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Article

Positive Solutions of a Singular Fractional Boundary Value Problem with *r*-Laplacian Operators

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Abstract: We investigate the existence and multiplicity of positive solutions for a system of Riemann–Liouville fractional differential equations with r-Laplacian operators and nonnegative singular nonlinearities depending on fractional integrals, supplemented with nonlocal uncoupled boundary conditions which contain Riemann–Stieltjes integrals and various fractional derivatives. In the proof of our main results we apply the Guo–Krasnosel'skii fixed point theorem of cone expansion and compression of norm type.

Keywords: Riemann–Liouville fractional differential equations; nonlocal boundary conditions; singular functions; positive solutions; multiplicity

MSC: 34A08; 34B10; 34B16; 34B18

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1. Introduction

We consider the system of fractional differential equations with r_1-Laplacian and r_2-Laplacian operators

$$\begin{cases} D_{0+}^{\gamma_1}\left(\varphi_{r_1}\left(D_{0+}^{\delta_1}u(\tau)\right)\right) = f\left(\tau, u(\tau), v(\tau), I_{0+}^{\sigma_1}u(\tau), I_{0+}^{\sigma_2}v(\tau)\right), \ \ \tau \in (0,1), \\ D_{0+}^{\gamma_2}\left(\varphi_{r_2}\left(D_{0+}^{\delta_2}v(\tau)\right)\right) = g\left(\tau, u(\tau), v(\tau), I_{0+}^{\varsigma_1}u(\tau), I_{0+}^{\varsigma_2}v(\tau)\right), \ \ \tau \in (0,1), \end{cases} \tag{1}$$

subject to the uncoupled nonlocal boundary conditions

$$\begin{cases} u^{(i)}(0) = 0, \ \ i = 0,\dots,p-2, \ \ D_{0+}^{\delta_1}u(0) = 0, \\ \varphi_{r_1}(D_{0+}^{\delta_1}u(1)) = \int_0^1 \varphi_{r_1}(D_{0+}^{\delta_1}u(\eta))\, d\mathcal{H}_0(\eta), \ \ D_{0+}^{\alpha_0}u(1) = \sum_{k=1}^{n}\int_0^1 D_{0+}^{\alpha_k}u(\eta)\, d\mathcal{H}_k(\eta), \\ v^{(i)}(0) = 0, \ \ i = 0,\dots,q-2, \ \ D_{0+}^{\delta_2}v(0) = 0, \\ \varphi_{r_2}(D_{0+}^{\delta_2}v(1)) = \int_0^1 \varphi_{r_2}(D_{0+}^{\delta_2}v(\eta))\, d\mathcal{K}_0(\eta), \ \ D_{0+}^{\beta_0}v(1) = \sum_{k=1}^{m}\int_0^1 D_{0+}^{\beta_k}v(\eta)\, d\mathcal{K}_k(\eta), \end{cases} \tag{2}$$

where $\gamma_1, \gamma_2 \in (1,2]$, $\delta_1 \in (p-1,p]$, $p \in \mathbb{N}$, $p \geq 3$, $\delta_2 \in (q-1,q]$, $q \in \mathbb{N}$, $q \geq 3$, $n, m \in \mathbb{N}$, $\sigma_1, \sigma_2, \varsigma_1, \varsigma_2 > 0$, $\alpha_k \in \mathbb{R}$, $k = 0,\dots,n$, $0 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_n \leq \alpha_0 < \delta_1 - 1$, $\alpha_0 \geq 1$, $\beta_k \in \mathbb{R}$, $k = 0,\dots,m$, $0 \leq \beta_1 < \beta_2 < \cdots < \beta_m \leq \beta_0 < \delta_2 - 1$, $\beta_0 \geq 1$, $\varphi_{r_i}(\eta) = |\eta|^{r_i-2}\eta$, $\varphi_{r_i}^{-1} = \varphi_{\varrho_i}$, $\varrho_i = \frac{r_i}{r_i-1}$, $i = 1,2$, $r_i > 1$, $i = 1,2$, $f, g : (0,1) \times \mathbb{R}_+^4 \to \mathbb{R}_+$ are continuous functions, singular at $\tau = 0$ and/or $\tau = 1$, ($\mathbb{R}_+ = [0,\infty)$), I_{0+}^{κ} is the Riemann–Liouville fractional integral of order κ (for $\kappa = \sigma_1, \sigma_2, \varsigma_1, \varsigma_2$), D_{0+}^{κ} is the Riemann-Liouville fractional derivative of order κ (for $\kappa = \gamma_1, \delta_1, \gamma_2, \delta_2, \alpha_0, \dots, \alpha_n, \beta_0, \dots, \beta_m$), and the integrals from the boundary conditions (2) are Riemann–Stieltjes integrals with $\mathcal{H}_i : [0,1] \to \mathbb{R}, \ i = 0,\dots,n$ and $\mathcal{K}_i : [0,1] \to \mathbb{R}, \ i = 0,\dots,m$ functions of bounded variation.

We give in this paper various conditions for the functions f and g such that problems (1) and (2) have at least one or two positive solutions. From a positive solution of (1) and (2) we understand a pair of functions $(u,v) \in (C([0,1],\mathbb{R}_+))^2$ satisfying the system (1) and the boundary conditions (2), with $u(\tau) > 0$ for all $\tau \in (0,1]$ or $v(\tau) > 0$ for all $\tau \in (0,1]$. In the proof of our main results we use the Guo–Krasnosel'skii fixed point theorem of cone expansion and compression of norm type. We now present some recent results which are connected with our problem. In [1], the authors studied the existence of multiple positive solutions for the system of nonlinear fractional differential equations with a p-Laplacian operator

$$\begin{cases} D_{0+}^{\beta_1}(\varphi_{p_1}(D_{0+}^{\alpha_1}x(\tau))) = f(\tau,x(\tau),y(\tau)), & \tau \in (0,1), \\ D_{0+}^{\beta_2}(\varphi_{p_2}(D_{0+}^{\alpha_2}y(\tau))) = g(\tau,x(\tau),y(\tau)), & \tau \in (0,1), \end{cases}$$

supplemented with the uncoupled boundary conditions

$$\begin{cases} x(0) = 0, \ D_{0+}^{\gamma_1}x(1) = \sum\limits_{i=1}^{m-2} \xi_{1i} D_{0+}^{\gamma_1}x(\eta_{1i}), \\ D_{0+}^{\alpha_1}x(0) = 0, \ \varphi_{p_1}(D_{0+}^{\alpha_1}x(1)) = \sum_{i=1}^{m-2} \zeta_{1i}\varphi_{p_1}(D_{0+}^{\alpha_1}x(\eta_{1i})), \\ y(0) = 0, \ D_{0+}^{\gamma_2}y(1) = \sum\limits_{i=1}^{m-2} \xi_{2i} D_{0+}^{\gamma_2}y(\eta_{2i}), \\ D_{0+}^{\alpha_2}y(0) = 0, \ \varphi_{p_2}(D_{0+}^{\alpha_2}y(1)) = \sum_{i=1}^{m-2} \zeta_{2i}\varphi_{p_2}(D_{0+}^{\alpha_2}y(\eta_{2i})), \end{cases}$$

