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The Darboux Transformation and N-Soliton Solutions of Coupled Cubic-Quintic Nonlinear Schrödinger Equation on a Time-Space Scale

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Abstract: The coupled cubic-quintic nonlinear Schrödinger (CQNLS) equation is a universal mathematical model describing many physical situations, such as nonlinear optics and Bose–Einstein condensate. In this paper, in order to simplify the process of similar analysis with different forms of the coupled CQNLS equation, this dynamic system is extended to a time-space scale based on the Lax pair and zero curvature equation. Furthermore, Darboux transformation of the coupled CQNLS dynamic system on a time-space scale is constructed, and the N-soliton solution is obtained. These results effectively combine the theory of differential equations with difference equations and become a bridge connecting continuous and discrete analysis.

Keywords: coupled cubic-quintic nonlinear Schrödinger equation; time-space scales; Darboux transformation; N-soliton solution

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1. Introduction

In recent years, many useful methods have been applied to obtain solutions of integrable systems, such as Darboux transformation [1–5], inverse scattering transformation [6], bilinear transformation [7], and Backlünd transformation [8]. Darboux transformation, originating from the work of Darboux in 1882 on the Sturm–Liouville equation, is a powerful method for constructing solutions for integrable systems. The basic idea of Darboux transformation is to construct the solution of integrable equations by solutions, which is called the eigenfunctions, of the linear partial differential equation associated with its Lax pair. Lots of literature can be found in this regard. In Ref. [9], the authors develop Darboux's idea to solve linear and nonlinear partial differential equations arising in soliton theory. In Ref. [1–5], various approaches have been proposed to construct a Darboux transformation for nonlinear partial differential equations, such as operator factorization, gauge transformation, and loop group transformation.

The time scale was introduced in 1988 by Stefan Hilger, and its main purpose is to unify continuous and discrete analysis [10], which is further researched and developed by Bohner and Peterson [11,12]. This theory is a powerful tool to unify various types of time-variable forms and simplifies the process of similar analysis on time scales with different forms, and can better solve complex models that contain multiple situations, such as continuous and discrete. Therefore, it is used widely in various fields due to unification and extension. For example, in biology, functional connectivity patterns can be studied on multiple time scales by simulating the neural dynamics of large scale inter-regional connection networks in the macaque cortex [13–15]. In physics, the fixed point index theory is used to establish a double fixed point theorem for a completely continuous operator in Banach space, and then an application of the two-point conjugate boundary problem is discussed [16–18]. In economics, the time scales model provides information for a problem for not evenly spaced intervals, for which the standard continuous and discrete models do not [19,20].

The coupled cubic-quintic nonlinear Schrödinger (CQNLS) equation is introduced as the following form

$$\begin{cases} q_t = \mathrm{i}\rho_1^2 q^3 r^2 - 2\rho_1 q q_x r - 2\rho_1 q^2 r_x - 2\mathrm{i} q^2 r + \mathrm{i} q_{xx}, \\ r_t = -\mathrm{i}\rho_1^2 q^2 r^3 - 2\rho_1 q r r_x - 2\rho_1 q_x r^2 + 2\mathrm{i} q r^2 - \mathrm{i} r_{xx}. \end{cases}$$

where ρ_1 is a real parameter [21,22]. This equation has been one of the universal mathematical models in the field of nonlinear science, which is applied widely into optics [23,24], Bose–Einstein condensation [25,26], and other fields [27]. Many scholars obtain soliton solutions of this equation through different methods, such as Darboux transformation, backscattering, and Backlünd transformation [28,29]. In Ref. [28], Nth-order rogue wave solutions of coupled CQNLS equation are obtained by generalized Darboux transformation. The dynamics of its general first and second-order rogue waves are further discussed and illustrated. In Ref. [29], the soliton solution of the coupled CQNLS equation is obtained through bilinear Backlünd transformation; this result has important applications for the ultrashort optical pulse propagation in non-Kerr media. In this paper, we extend the coupled CQNLS equation on a time-space scale based on the Lax pair and zero curvature [30–33]. This result simplifies the process of similar analysis with different forms and builds a bridge between differential equations and difference equations. Furthermore, this equation can be used in complex models with both discrete and continuous applications.

The structure of this article is as follows. Some preliminaries about the time-space scale are devoted in Section 2. The coupled CQNLS equation on a time-space scale is derived in Section 3. Darboux transformation of this equation on a time-space scale is constructed, and its N-soliton solution is obtained in Section 4. Lastly, the conclusion is given in Section 5.

2. Preliminaries

In this section, some definitions of the time-space scale are introduced first [34–37].

Definition 1. *A time-space scale is any non-empty closed subset of the real number $\mathbb{R}$, and it has topological and sequential relations induced by $\mathbb{R}$.*

Definition 2. *Assuming $\mathbb{T}$ and $\mathbb{X}$ are time and space scales, for $(t,x) \in \mathbb{T} \times \mathbb{X}$, the forward jump operators are respectively defined as*

$$\sigma : \mathbb{T} \to \mathbb{T},\ \rho : \mathbb{X} \to \mathbb{X},$$
$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\},\ \rho(x) = \inf\{y \in \mathbb{X} : y > x\}.$$

for $x \in \mathbb{X}$, the backward jump operator $\beta(x) : \mathbb{X} \to \mathbb{X}$ is defined as

$$\beta(x) = \rho^{-1}(x) := \sup\{y \in \mathbb{X} : y < x\}.$$

Definition 3. *The $\nabla-$derivative related to time and space variables is defined as*

$$\nabla_t f(t,x) = \lim_{p \to \mu(t)} \frac{f(t,x) - f^{\sigma}(t,x)}{p},$$
$$\nabla_x f(t,x) = \lim_{q \to v(x)} \frac{f(t,x) - f^{\rho}(t,x)}{q},$$

where the grayscale functions are defined as

$$\mu : \mathbb{T} \to [0,+\infty),\ \nu : \mathbb{X} \to [0,\infty),$$

$$\mu(t) = t - \sigma(t),\ \nu(x) = x - \rho(x).$$

Note that,

$$f^\sigma(t,x) := f(\sigma(t),x) = f(t,x) - \mu(t)\nabla_t f(t,x),$$
$$f^\rho(t,x) := f(t,\rho(x)) = f(t,x) - v(x)\nabla_x f(t,x).$$

