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# Further Integral Inequalities through Some Generalized Fractional Integral Operators

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**Abstract:** In this article, we utilize recent generalized fractional operators to establish some fractional inequalities in Hermite–Hadamard and Minkowski settings. It is obvious that many previously published inequalities can be derived as particular cases from our outcomes. Moreover, we articulate some flaws in the proofs of recently affiliated formulas by revealing the weak points and introducing more rigorous proofs amending and expanding the results.

**Keywords:** generalized fractional operators; integral inequalities; Hermite–Hadamard inequalities; Minkowski inequalities



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## 1. Introduction

Fractional calculus is a branch of mathematics that extends the principles of the classical derivative and integral to non-integer orders. It has attracted the interest of physicists, mathematicians, and engineers in recent decades [1–3]. Fractional derivatives can be used to describe the nonlinear oscillation of earthquakes, and to alleviate the inadequacy caused by the assumption of continuous traffic flow in a fluid-dynamic traffic model. Fractional derivatives are also used to simulate a variety of chemical processes, as well as mathematical biology and a variety of other physics and engineering problems [4–7]. Fractional integral operators were suggested to discuss multiple generalized integral inequalities [8–11]. In [12,13], it is shown that many fractional models provide more suitable results than similar analogous models with integer derivatives. This motivates the demand of more precise inequalities in dealing with such mathematical models employing fractional calculus. In the present work, we focus on the most notable inequality, namely the Hermite–Hadamard-type inequality for convex functions. Many scholars are interested in constructing general fractional types of Hermite–Hadamard inequalities. Guessab and Schmeisser [14] investigated the integral sharp inequalities of the Hermite–Hadamard type. Srivastava et al. [15] gave several improvements and extensions of several variables Hermite–Hadamard and Jensen inequalities.

The topic of fractional integral inequalities is extremely important in the subject of mathematics. Hermite–Hadamard inequality is one of the most well-known inequalities for convex functions and can be given as the form:

$$f\left(\frac{p+q}{2}\right) \leq \frac{1}{q-p}\int_p^q f(t)dt \leq \frac{f(p)+f(q)}{2}, \tag{1}$$

where $f : A \subseteq \mathbb{R} \to \mathbb{R}$ is a convex function and $p, q \in A$ with $p < q$. Additional studies about Hermite–Hadamard inequality can be found, for example, in [16,17]. Using fractional integrals, several scholars explored this inequality and published a number of generalizations and extensions [15,18,19]. We may start with recalling some well-known fractional concepts.

**Definition 1** ([20,21])**.** *The left and right fractional Riemann–Liouville integrals of a function $X$, of order $\omega$, are recognized, respectively, by:*

$$\Upsilon^{\omega}_{\xi+}X(\theta) = \frac{1}{\Gamma(\omega)}\int_{\xi}^{\theta}(\theta-\rho)^{\omega-1}X(\rho)d\rho \quad (\theta > \xi, Re(\omega) > 0), \tag{2}$$

*and*

$$\Upsilon^{\omega}_{\zeta-}X(\theta) = \frac{1}{\Gamma(\omega)}\int_{x}^{\zeta}(\rho-\theta)^{\omega-1}X(\rho)d\rho \quad (x < \zeta, Re(\omega) > 0), \tag{3}$$

*where $\Gamma$ is the traditional gamma function.*

Jarad et al. [22] generalized the fractional integrals of Riemann–Liouville through the following definition:

**Definition 2** ([22])**.** *The generalized left and right fractional integral operators of a function $X$, of order $\omega$, are given, respectively, by:*

$$^{\varepsilon}\Upsilon^{\omega}_{\xi+}X(\theta) = \frac{1}{\Gamma(\varepsilon)}\int_{\xi}^{\theta}\left(\frac{(\theta-\xi)^{\omega}-(\rho-\xi)^{\omega}}{\omega}\right)^{\varepsilon-1}\frac{X(\rho)}{(\rho-\xi)^{1-\omega}}d\rho, \quad \theta > \xi, \tag{4}$$

$$^{\varepsilon}\Upsilon^{\omega}_{\zeta-}X(\theta) = \frac{1}{\Gamma(\varepsilon)}\int_{\theta}^{\zeta}\left(\frac{(\zeta-\theta)^{\omega}-(\zeta-\rho)^{\omega}}{\omega}\right)^{\varepsilon-1}\frac{X(\rho)}{(\zeta-\rho)^{1-\omega}}d\rho, \quad \theta < \zeta, \tag{5}$$

*where $\varepsilon \in \mathbb{C}$ with $Re(\varepsilon) > 0$.*

In 2021, Hyder and Barakat [23] improved the fractional integral operators given in [22] and introduced more general definitions for the fractional integral operators as follows:

**Definition 3** ([23])**.** *The general improved left and right fractional integral operators of a function $X$, of order $\omega$, are given, respectively, by:*

$$^{\varepsilon}\Upsilon^{\omega}_{\xi+}X(\theta) = \frac{1}{\Gamma(\varepsilon)}\int_{\xi}^{\theta}W^{\varepsilon-1}(\theta-\xi,\rho-\xi,\omega)\frac{X(\rho)}{\varphi(\rho-\xi,\omega)}d\rho, \quad \theta > \xi, \tag{6}$$

$$^{\varepsilon}\Upsilon^{\omega}_{\zeta-}X(\theta) = \frac{1}{\Gamma(\varepsilon)}\int_{\theta}^{\zeta}W^{\varepsilon-1}(\zeta-\theta,\zeta-\rho,\omega)\frac{X(\rho)}{\varphi(\zeta-\rho,\omega)}d\rho, \quad \theta < \zeta, \tag{7}$$