where $\alpha_i, \beta_i \in (1,2]$, $\gamma_i \in (0,1]$, $\alpha_i + \beta_i \in (3,4]$, $\alpha_i > \gamma_i + 1$, $i = 1,2$, $\xi_{1i}, \eta_{1i}, \zeta_{1i}, \xi_{2i}, \eta_{2i}, \zeta_{2i} \in (0,1)$ for $i = 1,\dots,m-2$, and f and g are nonnegative and nonsingular functions. In the proof of the existence results they use the Leray–Schauder alternative theorem, the Leggett–Williams fixed point theorem and the Avery–Henderson fixed point theorem. In [2], the authors investigated the existence and multiplicity of positive solutions for the system of fractional differential equations with ϱ_1-Laplacian and ϱ_2-Laplacian operators

$$\begin{cases} D_{0+}^{\gamma_1}(\varphi_{\varrho_1}(D_{0+}^{\delta_1}x(\tau))) + f(\tau,x(\tau),y(\tau)) = 0, & \tau \in (0,1), \\ D_{0+}^{\gamma_2}(\varphi_{\varrho_2}(D_{0+}^{\delta_2}y(\tau))) + g(\tau,x(\tau),y(\tau)) = 0, & \tau \in (0,1), \end{cases} \tag{3}$$

subject to the uncoupled nonlocal boundary conditions

$$\begin{cases} x^{(j)}(0) = 0, \ j = 0,\dots,p-2; \ D_{0+}^{\delta_1}x(0) = 0, \ D_{0+}^{\alpha_0}x(1) = \sum\limits_{i=1}^{n} \int_0^1 D_{0+}^{\alpha_i}x(\tau)\,d\mathcal{H}_i(\tau), \\ y^{(j)}(0) = 0, \ j = 0,\dots,q-2; \ D_{0+}^{\delta_2}y(0) = 0, \ D_{0+}^{\beta_0}y(1) = \sum\limits_{i=1}^{m} \int_0^1 D_{0+}^{\beta_i}y(\tau)\,d\mathcal{K}_i(\tau), \end{cases}$$

where $\gamma_1, \gamma_2 \in (0,1]$, $\delta_1 \in (p-1,p]$, $\delta_2 \in (q-1,q]$, $p, q \in \mathbb{N}$, $p, q \geq 3$, $n, m \in \mathbb{N}$, $\alpha_i \in \mathbb{R}$ for all $i = 0,1,\dots,n$, $0 \leq \alpha_1 < \alpha_2 < \dots < \alpha_n \leq \alpha_0 < \delta_1 - 1$, $\alpha_0 \geq 1$, $\beta_i \in \mathbb{R}$ for all $i = 0,1,\dots,m$, $0 \leq \beta_1 < \beta_2 < \dots < \beta_m \leq \beta_0 < \delta_2 - 1$, $\beta_0 \geq 1$, $\varrho_1, \varrho_2 > 1$, the functions f and g are nonnegative and continuous, and they may be singular at $\tau = 0$ and/or $\tau = 1$, and $\mathcal{H}_i$, $i = 1,\dots,n$ and $\mathcal{K}_j$, $j = 1,\dots,m$ are functions of bounded variation. In the proof of the main existence results they applied the Guo–Krasnosel'skii fixed point theorem. In [3], the authors studied the existence and nonexistence of positive solutions for the system (3) with two positive parameters λ and μ, supplemented with the coupled nonlocal boundary conditions

$$\begin{cases} x^{(j)}(0) = 0, \ j = 0,\dots,p-2; \ D_{0+}^{\delta_1}x(0) = 0, \ D_{0+}^{\alpha_0}x(1) = \sum\limits_{i=1}^{n} \int_0^1 D_{0+}^{\alpha_i}y(\tau)\,d\mathcal{H}_i(\tau), \\ y^{(j)}(0) = 0, \ j = 0,\dots,q-2; \ D_{0+}^{\delta_2}y(0) = 0, \ D_{0+}^{\beta_0}y(1) = \sum\limits_{i=1}^{m} \int_0^1 D_{0+}^{\beta_i}x(\tau)\,d\mathcal{K}_i(\tau), \end{cases} \tag{4}$$

where $n, m \in \mathbb{N}$, $\alpha_i \in \mathbb{R}$ for all $i = 0, \ldots, n$, $0 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_n \leq \beta_0 < \delta_2 - 1$, $\beta_0 \geq 1$, $\beta_i \in \mathbb{R}$ for all $i = 0, \ldots, m$, $0 \leq \beta_1 < \beta_2 < \cdots < \beta_m \leq \alpha_0 < \delta_1 - 1$, $\alpha_0 \geq 1$, the functions $f, g \in C([0,1] \times \mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+)$, and the functions $\mathcal{H}_i$, $i = 1, \ldots, n$ and $\mathcal{K}_j$, $j = 1, \ldots, m$ are bounded variation functions. They presented sufficient conditions on the functions f and g, and intervals for the parameters λ and μ such that the problem (3) with these parameters and (4) has positive solutions. In [4], by using the Guo–Krasnosel'skii fixed point theorem, the authors investigated the existence and multiplicity of positive solutions for the nonlinear singular fractional differential equation

$$D_{0+}^{\alpha} w(\tau) + f(\tau, w(\tau), D_{0+}^{\alpha_1} w(\tau), \ldots, D_{0+}^{\alpha_{n-2}} w(\tau)) = 0, \quad \tau \in (0,1),$$

with the nonlocal boundary conditions

$$\begin{cases} w(0) = D_{0+}^{\gamma_1} w(0) = \cdots = D_{0+}^{\gamma_{n-2}} w(0) = 0, \\ D_{0+}^{\beta_1} w(1) = \int_0^{\eta} h(\tau) D_{0+}^{\beta_2} w(\tau)\, dA(\tau) + \int_0^1 a(\tau) D_{0+}^{\beta_3} w(\tau)\, dA(\tau), \end{cases}$$

where $\alpha \in (n-1, n]$, $n \geq 3$, $\alpha_k, \gamma_k \in (k-1, k]$, $k = 1, \ldots, n-2$, $\alpha - \gamma_j \in (n-j-1, n-j]$, $j = 1, \ldots, n-2$, $\alpha - \alpha_{n-2} - 1 \in (1,2]$, $\gamma_{n-2} \geq \alpha_{n-2}$, $\beta_1 \geq \beta_2$, $\beta_1 \geq \beta_3$, $\alpha \geq \beta_i + 1$, $\beta_i \geq \alpha_{n-2} + 1$, $i = 1, 2, 3$, $\beta_1 \leq n - 1$, the function $f : (0,1) \times \mathbb{R}_+^{n-1} \to \mathbb{R}_+$ is continuous, $a, h \in C((0,1), \mathbb{R}_+)$, and A is a function of bounded variation. In [5], the authors studied the existence of a unique positive solution for a system of three Caputo fractional equations with (p, q, r)-Laplacian operators subject to two-point boundary conditions, by using an n-fixed point theorem of ternary operators in partially ordered complete metric spaces. By relying on the properties of the Kuratowski noncompactness measure and the Sadovskii fixed point theorem; in [6], the authors obtained new existence results for the solutions of a Riemann–Liouville fractional differential equation with a p-Laplacian operator in a Banach space, supplemented with multi-point boundary conditions with fractional derivatives. In [7], the authors investigated the existence of solutions for a mixed fractional differential equation with $p(t)$-Laplacian operator and two-point boundary conditions at resonance, by applying the continuation theorem of coincidence degree theory. By using the Leggett–Williams fixed-point theorem, the authors studied in [8] the multiplicity of positive solutions for a Riemann–Liouville fractional differential equation with a p-Laplacian operator, subject to four-point boundary conditions. In [9], the authors established suitable criteria for the existence of positive solutions for a Riemann–Liouville fractional equation with a p-Laplacian operator and infinite-point boundary value conditions, by using the Krasnosel'skii fixed point theorem and Avery–Peterson fixed point theorem. By applying the Guo–Krasnosel'skii fixed point theorem the authors investigated in [10] the existence, multiplicity and the nonexistence of positive solutions for a mixed fractional differential equation with a generalized p-Laplacian operator and a positive parameter, supplemented with two-point boundary conditions. We also mention some recent monographs devoted to the investigation of boundary value problems for fractional differential equations and systems with many examples and applications, namely [11–15].