Definition 4. *Exponential function on a time scale is defined in the following form*

$$e_\alpha(x,x_0) := \exp\left(\int_{x_0}^{x} \zeta_{\mu(s)}(\alpha(s))\Delta s\right), e_\alpha(x) := e_\alpha(x,0),$$

where $\alpha : \mathbb{X} \to \mathbb{C}$ *is a given function and*

$$\zeta_h(z) := \frac{1}{h}\log(1+zh), h > 0, \zeta_0(z) := z.$$

This definition applies to the ν*-regressive functions* $\alpha = \alpha(x)$*, i.e., those satisfying*

$$1 + \nu(x)\alpha(x) \neq 0.$$

Such functions are usually called regressive.
In the constant discrete case ($\mathbb{X} = \varepsilon\mathbb{Z}, \varepsilon = const$)*, with* $\alpha = const$*, we have*

$$e_\alpha(x) = (1+\alpha\varepsilon)^{\frac{x}{\varepsilon}},$$

and in the case $\mathbb{X} = \mathbb{R}$*, we have*

$$e_\alpha(x) = \exp\int_0^x \alpha(\tau)d\tau.$$

Property 1. *The properties associated with* ∇*-derivatives are as follows*

$$\begin{cases} \nabla_t[f(t,x)g(t,x)] = f^\sigma(t,x)\nabla_t g(t,x) + \nabla_t f(t,x)g(t,x), \\ \nabla_x[f(t,x)g(t,x)] = f^\rho(t,x)\nabla_x g(t,x) + \nabla_x f(t,x)g(t,x). \end{cases}$$

3. The Coupled CQNLS Equation on a Time-Space Scale

In order to obtain the coupled CQNLS equation on a time-space scale, the ∇-dynamical system is considered as follows

$$\begin{cases} \nabla_x\varphi(t,x) &= U(t,x)\varphi(t,x), \\ \nabla_t\varphi(t,x) &= V(t,x)\varphi(t,x), \end{cases} \tag{1}$$

with

$$\begin{cases} U(t,x) = \begin{pmatrix} -i\lambda - \frac{i}{2}\rho_1 qr & q \\ r & i\lambda + \frac{i}{2}\rho_1 qr \end{pmatrix}, \\ V(t,x) = \begin{pmatrix} A & B \\ C & -A \end{pmatrix}, \end{cases}$$

where A, B, C are functions which contain spectral parameters λ and potential functions q, r.

According to the compatibility condition $\nabla_{xt}\varphi = \nabla_{tx}\varphi$, the zero curvature equation on a time-space scale is derived to

$$U^\sigma V + \nabla_t U - V^\rho U - \nabla_x V = 0. \tag{2}$$

By putting $U(t,x), V(t,x)$ into Equation (2), these equations are obtained

$$\begin{cases} \mathrm{i}\lambda(A^\rho - A) - \frac{\mathrm{i}}{2}\rho_1(qr)^\sigma A + q^\sigma C - \frac{\mathrm{i}}{2}\rho_1\nabla_t(qr) + \frac{\mathrm{i}}{2}\rho_1 qrA^\rho - rB^\rho - \nabla_x A = 0, \\ -\mathrm{i}\lambda(B^\rho + B) - \frac{\mathrm{i}}{2}\rho_1(qr)^\sigma B - q^\sigma A + \nabla_t q - qA^\rho - \frac{\mathrm{i}}{2}\rho_1 grB^\rho - \nabla_x B = 0, \\ \mathrm{i}\lambda(C + C^\rho) + r^\sigma A + + \frac{\mathrm{i}}{2}\rho_1(qr)^\sigma C + \nabla_t r + \frac{\mathrm{i}}{2}\rho_1 qrC^\rho + rA^\rho - \nabla_x C = 0, \\ -\mathrm{i}\lambda(A - A^\rho) + r^\sigma B - \frac{\mathrm{i}}{2}\rho_1(qr)^\sigma A + \frac{\mathrm{i}}{2}\rho_1\nabla_t(qr) - qC^\rho + \frac{\mathrm{i}}{2}\rho_1 grA^\rho + \nabla_x A = 0. \end{cases} \tag{3}$$

A, B, C are taken as quadratic polynomials of λ

$$A = \sum_{\mathrm{i}=0}^{2} a_{\mathrm{i}}\lambda^{\mathrm{i}},\ B = \sum_{\mathrm{i}=0}^{2} b_{\mathrm{i}}\lambda^{\mathrm{i}},\ C = \sum_{\mathrm{i}=0}^{2} c_{\mathrm{i}}\lambda^{\mathrm{i}}. \tag{4}$$

