\eta\right),s\in[0,\tau].$$

Suppose that $\tau\in(t_i,t_{i+1}]$, and then we let $J_0=[-b,0]$, $J_k=(t_{k-1},t_k]$, $k=1,2,\cdots,i$, $J_{i+1}=(t_i,\tau]$. Denote

$$\Omega=\left\{x\in PC(J,X):\|x\|_{PC([0,\tau],X)}\le R_0,\ \sup_{t\in[0,\tau]}\|x_t\|_{L[-b,0]}\le b(\|\phi\|_b+R_0);\ x(t)=\phi(t),\ t\in[-b,0]\right\}.$$

Then Ω is a closed convex set in $PC(J,X)$. By means of Lemma 1, we can define an operator $\mathcal{P}:PC(J,X)\to PC(J,X)$ by

$$(\mathcal{P}x)(t)=\begin{cases}\phi(0)+\Theta(t;x;u_x)+I(t;x)\\ +\int_0^t\wp(t-s)(\phi(0)+\Theta(s;x;u_x)+I(s;x))ds,\ \ t\in[0,\tau],\\ \phi(t),\ \ t\in[-b,0].\end{cases} \tag{8}$$

To simplify the proof of our main result, the following lemmas are needed.

Lemma 6. *Suppose that $f\in C(I\times X\times L([-b,0],X),X)$ and $u\in L^2(I,U)$. Then $\Theta(\cdot;x;u)\in C^\gamma(J_i,X)$, $i=1,2,\cdots,m+1$, and*

$$[|\Theta(\cdot;x;u)|]_{C^\gamma}\le\frac{2}{\gamma}\left(\|f\|_{C(I,X)}+\|B\|_{\mathcal{L}(U,X)}\cdot\sup_{t\in I}\|u(t)\|_U\right).$$

Proof. For $t\in J_i$ and $h>0$ such that $t+h\in J_i$, one has

$$\begin{aligned}&\|\Theta(t+h;x;u)-\Theta(t;x;u)\|\\ \le\ &\int_{t_{i-1}}^{t}((t-s)^{\gamma-1}-(t+h-s)^{\gamma-1})(\|f(s,x(s),x_s)\|+\|Bu(s)\|)ds\\ &+\int_t^{t+h}(t+h-s)^{\gamma-1}(\|f(s,x(s),x_s)\|+\|Bu(s)\|)ds\\ \le\ &\left(\frac{(t-t_{i-1})^\gamma-(t-t_{i-1}+h)^\gamma+h^\gamma}{\gamma}+\frac{h^\gamma}{\gamma}\right)\left(\|f\|_{C(I,X)}+\|B\|_{\mathcal{L}(U,X)}\cdot\sup_{t\in I}\|u(t)\|_U\right)\\ \le\ &\frac{2h^\gamma}{\gamma}\left(\|f\|_{C(I,X)}+\|B\|_{\mathcal{L}(U,X)}\cdot\sup_{t\in I}\|u(t)\|_U\right),\end{aligned}$$

which shows that $[|\Theta(\cdot;x;u)|]_{C^\gamma}\le\frac{2}{\gamma}\left(\|f\|_{C(I,X)}+\|B\|_{\mathcal{L}(U,X)}\cdot\sup_{t\in I}\|u(t)\|_U\right)$ and $\Theta(\cdot;x;u)\in C^\gamma(J_i,X)$, $i=1,2,\cdots,m+1$. This completes the proof. □

Lemma 7. *Assume that condition (H1) holds. Then the operator $T:PC(I,X)\to PC(I,X)$ defined by*

$$(Tx)(t)=\int_{t^*}^{t}(t-s)^{\gamma-1}f(s,x(s),x_s)ds,\quad\forall t^*,\ t\in I,$$

satisfies $\beta(T(D)(t))\le\lambda a^\gamma\beta(D)$ for any countable bounded set $D\subset PC(I,X)$.

Proof. No loss of generality, we may suppose that the bounded countable set $D = \{x_n\}_{n=1}^{\infty}$. By using Lemma 5, we have

$$\beta(\{(x_n)_s\}) \leq b\beta(\{x_n\}).$$

From Lemma 2 (II) and the Hölder inequality, it follows that

$$\begin{aligned}
\beta(T(D)(t)) &= \beta(\{T(x_n)(t)\}) \\
&\leq 2\int_{t^*}^{t}(t-s)^{\gamma-1}\beta(\{f(s,x_n(s),(x_n)_s)\})ds \\
&\leq 2\int_{t^*}^{t}(t-s)^{\gamma-1}l(s)(\beta(\{x_n\})+\beta(\{(x_n)_s\}))ds \\
&\leq 2(b+1)\int_{t^*}^{t}(t-s)^{\gamma-1}l(s)ds\cdot\beta(\{x_n\}) \\
&\leq 2(b+1)\left(\int_{t^*}^{t}[(t-s)^{\gamma-1}]^{\frac{1}{1-q}}ds\right)^{1-q}\left(\int_{t^*}^{t}l(s)^{\frac{1}{q}}ds\right)^{q}\cdot\beta(\{x_n\}) \\
&\leq 2(b+1)\left(\frac{1-q}{\gamma-q}\right)^{1-q}(t-t^*)^{\gamma-q}\|l\|_{L^{\frac{1}{q}}}\cdot\beta(\{x_n\}) \\
&\leq \lambda a^{\gamma}\beta(D).
\end{aligned}$$

This completes the proof. □

Lemma 8. *Assume that conditions (H1)(i), (H2), (H3)(i) and (H4) hold. Then $\{\mathcal{P}x : x \in \Omega\}$ is equicontinuous on each J_i $(i = 0, 1, \cdots, m+1)$.*