So in comparison with the above papers, the new characteristics of our problem (1) and (2) consist in a combination between the fractional orders $\gamma_1, \gamma_2 \in (1,2]$ with the arbitrary fractional orders δ_1, δ_2, the existence of the fractional integral terms in equations of (1), and the general uncoupled nonlocal boundary conditions with Riemann–Stieltjes integrals and fractional derivatives. In addition, one of its special feature is the singularity of the nonlinearities from the system (1), that is f, g become unbounded in the vicinity of 0 and/or 1 in the first variable (see Assumption $(I2)$ in Section 3).

The structure of this paper is as follows. In Section 2, some preliminary results including the properties of the Green functions associated to our problem (1) and (2) are presented. In Section 3 we discuss the existence and multiplicity of positive solutions for (1) and (2). Then two examples to illustrate our obtained theorems are given in Section 4, and Section 5 contains the conclusions for this paper.

2. Preliminary Results

We consider the fractional differential equation

$$D_{0+}^{\gamma_1}\Big(\varphi_{r_1}\Big(D_{0+}^{\delta_1}u(\tau)\Big)\Big) = x(\tau),\ \ \tau \in (0,1), \tag{5}$$

where $x \in C(0,1) \cap L^1(0,1)$, with the boundary conditions

$$\begin{cases} u^{(i)}(0) = 0,\ \ i = 0,\dots,p-2,\ \ D_{0+}^{\delta_1}u(0) = 0, \\ \varphi_{r_1}(D_{0+}^{\delta_1}u(1)) = \displaystyle\int_0^1 \varphi_{r_1}(D_{0+}^{\delta_1}u(\eta))\,d\mathcal{H}_0(\eta),\ \ D_{0+}^{\alpha_0}u(1) = \sum_{k=1}^{n}\int_0^1 D_{0+}^{\alpha_k}u(\eta)\,d\mathcal{H}_k(\eta). \end{cases} \tag{6}$$

We denote by

$$\mathfrak{a}_1 = 1 - \int_0^1 \eta^{\gamma_1-1}\,d\mathcal{H}_0(\eta),\ \ \mathfrak{a}_2 = \frac{\Gamma(\delta_1)}{\Gamma(\delta_1-\alpha_0)} - \sum_{i=1}^{n}\frac{\Gamma(\delta_1)}{\Gamma(\delta_1-\alpha_i)}\int_0^1 \eta^{\delta_1-\alpha_i-1}\,d\mathcal{H}_i(\eta). \tag{7}$$

Lemma 1. *If $\mathfrak{a}_1 \neq 0$ and $\mathfrak{a}_2 \neq 0$, then the unique solution $u \in C[0,1]$ of problem (5) and (6) is given by*

$$u(\tau) = \int_0^1 \mathcal{G}_2(\tau,\eta)\varphi_{\varrho_1}\left(\int_0^1 \mathcal{G}_1(\eta,\vartheta)x(\vartheta)\,d\vartheta\right)d\eta,\ \ \tau \in [0,1], \tag{8}$$

where

$$\mathcal{G}_1(\tau,\eta) = \mathfrak{g}_1(\tau,\eta) + \frac{\tau^{\gamma_1-1}}{\mathfrak{a}_1}\int_0^1 \mathfrak{g}_1(\vartheta,\eta)\,d\mathcal{H}_0(\vartheta),\ \ (\tau,\eta) \in [0,1]\times[0,1], \tag{9}$$

with

$$\mathfrak{g}_1(\tau,\eta) = \frac{1}{\Gamma(\gamma_1)}\begin{cases} \tau^{\gamma_1-1}(1-\eta)^{\gamma_1-1} - (\tau-\eta)^{\gamma_1-1},\ \ 0 \le \eta \le \tau \le 1, \\ \tau^{\gamma_1-1}(1-\eta)^{\gamma_1-1},\ \ 0 \le \tau \le \eta \le 1, \end{cases} \tag{10}$$

and

$$\mathcal{G}_2(\tau,\eta) = \mathfrak{g}_2(\tau,\eta) + \frac{\tau^{\delta_1-1}}{\mathfrak{a}_2}\sum_{i=1}^{n}\left(\int_0^1 \mathfrak{g}_{2i}(\vartheta,\eta)\,d\mathcal{H}_i(\vartheta)\right),\ \ (\tau,\eta) \in [0,1]\times[0,1], \tag{11}$$

with

$$\begin{aligned} &\mathfrak{g}_2(\tau,\eta) = \frac{1}{\Gamma(\delta_1)}\begin{cases} \tau^{\delta_1-1}(1-\eta)^{\delta_1-\alpha_0-1} - (\tau-\eta)^{\delta_1-1},\ \ 0 \le \eta \le \tau \le 1, \\ \tau^{\delta_1-1}(1-\eta)^{\delta_1-\alpha_0-1},\ \ 0 \le \tau \le \eta \le 1, \end{cases} \\ &\mathfrak{g}_{2i}(\tau,\eta) = \frac{1}{\Gamma(\delta_1-\alpha_i)}\begin{cases} \tau^{\delta_1-\alpha_i-1}(1-\eta)^{\delta_1-\alpha_0-1} - (\tau-\eta)^{\delta_1-\alpha_i-1},\ \ 0 \le \eta \le \tau \le 1, \\ \tau^{\delta_1-\alpha_i-1}(1-\eta)^{\delta_1-\alpha_0-1},\ \ 0 \le \tau \le \eta \le 1, \end{cases} \\ &i = 1,\dots,n. \end{aligned} \tag{12}$$

Proof. We denote by $\varphi_{r_1}(D_{0+}^{\delta_1}u(\tau)) = \phi_1(\tau)$, $\tau \in (0,1)$. Hence problems (5) and (6) are equivalent to the following two boundary value problems

$$(I)\qquad \begin{cases} D_{0+}^{\gamma_1}\phi_1(\tau) = x(\tau),\ \ \tau \in (0,1), \\ \phi_1(0) = 0,\ \ \phi_1(1) = \displaystyle\int_0^1 \phi_1(\eta)\,d\mathcal{H}_0(\eta), \end{cases}$$

and

$$(II)\qquad \begin{cases} D_{0+}^{\delta_1}u(\tau) = \varphi_{\varrho_1}(\phi_1(\tau)),\ \ \tau \in (0,1), \\ u^{(j)}(0) = 0,\ \ j = 0,\dots,p-2,\ \ D_{0+}^{\alpha_0}u(1) = \displaystyle\sum_{k=1}^{n}\int_0^1 D_{0+}^{\alpha_k}u(\eta)\,d\mathcal{H}_k(\eta). \end{cases}$$

By using Lemma 4.1.5 from [14], the unique solution $\phi_1 \in C[0,1]$ of problem (I) is

$$\phi_1(\tau) = -\int_0^1 \mathcal{G}_1(\tau,\vartheta)x(\vartheta)\,d\vartheta, \ \ \tau \in [0,1], \tag{13}$$

where $\mathcal{G}_1$ is given by (9). By using Lemma 2.4.2 from [12], the unique solution $u \in C[0,1]$ of problem (II) is

$$u(\tau) = -\int_0^1 \mathcal{G}_2(\tau,\eta)\varphi_{\varrho_1}(\phi_1(\eta))\,d\eta, \ \ \tau \in [0,1], \tag{14}$$

where $\mathcal{G}_2$ is given by (11). Combining the relations (13) and (14) we obtain the solution u of problem (5) and (6) which is given by relation (8). □