Take $a_2 = -2i$. By putting (4) into (3) and comparing coefficients of λ, the relations are obtained as follows

$$\begin{aligned}
&b_2 = 0,\ c_2 = 0,\\
&b_1 = -b_1^{\rho} + \mathrm{i}q^{\sigma}a_2 + \mathrm{i}qa_2^{\rho},\\
&c_1 = -c_1^{\rho} + \mathrm{i}r^{\sigma}a_2 + \mathrm{i}ra_2^{\rho},\\
&a_1 = \tfrac{1}{2}\nabla_x^{-1}(-r^{\sigma}b_1 + qc_1^{\rho} + q^{\sigma}c_1 - rb_1^{\rho}),\\
&b_0 = -b_0^{\rho} + \mathrm{i}q^{\sigma}a_1 + \mathrm{i}qa_1^{\rho} - \tfrac{1}{2}\rho_1 qrb_1^{\rho} - \tfrac{1}{2}\rho_1(qr)^{\sigma}b_1 + i\nabla_x b_1,\\
&c_0 = -c_0^{\rho} + \mathrm{i}r^{\sigma}a_1 + \mathrm{i}ra_1^{\rho} - \tfrac{1}{2}\rho_1 qrc_1^{\rho} - \tfrac{1}{2}\rho_1(qr)^{\sigma}c_1 + i\nabla_x c_1,
\end{aligned} \tag{5}$$

and

$$\begin{aligned}
&\nabla_t q = q^{\sigma}a_0 + qa_0^{\rho} + \tfrac{\mathrm{i}}{2}\rho_1 qrb_0^{\rho} + \tfrac{\mathrm{i}}{2}\rho_1(qr)^{\sigma}b_0 + \nabla_x b_0,\\
&\nabla_t r = -r^{\sigma}a_0 - ra_0^{\rho} - \tfrac{\mathrm{i}}{2}\rho_1 qrc_0^{\rho} - \tfrac{\mathrm{i}}{2}\rho_1(qr)^{\sigma}c_0 + \nabla_x c_0.
\end{aligned} \tag{6}$$

According to the derivative rule on a time-space scale, $a_i,\ b_i,\ c_i (i = 0, 1)$ are obtained as follows

$$\begin{aligned}
&a_1 = 0,\\
&b_1 = 2(2 - \nu(x)\nabla_x)^{-1}(q + q^{\sigma}),\\
&c_1 = 2(2 - \nu(x)\nabla_x)^{-1}(r + r^{\sigma}),\\
&b_0 = 2M_1(q + q^{\sigma}),\\
&c_0 = 2M_1(r + r^{\sigma}),\\
&a_0 = 2M_2M_1[M_4 + M_3],
\end{aligned} \tag{7}$$

with

$$\begin{aligned}
M_1 &= \left[(2 - \nu(x)\nabla_x)^{-1}\right]^2\left(i\nabla_x - \frac{1}{2}\rho_1 qr(1 - \nu(x)\nabla_x) - \frac{1}{2}\rho_1(qr)^{\sigma}\right),\\
M_2 &= [2\nabla_x + i\rho_1 q^{\sigma}(r - r^{\sigma}) + i\rho_1 r(1 - \nu(x)\nabla_x)(q - q^{\sigma})]^{-1},\\
M_3 &= \frac{1}{2}\rho_1^2(qr)^{\sigma}qr - \frac{1}{2}\rho_1^2(qr)^{\sigma}q^{\sigma}r^{\sigma} - i\rho_1\nabla_x(2q^{\sigma}r + qr + q^{\sigma}r^{\sigma}) + q^{\sigma}r - qr^{\sigma},\\
M_4 &= (1 - \nu(x)\nabla_x)\left(\frac{1}{2}\rho_1^2q^2r^2 - \frac{1}{2}\rho_1^2q^{\sigma}r^{\sigma}qr + qr^{\sigma} - q^{\sigma}r\right).
\end{aligned}$$

Then, Equation (6) is the coupled CQNLS dynamical system on a time-space scale, where a_0, b_0, c_0 are defined by Equation (7). Next, several special cases of the coupled CQNLS dynamical system will be obtained.

Case I

Consider the case $\mathbb{T} \times \mathbb{X} = \mathbb{R} \times \mathbb{R}$. We get $\mu(t) = \nu(x) = 0$. Then, (7) can be converted to

$$\begin{aligned}
&a_1 = 0,\\
&b_1 = 2q,\\
&c_1 = 2r,\\
&b_0 = -\rho_1 q^2 r + \mathrm{i}q_x,\\
&c_0 = -\rho_1 qr^2 - \mathrm{i}r_x,\\
&a_0 = \mathrm{i}\rho_1^2q^2r^2 + \frac{1}{2}\rho_1 q_x r - \frac{1}{2}\rho_1 qr_x - \mathrm{i}qr,
\end{aligned}$$

System (6) is transformed into the coupled CQNLS equation, i.e.,

$$\begin{cases} q_t = \mathrm{i}\rho_1^2 q^3 r^2 - 2\rho_1 q q_x r - 2\rho_1 q^2 r_x - 2\mathrm{i}q^2 r + \mathrm{i}q_{xx}, \\ r_t = -\mathrm{i}\rho_1^2 q^2 r^3 - 2\rho_1 q r r_x - 2\rho_1 q_x r^2 + 2\mathrm{i}q r^2 - \mathrm{i}r_{xx}. \end{cases}$$

Case II

Consider the case $\mathbb{T} \times \mathbb{X} = \mathbb{R} \times \mathbb{Z}$. We get $\mu(t) = 0,\ \nu(x) = 1$. Then,

$$\begin{aligned} &f^{\sigma}(x,t) = f(x,t), \\ &f^{\rho}(x,t) = Ef(x,t) = f(x,t) - (1-E)f(x,t), \end{aligned}$$

where E is the shift operator. By calculation, Equation (7) can be converted to

$$\begin{aligned} &a_1(x) = 0, \\ &b_1(x) = 4(1+E)^{-1}q(x), \\ &c_1(x) = 4(1+E)^{-1}r(x), \\ &b_0(x) = (1+E)^{-2}(-\tfrac{1}{2}\rho_1 Eq(x)r(x) - \tfrac{1}{2}\rho_1 q(x)r(x) + i - iE)4q(x), \\ &c_0(x) = (1+E)^{-2}(-\tfrac{1}{2}\rho_1 Eq(x)r(x) - \tfrac{1}{2}\rho_1 q(x)r(x) + i - iE)4r(x), \\ &a_0(x) = (1+E)^{-2}2i\rho_1[\rho_1 q(x)r(x)(1+E) - 2i(1-E)]q(x)r(x). \end{aligned}$$