*where*

$$W(\theta,\rho,\omega) = \int_{\rho}^{\theta}\frac{d\nu}{\varphi(\nu,\omega)}, \tag{8}$$

*and $\varphi$ is either a continuous function from $\mathbb{R}_+ \times (0,1]$ into $\mathbb{R}$, satisfying the conditions $\varphi(\theta,1) = 1$ for all $\theta \in \mathbb{R}_+$, $\varphi(\theta,\omega) > 0$ for all $(\theta,\omega) \in \mathbb{R}_+ \times (0,1]$, and $\varphi(\cdot,\omega_1) \neq \varphi(\cdot,\omega_2)$ for all $\omega_1, \omega_2 \in (0,1]$ such that $\omega_1 \neq \omega_2$, or the constant function $\varphi(\theta,\omega) = 1$. Additionally, the one-sided improved fractional operator can be set as the form*

$$^{\varepsilon}\Upsilon^{\omega}X(\theta) = \frac{1}{\Gamma(\varepsilon)}\int_{0}^{\theta}W^{\varepsilon-1}(\theta,\rho,\omega)\frac{X(\rho)}{\varphi(\rho,\omega)}d\rho. \tag{9}$$

**Remark 1.** *If $\varphi(\theta,\omega) = \theta^{1-\omega}$, then $W(\theta,\rho,\omega) = \frac{\theta^{\omega}-\rho^{\omega}}{\omega}$ and the fractional operators in (6) and (7) reduce to those defined in (4) and (5), respectively. Moreover, if $\varphi(\theta,\omega) = \theta^{1-\omega}$, $\xi = \zeta = 0$,* *and $\omega = 1$, then $W(\theta, \rho, \omega) = \theta - \rho$, and the fractional operators in (6) and (7) reduce to those defined in (2) and (3), respectively. Furthermore, as special cases of fractional operators (6) and (7), many fractional operators in the literature [24–26] can be obtained. Therefore, the fractional operators (6) and (7) can be used to obtain new general results related to Hermite–Hadamard and Minkowski inequalities.*

Recently, Tariq et al [27] explored a new kind of Hermite–Hadamard inequality by Raina type function and some generalized convex functions. Their contribution lies in choosing different convex-type functions to build a new version of Hermite–Hadamard inequality, while the contribution of the present work lies in employing recent and general fractional integrals to construct a novel fractional Hermite–Hadamard and Minkowski inequalities.

In this paper, we employ recently developed generalized fractional operators to construct novel fractional inequalities for integrable and non-negative functions. These inequalities concern the Hermite–Hadamard and Minkowski inequalities. Our outcomes can be compared by the previous results established in [28,29]. The inequalities obtained in these references can be derived as particular cases. Additionally, in this work we show that the inequality of [29] (Theorem 2.5) is incorrect as we explain in the following. Finally, this paper is organized as follows: Section 2 contains the main results and Section 3 provides concluding remarks.

## 2. Main Results

In this section, we use new developed fractional integral operators to construct generalized fractional inequalities in Hermite–Hadamard and Minkowski settings. The following theorems are presented to achieve this claim.

**Theorem 1.** *Let $\varepsilon, \omega > 0$, $s \geq 1$, and let $X$ and $Y$ be two functions on $[0, \infty)$ such that, for all $\theta > 0$, $X(\theta), Y(\theta) > 0$, ${}^{\varepsilon}\Upsilon^{\omega} X^s(\theta) < \infty$, and ${}^{\varepsilon}\Upsilon^{\omega} Y^s(\theta) < \infty$. If $0 < j \leq \frac{X(\rho)}{Y(\rho)} \leq J$, $\rho \in [0, \theta]$, then the next inequality holds:*

$$\left({}^{\varepsilon}\Upsilon^{\omega} X^s(\theta)\right)^{1/s} + \left({}^{\varepsilon}\Upsilon^{\omega} Y^s(\theta)\right)^{1/s} \leq \frac{1 + (j+2)J}{(j+1)(J+1)} \left({}^{\varepsilon}\Upsilon^{\omega} (X+Y)^s(\theta)\right)^{1/s}. \tag{10}$$

**Proof.** According to the condition $\frac{X(\rho)}{Y(\rho)} \leq J$, $\rho \in [0, \theta]$, $\theta > 0$, we get

$$(J+1)^s X^s(\rho) \leq J^s (X+Y)^s(\rho). \tag{11}$$

Therefore, we have

$$\frac{(J+1)^s}{\Gamma(\varepsilon)} W^{\varepsilon-1}(\theta, \rho, \omega) \frac{X^s(\rho)}{\varphi(\rho, \omega)} \leq \frac{J^s}{\Gamma(\varepsilon)} W^{\varepsilon-1}(\theta, \rho, \omega) \frac{(X+Y)^s(\rho)}{\varphi(\rho, \omega)}. \tag{12}$$

Integrating the inequality (12) from 0 to $\theta$ with respect to $\rho$ and using (9), we get

$${}^{\varepsilon}\Upsilon^{\omega} X^s(\theta) \leq \frac{J^s}{(J+1)^s} \, {}^{\varepsilon}\Upsilon^{\omega} (X+Y)^s(\theta). \tag{13}$$