We consider now the fractional differential equation

$$D_{0+}^{\gamma_2}\left(\varphi_{r_2}\left(D_{0+}^{\delta_2}v(\tau)\right)\right) = y(\tau), \ \ \tau \in (0,1), \tag{15}$$

where $y \in C(0,1) \cap L^1(0,1)$, with the boundary conditions

$$\begin{cases} v^{(i)}(0) = 0, \ \ i = 0,\ldots,q-2, \ \ D_{0+}^{\delta_2}v(0) = 0, \\ \varphi_{r_2}(D_{0+}^{\delta_2}v(1)) = \int_0^1 \varphi_{r_2}(D_{0+}^{\delta_2}v(\eta))\,d\mathcal{K}_0(\eta), \ \ D_{0+}^{\beta_0}v(1) = \sum_{k=1}^m \int_0^1 D_{0+}^{\beta_k}v(\eta)\,d\mathcal{K}_k(\eta). \end{cases} \tag{16}$$

We denote by

$$\mathfrak{b}_1 = 1 - \int_0^1 \eta^{\gamma_2 - 1}\,d\mathcal{K}_0(\eta), \ \ \mathfrak{b}_2 = \frac{\Gamma(\delta_2)}{\Gamma(\delta_2 - \beta_0)} - \sum_{i=1}^m \frac{\Gamma(\delta_2)}{\Gamma(\delta_2 - \beta_i)} \int_0^1 \eta^{\delta_2 - \beta_i - 1}\,d\mathcal{K}_i(\eta). \tag{17}$$

Similar to Lemma 1 we obtain the next result.

Lemma 2. *If $\mathfrak{b}_1 \neq 0$ and $\mathfrak{b}_2 \neq 0$, then the unique solution $v \in C[0,1]$ of problem (15) and (16) is given by*

$$v(\tau) = \int_0^1 \mathcal{G}_4(\tau,\eta)\varphi_{\varrho_2}\left(\int_0^1 \mathcal{G}_3(\eta,\vartheta)y(\vartheta)\,d\vartheta\right)d\eta, \ \ \tau \in [0,1], \tag{18}$$

where

$$\mathcal{G}_3(\tau,\eta) = \mathfrak{g}_3(\tau,\eta) + \frac{\tau^{\gamma_2-1}}{\mathfrak{b}_1}\int_0^1 \mathfrak{g}_3(\vartheta,\eta)\,d\mathcal{K}_0(\vartheta), \ \ (\tau,\eta) \in [0,1]\times[0,1], \tag{19}$$

with

$$\mathfrak{g}_3(\tau,\eta) = \frac{1}{\Gamma(\gamma_2)}\begin{cases} \tau^{\gamma_2-1}(1-\eta)^{\gamma_2-1} - (\tau-\eta)^{\gamma_2-1}, & 0 \le \eta \le \tau \le 1, \\ \tau^{\gamma_2-1}(1-\eta)^{\gamma_2-1}, & 0 \le \tau \le \eta \le 1, \end{cases} \tag{20}$$

and

$$\mathcal{G}_4(\tau,\eta) = \mathfrak{g}_4(\tau,\eta) + \frac{\tau^{\delta_2-1}}{\mathfrak{b}_2}\sum_{i=1}^m\left(\int_0^1 \mathfrak{g}_{4i}(\vartheta,\eta)\,d\mathcal{K}_i(\vartheta)\right), \ \ (\tau,\eta) \in [0,1]\times[0,1], \tag{21}$$

with

$$\begin{aligned} &\mathfrak{g}_4(\tau,\eta) = \frac{1}{\Gamma(\delta_2)}\begin{cases} \tau^{\delta_2-1}(1-\eta)^{\delta_2-\beta_0-1} - (\tau-\eta)^{\delta_2-1}, & 0 \le \eta \le \tau \le 1, \\ \tau^{\delta_2-1}(1-\eta)^{\delta_2-\beta_0-1}, & 0 \le \tau \le \eta \le 1, \end{cases} \\ &\mathfrak{g}_{4i}(\tau,\eta) = \frac{1}{\Gamma(\delta_2-\beta_i)}\begin{cases} \tau^{\delta_2-\beta_i-1}(1-\eta)^{\delta_2-\beta_0-1} - (\tau-\eta)^{\delta_2-\beta_i-1}, & 0 \le \eta \le \tau \le 1, \\ \tau^{\delta_2-\beta_i-1}(1-\eta)^{\delta_2-\beta_0-1}, & 0 \le \tau \le \eta \le 1, \end{cases} \\ &i = 1,\ldots,m. \end{aligned} \tag{22}$$

Lemma 3. *We assume that* $\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{b}_1, \mathfrak{b}_2 > 0$, $\mathcal{H}_i$, $i = 0, \dots, n$, *and* $\mathcal{K}_j$, $j = 0, \dots, m$ *are non-decreasing functions. Then the functions* $\mathcal{G}_i$, $i = 1, \dots, 4$ *given by (9), (11), (19) and (21) have the properties*

(a) $\mathcal{G}_i : [0,1] \times [0,1] \to [0, \infty)$, $i = 1, \dots, 4$ *are continuous functions;*

(b) $\mathcal{G}_1(\tau, \eta) \le \mathcal{J}_1(\eta)$, $\forall (\tau, \eta) \in [0,1] \times [0,1]$, *where*

$$\mathcal{J}_1(\eta) = \mathfrak{h}_1(\eta) + \frac{1}{\mathfrak{a}_1} \int_0^1 \mathfrak{g}_1(\vartheta, \eta)\, d\mathcal{H}_0(\vartheta), \ \forall \eta \in [0,1],$$

with $\mathfrak{h}_1(\eta) = \frac{1}{\Gamma(\gamma_1)}(1-\eta)^{\gamma_1 - 1}$, $\eta \in [0,1]$;

(c) $\mathcal{G}_2(\tau, \eta) \le \mathcal{J}_2(\eta)$, $\forall (\tau, \eta) \in [0,1] \times [0,1]$, *where*

$$\mathcal{J}_2(\eta) = \mathfrak{h}_2(\eta) + \frac{1}{\mathfrak{a}_2} \sum_{i=1}^{n} \int_0^1 \mathfrak{g}_{2i}(\vartheta, \eta)\, d\mathcal{H}_i(\vartheta), \ \forall \eta \in [0,1],$$

with $\mathfrak{h}_2(\eta) = \frac{1}{\Gamma(\delta_1)}(1-\eta)^{\delta_1 - \alpha_0 - 1}(1 - (1-\eta)^{\alpha_0})$, $\eta \in [0,1]$;

(d) $\mathcal{G}_2(\tau, \eta) \ge \tau^{\delta_1 - 1} \mathcal{J}_2(\eta)$, $\forall (\tau, \eta) \in [0,1] \times [0,1]$;

(e) $\mathcal{G}_3(\tau, \eta) \le \mathcal{J}_3(\eta)$, $\forall (\tau, \eta) \in [0,1] \times [0,1]$, *where*

$$\mathcal{J}_3(\eta) = \mathfrak{h}_3(\eta) + \frac{1}{\mathfrak{b}_1} \int_0^1 \mathfrak{g}_3(\vartheta, \eta)\, d\mathcal{K}_0(\vartheta), \ \forall \eta \in [0,1],$$

with $\mathfrak{h}_3(\eta) = \frac{1}{\Gamma(\gamma_2)}(1-\eta)^{\gamma_2 - 1}$, $\eta \in [0,1]$;