Equation (6) can be transformed into

$$\begin{cases} q_t(x) = (1+E)^{-2}[i(1+E)\rho_1 q(x)r(x) + 2(1-E)][(1+E)\rho_1 q(x)r(x) + 2i(1-E)]q(x), \\ r_t(x) = -(1+E)^{-2}[i(1+E)\rho_1 q(x)r(x) + 2(1+E)][(1+E)\rho_1 q(x)r(x) - 2i(1-E)]r(x), \end{cases}$$

which is a semi-discrete coupled CQNLS equation.

4. Darboux Transformation of CQNLS Equation on a Time-Space Scale

In this section, the generalized Darboux transformation on a time-space scale will be constructed, and the N-soliton solution of the CQNLS equation will be obtained.

Firstly, $U,\ V$ of the CQNLS equation are rewritten as follows

$$\begin{aligned} U &= -i\lambda\sigma_3 + Q - \frac{1}{2}i\rho_1 Q^2\sigma_3, \\ V &= -2i\lambda^2\sigma_3 + B_1\lambda + a_0\sigma_3 + B_0, \end{aligned} \tag{8}$$

with $Q = \begin{pmatrix} 0 & q \\ -q^* & 0 \end{pmatrix}$, $B_1 = \begin{pmatrix} 0 & b_1 \\ c_1 & 0 \end{pmatrix}$, $B_0 = \begin{pmatrix} 0 & b_0 \\ c_0 & 0 \end{pmatrix}$, σ_3 is the Pauli matrix.

Proposition 1. *The Lax pair of the CQNLS equation is invariant under the following Darboux transformation*

$$\begin{cases} \varphi[1] = T[1]\varphi = (\lambda I - S)\varphi, \\ q[1] = q - is_{12} - is^{\rho}{}_{12}, \end{cases} \tag{9}$$

where

$$S = H\Lambda H^{-1},\ \Lambda = diag(\lambda_1, \lambda_1^*),$$

H satisfies

$$\begin{cases} \nabla_x H = -i\sigma_3 H\Lambda + QH - \frac{1}{2}i\rho_1 Q^2\sigma_3 H, \\ \nabla_t H = -2i\sigma_3 H\Lambda^2 + B_1 H\Lambda + a_0\sigma_3 + B_0 H. \end{cases}$$

Then, $\varphi[1]$ satisfies the linear spectrum problem as follows

$$\begin{cases} \nabla_x\varphi[1] = U[1]\varphi[1], \\ \nabla_t\varphi[1] = V[1]\varphi[1], \end{cases} \tag{10}$$

where

$$\begin{aligned} U[1] &= -i\lambda\sigma_3 + Q[1] - \frac{1}{2}i\rho_1 Q[1]^2\sigma_3, \\ V[1] &= -2i\lambda^2\sigma_3 + B_1[1]\lambda + a_0[1]\sigma_3 + B_0[1], \end{aligned} \tag{11}$$

$$Q[1] = Q + iS\sigma_3 + iS^\rho\sigma_3,\ B_1[1] = \begin{pmatrix} 0 & b_1[1] \\ c_1[1] & 0 \end{pmatrix},\ B_0 = \begin{pmatrix} 0 & b_0[1] \\ c_0[1] & 0 \end{pmatrix}.$$

Proof. Assume a gauge transformation

$$\varphi[1] = T[1]\varphi,\ T[1] = T_0 + T_1\lambda, \tag{12}$$

where $T_0 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $T_1 = \begin{pmatrix} a_{11} & b_{11} \\ c_{11} & d_{11} \end{pmatrix}$. □

By substituting Equation (12) into Equation (1), the constraint relation on the x part is obtained

$$\nabla_x T[1] + T[1]^\rho U - U[1]T[1] = 0. \tag{13}$$

Substituting Equations (8), (11), and (12) into Equation (13) and comparing the coefficients of λ, we get

$$\begin{aligned} & i{a_{11}}^\rho = ia_{11},\ i{b_{11}}^\rho = -ib_{11},\ i{c_{11}}^\rho = -ic_{11}, \\ & \nabla_x a = -\frac{1}{2}i\rho_1 qq^* a^\rho + q^* b^\rho + \frac{1}{2}i\rho_1 q[1]q^*[1]a + q[1]c, \\ & \nabla_x b = \frac{1}{2}i\rho_1 qq^* b^\rho - qa^\rho + \frac{1}{2}i\rho_1 q[1]q^*[1]b + q[1]d, \\ & \nabla_x c = -\frac{1}{2}i\rho_1 qq^* c^\rho + q^* d^\rho - \frac{1}{2}i\rho_1 q[1]q^*[1]c - q[1]^* a, \\ & \nabla_x d = +\frac{1}{2}i\rho_1 qq^* d^\rho - qc^\rho - \frac{1}{2}i\rho_1 q[1]q^*[1]d - q[1]^* b, \\ & \nabla_x a_{11} = ia^\rho - \frac{1}{2}i\rho_1 qq^* {a_{11}}^\rho + q^* {b_{11}}^\rho - ia + \frac{1}{2}i\rho_1 q[1]q^*[1]a_{11} + q[1]c_{11}, \\ & \nabla_x b_{11} = -ib^\rho + \frac{1}{2}i\rho_1 qq^* {b_{11}}^\rho - q{a_{11}}^\rho - ib + \frac{1}{2}i\rho_1 q[1]q^*[1]b_{11} + q[1]d_{11}, \\ & \nabla_x c_{11} = ic^\rho - \frac{1}{2}i\rho_1 qq^* {c_{11}}^\rho + q^* {d_{11}}^\rho + ic - \frac{1}{2}i\rho_1 q[1]q^*[1]c_{11} - q^*[1]a_{11}, \\ & \nabla_x d_{11} = -id^\rho + \frac{1}{2}i\rho_1 qq^* {d_{11}}^\rho - q{c_{11}}^\rho + id - \frac{1}{2}i\rho_1 q[1]q^*[1]d_{11} - q[1]^* b_{11}. \end{aligned} \tag{14}$$