Hence,

$$\left({}^{\varepsilon}\Upsilon^{\omega} X^s(\theta)\right)^{1/s} \leq \frac{J}{J+1} \left({}^{\varepsilon}\Upsilon^{\omega} (X+Y)^s(\theta)\right)^{1/s}. \tag{14}$$

Now, according to the condition $\frac{X(\rho)}{Y(\rho)} \geq j$, we have

$$\left(1 + \frac{1}{j}\right)^s Y^s(\rho) \leq \left(\frac{1}{j}\right)^s (X+Y)^s(\rho), \tag{15}$$

and

$$\frac{1}{\Gamma(\varepsilon)}\left(1+\frac{1}{j}\right)^{s} W^{\varepsilon-1}(\theta,\rho,\omega)\frac{Y^{s}(\rho)}{\varphi(\rho,\omega)} \leq \frac{1}{\Gamma(\varepsilon)}\left(\frac{1}{j}\right)^{s} W^{\varepsilon-1}(\theta,\rho,\omega)\frac{(X+Y)^{s}(\rho)}{\varphi(\rho,\omega)}. \tag{16}$$

Integrating the inequality (16) from 0 to $\theta$ with respect to $\rho$ and using (9), we get

$$({}^{\varepsilon}\Upsilon^{\omega} Y^{s}(\theta))^{1/s} \leq \frac{1}{j+1}({}^{\varepsilon}\Upsilon^{\omega}(X+Y)^{s}(\theta))^{1/s}. \tag{17}$$

Thus, the desired inequality (10) can be acquired by adding the inequalities (14) and (17). □

**Theorem 2.** *Let $\omega > 0, s \geq 1, \varepsilon \in \mathbb{C}, Re(\varepsilon) > 0$, and let $X$ and $Y$ be two functions on $[0,\infty)$ such that, for all $\theta > 0$, $X(\theta), Y(\theta) > 0$, ${}^{\varepsilon}\Upsilon^{\omega} X^{s}(\theta) < \infty$, and ${}^{\varepsilon}\Upsilon^{\omega} Y^{s}(\theta) < \infty$. If $0 < j \leq \frac{X(\rho)}{Y(\rho)} \leq J$, $\rho \in [0,\theta]$, then the next inequality holds:*

$$({}^{\varepsilon}\Upsilon^{\omega} X^{s}(\theta))^{2/s} + ({}^{\varepsilon}\Upsilon^{\omega} Y^{s}(\theta))^{2/s} \geq \left(\frac{(J+1)(j+1)}{J} - 2\right)({}^{\varepsilon}\Upsilon^{\omega}(X+Y)^{s}(\theta))^{1/s}. \tag{18}$$

**Proof.** Multiplying the inequalities (14) and (17), we have

$$\left(\frac{(J+1)(j+1)}{J}\right)({}^{\varepsilon}\Upsilon^{\omega} X^{s}(\theta))^{1/s}({}^{\varepsilon}\Upsilon^{\omega} Y^{s}(\theta))^{1/s} \leq ({}^{\varepsilon}\Upsilon^{\omega}(X+Y)^{s}(\theta))^{2/s}. \tag{19}$$

According to Minkowski inequality, we get

$$\begin{aligned} ({}^{\varepsilon}\Upsilon^{\omega}(X+Y)^{s}(\theta))^{2/s} &\leq \left(({}^{\varepsilon}\Upsilon^{\omega} X^{s}(\theta))^{1/s} + ({}^{\varepsilon}\Upsilon^{\omega} Y^{s}(\theta))^{1/s}\right)^{2} \\ &= ({}^{\varepsilon}\Upsilon^{\omega} X^{s}(\theta))^{2/s} + ({}^{\varepsilon}\Upsilon^{\omega} Y^{s}(\theta))^{2/s} \\ &+ 2({}^{\varepsilon}\Upsilon^{\omega} X^{s}(\theta))^{1/s}({}^{\varepsilon}\Upsilon^{\omega} Y^{s}(\theta))^{1/s}. \end{aligned} \tag{20}$$

Therefore, by the inequalities (19) and (20), the wanted inequality (18) can be obtained. □

**Lemma 1** ([30])**.** *Assume $X(\theta)$ is a concave function for $\theta \in [p,q]$, then the next inequalities hold:*

$$X(p) + X(q) \leq X(p+q-\theta) + X(\theta) \leq 2X\left(\frac{p+q}{2}\right). \tag{21}$$

**Theorem 3.** *Let $\varepsilon, \omega > 0$, $\varepsilon \in \mathbb{C}$, $k, l > 1$, and let $X$ and $Y$ be two functions on $[0,\infty)$ such that $X(\theta), Y(\theta) > 0$ for $\theta > 0$. If $X^{k}$ and $Y^{l}$ are concave functions on $[0,\infty)$, then the next inequality holds:*

$$\begin{aligned} &\frac{1}{2^{k+l}}(X(0) + X(\omega W(\theta,0,\omega)))^{k}(Y(0) + Y(\omega W(\theta,0,\omega)))^{l}\left({}^{\varepsilon}\Upsilon^{\omega}\left(\omega^{\varepsilon-1}W^{\varepsilon-1}(\theta,0,\omega)\right)\right)^{2} \\ \leq\ & {}^{\varepsilon}\Upsilon^{\omega}\left(\omega^{\varepsilon-1}W^{\varepsilon-1}(\theta,0,\omega)X^{k}(\omega W(\theta,0,\omega))\right){}^{\varepsilon}\Upsilon^{\omega}\left(\omega^{\varepsilon-1}W^{\varepsilon-1}(\theta,0,\omega)Y^{l}(\omega W(\theta,0,\omega))\right). \end{aligned} \tag{22}$$