Proof. The first step is to demonstrate that $\mathcal{P}(\Omega) \subseteq \Omega$. From (H2), one has

$$\begin{aligned}
&\|u_x(t)\|_U \\
\leq\ & M_2\Bigg(\|x_1\|+\|\phi(0)\|_{\mathcal{D}}+\sum_{0<t_k<\tau}\|I_k(x(t_k))\|_{\mathcal{D}}+\int_0^{\tau}\varphi(\tau-s)(\|\phi(0)\|_{\mathcal{D}}+\sum_{0<t_k<s}\|I_k(x(t_k))\|_{\mathcal{D}})ds\Bigg) \\
&+\tfrac{M_2}{\Gamma(\gamma)}\Bigg(\sum_{0<t_k<\tau}\int_{t_{k-1}}^{t_k}(t_k-s)^{\gamma-1}\|f(s,x(s),x_s)\|_{\mathcal{D}}ds+\int_{t_k}^{\tau}(\tau-s)^{\gamma-1}\|f(s,x(s),x_s)\|_{\mathcal{D}}ds\Bigg) \\
&+\tfrac{M_2}{\Gamma(\gamma)}\int_0^{\tau}\varphi(\tau-s)\Bigg(\sum_{0<t_k<s}\int_{t_{k-1}}^{t_k}(t_k-\eta)^{\gamma-1}\|f(\eta,x(\eta),x_\eta)\|_{\mathcal{D}}d\eta+\int_{t_k}^{s}(s-\eta)^{\gamma-1}\|f(\eta,x(\eta),x_\eta)\|_{\mathcal{D}}d\eta\Bigg)ds \\
\leq\ & M_2\Big(\|x_1\|+\|\phi(0)\|_{\mathcal{D}}+mC_0+\|\varphi\|_{L^1(I)}(\|\phi(0)\|_{\mathcal{D}}+mC_0)\Big)+M_2\tfrac{(m+1)M_0a^{\gamma}}{\Gamma(\gamma+1)}+M_2\|\varphi\|_{L^1(I)}\tfrac{(m+1)M_0a^{\gamma}}{\Gamma(\gamma+1)} \\
\leq\ & M_2\Big((\|\varphi\|_{L^1(I)}+1)(\|\phi(0)\|_{\mathcal{D}}+\tfrac{(m+1)M_0a^{\gamma}}{\Gamma(\gamma+1)}+mC_0)+\|x_1\|\Big)=M_3,\ \ t\in I.
\end{aligned}$$

For any $x \in \Omega$ and $t \in [0, \tau]$, we obtain from (H2)

$$\begin{aligned}
\|\Theta(t;x;u_x)\|_{\mathcal{D}} \leq\ & \frac{1}{\Gamma(\gamma)}\left(\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{\gamma-1}\|f(s,x(s),x_s)\|_{\mathcal{D}}ds+\int_{t_k}^{t}(t-s)^{\gamma-1}\|f(s,x(s),x_s)\|_{\mathcal{D}}ds\right) \\
&+\frac{1}{\Gamma(\gamma)}\left(\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{\gamma-1}\|Bu_x(s)\|_{\mathcal{D}}ds+\int_{t_k}^{t}(t-s)^{\gamma-1}\|Bu_x(s)\|_{\mathcal{D}}ds\right) \\
\leq\ & \frac{(m+1)M_0\tau^{\gamma}}{\Gamma(\gamma+1)}+\frac{(m+1)M_1M_3\tau^{\gamma}}{\Gamma(\gamma+1)}=\frac{(m+1)(M_0+M_1M_3)\tau^{\gamma}}{\Gamma(\gamma+1)}.
\end{aligned} \tag{9}$$

Thus, this together with (7) shows

$$
\begin{aligned}
\|(\mathcal{P}x)(t)\| &\leq \|\phi(0)\|_{\mathcal{D}} + \|\Theta(t;x;u_x)\|_{\mathcal{D}} + \|I(t;x)\|_{\mathcal{D}} \\
&\quad + \int_0^t \varphi(t-s)(\|\phi(0)\|_{\mathcal{D}} + \|\Theta(s;x;u_x)\|_{\mathcal{D}} + \|I(s;x)\|_{\mathcal{D}})ds \\
&\leq \|\phi(0)\|_{\mathcal{D}} + \frac{(m+1)(M_0+M_1M_3)\tau^\gamma}{\Gamma(\gamma+1)} + mC_0 \\
&\quad + \|\varphi\|_{L^1(I)}\left(\|\phi(0)\|_{\mathcal{D}} + \frac{(m+1)(M_0+M_1M_3)\tau^\gamma}{\Gamma(\gamma+1)} + mC_0\right) \\
&\leq (\|\phi(0)\|_{\mathcal{D}} + mC_0)\left(\|\varphi\|_{L^1(I)} + 1\right) + \frac{(m+1)(M_0+M_1M_3)\left(\|\varphi\|_{L^1(I)}+1\right)}{\Gamma(\gamma+1)}\tau^\gamma \\
&\leq R_0.
\end{aligned}
$$

On the other hand,

$$
\|(\mathcal{P}x)_t\|_{L[-b,0]} = \int_{-b}^0 \|(\mathcal{P}x)_t(\theta)\|d\theta = \begin{cases} \int_{t-b}^0 \|\phi(s)\|ds + \int_0^t \|(\mathcal{P}x)(s)\|ds, & t \leq b, \\ \int_{t-b}^t \|(\mathcal{P}x)(s)\|ds, & t \geq b, \end{cases}
$$

which means

$$
\|(\mathcal{P}x)_t\|_{L[-b,0]} \leq b\|\phi\|_b + b\|\mathcal{P}x\|_{PC([0,t],X)}.
$$

Then, we have

$$
\sup_{t\in[0,\tau]} \|(\mathcal{P}x)_t\|_{L[-b,0]} \leq b(\|\phi\|_b + R_0).
$$

It is obvious that $(\mathcal{P}x)(t) = \phi(t)$ for any $t \in [-b,0]$. Then the fact $\mathcal{P}(\Omega) \subseteq \Omega$ is thus proved.

Next, we shall prove that $\{\mathcal{P}x : x \in \Omega\}$ is equicontinuous on each J_i. For any $x \in \Omega$ and $\xi_1, \xi_2 \in J_i$ with $\xi_1 < \xi_2$, the discussion can be divided into two cases.