Taking $a_{11} = d_{11} = 1$, we obtain

$$\begin{aligned} q[1] &= q + ib^\rho + ib, \\ q[1]^* &= q^* + ic^\rho + ic. \end{aligned} \tag{15}$$

Then, matrix S is constructed as

$$S = \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix} = -T_0.$$

Then,

$$\begin{aligned} T[1] &= \lambda I - S, \\ q[1] &= q - is_{12} - is_{12}^\rho. \end{aligned}$$

From the constant term of Equation (13), we obtain

$$\nabla_x S = -S^\rho Q + \frac{1}{2}i\rho_1 S^\rho Q^2\sigma_3 + QS - iS^2\sigma_3 - iS^\rho S\sigma_3 + \frac{1}{2}i\rho_1 Q^2 S\sigma_3\,. \tag{16}$$

If H is a solution of Equation (1), then H satisfies

$$\nabla_x H = -i\sigma_3 H\Lambda + QH - \frac{1}{2}i\rho_1 Q^2\sigma_3 H, \tag{17}$$

where $\Lambda = diag(\lambda, \lambda)$.

Take

$$S = H\Lambda H^{-1}\,, \tag{18}$$

then,

$$\begin{aligned}\nabla_x S &= \nabla_x\left(H\Lambda H^{-1}\right)\\ &= -i\sigma_3 S^2 + QS + \frac{1}{2}i\rho_1 Q^2 S\sigma_3 - iS^\rho S\sigma_3 - S^\rho Q + \frac{1}{2}i\rho_1 S^\rho Q^2\sigma_3.\end{aligned}$$

In addition, the gauge transformation (9) satisfies another constraint relation concerning t part

$$\nabla_t T[1] + T[1]^\sigma V - V[1]T[1] = 0. \tag{19}$$

By substituting Equation (12) into Equation (19), the following formula is derived

$$-\nabla_t S + (\lambda I - S^\sigma)V = V[1](\lambda I - S), \tag{20}$$

where V and $V[1]$ are obtained from Equations (8) and (11).

By comparing coefficients of λ, Equations (21)–(24) are obtained as follows

$$B_1 + 2iS^\sigma\sigma_3 = 2i\sigma_3 S + B_1[1], \tag{21}$$

$$a_0\sigma_3 + B_0 - S^\sigma B_1 = -B_1[1]S + a_0[1]\sigma_3 + B_0[1], \tag{22}$$

$$\nabla_t S + S^\sigma(a_0\sigma_3 + B_0) = (a_0[1]\sigma_3 + B_0[1])S. \tag{23}$$

when $T[1] = \lambda I - S$, $q[1] = q - is_{12} - is^\rho{}_{12}$, Equations (21)–(24) are constant. The proof is completed. Having the explicit form of the Darboux transformation, we are ready to construct the exact solutions of the CQNLS equation.

Matrix H and Λ are constructed as

$$H = \begin{pmatrix} \psi_1 & \phi_1^* \\ \phi_1 & -\psi_1^* \end{pmatrix},\ \Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1^* \end{pmatrix},$$

where the column vector $(\psi_1, \phi_1)^T$ is a set of solutions to the Lax pair (1) when $\lambda = \lambda_1$, column vector $\left(\phi_1^*, -\psi_1^*\right)^T$ is a set of solution when $\lambda = \lambda_1^*$. Then one soliton solution will be obtained as follows.

First, rewrite Equation (1) as

$$\begin{cases} \nabla_x\varphi[0] = \begin{pmatrix} -i\lambda + \frac{i}{2}\rho_1 q[0]q[0]^* & q[0] \\ -q[0]^* & i\lambda - \frac{i}{2}\rho_1 q[0]q[0]^* \end{pmatrix}\varphi[0], \\ \nabla_t\varphi[0] = \begin{pmatrix} -2i\lambda^2 + a_0[0] & b_1[0]\lambda + b_0[0] \\ c_1[0]\lambda + c_0[0] & 2i\lambda^2 - a_0[0] \end{pmatrix}\varphi[0]. \end{cases} \tag{24}$$

Then,

$$
\begin{aligned}
\varphi[1] &= T[1]\varphi[0] \\
&= (\lambda I - S[0])\varphi[0] \\
&= \begin{pmatrix} \lambda - s_{11}[0] & -s_{12}[0] \\ -s_{21} & \lambda - s_{22}[0] \end{pmatrix}\varphi[0], \\
q[1] &= q[0] - is_{12}[0] - is_{12}[0]^{\rho} \\
&= q[0] + i(\lambda_1 - \lambda_1^*)\frac{\psi_1[0]\phi_1[0]^*}{\psi_1[0]^2 + \phi_1[0]^2} + i(\lambda_1 - \lambda_1^*)\frac{\psi_1^{\rho}[0]\phi_1^{\rho}[0]^*}{\psi_1^{\rho}[0]^2 + \phi_1^{\rho}[0]^2}.
\end{aligned} \tag{25}
$$

where

$$
S[0] = \frac{1}{\psi_1[0]^2 + \phi_1[0]^2}\begin{pmatrix} -\lambda_1|\psi_1[0]|^2 - \lambda_1^*|\phi_1[0]|^2 & -(\lambda_1 - \lambda_1^*)\psi_1[0]\phi_1[0]^* \\ -(\lambda_1 - \lambda_1^*)\psi_1[0]^*\phi_1[0] & -\lambda_1|\phi_1[0]|^2 - \lambda_1^*|\psi_1[0]|^2 \end{pmatrix},
$$