**Proof.** From the concavity of the functions $X^{k}, Y^{l}$, and Lemma 1, for $\theta > 0, \omega > 0, \rho \in [0,\theta]$ we have

$$X^{k}(0) + X^{k}(\omega W(\theta,0,\omega)) \leq X^{k}(\omega W(\theta,\rho,\omega)) + X^{k}(\omega W(\rho,0,\omega)) \leq 2X^{k}(\frac{\omega}{2}W(\theta,0,\omega)), \tag{23}$$

$$Y^{l}(0) + Y^{l}(\omega W(\theta,0,\omega)) \leq Y^{l}(\omega W(\theta,\rho,\omega)) + Y^{l}(\omega W(\rho,0,\omega)) \leq 2Y^{l}(\frac{\omega}{2}W(\theta,0,\omega)). \tag{24}$$

Multiplying the inequalities (23) and (24) by $\frac{1}{\Gamma(\varepsilon)\varphi(\rho,\omega)}(\omega W(\theta,\rho,\omega)W(\rho,0,\omega))^{\varepsilon-1}$ and integrating the outcoming inequalities from 0 to $\theta$, we get

$$\begin{aligned}
&\frac{X^k(0)+X^k(\omega W(\theta,0,\omega))}{\Gamma(\varepsilon)}\int_0^\theta(\omega W(\theta,\rho,\omega)W(\rho,0,\omega))^{\varepsilon-1}\frac{d\rho}{\varphi(\rho,\omega)}\\
\le\;& \frac{1}{\Gamma(\varepsilon)}\int_0^\theta(\omega W(\theta,\rho,\omega)W(\rho,0,\omega))^{\varepsilon-1}\frac{X^k(\omega W(\theta,\rho,\omega)}{\varphi(\rho,\omega)}d\rho\\
+\;& \frac{1}{\Gamma(\varepsilon)}\int_0^\theta(\omega W(\theta,\rho,\omega)W(\rho,0,\omega))^{\varepsilon-1}\frac{X^k(\omega W(\rho,0,\omega)}{\varphi(\rho,\omega)}d\rho\\
\le\;& \frac{2X^k\left(\frac{\omega}{2}W(\theta,0,\omega)\right)}{\Gamma(\varepsilon)}\int_0^\theta(\omega W(\theta,\rho,\omega)W(\rho,0,\omega))^{\varepsilon-1}\frac{d\rho}{\varphi(\rho,\omega)},
\end{aligned}\tag{25}$$

and

$$\begin{aligned}
&\frac{Y^l(0)+Y^l(\omega W(\theta,0,\omega))}{\Gamma(\varepsilon)}\int_0^\theta(\omega W(\theta,\rho,\omega)W(\rho,0,\omega))^{\varepsilon-1}\frac{d\rho}{\varphi(\rho,\omega)}\\
\le\;& \frac{1}{\Gamma(\varepsilon)}\int_0^\theta(\omega W(\theta,\rho,\omega)W(\rho,0,\omega))^{\varepsilon-1}\frac{Y^l(\omega W(\theta,\rho,\omega)}{\varphi(\rho,\omega)}d\rho\\
+\;& \frac{1}{\Gamma(\varepsilon)}\int_0^\theta(\omega W(\theta,\rho,\omega)W(\rho,0,\omega))^{\varepsilon-1}\frac{Y^l(\omega W(\rho,0,\omega)}{\varphi(\rho,\omega)}d\rho\\
\le\;& \frac{2Y^l\left(\frac{\omega}{2}W(\theta,0,\omega)\right)}{\Gamma(\varepsilon)}\int_0^\theta(\omega W(\theta,\rho,\omega)W(\rho,0,\omega))^{\varepsilon-1}\frac{d\rho}{\varphi(\rho,\omega)}.
\end{aligned}\tag{26}$$

If we set $W(\theta,\rho,\omega)=W(\eta,0,\omega)$, then we have

$$\frac{1}{\Gamma(\varepsilon)}\int_0^\theta(\omega W(\theta,\rho,\omega)W(\rho,0,\omega))^{\varepsilon-1}\frac{d\rho}{\varphi(\rho,\omega)}=^{\varepsilon}\Upsilon^{\omega}\Big((\omega W(\theta,0,\omega))^{\varepsilon-1}\Big),\tag{27}$$

$$\begin{aligned}
&\frac{1}{\Gamma(\varepsilon)}\int_0^\theta(\omega W(\theta,\rho,\omega)W(\rho,0,\omega))^{\varepsilon-1}\frac{X^k(\omega W(\theta,\rho,\omega)}{\varphi(\rho,\omega)}d\rho\\
=\;& {}^{\varepsilon}\Upsilon^{\omega}\Big((\omega W(\theta,0,\omega))^{\varepsilon-1}X^k(\omega W(\theta,0,\omega)\Big),
\end{aligned}\tag{28}$$

and

$$\begin{aligned}
&\frac{1}{\Gamma(\varepsilon)}\int_0^\theta(\omega W(\theta,\rho,\omega)W(\rho,0,\omega))^{\varepsilon-1}\frac{Y^l(\omega W(\theta,\rho,\omega)}{\varphi(\rho,\omega)}d\rho\\
=\;& {}^{\varepsilon}\Upsilon^{\omega}\Big((\omega W(\theta,0,\omega))^{\varepsilon-1}Y^l(\omega W(\theta,0,\omega)\Big).
\end{aligned}\tag{29}$$