Case (i): If $\xi_1, \xi_2 \in J_0$, then from the continuity of $\phi(\cdot)$, we have

$$
\|(\mathcal{P}x)(\xi_2) - (\mathcal{P}x)(\xi_1)\| = \|\phi(\xi_2) - \phi(\xi_1)\| \to 0, \text{ as } |\xi_1 - \xi_2| \to 0.
$$

Case (ii): If $\xi_1, \xi_2 \in J_i, i > 0$, then

$$
\begin{aligned}
&(\mathcal{P}x)(\xi_2) - (\mathcal{P}x)(\xi_1) \\
= &\ \Theta(\xi_2;x;u_x) - \Theta(\xi_1;x;u_x) + I(\xi_2;x) - I(\xi_1;x) \\
&+ \int_0^{\xi_2} \wp(\xi_2 - s)\phi(0)ds - \int_0^{\xi_1} \wp(\xi_1 - s)\phi(0)ds \\
&+ \int_0^{\xi_2} \wp(\xi_2 - s)\Theta(s;x;u_x)ds - \int_0^{\xi_1} \wp(\xi_1 - s)\Theta(s;x;u_x)ds \\
&+ \int_0^{\xi_2} \wp(\xi_2 - s)I(s;x)ds - \int_0^{\xi_1} \wp(\xi_1 - s)I(s;x)ds.
\end{aligned}
$$

Denote by

$\Lambda_1 = \Theta(\xi_2;x;u_x) - \Theta(\xi_1;x;u_x) + I(\xi_2;x) - I(\xi_1;x)$;

$\Lambda_2 = \int_0^{\xi_2} \wp(\xi_2 - s)\phi(0)ds - \int_0^{\xi_1} \wp(\xi_1 - s)\phi(0)ds$;

$\Lambda_3 = \int_0^{\xi_2} \wp(\xi_2 - s)(\Theta(s;x;u_x) + I(s;x))ds - \int_0^{\xi_1} \wp(\xi_1 - s)(\Theta(s;x;u_x) + I(s;x))ds.$

Then we have

$$
\|(\mathcal{P}x)(\xi_2) - (\mathcal{P}x)(\xi_1)\| \leq \|\Lambda_1\| + \|\Lambda_2\| + \|\Lambda_3\|.
$$

In the following, we prove that $\|\Lambda_i\| \to 0$ independently of $x \in \Omega$ as $|\xi_1 - \xi_2| \to 0,\ i = 1,2,3$. For Λ_1, we have

$$\begin{aligned}
\|\Lambda_1\| &\leq \left\|\int_{t_{i-1}}^{\xi_2}(\xi_2-s)^{\gamma-1}f(s,x(s),x_s)ds-\int_{t_{i-1}}^{\xi_1}(\xi_1-s)^{\gamma-1}f(s,x(s),x_s)ds\right\| \\
&\quad+\left\|\int_{t_{i-1}}^{\xi_2}(\xi_2-s)^{\gamma-1}Bu_x(s)ds-\int_{t_{i-1}}^{\xi_1}(\xi_1-s)^{\gamma-1}Bu_x(s)ds\right\| \\
&\leq \int_{t_{i-1}}^{\xi_1}[(\xi_1-s)^{\gamma-1}-(\xi_2-s)^{\gamma-1}]\|f(s,x(s),x_s)\|_{\mathcal{D}}ds+\int_{\xi_1}^{\xi_2}(\xi_2-s)^{\gamma-1}\|f(s,x(s),x_s)\|_{\mathcal{D}}ds \\
&\quad+\int_{t_{i-1}}^{\xi_1}[(\xi_1-s)^{\gamma-1}-(\xi_2-s)^{\gamma-1}]\|Bu_x(s)\|_{\mathcal{D}}ds+\int_{\xi_1}^{\xi_2}(\xi_2-s)^{\gamma-1}\|Bu_x(s)\|_{\mathcal{D}}ds \\
&\leq \frac{M_0}{\gamma}[(\xi_1-t_{i-1})^{\gamma}-(\xi_2-t_{i-1})^{\gamma}+(\xi_2-\xi_1)^{\gamma}]+\frac{M_0}{\gamma}(\xi_2-\xi_1)^{\gamma} \\
&\quad+\frac{M_1M_3}{\gamma}[(\xi_1-t_{i-1})^{\gamma}-(\xi_2-t_{i-1})^{\gamma}+(\xi_2-\xi_1)^{\gamma}]+\frac{M_1M_3}{\gamma}(\xi_2-\xi_1)^{\gamma} \\
&\quad\to 0,\ as\ |\xi_1-\xi_2|\to 0.
\end{aligned}$$

For Λ_2, we can rewrite it as

$$\begin{aligned}
\Lambda_2 &= \int_0^{\xi_2}\wp(\xi_2-s)\phi(0)ds-\int_0^{\xi_1}\wp(\xi_1-s)\phi(0)ds \\
&= \int_0^{\xi_2-\xi_1}\wp(\xi_2-s)\phi(0)ds+\int_{\xi_2-\xi_1}^{\xi_2}\wp(\xi_2-s)\phi(0)ds-\int_0^{\xi_1}\wp(\xi_1-s)\phi(0)ds \\
&= \int_0^{\xi_2-\xi_1}\wp(\xi_2-s)\phi(0)ds.
\end{aligned}$$

By Definition 4, one gets

$$\|\Lambda_2\| \leq \|\phi(0)\|_{\mathcal{D}}\int_0^{\xi_2-\xi_1}\varphi(\xi_2-s)ds \to 0,\ as\ |\xi_1-\xi_2|\to 0.$$