(f) $\mathcal{G}_4(\tau, \eta) \le \mathcal{J}_4(\eta)$, $\forall (\tau, \eta) \in [0,1] \times [0,1]$, *where*

$$\mathcal{J}_4(\eta) = \mathfrak{h}_4(\eta) + \frac{1}{\mathfrak{b}_2} \sum_{i=1}^{m} \int_0^1 \mathfrak{g}_{4i}(\vartheta, \eta)\, d\mathcal{K}_i(\vartheta), \ \forall \eta \in [0,1],$$

with $\mathfrak{h}_4(\eta) = \frac{1}{\Gamma(\delta_2)}(1-\eta)^{\delta_2 - \beta_0 - 1}(1 - (1-\eta)^{\beta_0})$, $\eta \in [0,1]$;

(g) $\mathcal{G}_4(\tau, \eta) \ge \tau^{\delta_2 - 1} \mathcal{J}_4(\eta)$, $\forall (\tau, \eta) \in [0,1] \times [0,1]$.

Proof. (a) Based on the continuity of functions $\mathfrak{g}_1, \mathfrak{g}_2, \mathfrak{g}_{2i}$, $i = 1, \dots, n$, $\mathfrak{g}_3$, $\mathfrak{g}_4$, $\mathfrak{g}_{4i}$, $i = 1, \dots, m$ (given by (10), (12), (20) and (22)), we obtain that the functions $\mathcal{G}_i$, $i = 1, \dots, 4$ are continuous.

(b) By the definition of $\mathfrak{g}_1$ we find

$$\begin{aligned} \mathcal{G}_1(\tau, \eta) &\le \frac{1}{\Gamma(\gamma_1)}(1-\eta)^{\gamma_1 - 1} + \frac{1}{\mathfrak{a}_1} \int_0^1 \mathfrak{g}_1(\vartheta, \eta)\, d\mathcal{H}_0(\vartheta) \\ &= \mathfrak{h}_1(\eta) + \frac{1}{\mathfrak{a}_1} \int_0^1 \mathfrak{g}_1(\vartheta, \eta)\, d\mathcal{H}_0(\vartheta) = \mathcal{J}_1(\eta), \ \forall \tau, \eta \in [0,1]. \end{aligned}$$

(c–d) Using our assumptions and the properties of function $\mathfrak{g}_2$ from Lemma 2.1.3 from [12], namely $\mathfrak{g}_2(\tau, \eta) \le \frac{1}{\Gamma(\delta_1)}(1-\eta)^{\delta_1 - \alpha_0 - 1}(1 - (1-\eta)^{\alpha_0}) = \mathfrak{h}_2(\eta)$ and $\mathfrak{g}_2(\tau, \eta) \ge \tau^{\delta_1 - 1}\mathfrak{h}_2(\eta)$ for all τ, $\eta \in [0,1]$, we deduce

$$\begin{aligned} &\mathcal{G}_2(\tau, \eta) \le \mathfrak{h}_2(\eta) + \frac{1}{\mathfrak{a}_2} \sum_{i=1}^{n} \int_0^1 \mathfrak{g}_{2i}(\vartheta, \eta)\, d\mathcal{H}_i(\vartheta) = \mathcal{J}_2(\eta), \\ &\mathcal{G}_2(\tau, \eta) \ge \tau^{\delta_1 - 1}\left(\mathfrak{h}_2(\eta) + \frac{1}{\mathfrak{a}_2} \sum_{i=1}^{n} \mathfrak{g}_{2i}(\vartheta, \eta)\, d\mathcal{H}_i(\vartheta) \right) = \tau^{\delta_1 - 1}\mathcal{J}_2(\eta), \ \forall \tau, \eta \in [0,1]. \end{aligned}$$

(e) By the definition of $\mathfrak{g}_3$ we obtain

$$\begin{aligned}\mathcal{G}_3(\tau,\eta) &\leq \frac{1}{\Gamma(\gamma_2)}(1-\eta)^{\gamma_2-1} + \frac{1}{\mathfrak{b}_1}\int_0^1 \mathfrak{g}_3(\vartheta,\eta)\,d\mathcal{K}_0(\vartheta)\\ &= \mathfrak{h}_3(\eta) + \frac{1}{\mathfrak{b}_1}\int_0^1 \mathfrak{g}_3(\vartheta,\eta)\,d\mathcal{K}_0(\vartheta) = \mathcal{J}_3(\eta),\ \ \forall\,\tau,\eta\in[0,1].\end{aligned}$$

(f–g) Using the assumptions of this lemma and the properties of function $\mathfrak{g}_4$ from Lemma 2.1.3 from [12], namely $\mathfrak{g}_4(\tau,\eta) \leq \frac{1}{\Gamma(\delta_2)}(1-\eta)^{\delta_2-\beta_0-1}(1-(1-\eta)^{\beta_0}) = \mathfrak{h}_4(\eta)$ and $\mathfrak{g}_4(\tau,\eta) \geq \tau^{\delta_2-1}\mathfrak{h}_4(\eta)$ for all τ, $\eta\in[0,1]$, we find

$$\begin{aligned}&\mathcal{G}_4(\tau,\eta) \leq \mathfrak{h}_4(\eta) + \frac{1}{\mathfrak{b}_2}\sum_{i=1}^{m}\int_0^1 \mathfrak{g}_{4i}(\vartheta,\eta)\,d\mathcal{K}_i(\vartheta) = \mathcal{J}_4(\eta),\\ &\mathcal{G}_4(\tau,\eta) \geq \tau^{\delta_2-1}\left(\mathfrak{h}_4(\eta) + \frac{1}{\mathfrak{b}_2}\sum_{i=1}^{m}\mathfrak{g}_{4i}(\vartheta,\eta)\,d\mathcal{K}_i(\vartheta)\right) = \tau^{\delta_2-1}\mathcal{J}_4(\eta),\ \ \forall\,\tau,\eta\in[0,1].\end{aligned}$$

□

Lemma 4. *We assume that* $\mathfrak{a}_1,\mathfrak{a}_2,\mathfrak{b}_1,\mathfrak{b}_2>0$, $\mathcal{H}_i$, $i=0,\ldots,n$, *and* $\mathcal{K}_j$, $j=0,\ldots,m$ *are nondecreasing functions,* $x,y\in C(0,1)\cap L^1(0,1)$ *with* $x(\tau)\geq 0$, $y(\tau)\geq 0$ *for all* $\tau\in(0,1)$. *Then the solutions* u *and* v *of problems (5), (6) and (15), (16), respectively, satisfy the inequalities* $u(\tau)\geq 0$, $v(\tau)\geq 0$ *for all* $\tau\in[0,1]$ *and* $u(\tau)\geq\tau^{\delta_1-1}u(s)$ *and* $v(\tau)\geq\tau^{\delta_2-1}v(s)$ *for all* $\tau,s\in[0,1]$.