Therefore, by (25), (27), and (28), we get

$$\begin{aligned}
&\Big(X^k(0)+X^k(\omega W(\theta,0,\omega)\Big)\Big({}^{\varepsilon}\Upsilon^{\omega}\Big((\omega W(\theta,0,\omega))^{\varepsilon-1}\Big)\Big)\\
\le\;& 2^{\varepsilon}\Upsilon^{\omega}\Big((\omega W(\theta,0,\omega))^{\varepsilon-1}X^k(\omega W(\theta,0,\omega)\Big)\\
\le\;& 2X^k\Big(\frac{\omega}{2}W(\theta,0,\omega)\Big)\Big({}^{\varepsilon}\Upsilon^{\omega}\Big((\omega W(\theta,0,\omega))^{\varepsilon-1}\Big)\Big).
\end{aligned}\tag{30}$$

Additionally, by (26), (27), and (29), we have

$$\begin{aligned}&\Big(Y^l(0)+Y^l(\omega W(\theta,0,\omega))\Big)\Big({}^{\varepsilon}\Upsilon^{\omega}\Big((\omega W(\theta,0,\omega))^{\varepsilon-1}\Big)\Big)\\ \leq\ & 2^{\varepsilon}\Upsilon^{\omega}\Big((\omega W(\theta,0,\omega))^{\varepsilon-1}Y^l(\omega W(\theta,0,\omega)\Big)\\ \leq\ & 2Y^l\Big(\frac{\omega}{2}W(\theta,0,\omega)\Big)\Big({}^{\varepsilon}\Upsilon^{\omega}\Big((\omega W(\theta,0,\omega))^{\varepsilon-1}\Big)\Big).\end{aligned}\tag{31}$$

Hence, by (30) and (31), it follows that

$$\begin{aligned}&\Big(X^k(0)+X^k(\omega W(\theta,0,\omega))\Big)\Big(Y^l(0)+Y^l(\omega W(\theta,0,\omega))\Big)\Big({}^{\varepsilon}\Upsilon^{\omega}\Big((\omega W(\theta,0,\omega))^{\varepsilon-1}\Big)\Big)^2\\ \leq\ & 4\Big({}^{\varepsilon}\Upsilon^{\omega}\Big((\omega W(\theta,0,\omega))^{\varepsilon-1}X^k(\omega W(\theta,0,\omega))\Big)\Big)\\ \times\ & \Big({}^{\varepsilon}\Upsilon^{\omega}\Big((\omega W(\theta,0,\omega))^{\varepsilon-1}Y^l(\omega W(\theta,0,\omega))\Big)\Big).\end{aligned}\tag{32}$$

As $X(\theta)Y(\theta)>0$ for $\theta>0.$, then for $\omega>0,k\geq 0$, and $l\geq 0$, we have

$$\left(\frac{X^k(0)+X^k(\omega W(\theta,0,\omega)}{2}\right)^{1/k}\geq\frac{X(0)+X(\omega W(\theta,0,\omega)}{2},\tag{33}$$

and

$$\left(\frac{Y^l(0)+Y^l(\omega W(\theta,0,\omega)}{2}\right)^{1/l}\geq\frac{Y(0)+Y(\omega W(\theta,0,\omega)}{2}.\tag{34}$$

Therefore,

$$\begin{aligned}&\left(\frac{X^k(0)+X^k(\omega W(\theta,0,\omega)}{2}\right)\Big({}^{\varepsilon}\Upsilon^{\omega}\Big((\omega W(\theta,0,\omega))^{\varepsilon-1}\Big)\Big)\\ \geq\ & \left(\frac{X(0)+X(\omega W(\theta,0,\omega)}{2}\right)^k\Big({}^{\varepsilon}\Upsilon^{\omega}\Big((\omega W(\theta,0,\omega))^{\varepsilon-1}\Big)\Big),\end{aligned}\tag{35}$$

and

$$\begin{aligned}&\left(\frac{Y^l(0)+Y^l(\omega W(\theta,0,\omega)}{2}\right)\Big({}^{\varepsilon}\Upsilon^{\omega}\Big((\omega W(\theta,0,\omega))^{\varepsilon-1}\Big)\Big)\\ \geq\ & \left(\frac{X(0)+X(\omega W(\theta,0,\omega)}{2}\right)^l\Big({}^{\varepsilon}\Upsilon^{\omega}\Big((\omega W(\theta,0,\omega))^{\varepsilon-1}\Big)\Big).\end{aligned}\tag{36}$$

According to inequalities (35) and (36), we acquire

$$\begin{aligned}&\Big(X^k(0)+X^k(\omega W(\theta,0,\omega))\Big)\Big(Y^l(0)+Y^l(\omega W(\theta,0,\omega))\Big)\Big({}^{\varepsilon}\Upsilon^{\omega}\Big((\omega W(\theta,0,\omega))^{\varepsilon-1}\Big)\Big)^2\\ \geq\ & 2^{2-k-l}(X(0)+X(\omega W(\theta,0,\omega))^k(Y(0)+Y(\omega W(\theta,0,\omega))^l\\ \times\ & \Big({}^{\varepsilon}\Upsilon^{\omega}\Big((\omega W(\theta,0,\omega))^{\varepsilon-1}\Big)\Big)^2.\end{aligned}\tag{37}$$