From the proof process of Lemma 6 and (9), it follows that

$$\begin{aligned}
\|\Lambda_3\| &= \left\|\int_0^{\xi_2}\wp(\xi_2-s)(\Theta(s;x;u_x)+I(s;x))ds-\int_0^{\xi_1}\wp(\xi_1-s)(\Theta(s;x;u_x)+I(s;x))ds\right\| \\
&\leq \left\|\int_0^{\xi_2-\xi_1}\wp(\xi_2-s)\Theta(s;x;u_x)ds+\int_{\xi_2-\xi_1}^{\xi_2}\wp(\xi_2-s)\Theta(s;x;u_x)ds-\int_0^{\xi_1}\wp(\xi_1-s)\Theta(s;x;u_x)ds\right\| \\
&\quad+\left\|\int_0^{\xi_2}\wp(\xi_2-s)I(s;x)ds-\int_0^{\xi_1}\wp(\xi_1-s)I(s;x)ds\right\| \\
&\leq \left\|\int_0^{\xi_2-\xi_1}\wp(\xi_2-s)\Theta(s;x;u_x)ds+\int_0^{\xi_1}\wp(s)\Theta(\xi_2-s;x;u_x)ds-\int_0^{\xi_1}\wp(s)\Theta(\xi_1-s;x;u_x)ds\right\| \\
&\quad+\left\|\int_0^{\xi_2-\xi_1}\wp(\xi_2-s)I(s;x)ds+\int_0^{\xi_1}\wp(s)[I(\xi_2-s;x)-I(\xi_1-s;x)]ds\right\| \\
&\leq \int_0^{\xi_2-\xi_1}\varphi(\xi_2-s)\|\Theta(s;x;u_x)\|_{\mathcal{D}}ds+\int_0^{\xi_1}\varphi(s)\|\Theta(\xi_2-s;x;u_x)-\Theta(\xi_1-s;x;u_x)\|_{\mathcal{D}}ds \\
&\quad+\int_0^{\xi_2-\xi_1}\varphi(\xi_2-s)\|I(s;x)\|_{\mathcal{D}}ds+\int_0^{\xi_1}\varphi(s)\|I(\xi_2-s;x)-I(\xi_1-s;x)\|_{\mathcal{D}}ds \\
&\leq \int_0^{\xi_2-\xi_1}\varphi(\xi_2-s)ds\cdot\frac{(m+1)(M_0+M_1M_3)\tau^{\gamma}}{\Gamma(\gamma+1)}+\int_0^{\xi_1}\varphi(s)ds\cdot\frac{2(M_0+M_1M_3)}{\gamma}(\xi_2-\xi_1)^{\gamma} \\
&\quad+mC_0\int_0^{\xi_2-\xi_1}\varphi(\xi_2-s)ds \\
&\quad\to 0,\ as\ |\xi_1-\xi_2|\to 0.
\end{aligned}$$

To sum up, it can be concluded that $\|(\mathcal{P}x)(\xi_2)-(\mathcal{P}x)(\xi_1)\|\to 0$, as $|\xi_1-\xi_2|\to 0$, for all $x\in\Omega$. Consequently, $\{\mathcal{P}x : x\in\Omega\}$ is equicontinuous on each J_i $(i=0,1,\cdots,m+1)$. □

Lemma 9. *Assume that conditions (H1)(i), (H2), (H3)(i) and (H4) hold. Then the operator* $\mathcal{P}:\Omega\to\Omega$ *is continuous.*

Proof. Since $\mathcal{P}(\Omega)\subseteq\Omega$ from Lemma 8, we only need to prove that $\mathcal{P}$ is continuous. Suppose $\{y_n\}$ to be a sequence satisfying $y_n\to y$ in Ω as $n\to\infty$.

From condition (H3), it is easy to see that

$$\sum_{0<t_k<t}\|I_k(y_n(t_k))-I_k(y(t_k))\|_{\mathcal{D}}\to 0,\ \ t\in[0,\tau],\ as\ \ n\to+\infty.$$

From condition (H1) and Lebesgue dominated convergence theorem, it follows that

$$\int_{t^*}^{t}(t-s)^{\gamma-1}\|f(s,y_n(s),(y_n)_s)-f(s,y(s),y_s)\|_{\mathcal{D}}ds\to 0,\ \ \forall t^*,t\in[0,\tau],\ as\ \ n\to+\infty.$$

Therefore, one can obtain

$$\begin{aligned}
&\|Bu_{y_n}(s)-Bu_y(s)\|_{\mathcal{D}}\\
\leq\ & M_1M_2\Big[\|\Theta_f(\tau;y_n)-\Theta_f(\tau;y)\|_{\mathcal{D}}+\|I(\tau;y_n)-I(\tau;y)\|_{\mathcal{D}}\\
&+\int_0^{\tau}\varphi(\tau-s)(\|\Theta_f(s;y_n)-\Theta_f(s;y)\|_{\mathcal{D}}+\|I(s;y_n)-I(s;y)\|_{\mathcal{D}})ds\Big]\\
\leq\ & M_1M_2\Bigg[\sum_{0<t_k<\tau}\int_{t_{k-1}}^{t_k}(t_k-s)^{\gamma-1}\|f(s,y_n(s),(y_n)_s)-f(s,y(s),y_s)\|_{\mathcal{D}}ds\\
&+\int_{t_k}^{\tau}(\tau-s)^{\gamma-1}\|f(s,y_n(s),(y_n)_s)-f(s,y(s),y_s)\|_{\mathcal{D}}ds+\sum_{0<t_k<\tau}\|I_k(y_n(t_k))-I_k(y(t_k))\|_{D}\\
&+\int_0^{\tau}\varphi(\tau-s)\Bigg(\sum_{0<t_k<s}\int_{t_{k-1}}^{t_k}(t_k-\eta)^{\gamma-1}\|f(\eta,y_n(\eta),(y_n)_\eta)-f(\eta,y(\eta),y_\eta)\|_{\mathcal{D}}d\eta\\
&+\int_{t_k}^{s}(s-\eta)^{\gamma-1}\|f(\eta,y_n(\eta),(y_n)_\eta)-f(\eta,y(\eta),y_\eta)\|_{\mathcal{D}}d\eta+\sum_{0<t_k<s}\|I_k(y_n(t_k))-I_k(y(t_k))\|_{\mathcal{D}}\Bigg)ds\Bigg]\\
&\to 0,\ as\ \ n\to+\infty.
\end{aligned}$$