$\varphi_1[0] = \begin{pmatrix} \psi_1[0] \\ \phi_1[0] \end{pmatrix}$ is a solution to (24) when $\lambda = \lambda_1$. At the same time, $\varphi[1]$ satisfies

$$
\begin{cases}
\nabla_x \varphi[1] = \begin{pmatrix} -i\lambda + \frac{\mathrm{i}}{2}\rho_1 q[1]q[1]^* & q[1] \\ -q[1]^* & i\lambda - \frac{\mathrm{i}}{2}\rho_1 q[1]q[1]^* \end{pmatrix}\varphi[1], \\
\nabla_t \varphi[1] = \begin{pmatrix} a_2[1]\lambda^2 + a_0[1] & b_1[1]\lambda + b_0[1] \\ c_1[1]\lambda + c_0[1] & -a_2[1]\lambda^2 - a_0[1] \end{pmatrix}\varphi[1].
\end{cases} \tag{26}
$$

Taking "seed solution" $q = 0$, we derive

$$
\begin{aligned}
\psi_1 &= e_{-i\lambda}(x,0)e_{-2i\lambda^2}(t,0), \\
\phi_1 &= e_{i\lambda}(x,0)e_{2i\lambda^2}(t,0), \\
\psi_1^{\rho} &= [1 - i\lambda v(x)]e_{-i\lambda}(x,0)e_{-2i\lambda^2}(t,0), \\
\phi_1^{\rho} &= [1 + i\lambda v(x)]e_{i\lambda}(x,0)e_{2i\lambda^2}(t,0).
\end{aligned}
$$

Then, one soliton solution is obtained

$$
q[1] = q[0] + i(\lambda_1 - \lambda_1^*)\frac{M_1}{M_1 + M_2} + i(\lambda_1 - \lambda_1^*)\frac{[1 - i\lambda v(x)]^2 M_1}{[1 - i\lambda v(x)]^2 M_1 + [1 + i\lambda v(x)]^2 M_2},
$$

with

$$
\begin{aligned}
M_1 &= e_{-2i\lambda}(x,0)e_{-4i\lambda^2}(t,0), \\
M_2 &= e_{2i\lambda}(x,0)e_{4i\lambda^2}(t,0).
\end{aligned}
$$

Next, the second Darboux transformation is constructed and two soliton solutions are obtained in a similar way

$$
\begin{aligned}
\varphi[2] &= T[2]\varphi[1] \\
&= (\lambda I - S[1])\varphi[1] \\
&= \begin{pmatrix} \lambda - s_{11}[1] & -s_{12}[1] \\ -s_{21}[1] & \lambda - s_{22}[1] \end{pmatrix}\varphi[1] \\
&= T[2]T[1]\varphi[0], \\
q[2] &= q[1] - is_{12}[1] - is_{12}[1]^{\rho} \\
&= q[1] + i(\lambda_2 - \lambda_2^*)\frac{\psi_2[1]\phi_2[1]^*}{\psi_2[1]^2 + \phi_2[1]^2} + i(\lambda_2 - \lambda_2^*)\frac{\psi_2^{\rho}[1]\phi_2^{\rho}[1]^*}{\psi_2^{\rho}[1]^2 + \phi_2^{\rho}[1]^2},
\end{aligned}
$$

where

$$
\varphi_2[1] = \begin{pmatrix} \psi_2[1] \\ \phi_2[1] \end{pmatrix} = \begin{pmatrix} \lambda_2 - s_{11}[1] & -s_{12}[1] \\ -s_{21}[1] & \lambda_2 - s_{22}[1] \end{pmatrix}\begin{pmatrix} \psi_2 \\ \phi_2 \end{pmatrix}.
$$

$\varphi_2[1]$is a solution to Equation (26) when $\lambda = \lambda_2$ and $(\psi_2, \phi_2)^T$ is a solution to (24) when $\lambda = \lambda_2$, $\varphi[2]$ satisfies

$$\begin{cases} \nabla_x \varphi[2] = \begin{pmatrix} -i\lambda + \frac{i}{2}\rho_1 q[2]q[2]^* & q[2] \\ -q[2]^* & i\lambda - \frac{i}{2}\rho_1 q[2]q[2]^* \end{pmatrix} \varphi[2], \\ \nabla_t \varphi[2] = \begin{pmatrix} a_2[2]\lambda^2 + a_0[2] & b_1[2]\lambda + b_0[2] \\ c_1[2]\lambda + c_0[2] & -a_2[2]\lambda^2 - a_0[2] \end{pmatrix} \varphi[2]. \end{cases} \tag{27}$$