Proof. Based on the assumptions of this lemma, we obtain that the solutions u and v of problems (5), (6) and (15), (16), respectively, are nonnegative, that is $u(\tau)\geq 0$, $v(\tau)\geq 0$ for all $\tau\in[0,1]$. In addition, by using Lemma 3, we deduce

$$\begin{aligned}&u(\tau) \geq \tau^{\delta_1-1}\int_0^1 \mathcal{J}_2(\eta)\varphi_{\varrho_1}\left(\int_0^1 \mathcal{G}_1(\eta,\vartheta)x(\vartheta)\,d\vartheta\right)d\eta\\ &\geq \tau^{\delta_1-1}\int_0^1 \mathcal{G}_2(s,\eta)\varphi_{\varrho_1}\left(\int_0^1 \mathcal{G}_1(\eta,\vartheta)x(\vartheta)\,d\vartheta\right)d\eta\\ &= \tau^{\delta_1-1}u(s),\\ &v(\tau) \geq \tau^{\delta_2-1}\int_0^1 \mathcal{J}_4(\eta)\varphi_{\varrho_2}\left(\int_0^1 \mathcal{G}_3(\eta,\vartheta)y(\vartheta)\,d\vartheta\right)d\eta\\ &\geq \tau^{\delta_2-1}\int_0^1 \mathcal{G}_4(s,\eta)\varphi_{\varrho_2}\left(\int_0^1 \mathcal{G}_3(\eta,\vartheta)y(\vartheta)\,d\vartheta\right)d\eta\\ &= \tau^{\delta_2-1}v(s),\end{aligned}$$

for all τ, $s\in[0,1]$. □

We present finally in this section the Guo–Krasnosel'skii fixed point theorem, which we will use in the proofs of our main results.

Theorem 1. *([16]). Let* $\mathcal{X}$ *be a real Banach space with the norm* $\|\cdot\|$, *and let* $\mathcal{C}\subset X$ *be a cone in* $\mathcal{X}$. *Assume* Ω_1 *and* Ω_2 *are bounded open subsets of* $\mathcal{X}$ *with* $0\in\Omega_1$, $\overline{\Omega}_1\subset\Omega_2$ *and let* $\mathcal{A}:\mathcal{C}\cap(\overline{\Omega}_2\setminus\Omega_1)\to\mathcal{C}$ *be a completely continuous operator such that, either*

(i) $\|\mathcal{A}u\|\leq\|u\|$, $\forall\,u\in\mathcal{C}\cap\partial\Omega_1$, *and* $\|\mathcal{A}u\|\geq\|u\|$, $\forall\,u\in\mathcal{C}\cap\partial\Omega_2$; *or*

(ii) $\|\mathcal{A}u\|\geq\|u\|$, $\forall\,u\in\mathcal{C}\cap\partial\Omega_1$, *and* $\|\mathcal{A}u\|\leq\|u\|$, $\forall\,u\in\mathcal{C}\cap\partial\Omega_2$.

Then $\mathcal{A}$ *has at least one fixed point in* $\mathcal{C}\cap(\overline{\Omega}_2\setminus\Omega_1)$.

3. Existence of Positive Solutions

According to Lemmas 1 and 2, the pair of functions (u,v) is a solution of problem (1) and (2) if and only if (u,v) is a solution of the system

$$\begin{cases} u(\tau) = \displaystyle\int_0^1 \mathcal{G}_2(\tau,\zeta)\varphi_{\varrho_1}\left(\int_0^1 \mathcal{G}_1(\zeta,\vartheta)f\big(\vartheta,u(\vartheta),v(\vartheta),I_{0+}^{\sigma_1}u(\vartheta),I_{0+}^{\sigma_2}v(\vartheta)\big)d\vartheta\right)d\zeta, \\ v(\tau) = \displaystyle\int_0^1 \mathcal{G}_4(\tau,\zeta)\varphi_{\varrho_2}\left(\int_0^1 \mathcal{G}_3(\zeta,\vartheta)g\big(\vartheta,u(\vartheta),v(\vartheta),I_{0+}^{\varsigma_1}u(\vartheta),I_{0+}^{\varsigma_2}v(\vartheta)\big)d\vartheta\right)d\zeta, \end{cases}$$

for all $\tau \in [0,1]$. We introduce the Banach space $\mathcal{X} = C[0,1]$ with supreme norm $\|u\| = \sup_{\tau\in[0,1]} |u(\tau)|$, and the Banach space $\mathcal{Y} = \mathcal{X} \times \mathcal{X}$ with the norm $\|(u,v)\|_{\mathcal{Y}} = \|u\| + \|v\|$. We define the cone

$$\mathcal{P} = \{(u,v) \in \mathcal{Y},\ u(\tau) \geq 0,\ v(\tau) \geq 0,\ \forall\, \tau \in [0,1]\}.$$

We also define the operators $\mathcal{A}_1, \mathcal{A}_2 : \mathcal{Y} \to \mathcal{X}$ and $\mathcal{A} : \mathcal{Y} \to \mathcal{Y}$ by

$$\mathcal{A}_1(u,v)(\tau) = \int_0^1 \mathcal{G}_2(\tau,\zeta)\varphi_{\varrho_1}\left(\int_0^1 \mathcal{G}_1(\zeta,\vartheta)f\big(\vartheta,u(\vartheta),v(\vartheta),I_{0+}^{\sigma_1}u(\vartheta),I_{0+}^{\sigma_2}v(\vartheta)\big)d\vartheta\right)d\zeta,$$
$$\mathcal{A}_2(u,v)(\tau) = \int_0^1 \mathcal{G}_4(\tau,\zeta)\varphi_{\varrho_2}\left(\int_0^1 \mathcal{G}_3(\zeta,\vartheta)g\big(\vartheta,u(\vartheta),v(\vartheta),I_{0+}^{\varsigma_1}u(\vartheta),I_{0+}^{\varsigma_2}v(\vartheta)\big)d\vartheta\right)d\zeta,$$

for $\tau \in [0,1]$ and $(u,v) \in \mathcal{Y}$, and $\mathcal{A}(u,v) = (\mathcal{A}_1(u,v), \mathcal{A}_2(u,v))$, $(u,v) \in \mathcal{Y}$. We see that (u,v) is a solution of problem (1) and (2) if and only if (u,v) is a fixed point of operator $\mathcal{A}$.

We introduce now the basic assumptions that we will use in this section.

$(I1)$ $\gamma_1,\gamma_2 \in (1,2]$, $\delta_1 \in (p-1,p]$, $p \in \mathbb{N}$, $p \geq 3$, $\delta_2 \in (q-1,q]$, $q \in \mathbb{N}$, $q \geq 3$, $n,m \in \mathbb{N}$, $\sigma_1,\sigma_2,\varsigma_1,\varsigma_2 > 0$, $\alpha_j \in \mathbb{R}$, $j = 0,\ldots,n$, $0 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_n \leq \alpha_0 < \delta_1 - 1$, $\alpha_0 \geq 1$, $\beta_j \in \mathbb{R}$, $j = 0,\ldots,m$, $0 \leq \beta_1 < \beta_2 < \cdots < \beta_m \leq \beta_0 < \delta_2 - 1$, $\beta_0 \geq 1$, $\varphi_{r_i}(\tau) = |\tau|^{r_i-2}\tau$, $\varphi_{r_i}^{-1} = \varphi_{\varrho_i}$, $\varrho_i = \frac{r_i}{r_i-1}$, $i = 1,2$, $r_i > 1$, $i = 1,2$, $\mathcal{H}_i : [0,1] \to \mathbb{R}$, $i = 0,\ldots,n$, and $\mathcal{K}_j : [0,1] \to \mathbb{R}$, $j = 0,\ldots,m$ are nondecreasing functions, $\mathfrak{a}_1,\mathfrak{a}_2,\mathfrak{b}_1,\mathfrak{b}_2 > 0$ (given by (7) and (17)).