Then, for each $t\in[0,\tau]$, one has

$$\begin{aligned}
&\|(\mathcal{P}y_n)(t)-(\mathcal{P}y)(t)\|\\
\leq\ & \|\Theta(t;y_n;u_{y_n})-\Theta(t;y;u_y)\|_{\mathcal{D}}+\|I(t;y_n)-I(t;y)\|_{\mathcal{D}}\\
&+\int_0^t\|\wp(t-s)[\Theta(s;y_n;u_{y_n})-\Theta(s;y;u_y)]\|ds+\int_0^t\|\wp(t-s)[I(s;y_n)-I(s;y)]\|ds\\
\leq\ & \sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{\gamma-1}\|f(s,y_n(s),(y_n)_s)-f(s,y(s),y_s)\|_{\mathcal{D}}ds\\
&+\int_{t_k}^{t}(t-s)^{\gamma-1}\|f(s,y_n(s),(y_n)_s)-f(s,y(s),y_s)\|_{\mathcal{D}}ds+\sum_{0<t_k<t}\|I_k(y_n(t_k))-I_k(y(t_k))\|_{\mathcal{D}}\\
&+\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{\gamma-1}\|Bu_{y_n}(s)-Bu_y(s)\|_{\mathcal{D}}ds+\int_{t_k}^{t}(t-s)^{\gamma-1}\|Bu_{y_n}(s)-Bu_y(s)\|_{\mathcal{D}}ds\\
&+\int_0^t\varphi(t-s)\Bigg(\sum_{0<t_k<s}\int_{t_{k-1}}^{t_k}(t_k-\eta)^{\gamma-1}\|f(\eta,y_n(\eta),(y_n)_\eta)-f(\eta,y(\eta),y_\eta)\|_{\mathcal{D}}d\eta\\
&+\int_{t_k}^{s}(s-\eta)^{\gamma-1}\|f(\eta,y_n(\eta),(y_n)_\eta)-f(\eta,y(\eta),y_\eta)\|_{\mathcal{D}}d\eta\Bigg)ds\\
&+\int_0^t\varphi(t-s)\Bigg(\sum_{0<t_k<s}\int_{t_{k-1}}^{t_k}(t_k-\eta)^{\gamma-1}\|Bu_{y_n}(\eta)-Bu_y(\eta)\|_{\mathcal{D}}d\eta\\
&+\int_{t_k}^{s}(s-\eta)^{\gamma-1}\|Bu_{y_n}(\eta)-Bu_y(\eta)\|_{\mathcal{D}}d\eta\Bigg)ds+\int_0^t\varphi(t-s)\Bigg(\sum_{0<t_k<s}\|I_k(y_n(t_k))-I_k(y(t_k))\|_{\mathcal{D}}\Bigg)ds\\
&\to 0,\ as\ \ n\to+\infty.
\end{aligned}$$

By means of the similar proof of equicontinuous for $\{\mathcal{P}x : x\in\Omega\}$ in Lemma 8 and the Ascoli-Arzelà theorem, it is easy to get $\|\mathcal{P}y_n-\mathcal{P}y\|_{PC(J,X)}\to 0$, as $n\to+\infty$, i.e., $\mathcal{P}$ is continuous on Ω. The conclusion follows. □

Now, it is in the position to present our main theorem of this work.

Theorem 1. *Assume that hypotheses (H1)–(H4) hold, then the fractional evolution Equations (1) satisfies exact controllability on I.*

Proof. From (8) and Lemma 1, we know that it is suffices to show that under control u_x the operator $\mathcal{P}$ has a fixed point x which is a mild solution of (1) on J. Simple verification implies the fact $x(\tau) = (\mathcal{P}x)(\tau) = x_1$ which can show that system (1) is exactly controllable on I. For this purpose, we shall take advantage of Mönch fixed point theorem.

The continuity of operator $\mathcal{P} : \Omega \to \Omega$ is given by Lemma 9. Take $B = \overline{co}\mathcal{P}(\Omega)$. It is not difficult to check that $\mathcal{P}(B) \subseteq B$. Suppose $D_0 \subset B$ to be a bounded countable set satisfying $D_0 \subset \overline{co}(\{x_0\} \bigcup \mathcal{P}(D_0))$, we shall prove that $\beta(D_0) = 0$. From Lemma 8, it is easy to derive that $\mathcal{P}(D_0)$ is equicontinuous on J_i, $i = 0, 1, \cdots, m+1$. Notice that $D_0 \subset \overline{co}(\{x_0\} \bigcup \mathcal{P}(D_0))$, so D_0 is also equicontinuous on each J_i.

For any $x \in D_0$, denote

$$(\mathcal{P}x)(t) = \begin{cases} (\mathcal{P}_1x)(t) + (\mathcal{P}_2x)(t) + (\mathcal{P}_3x)(t) + (\mathcal{P}_4x)(t), & t \in [0, \tau], \\ \phi(t), & t \in [-b, 0], \end{cases}$$

where

$$(\mathcal{P}_1x)(t) = \phi(0) + I(t; x);$$
$$(\mathcal{P}_2x)(t) = \Theta(t; x; u_x);$$
$$(\mathcal{P}_3x)(t) = \int_0^t \dot{\wp}(t-s)\Theta(s; x; u_x)ds;$$
$$(\mathcal{P}_4x)(t) = \int_0^t \dot{\wp}(t-s)(\phi(0) + I(s; x))ds.$$

Without loss of generality, let $D_0 = \{z_n\}_{n=1}^{\infty}$. Then it is not difficult to obtain that

$$\begin{aligned} \beta(\mathcal{P}_1(D_0)(t)) &= \beta\left(\left\{\phi(0) + \sum_{0<t_k<t} I_k(z_n(t_k))\right\}\right) \\ &\leq \sum_{i=1}^{m} k_i \beta(\{z_n\}) \\ &= \sum_{i=1}^{m} k_i \cdot \beta(D_0), \ t \in [0, \tau]. \end{aligned} \tag{10}$$

From hypothesis (H1)(ii) and Lemma 5, for any $s \in I$, we get

$$\begin{aligned} \beta(\{f(s, z_n(s), (z_n)_s)\}) &\leq l(s)(\beta(\{z_n(s)\}) + \beta(\{(z_n)_s\})) \\ &\leq l(s)(\beta(\{z_n(s)\}) + b\beta(\{z_n\})) \\ &\leq l(s)(\beta(D_0(s)) + b\beta(D_0)) \\ &\leq l(s)(b+1)\beta(D_0). \end{aligned}$$