Then, the N-soliton solution is obtained

$$\begin{aligned} \varphi[N] &= T[N]\varphi[N-1] \\ &= (\lambda I - S[N-1])\varphi[N-1] \\ &= \begin{pmatrix} \lambda - s_{11}[N-1] - s_{12}[N-1] & \\ -S_{21}[N-1] & \lambda - s_{22}[N-1] \end{pmatrix} \varphi[N-1] \\ &= T[N]\cdots T[3]T[2]T[1]\varphi[0], \\ q[N] &= q[N-1] + i(\lambda_N - \lambda_N^*)\frac{\psi_N[N-1]\phi_N[N-1]^*}{\psi_N[N-1]^2 + \phi_N[N-1]^2} + i(\lambda_N - \lambda_N^*)\frac{\psi_N^\rho[N-1]\phi_N^\rho[N-1]^*}{\psi_N^\rho[N-1]^2 + \phi_N^\rho[N-1]^2} \\ &= q[0] + i\sum_{j=1}^{N}\left(\lambda_j - \lambda_j^*\right)\frac{\psi_j[j-1]\phi_j[j-1]^*}{\psi_j[j-1]^2 + \phi_j[j-1]^2} + i\sum_{j=1}^{N}\left(\lambda_j - \lambda_j^*\right)\frac{\psi_j^\rho[j-1]\phi_j^\rho[j-1]^*}{\psi_j^\rho[j-1]^2 + \phi_j^\rho[j-1]^2}. \end{aligned}$$

Finally, three different cases of Darboux transformation are discussed, their N-soliton solutions are given separately.

Case I

Considering the case $\mathbb{T} \times \mathbb{X} = \mathbb{R} \times \mathbb{R}$, we have

$$f^\rho = f(t,x), \quad f^\sigma = f(t,x).$$

Equation (15) can be simplified into

$$q[1] = q + 2ib, \quad q[1]^* = q^* + 2ic. \tag{28}$$

when $q[0] = 0$ and $\lambda = \xi + i\eta$, ξ and η are real constants, we derive

$$\psi_1 = e^{(\eta - i\xi)x + \left[4\xi\eta - 2i\left(\xi^2 - \eta^2\right)\right]t}, \quad \phi_1 = e^{-(\eta - i\xi)x - \left[4\xi\eta - 2i\left(\xi^2 - \eta^2\right)\right]t}.$$

Then one-soliton solution of the CQNLS equation is obtained

$$q[1] = -\frac{2i}{|\psi_1|^2 + |\phi_1|^2}(\lambda_1 - \lambda_1^*)\psi_1\phi_1^* = 2\eta e^{-2i\xi x - 4i\left(\xi^2 - \eta^2\right)t}\operatorname{sech}(2\eta x + 8\xi\eta t) \tag{29}$$

The dynamic of the one-soliton solution is presented in Figure 1.

Then, the N-soliton solution can be obtained

$$q[N] = q[N-1] + 2i\left(\lambda_N - \lambda_N^*\right)\frac{\psi_N[N-1]\phi_N[N-1]^*}{|\psi_N[N-1]|^2 + |\phi_N[N-1]|^2} = 2i\sum_{j=1}^{N}\left(\lambda_j - \lambda_j^*\right)\frac{\psi_j[j-1]\phi_j[j-1]^*}{|\psi_j[j-1]|^2 + |\phi_j[j-1]|^2}.$$

Case II

Considering the case $\mathbb{T} \times \mathbb{X} = \mathbb{R} \times \mathbb{C}$, we have

$$\mu(t) = 0,\ \nu(x) = \begin{cases} \frac{1}{3^{n+1}},\ x = \sum_{k=1}^{n} \frac{e_k}{3^k} + \frac{1}{3^{n+1}} \in \mathbb{L}, \\ 0,\ x \in \mathbb{C}\backslash\mathbb{L}, \end{cases}$$

where $\mathbb{C}$ is a Cantor set and $\mathbb{L}$ is a set that contains the left discrete elements of $\mathbb{C}$,

$$\mathbb{L} = \left\{ \sum_{k=1}^{n} \frac{e_k}{3^k} + \frac{1}{3^{n+1}} :\ n \in N,\ e_k \in \{0,2\},\ 1 \le k \le n \right\}.$$

when $\lambda = \xi + i\eta$, one-soliton solution is obtained by the first iteration.

$$q[1] = \begin{cases} -2\eta \frac{N_1}{N_1+N_2} - 2\eta \frac{\left[1-3^{-n-1}(i\xi-\eta)\right]^2 N_1}{\left[1-3^{-n-1}(i\xi-\eta)\right]^2 N_1 + \left[1+3^{-n-1}(i\xi-\eta)\right]^2 N_2},\ x \in \mathbb{L},\ t \in \mathbb{R}, \\ 2\eta e^{-2i\xi x - 4i\left(\xi^2-\eta^2\right)t} \operatorname{sech}(2\eta x + 8\xi\eta t),\ x \in \mathbb{C}\backslash\mathbb{L},\ t \in \mathbb{R}, \end{cases} \tag{30}$$

with

$$N_1 = e_{-2(i\xi-\eta)}(x,0) e_{-4i(\xi^2+2i\xi\eta-\eta^2)}(t,0),$$
$$N_2 = e_{2(i\xi-\eta)}(x,0) e_{4i(\xi^2+2i\xi\eta-\eta^2)}(t,0).$$

Then, the N-soliton solution is obtained

$$\begin{cases} q[N] = N_3 + N_4, x \in \mathbb{L}, t \in \mathbb{R}, \\ q[N] = 2N_3, x \in \mathbb{C}\backslash\mathbb{L}, t \in \mathbb{R}, \end{cases}$$

where

$$N_3 = i \sum_{j=1}^{N} \left(\lambda_j - \lambda_j^*\right) \frac{\psi_j[j-1]\phi_j[j-1]^*}{\left|\psi_j[j-1]\right|^2 + \left|\phi_j[j-1]\right|^2},$$
$$N_4 = i \sum_{j=1}^{N} \left(\lambda_j - \lambda_j^*\right) \frac{\left(\psi_j[j-1] - 3^{-n-1}\nabla_x \psi_j[j-1]\right)\left(\phi_j^*[j-1] - 3^{-n-1}\nabla_x \phi_j^*[j-1]\right)}{\left|\psi_j[j-1] - 3^{-n-1}\nabla_x \psi_j[j-1]\right|^2 + \left|\phi_j[j-1] - 3^{-n-1}\nabla_x \phi_j[j-1]\right|^2}.$$