$(I2)$ The functions $f,g \in C((0,1)\times\mathbb{R}_+^4,\mathbb{R}_+)$ and there exist the functions $\psi_1,\psi_2 \in C((0,1),\mathbb{R}_+)$ and $\chi_1,\chi_2 \in C([0,1]\times\mathbb{R}_+^4,\mathbb{R}_+)$ with $\Lambda_1 = \int_0^1 (1-\tau)^{\gamma_1-1}\psi_1(\tau)\,d\tau \in (0,\infty)$, $\Lambda_2 = \int_0^1 (1-\tau)^{\gamma_2-1}\psi_2(\tau)\,d\tau \in (0,\infty)$, such that

$$f(\tau,z_1,z_2,z_3,z_4) \leq \psi_1(\tau)\chi_1(\tau,z_1,z_2,z_3,z_4),$$
$$g(\tau,z_1,z_2,z_3,z_4) \leq \psi_2(\tau)\chi_2(\tau,z_1,z_2,z_3,z_4),$$

for any $\tau \in (0,1)$, $z_i \in \mathbb{R}_+$, $i = 1,\ldots,4$.

Lemma 5. *We assume that assumptions $(I1)$ and $(I2)$ are satisfied. Then operator $\mathcal{A} : \mathcal{P} \to \mathcal{P}$ is completely continuous.*

Proof. We denote by $M_1 = \int_0^1 \mathcal{J}_1(\eta)\psi_1(\eta)\,d\eta$, $M_2 = \int_0^1 \mathcal{J}_3(\eta)\psi_2(\eta)\,d\eta$. By using $(I2)$ and Lemma 3, we deduce that $M_1 > 0$ and $M_2 > 0$. In addition we find

$$
\begin{aligned}
&M_1 = \int_0^1 \mathcal{J}_1(\eta)\psi_1(\eta)\,d\eta = \int_0^1 \left(\mathfrak{h}_1(\eta) + \frac{1}{\mathfrak{a}_1}\int_0^1 \mathfrak{g}_1(\zeta,\eta)\,d\mathcal{H}_0(\zeta)\right)\psi_1(\eta)\,d\eta \\
&\le \frac{1}{\Gamma(\gamma_1)}\int_0^1 (1-\eta)^{\gamma_1-1}\psi_1(\eta)\,d\eta + \frac{1}{\mathfrak{a}_1}\int_0^1\left(\int_0^1 \frac{1}{\Gamma(\gamma_1)}\zeta^{\gamma_1-1}(1-\eta)^{\gamma_1-1}\,d\mathcal{H}_0(\zeta)\right)\psi_1(\eta)d\eta \\
&= \left[1 + \frac{1}{\mathfrak{a}_1}\left(\int_0^1 \zeta^{\gamma_1-1}\,d\mathcal{H}_0(\zeta)\right)\right]\frac{1}{\Gamma(\gamma_1)}\int_0^1 (1-\eta)^{\gamma_1-1}\psi_1(\eta)\,d\eta < \infty, \\
&M_2 = \int_0^1 \mathcal{J}_3(\eta)\psi_2(\eta)\,d\eta = \int_0^1 \left(\mathfrak{h}_3(\eta) + \frac{1}{\mathfrak{b}_1}\int_0^1 \mathfrak{g}_3(\zeta,\eta)\,d\mathcal{K}_0(\zeta)\right)\psi_2(\eta)\,d\eta \\
&\le \frac{1}{\Gamma(\gamma_2)}\int_0^1 (1-\eta)^{\gamma_2-1}\psi_2(\eta)\,d\eta + \frac{1}{\mathfrak{b}_1}\int_0^1\left(\int_0^1 \frac{1}{\Gamma(\gamma_2)}\zeta^{\gamma_2-1}(1-\eta)^{\gamma_2-1}\,d\mathcal{K}_0(\zeta)\right)\psi_2(\eta)d\eta \\
&= \left[1 + \frac{1}{\mathfrak{b}_1}\left(\int_0^1 \zeta^{\gamma_2-1}\,d\mathcal{K}_0(\zeta)\right)\right]\frac{1}{\Gamma(\gamma_2)}\int_0^1 (1-\eta)^{\gamma_2-1}\psi_2(\eta)\,d\eta < \infty.
\end{aligned}
$$

Also, by Lemma 3 we conclude that $\mathcal{A}$ maps $\mathcal{P}$ into $\mathcal{P}$.

We will prove that $\mathcal{A}$ maps bounded sets into relatively compact sets. Let $\mathcal{E} \subset \mathcal{P}$ be an arbitrary bounded set. Then there exists $\Xi_1 > 0$ such that $\|(u,v)\|_{\mathcal{Y}} \le \Xi_1$ for all $(u,v) \in \mathcal{E}$. By the continuity of χ_1 and χ_2, we deduce that there exists $\Xi_2 > 0$ such that $\Xi_2 = \max\{\sup_{\tau\in[0,1], z_i\in[0,\omega], i=1,\dots,4} \chi_1(\tau, z_1, z_2, z_3, z_4), \sup_{\tau\in[0,1], z_i\in[0,\omega], i=1,\dots,4} \chi_2(\tau, z_1, z_2, z_3, z_4)\}$, where $\omega = \Xi_1 \max\left\{1, \frac{1}{\Gamma(\sigma_1+1)}, \frac{1}{\Gamma(\sigma_2+1)}, \frac{1}{\Gamma(\varsigma_1+1)}, \frac{1}{\Gamma(\varsigma_2+1)}\right\}$. Based on the inequality $|I_{0+}^{\xi} w(\eta)| \le \frac{\|w\|}{\Gamma(\xi+1)}$, for $\xi > 0$ and $w \in C[0,1]$, and by Lemma 3, we find for any $(u,v) \in \mathcal{E}$ and $\eta \in [0,1]$

$$
\begin{aligned}
&\mathcal{A}_1(u,v)(\eta) \le \int_0^1 \mathcal{J}_2(\zeta)\varphi_{\varrho_1}\left(\int_0^1 \mathcal{J}_1(\tau)\psi_1(\tau)\chi_1(\tau, u(\tau), v(\tau), I_{0+}^{\sigma_1}u(\tau), I_{0+}^{\sigma_2}v(\tau))d\tau\right)d\zeta \\
&\le \Xi_2^{\varrho_1-1}\varphi_{\varrho_1}\left(\int_0^1 \mathcal{J}_1(\tau)\psi_1(\tau)\,d\tau\right)\int_0^1 \mathcal{J}_2(\zeta)\,d\zeta = M_1^{\varrho_1-1}\Xi_2^{\varrho_1-1}M_3, \\
&\mathcal{A}_2(u,v)(\eta) \le \int_0^1 \mathcal{J}_4(\zeta)\varphi_{\varrho_2}\left(\int_0^1 \mathcal{J}_3(\tau)\psi_2(\tau)\chi_2(\tau, u(\tau), v(\tau), I_{0+}^{\varsigma_1}u(\tau), I_{0+}^{\varsigma_2}v(\tau))d\tau\right)d\zeta \\
&\le \Xi_2^{\varrho_2-1}\varphi_{\varrho_2}\left(\int_0^1 \mathcal{J}_3(\tau)\psi_2(\tau)\,d\tau\right)\int_0^1 \mathcal{J}_4(\zeta)\,d\zeta = M_2^{\varrho_2-1}\Xi_2^{\varrho_2-1}M_4,
\end{aligned}
$$

where $M_3 = \int_0^1 \mathcal{J}_2(\zeta)\,d\zeta$ and $M_4 = \int_0^1 \mathcal{J}_4(\zeta)\,d\zeta$.