Then this implies from Lemma 2 and Lemma 7 that

$$\begin{aligned} &\beta\Big(\Big\{\Theta_f(t; z_n)\Big\}\Big) \\ = \ &\beta\left(\left\{\frac{1}{\Gamma(\gamma)}\left(\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{\gamma-1}f(s, z_n(s), (z_n)_s)ds + \int_{t_k}^{t}(t-s)^{\gamma-1}f(s, z_n(s), (z_n)_s)ds\right)\right\}\right) \\ \leq \ &\frac{2}{\Gamma(\gamma)}\left(\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{\gamma-1}\beta(\{f(s, z_n(s), (z_n)_s)\})ds + \int_{t_k}^{t}(t-s)^{\gamma-1}\beta(\{f(s, z_n(s), (z_n)_s)\})ds\right) \\ \leq \ &\frac{2(m+1)}{\Gamma(\gamma)}\lambda\tau^{\gamma}\beta(D_0), \ t \in [0, \tau], \end{aligned}$$

which together with (H2) (ii) indicates

$$\begin{aligned}
&\beta(\{u_{z_n}(s)\}) \\
\leq\ & k(s)\left(\beta(\{(\mathcal{P}_1 z_n)(\tau)\}) + \beta\Big(\Big\{\Theta_f(\tau;z_n)\Big\}\Big) + 2\int_0^\tau \beta\Big(\Big\{\wp(\tau-s)\Big((\mathcal{P}_1 z_n)(s) + \Theta_f(s;z_n)\Big)\Big\}\Big)ds\right) \\
\leq\ & k(s)\left(\beta(\mathcal{P}_1(D_0)(\tau)) + \beta\Big(\Big\{\Theta_f(\tau;z_n)\Big\}\Big) + 2\int_0^\tau \varphi(\tau-s)\Big(\beta(\mathcal{P}_1(D_0)(s)) + \beta\Big(\Big\{\Theta_f(s;z_n)\Big\}\Big)\Big)ds\right) \\
\leq\ & k(s)\left(\left(\sum_{i=1}^m k_i + \frac{2(m+1)}{\Gamma(\gamma)}\lambda\tau^\gamma\right)\beta(D_0) + 2\|\varphi\|_{L^1(I)}\left(\sum_{i=1}^m k_i + \frac{2(m+1)}{\Gamma(\gamma)}\lambda\tau^\gamma\right)\beta(D_0)\right) \\
\leq\ & k(s)\left(\sum_{i=1}^m k_i + \frac{2(m+1)}{\Gamma(\gamma)}\lambda\tau^\gamma\right)\Big(1+2\|\varphi\|_{L^1(I)}\Big)\beta(D_0),\ s\in[0,\tau].
\end{aligned}$$

In addition, by using Hölder inequality, we have

$$\begin{aligned}
&\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{\gamma-1}k(s)ds + \int_{t_k}^t (t-s)^{\gamma-1}k(s)ds \\
\leq\ & m\left(\frac{1-p}{\gamma-p}\right)^{1-p}\|k\|_{L^{\frac{1}{p}}}a^\gamma + \left(\frac{1-p}{\gamma-p}\right)^{1-p}\|k\|_{L^{\frac{1}{p}}}a^\gamma \\
=\ & \rho.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\beta(\mathcal{P}_2(D_0)(t)) \\
\leq\ & \beta\Big(\Big\{\Theta_f(t;z_n)\Big\}\Big) + \beta\left(\left\{\frac{1}{\Gamma(\gamma)}\left(\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{\gamma-1}Bu_{z_n}(s)ds + \int_{t_k}^t(t-s)^{\gamma-1}Bu_{z_n}(s)ds\right)\right\}\right) \\
\leq\ & \beta\Big(\Big\{\Theta_f(t;z_n)\Big\}\Big) + \frac{2M_1}{\Gamma(\gamma)}\left(\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{\gamma-1}\beta(\{u_{z_n}(s)\})ds + \int_{t_k}^t(t-s)^{\gamma-1}\beta(\{u_{z_n}(s)\})ds\right) \\
\leq\ & \beta\Big(\Big\{\Theta_f(t;z_n)\Big\}\Big) \\
&+\frac{2M_1}{\Gamma(\gamma)}\left(\sum_{i=1}^m k_i + \frac{2(m+1)}{\Gamma(\gamma)}\lambda\tau^\gamma\right)\Big(1+2\|\varphi\|_{L^1(I)}\Big)\left(\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{\gamma-1}k(s)ds + \int_{t_k}^t(t-s)^{\gamma-1}k(s)ds\right)\beta(D_0) \quad (11)\\
\leq\ & \frac{2(m+1)}{\Gamma(\gamma)}\lambda\tau^\gamma\beta(D_0) + \frac{2\rho M_1}{\Gamma(\gamma)}\left(\sum_{i=1}^m k_i + \frac{2(m+1)}{\Gamma(\gamma)}\lambda\tau^\gamma\right)\Big(1+2\|\varphi\|_{L^1(I)}\Big)\beta(D_0) \\
\leq\ & \frac{1+2\rho M_1\Big(1+2\|\varphi\|_{L^1(I)}\Big)}{\Gamma(\gamma)}\left(2(m+1)\lambda\tau^\gamma + \sum_{i=1}^m k_i\right)\beta(D_0) \\
=\ & \left(2(m+1)M\lambda\tau^\gamma + M\sum_{i=1}^m k_i\right)\beta(D_0).
\end{aligned}$$

From Lemma 2, for $t\in[0,\tau]$, we have

$$\begin{aligned}
\beta(\mathcal{P}_3(D_0)(t)) &= \beta\left(\left\{\int_0^t \wp(t-s)\Theta(s;z_n;u_{z_{n_x}})ds\right\}\right) \\
&\leq 2\int_0^t \beta\big(\{\wp(t-s)\Theta(s;z_n;u_{z_{n_x}})\}\big)ds \\
&\leq 2\int_0^t \varphi(t-s)\beta(\mathcal{P}_2(D_0)(s))ds \\
&\leq 2\|\varphi\|_{L^1(I)}\left(2(m+1)M\lambda\tau^\gamma + M\sum_{i=1}^m k_i\right)\beta(D_0).
\end{aligned} \quad (12)$$