Hence, by merging inequalities (32) and (37), we obtain the required inequality (22). □

**Theorem 4.** *Let $\varepsilon,\omega,\nu > 0$, $\varepsilon,\nu \in \mathbb{C}$, $k,l > 1$, and let $X$ and $Y$ be two functions on $[0,\infty)$ such that $X(\theta), Y(\theta) > 0$ for $\theta > 0$. If $X^k, Y^l$ are concave functions on $[0,\infty)$. Then, the next inequality holds:*

$$\begin{aligned}
&\frac{1}{2^{k+l-2}}(X(0)+X(\theta))^k(Y(0)+Y(\theta))^l\Big({}^{\varepsilon}\Upsilon^{\omega}\Big(\omega^{\nu-1}W^{\nu-1}(\theta,0,\omega)\Big)\Big)^2 \\
\leq\;& \Bigg[\frac{\omega^{\nu-\varepsilon}\Gamma(\nu)}{\Gamma(\varepsilon)}\,{}^{\nu}\Upsilon^{\omega}\Big(\omega^{\varepsilon-1}W^{\varepsilon-1}(\theta,0,\omega)X^k(\omega W(\theta,0,\omega))\Big) \\
+\;& {}^{\varepsilon}\Upsilon^{\omega}\Big(\omega^{\nu-1}W^{\nu-1}(\theta,0,\omega)X^k(\omega W(\theta,0,\omega))\Big)\Bigg] \\
\times\;& \Bigg[\frac{\omega^{\nu-\varepsilon}\Gamma(\nu)}{\Gamma(\varepsilon)}\,{}^{\nu}\Upsilon^{\omega}\Big(\omega^{\varepsilon-1}W^{\varepsilon-1}(\theta,0,\omega)Y^l(\omega W(\theta,0,\omega))\Big) \\
+\;& {}^{\varepsilon}\Upsilon^{\omega}\Big(\omega^{\nu-1}W^{\nu-1}(\theta,0,\omega)Y^l(\omega W(\theta,0,\omega))\Big)\Bigg].
\end{aligned} \tag{38}$$

**Proof.** By multiplying inequalities (23) and (24) with $\frac{1}{\Gamma(\varepsilon)\varphi(\rho,\omega)}W^{\varepsilon-1}(\theta,\rho,\omega)\Big(\omega W(\rho,0,\omega)\Big)^{\nu-1}$, then integrating the resultant inequalities with respect to $\rho$ from 0 to $\theta$, we obtain

$$\begin{aligned}
&\frac{X^k(0)+X^k(\omega W(\theta,0,\omega))}{\Gamma(\varepsilon)}\int_0^{\theta}W^{\varepsilon-1}(\theta,\rho,\omega)(\omega W(\rho,0,\omega))^{\nu-1}\frac{d\rho}{\varphi(\rho,\omega)} \\
\leq\;& \frac{1}{\Gamma(\varepsilon)}\int_0^{\theta}W^{\varepsilon-1}(\theta,\rho,\omega)(\omega W(\rho,0,\omega))^{\nu-1}\frac{X^k(\omega W(\theta,\rho,\omega))}{\varphi(\rho,\omega)}d\rho \\
+\;& \frac{1}{\Gamma(\varepsilon)}\int_0^{\theta}W^{\varepsilon-1}(\theta,\rho,\omega)(\omega W(\rho,0,\omega))^{\nu-1}\frac{X^k(\omega W(\rho,0,\omega))}{\varphi(\rho,\omega)}d\rho \\
\leq\;& \frac{2X^k\big(\frac{\omega}{2}W(\theta,0,\omega)\big)}{\Gamma(\varepsilon)}\int_0^{\theta}W^{\varepsilon-1}(\theta,\rho,\omega)(\omega W(\rho,0,\omega))^{\nu-1}\frac{d\rho}{\varphi(\rho,\omega)},
\end{aligned} \tag{39}$$

and

$$\begin{aligned}
&\frac{Y^l(0)+Y^l(\omega W(\theta,0,\omega))}{\Gamma(\varepsilon)}\int_0^{\theta}W^{\varepsilon-1}(\theta,\rho,\omega)(\omega W(\rho,0,\omega))^{\nu-1}\frac{d\rho}{\varphi(\rho,\omega)} \\
\leq\;& \frac{1}{\Gamma(\varepsilon)}\int_0^{\theta}W^{\varepsilon-1}(\theta,\rho,\omega)(\omega W(\rho,0,\omega))^{\nu-1}\frac{Y^l(\omega W(\theta,\rho,\omega))}{\varphi(\rho,\omega)}d\rho \\
+\;& \frac{1}{\Gamma(\varepsilon)}\int_0^{\theta}W^{\varepsilon-1}(\theta,\rho,\omega)(\omega W(\rho,0,\omega))^{\nu-1}\frac{Y^l(\omega W(\rho,0,\omega))}{\varphi(\rho,\omega)}d\rho \\
\leq\;& \frac{2Y^l\big(\frac{\omega}{2}W(\theta,0,\omega)\big)}{\Gamma(\varepsilon)}\int_0^{\theta}W^{\varepsilon-1}(\theta,\rho,\omega)(\omega W(\rho,0,\omega))^{\nu-1}\frac{d\rho}{\varphi(\rho,\omega)}.
\end{aligned} \tag{40}$$