Case III

Considering the case $\mathbb{T} \times \mathbb{X} = \mathbb{R} \times \mathbb{K}_p$, we have

$$\mu(t) = 0,$$
$$v(x) = \begin{cases} 0, x = 0, \\ (1-p^{-1})x, x = p^k \in p^{\mathbb{Z}}. \end{cases}$$

where $p > 1$, $p^{\mathbb{Z}} = \{p^k :\ k \in \mathbb{Z}\}$ and $K_p = p^{\mathbb{Z}} \bigcup \{0\}$.

By the iteration of the Darboux transformation, when $\lambda = \xi + i\eta$, one-soliton solution can be presented as

$$q[1] = \begin{cases} -2\eta \frac{e^{2(p-1)N_5}}{e^{2(p-1)N_5} + e^{2(p-1)N_6}} - 2\eta \frac{e^{2\left(1-p^{-1}\right)N_5}}{e^{2\left(1-p^{-1}\right)N_5} + e^{2\left(1-p^{-1}\right)N_6}},\ x \in p^{\mathbb{Z}},\ t \in \mathbb{R}, \\ 2\eta e^{-2i\xi x - 4i\left(\xi^2-\eta^2\right)t} \operatorname{sech}(2\eta x + 8\xi\eta t),\ x = 0,\ t \in \mathbb{R}, \end{cases} \tag{31}$$

where

$$N_5 = \sum_{x \in (0,p^k]} (\eta - i\xi)x - 2i\left(\xi^2 + 2i\xi\eta - \eta^2\right),$$
$$N_6 = \sum_{x \in (0,p^k]} (i\xi - \eta)x + 2i\left(\xi^2 + 2i\xi\eta - \eta^2\right).$$

Then, the N-soliton solution is obtained

$$q[N] = \begin{cases} i\sum_{j=1}^{N}\left(\lambda_j - \lambda_j^*\right)\frac{\psi_j[j-1]\phi_j[j-1]^*}{|\psi_j[j-1]^2+|\phi_j[j-1]^2} + i\sum_{j=1}^{N}\left(\lambda_j - \lambda_j^*\right)M_{13},\ x \in p^{\mathbb{Z}},\ t \in \mathbb{R}, \\ 2i\sum_{i=1}^{N}\left(\lambda_j - \lambda_j^*\right)\frac{\psi_j[j-1]\phi_j[j-1]^*}{|\psi_j[j-1]^2+|\phi_j[j-1]^2},\ x = 0,\ t \in \mathbb{R}, \end{cases}$$

where

$$M_{13} = \frac{\left[\psi_j[j-1] - (1-p^{-1})x\nabla_x\psi_j[j-1]\right]\left[\phi_j^*[j-1] - (1-p^{-1})x\nabla_x\phi_j^*[j-1]\right]}{\left|\psi_j[j-1] - (1-p^{-1})x\nabla_x\psi_j[j-1]\right|^2 + \left|\phi_j[j-1] - (1-p^{-1})x\nabla_x\phi_j[j-1]\right|^2}.$$

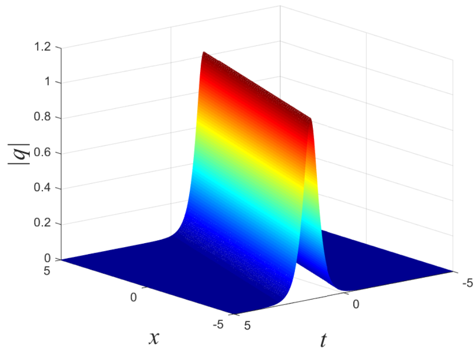

Figure 1. One-soliton solution with $\xi = 0.825$, $\eta = 0.512$.

5. Conclusions

In this paper, starting from the ∇-dynamical system, the specific form of the coupled CQNLS equation on a time-space scale is derived by the zero curvature equation and different forms of this equation in continuous time scales and discrete time scale are discussed separately. In addition, the Darboux transformation of the coupled CQNLS equation on a time-space scale is elaborately constructed; its one, two, and N-soliton solutions are further investigated. These results effectively combine the theory of differential equations with the theory of difference equations, which can simplify the process of similar analysis on different time and space scales, and can better solve complex models that include continuous and discrete situations.

The extension to arbitrary time or space scales provides access to a wider range of nonlinear integrable dynamic equations. By taking the seed solution $q = 0$, $\lambda = \xi + i\eta$, one-solution of the CQNLS equation is obtained on three different time-space scales ($\mathbb{X} = \mathbb{R}, \mathbb{X} = \mathbb{C}$, and $\mathbb{X} = \mathbb{K}_p$). In one case, the exact solution (29) and its dynamic figure are obtained when $x \in \mathbb{R}$. In the other cases, when $x \in \mathbb{C}\backslash\mathbb{L}$ and $x = 0$, exact solutions (30) and (31) are obtained and are similar to Equation (29). Nevertheless, when $x \in \mathbb{L}$ and $x \in p^{\mathbb{Z}}$, the structures of solutions are more complicated at discontinuity points. Due to the limitations of the computer, it was difficult to obtain their dynamic figures at this stage. We will find the most effective way to reduce structures of solutions on $\mathbb{C}$ and $\mathbb{K}_p$, then extend the nonlocal symmetry reduction [38] to a time-space scale, which is the focus of our future work.

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