In view of (10), for each $t \in [0, \tau]$, we obtain

$$\begin{aligned} \beta(\mathcal{P}_4(D_0)(t)) &= \beta\left(\left\{\int_0^t \wp(t-s)(\phi(0)+I(s;z_n))ds\right\}\right) \\ &\leq 2\int_0^t \beta(\{\wp(t-s)(\phi(0)+I(s;z_n))\})ds \\ &\leq 2\int_0^t \varphi(t-s)\beta(\mathcal{P}_1(D_0)(s))ds \\ &\leq \left(2\|\varphi\|_{L^1(I)}\sum_{i=1}^m k_i\right)\beta(D_0). \end{aligned} \tag{13}$$

Therefore, by (10), (11), (12) and (13), we can get

$$\begin{aligned} \beta(\mathcal{P}(D_0)(t)) &\leq \beta(\mathcal{P}_1(D_0)(t)) + \beta(\mathcal{P}_2(D_0)(t)) + \beta(\mathcal{P}_3(D_0)(t)) + \beta(\mathcal{P}_4(D_0)(t)) \\ &\leq \sum_{i=1}^m k_i \cdot \beta(D_0) + \left(2(m+1)M\lambda\tau^\gamma + M\sum_{i=1}^m k_i\right)\beta(D_0) \\ &\quad +2\|\varphi\|_{L^1(I)}\left(2(m+1)M\lambda\tau^\gamma + M\sum_{i=1}^m k_i\right)\beta(D_0) + \left(2\|\varphi\|_{L^1(I)}\sum_{i=1}^m k_i\right)\beta(D_0) \\ &\leq \left(1+2\|\varphi\|_{L^1(I)}\right)\left(2(m+1)M\lambda\tau^\gamma + (1+M)\sum_{i=1}^m k_i\right)\beta(D_0) \\ &\leq \left(2\left(1+2\|\varphi\|_{L^1(I)}\right)(m+1)\lambda M\tau^\gamma + \left(1+2\|\varphi\|_{L^1(I)}\right)(1+M)\sum_{i=1}^m k_i\right)\beta(D_0). \end{aligned} \tag{14}$$

Besides, from the equicontinuity of $\mathcal{P}(D_0)$ on each J_i and Proposition 7.3 of [31] about the measures of noncompactness, it follows that

$$\beta(\mathcal{P}(D_0)) = \max_{0\leq i\leq m+1}\max_{t\in J_i}\beta((\mathcal{P}D_0)(t)). \tag{15}$$

Consequently, by (7), (14) and (15), we can obtain

$$\beta(D_0) \leq \beta(\overline{co}(\{x_0\}\cup\mathcal{P}(D_0))) \leq \beta(\mathcal{P}(D_0)) < \beta(D_0),$$

which indicates $\beta(D_0) = 0$. By lemma 2 (I)(i), we know that D_0 is relatively compact. Then from Lemma 3, $\mathcal{P}$ has at least one fixed point $x \in B$, which means that system (1) is exactly controllable on I. The conclusion follows. □

Remark 3. *Resolvent operator is a generalization of C_0 semigroup and then has more extensive applications [28]. For instance, for the special case that scalar kernel is taken as 1, the resolvent operator $\wp(t)$ becomes the C_0 semigroup e^{At} generated by A. We refer the readers to [28,32], in which examples are presented to show that they can not generate a C_0 semigroup but admit a resolvent operator. Then we improve and generalize some analogous results of fractional evolution systems.*

4. Examples

Example 1. *Consider the following fractional partial differential evolution system of the form*

$$\begin{cases} \dfrac{\partial^{\frac{3}{5}}}{\partial t^{\frac{3}{5}}}x(t,\xi) = \dfrac{\partial}{\partial\xi}x(t,\xi) + \dfrac{\mu(t)x_t(\xi)}{1+|x_t(\xi)|} + \displaystyle\int_{-b}^t \sigma(t-s)x(s,\xi)ds + \delta(\xi)\omega(t,\xi), \ (t,\xi)\in[0,a]\times(0,1), \\ x(t,0) = x(t,1) = 0, \ t\in[0,a], \\ \Delta x(t_i,\xi) = \dfrac{sin(x(t_i,\xi))}{2+\zeta_i(1+|x(t_i,\xi)|)}, \ i = 1,2,\cdots,m, \\ x(t,\xi) = \phi(t,\xi), \ (t,\xi)\in[-b,0]\times[0,1], \end{cases} \tag{16}$$

where $\mu \in C([0,a],R)$, $\sigma \in L([-b,a+b],R)$, δ is a characteristic function of certain subinterval $D \subset [0,1]$, $\omega \in C([0,a]\times[0,1],R)$, $\zeta_i \in C(R,R^+)$, $i=1,2,\cdots,m$, and $\phi \in C([-b,0]\times[0,1],R)$ which satisfies $\phi(t,0)=\phi(t,1)=0$ for $t\in[-b,0]$.

Define $X = C([0,1],R)$, $\mathcal{D} = \{x \in X : x' \in X, x(0) = x(1) = 0\}$, $Ax = x'$ for $x \in \mathcal{D}$. Thus, A is an infinitesimal generator of a noncompact semigroup $\{T(t) : t \geq 0\}$ which is given by $T(t)x(s) = x(t+s)$ for $x \in X$. From the subordinate principle (see Chapter 3, [33]), it follows that A is the infinitesimal generator of a strongly continuous differentiable bounded linear operators family $\{\wp(t)\}_{t\geq 0}$ $(0<\gamma<1)$ such that $\wp(0) = I$, and

$$\wp(t) = \int_0^\infty \varphi_{t,\gamma}(s)T(s)ds,\ t>0,$$

where $\varphi_{t,\gamma}(s) = t^{-\gamma}\Phi_\gamma(st^{-\gamma})$, and

$$\Phi_\gamma(z) = \sum_{n=0}^{\infty}\frac{(-z)^n}{n!\Gamma(-\gamma n+1-\gamma)} = \frac{1}{2\pi i}\int_{\Gamma_0}\varrho^{\gamma-1}exp(\varrho - z\varrho^\gamma)d\varrho,\ \ 0<\gamma<1,$$

where Γ_0 is a contour which starts and ends at $-\infty$ and encircles the origin once counter-clockwise.