If we set $W(\theta,\rho,\omega) = W(\eta,0,\omega)$, then we have

$$\frac{1}{\Gamma(\varepsilon)}\int_0^{\theta}W^{\varepsilon-1}(\theta,\rho,\omega)(\omega W(\rho,0,\omega))^{\nu-1}\frac{d\rho}{\varphi(\rho,\omega)} =^{\varepsilon}\Upsilon^{\omega}\Big((\omega W(\theta,0,\omega))^{\nu-1}\Big), \tag{41}$$

$$\begin{aligned}
&\frac{1}{\Gamma(\varepsilon)}\int_0^{\theta}W^{\varepsilon-1}(\theta,\rho,\omega)(\omega W(\rho,0,\omega))^{\nu-1}\frac{X^k(\omega W(\theta,\rho,\omega))}{\varphi(\rho,\omega)}d\rho \\
=\;& \frac{\omega^{\nu-\varepsilon}\Gamma(\nu)}{\Gamma(\varepsilon)}\,{}^{\nu}\Upsilon^{\omega}\Big((\omega W(\theta,0,\omega))^{\varepsilon-1}X^k(\omega W(\theta,0,\omega))\Big),
\end{aligned} \tag{42}$$

and

$$
\begin{aligned}
& \frac{1}{\Gamma(\varepsilon)} \int_0^\theta W^{\varepsilon-1}(\theta,\rho,\omega)(\omega W(\rho,0,\omega))^{\nu-1} \frac{Y^l(\omega W(\theta,\rho,\omega)}{\varphi(\rho,\omega)} d\rho \\
= & \frac{\omega^{\nu-\varepsilon}\Gamma(\nu)}{\Gamma(\varepsilon)} {}^{\nu}\Upsilon^{\omega}\Big((\omega W(\theta,0,\omega))^{\varepsilon-1} Y^l(\omega W(\theta,0,\omega)\Big),
\end{aligned} \tag{43}
$$

Therefore, by (38), (41), and (42), we get

$$
\begin{aligned}
& \Big(X^k(0) + X^k(\omega W(\theta,0,\omega)\Big)\Big({}^{\varepsilon}\Upsilon^{\omega}\Big((\omega W(\theta,0,\omega))^{\nu-1}\Big)\Big) \\
\leq & \frac{\omega^{\nu-\varepsilon}\Gamma(\nu)}{\Gamma(\varepsilon)} {}^{\nu}\Upsilon^{\omega}\Big((\omega W(\theta,0,\omega))^{\varepsilon-1} X^k(\omega W(\theta,0,\omega)\Big) \\
+ & {}^{\varepsilon}\Upsilon^{\omega}\Big((\omega W(\theta,0,\omega))^{\nu-1} X^k(\omega W(\theta,0,\omega)\Big).
\end{aligned} \tag{44}
$$

Additionally, by (40), (41), and (43), we get

$$
\begin{aligned}
& \Big(Y^l(0) + Y^l(\omega W(\theta,0,\omega)\Big)\Big({}^{\varepsilon}\Upsilon^{\omega}\Big((\omega W(\theta,0,\omega))^{\nu-1}\Big)\Big) \\
\leq & \frac{\omega^{\nu-\varepsilon}\Gamma(\nu)}{\Gamma(\varepsilon)} {}^{\nu}\Upsilon^{\omega}\Big((\omega W(\theta,0,\omega))^{\varepsilon-1} Y^l(\omega W(\theta,0,\omega)\Big) \\
+ & {}^{\varepsilon}\Upsilon^{\omega}\Big((\omega W(\theta,0,\omega))^{\nu-1} Y^l(\omega W(\theta,0,\omega)\Big).
\end{aligned} \tag{45}
$$

By multiplying the inequalities (44) and (45) we get

$$
\begin{aligned}
& \Big(X^k(0) + X^k(\omega W(\theta,0,\omega))\Big)\Big(Y^l(0) + Y^l(\omega W(\theta,0,\omega))\Big)\Big({}^{\varepsilon}\Upsilon^{\omega}\Big((\omega W(\theta,0,\omega))^{\nu-1}\Big)\Big)^2 \\
\leq & \Bigg[\frac{\omega^{\nu-\varepsilon}\Gamma(\nu)}{\Gamma(\varepsilon)} {}^{\nu}\Upsilon^{\omega}\Big((\omega W(\theta,0,\omega))^{\varepsilon-1} X^k(\omega W(\theta,0,\omega)\Big) \\
+ & {}^{\varepsilon}\Upsilon^{\omega}\Big((\omega W(\theta,0,\omega))^{\nu-1} X^k(\omega W(\theta,0,\omega)\Big)\Bigg]
\end{aligned} \tag{46}
$$

$$
\begin{aligned}
\leq & \Bigg[\frac{\omega^{\nu-\varepsilon}\Gamma(\nu)}{\Gamma(\varepsilon)} {}^{\nu}\Upsilon^{\omega}\Big((\omega W(\theta,0,\omega))^{\varepsilon-1} Y^l(\omega W(\theta,0,\omega)\Big) \\
+ & {}^{\varepsilon}\Upsilon^{\omega}\Big((\omega W(\theta,0,\omega))^{\nu-1} Y^l(\omega W(\theta,0,\omega)\Big)\Bigg].
\end{aligned} \tag{47}
$$