Let

$$\begin{aligned}
&D^{\frac{3}{5}}x(t)(\xi) = \frac{\partial^{\frac{3}{5}}}{\partial t^{\frac{3}{5}}}x(t,\xi),\\
&x(t)(\xi) = x(t,\xi),\\
&Bu(t)(\xi) = \delta(\xi)\omega(t,\xi),\\
&\phi(t)(\xi) = \phi(t,\xi),\\
&f(t,x(t),x_t)(\xi) = \frac{\mu(t)x_t(\xi)}{1+|x_t(\xi)|} + \int_{-b}^{t}\sigma(t-s)x(s,\xi)ds,\\
&I_i(x(t_i))(\xi) = \frac{sin(x(t_i,\xi))}{2+\zeta_i(1+|x(t_i,\xi)|)},\ i=1,2,\cdots,m.
\end{aligned}$$

Then problem (16) can be regarded as

$$\begin{cases}
D^\gamma x(t) = Ax(t) + f(t,x(t),x_t) + Bu(t),\ \ a.e.\ t\in I := [0,a],\\
\Delta x(t_i) = x(t_i^+) - x(t_i^-) = I_i(x(t_i)),\ \ i=1,2,\cdots,m,\\
x(t) = \phi(t),\ \ t\in[-b,0],
\end{cases}$$

and it is not difficult to check that all the hypotheses of Theorem 1 are satisfied. Then system (16) satisfies exact controllability on $[0,a]$.

Example 2. *consider the following fractional partial differential evolution system of the form*

$$\begin{cases}
\dfrac{\partial^{\frac{1}{3}}}{\partial t^{\frac{1}{3}}}x(t,y) = \Delta x(t,y) + \dfrac{e^{-2t}}{3+e^t}x(t+\theta,y) + \delta(y)u(t,y),\ \ (t,y)\in[0,a]\times\Omega,\\
x(t,y) = 0,\ \ (t,y)\in[0,a]\times\partial\Omega,\\
\Delta x(t_i,y) = \dfrac{sin(x(t_i,y))}{e+\varpi_i(1+|x(t_i,y)|)},\ \ i=1,2,\cdots,m,\\
x(t,y) = \phi(t,y),\ \ (t,y)\in[-b,0]\times\Omega,
\end{cases} \tag{17}$$

where $\Omega \subset R^N$ represents a bounded domain with smooth boundary $\partial\Omega$, Δ denotes the Laplace operator, δ stands for the characteristic function of certain subdomain $D\subset\Omega$, $u\in L^2([0,a]\times\Omega)$, $\varpi_i \in C(R,R^+)$, $i=1,2,\cdots,m$, $\phi\in C^{2,1}([-b,0]\times\overline{\Omega})$ which satisfies $\phi(t,y)\equiv 0$ for $(t,y)\in[-b,0]\times\partial\Omega$, and $\theta\in[-b,0]$.

Let $X = L^P(\Omega)$ and the operator $A : \mathcal{D}\subset X \to X$ defined as $Ax = \Delta x$ with domain $\mathcal{D} = \left\{W^{2,N}(\Omega)\bigcap W_0^{1,N}(\Omega)\right\}$. Then, A generates a uniformly bounded analytic semigroup.

Define $\gamma = \frac{1}{3}$, $x(t)(y) = x(t,y)$, $u(t)(y) = u(t,y)$, $Bu(t)(y) = \delta(y)u(t,y)$, $\phi(t)(y) = \phi(t,y)$, and

$$I_i(x(t_i))(y) = \frac{sin(x(t_i,y))}{e + \varpi_i(1 + |x(t_i,y)|)}, \ i = 1,2,\cdots,m.$$

Let

$$f(t,x(t),x_t)(y) = \frac{e^{-2t}}{3+e^t}x(t+\theta,y).$$

Then problem (17) can be regarded as

$$\begin{cases} D^\gamma x(t) = Ax(t) + f(t,x(t),x_t) + Bu(t), \ \ a.e.\ t \in I := [0,a], \\ \Delta x(t_i) = x(t_i^+) - x(t_i^-) = I_i(x(t_i)), \ \ i = 1,2,\cdots,m, \\ x(t) = \phi(t), \ \ t \in [-b,0], \end{cases}$$

It is not difficult to check that all the hypotheses of Theorem 1 are satisfied. Then system (17) satisfies exact controllability on $[0,a]$.

5. Conclusions

This paper derives some new controllability results for a class of fractional impulsive evolution equations with time delay in Banach spaces by using resolvent operator theory and the theory of nonlinear functional analysis. In detail, from the point of view of the restrictions imposed on nonlinearity and impulse terms, exact controllability of the addressed system can be guaranteed even if the nonlinearity and impulse items are only continuous rather than Lipschitz continuous and other restrictive conditions. In order to avoid the limitation that the exact controllability only apply to the finite dimensional space due to the compact semigroup, we substitute the differentiability of resolvent operator for the compactness of semigroup in the present work. With the properly defined delay item in a corresponding complete space, we have solved the disturbances of time delay to the investigation of exact controllability for the considered system.

Further investigation about the nonlocal controllability for a class of fractional impulsive integrodifferential evolution inclusions with time delay and nonlocal conditions will be carried on:

$$\begin{cases} D^\gamma x(t) \in Ax(t) + f(t,x_t,\int_0^t k(t,s,x_s)ds) + Bu(t), \ \ a.e.\ t \in I := [0,a], \\ \Delta x(t_i) = x(t_i^+) - x(t_i^-) = I_i(x(t_i)), \ \ i = 1,2,\cdots,m, \\ x(t) + g_t(x) = \phi(t), \ \ t \in [-b,0], \end{cases} \tag{18}$$

where $g_t : C([-b,a],E) \to E$ is given function. Compared with the classical initial condition $x(0) = x_0$ and other nonlocal items [25,34,35], this nonlocal condition has better application effect in physics. In practical applications, it may be given by $g_t(x) = \sum_{i=1}^k c_i x(\tau_i + t)$, $t \in [-b,0]$, where c_i $(i = 1,2,\cdots,k)$ are given constants and $0 < \tau_1 < \tau_2 < \cdots < \tau_n \leq a$. At time $t = 0$, we have $g_0(x) = \sum_{i=1}^k c_i x(\tau_i)$, which is exactly the cases in [25,34,35].

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