According to inequalities (35) and (36), we acquire

$$
\begin{aligned}
& \Big(X^k(0) + X^k(\omega W(\theta,0,\omega))\Big)\Big(Y^l(0) + Y^l(\omega W(\theta,0,\omega))\Big)\Big({}^{\varepsilon}\Upsilon^{\omega}\Big((\omega W(\theta,0,\omega))^{\varepsilon-1}\Big)\Big)^2 \\
\geq & 2^{2-k-l}(X(0) + X(\omega W(\theta,0,\omega)))^k (Y(0) + Y(\omega W(\theta,0,\omega)))^l \\
\times & \Big({}^{\varepsilon}\Upsilon^{\omega}\Big((\omega W(\theta,0,\omega))^{\varepsilon-1}\Big)\Big)^2.
\end{aligned} \tag{48}
$$

Hence, by merging inequalities (46) and (48), we obtain the desired inequality (38). □

## 3. Corrigendum to a Recently Published Result

We highlighted that the proof of Theorem 2.5 in [29] is questionable. Indeed, those authors mainly built their proof on the following allegations:

$$
\frac{1}{\Gamma(\varepsilon)} \int_0^\theta \left(\frac{\theta^\omega - \rho^\omega}{\omega}\right)^{\varepsilon-1} \rho^{\omega\nu-1} X^k(\theta^\omega - \rho^\omega) d\rho = \frac{\Gamma(\nu)}{\Gamma(\varepsilon)} {}^{\nu}\Upsilon^{\omega}\Big(\theta^{\omega(\varepsilon-1)} X^k(\theta^\omega)\Big),
$$

and

$$\frac{1}{\Gamma(\varepsilon)}\int_0^\theta \left(\frac{\theta^\omega-\rho^\omega}{\omega}\right)^{\varepsilon-1}\rho^{\omega\nu-1}Y^l(\theta^\omega-\rho^\omega)d\rho=\frac{\Gamma(\nu)}{\Gamma(\varepsilon)}\,{}^{\nu}\Upsilon^\omega\left(\theta^{\omega(\varepsilon-1)}Y^l(\theta^\omega)\right).$$

Obviously, the authors used one-sided generalized fractional integral operator; they replaced $\omega^{\varepsilon-1}$ by $\omega^{\nu-1}$ to deduce the above essential equations and consequently completed their proof. In our opinion, this is dubious. In fact, their claimed theorem is just a particular case of Theorem 2.4 here, and a more precise version of Theorem 2.5 in [29] is reachable through the following Corollary.

**Corollary 1.** *Let $\varepsilon,\omega,\nu>0$, $\varepsilon,\nu\in\mathbb{C}$, $k,l>1$, and let $X$ and $Y$ be two functions on $[0,\infty)$ such that $X(\theta),Y(\theta)>0$ for $\theta>0$. If$X^k$ and $Y^l$ are concave functions on $[0,\infty)$, then the next inequality holds:*

$$\begin{aligned}&\frac{1}{2^{k+l-2}}(X(0)+X(\theta))^k(Y(0)+Y(\theta))^l\left({}^{\varepsilon}\Upsilon^\omega(\theta^{\omega(\nu-1)})\right)^2\\ &\le\left[\frac{\omega^{\nu-\varepsilon}\Gamma(\nu)}{\Gamma(\varepsilon)}\,{}^{\nu}\Upsilon^\omega\left(\theta^{\omega(\varepsilon-1)}X^k(\theta^\omega)\right)+{}^{\varepsilon}\Upsilon^\omega(\theta^{(\nu-1)\omega}X^k(\theta^\omega))\right]\\ &\times\left[\frac{\omega^{\nu-\varepsilon}\Gamma(\nu)}{\Gamma(\varepsilon)}\,{}^{\nu}\Upsilon^\omega\left(\theta^{\omega(\varepsilon-1)}Y^l(\theta^\omega)\right)+{}^{\varepsilon}\Upsilon^\omega(\theta^{(\nu-1)\omega}Y^l(\theta^\omega))\right].\end{aligned}\tag{49}$$

**Proof.** Using the evidence of Theorem 2.4 as a guide, if we put $\varphi(\theta,\omega)=\theta^{1-\omega}$, then $\omega W(\theta,\rho,\omega)=\theta^\omega-\rho^\omega$. Consequently, we get the proof. □

## 4. Concluding Remarks

In this study, Hermite–Hadamard and Minkowski inequalities have been established in the context of newly generalized fractional integral operators. Throughout the paper, if we set $\varphi(\theta,\omega)=\theta^{1-\omega}$, then $W(\theta,\rho,\omega)=\frac{\theta^\omega-\rho^\omega}{\omega}$, and the acquired outcomes will reduced to the integral inequalities gained by Nisar et al. [29]. Furthermore, if $\omega=1$, then all the outcomes will be approached to the fractional inequalities introduced by Dahmani [30]. Moreover, if we put $\omega=\varepsilon=1$, then all the outcomes will be reduced to the traditional inequalities introduced in [28]. In addition, numerous research directions related to the integral inequalities can be considered in the frame of the general improved fractional integral (6) and (7). Under these operators, it is expected there will be more investigations into the Hermite–Hadamard inequality with differentiable $h$-convex functions [31], Hermite–Hadamard inequality for $s$-Convex functions [32], the binary Brunn–Minkowski inequality [33], and others.

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