Then $\|\mathcal{A}_1(u,v)\| \le M_1^{\varrho_1-1}\Xi_2^{\varrho_1-1}M_3$, $\|\mathcal{A}_2(u,v)\| \le M_2^{\varrho_2-1}\Xi_2^{\varrho_2-1}M_4$ for all $(u,v) \in \mathcal{E}$, and $\|\mathcal{A}(u,v)\|_{\mathcal{Y}} \le M_1^{\varrho_1-1}\Xi_2^{\varrho_1-1}M_3 + M_2^{\varrho_2-1}\Xi_2^{\varrho_2-1}M_4$ for all $(u,v) \in \mathcal{E}$, that is $\mathcal{A}_1(\mathcal{E})$, $\mathcal{A}_2(\mathcal{E})$ and $\mathcal{A}(\mathcal{E})$ are bounded.

We will show that $\mathcal{A}(\mathcal{E})$ is equicontinuous. By using Lemma 1, for $(u,v) \in \mathcal{E}$ and $\eta \in [0,1]$ we obtain

$$
\begin{aligned}
&\mathcal{A}_1(u,v)(\eta) = \int_0^1 \left(\mathfrak{g}_2(\eta,\zeta) + \frac{\eta^{\delta_1-1}}{\mathfrak{a}_2}\sum_{i=1}^n \left(\int_0^1 \mathfrak{g}_{2i}(\tau,\zeta)\,d\mathcal{H}_i(\tau)\right)\right)\varphi_{\varrho_1}\left(\int_0^1 \mathcal{G}_1(\zeta,\vartheta)\right. \\
&\quad \left.\times f(\vartheta, u(\vartheta), v(\vartheta), I_{0+}^{\sigma_1}u(\vartheta), I_{0+}^{\sigma_2}v(\vartheta))\,d\vartheta\right)d\zeta \\
&= \int_0^\eta \frac{1}{\Gamma(\delta_1)}[\eta^{\delta_1-1}(1-\zeta)^{\delta_1-\alpha_0-1} - (\eta-\zeta)^{\delta_1-1}] \\
&\quad \times\varphi_{\varrho_1}\left(\int_0^1 \mathcal{G}_1(\zeta,\vartheta)f(\vartheta, u(\vartheta), v(\vartheta), I_{0+}^{\sigma_1}u(\vartheta), I_{0+}^{\sigma_2}v(\vartheta))\,d\vartheta\right)d\zeta \\
&\quad + \int_\eta^1 \frac{1}{\Gamma(\delta_1)}\eta^{\delta_1-1}(1-\zeta)^{\delta_1-\alpha_0-1}\varphi_{\varrho_1}\left(\int_0^1 \mathcal{G}_1(\zeta,\vartheta)f(\vartheta, u(\vartheta), v(\vartheta),\right. \\
&\quad \left. I_{0+}^{\sigma_1}u(\vartheta), I_{0+}^{\sigma_2}v(\vartheta))\,d\vartheta\right)d\zeta \\
&\quad + \frac{\eta^{\delta_1-1}}{\mathfrak{a}_2}\int_0^1 \sum_{i=1}^n \left(\int_0^1 \mathfrak{g}_{2i}(\tau,\zeta)\,d\mathcal{H}_i(\tau)\right)\varphi_{\varrho_1}\left(\int_0^1 \mathcal{G}_1(\zeta,\vartheta)f(\vartheta, u(\vartheta), v(\vartheta),\right. \\
&\quad \left. I_{0+}^{\sigma_1}u(\vartheta), I_{0+}^{\sigma_2}v(\vartheta))\,d\vartheta\right)d\zeta.
\end{aligned}
$$

Then for any $\eta \in (0,1)$ we deduce

$$
\begin{aligned}
&(\mathcal{A}_1(u,v))'(\eta) = \int_0^\eta \frac{1}{\Gamma(\delta_1)}[(\delta_1-1)\eta^{\delta_1-2}(1-\zeta)^{\delta_1-\alpha_0-1} - (\delta_1-1)(\eta-\zeta)^{\delta_1-2}] \\
&\quad \times \varphi_{\varrho_1}\left(\int_0^1 \mathcal{G}_1(\zeta,\vartheta) f(\vartheta,u(\vartheta),v(\vartheta),I_{0+}^{\sigma_1}u(\vartheta),I_{0+}^{\sigma_2}v(\vartheta))\,d\vartheta\right)d\zeta \\
&\quad + \int_\eta^1 \frac{1}{\Gamma(\delta_1)}(\delta_1-1)\eta^{\delta_1-2}(1-\zeta)^{\delta_1-\alpha_0-1}\varphi_{\varrho_1}\bigg(\int_0^1 \mathcal{G}_1(\zeta,\vartheta) f(\vartheta,u(\vartheta),v(\vartheta), \\
&\quad I_{0+}^{\sigma_1}u(\vartheta),I_{0+}^{\sigma_2}v(\vartheta))\,d\vartheta\bigg)d\zeta \\
&\quad + \frac{(\delta_1-1)\eta^{\delta_1-2}}{\mathfrak{a}_2}\int_0^1 \sum_{i=1}^n \left(\int_0^1 \mathfrak{g}_{2i}(\tau,\zeta)\,d\mathcal{H}_i(\tau)\right)\varphi_{\varrho_1}\bigg(\int_0^1 \mathcal{G}_1(\zeta,\vartheta) f(\vartheta,u(\vartheta),v(\vartheta), \\
&\quad I_{0+}^{\sigma_1}u(\vartheta),I_{0+}^{\sigma_2}v(\vartheta))\,d\vartheta\bigg)d\zeta.
\end{aligned}
$$

So for any $\eta \in (0,1)$ we find

$$
\begin{aligned}
&|(\mathcal{A}_1(u,v))'(\eta)| \le \frac{1}{\Gamma(\delta_1-1)}\int_0^\eta [\eta^{\delta_1-2}(1-\zeta)^{\delta_1-\alpha_0-1} + (\eta-\zeta)^{\delta_1-2}] \\
&\quad \times \varphi_{\varrho_1}\left(\int_0^1 \mathcal{J}_1(\vartheta)\psi_1(\vartheta)\chi_1(\vartheta,u(\vartheta),v(\vartheta),I_{0+}^{\sigma_1}u(\vartheta),I_{0+}^{\sigma_2}v(\vartheta))\,d\vartheta\right)d\zeta \\
&\quad + \frac{1}{\Gamma(\delta_1-1)}\int_\eta^1 \eta^{\delta_1-2}(1-\zeta)^{\delta_1-\alpha_0-1}\varphi_{\varrho_1}\bigg(\int_0^1 \mathcal{J}_1(\vartheta)\psi_1(\vartheta)\chi_1(\vartheta,u(\vartheta),v(\vartheta), \\
&\quad I_{0+}^{\sigma_1}u(\vartheta),I_{0+}^{\sigma_2}v(\vartheta))\,d\vartheta\bigg)d\zeta \\
&\quad + \frac{(\delta_1-1)\eta^{\delta_1-2}}{\mathfrak{a}_2}\int_0^1 \sum_{i=1}^n \left(\int_0^1 \mathfrak{g}_{2i}(\tau,\zeta)\,d\mathcal{H}_i(\tau)\right)\varphi_{\varrho_1}\bigg(\int_0^1 \mathcal{J}_1(\vartheta)\chi_1(\vartheta,u(\vartheta),v(\vartheta), \\
&\quad I_{0+}^{\sigma_1}u(\vartheta),I_{0+}^{\sigma_2}v(\vartheta))\,d\vartheta\bigg)d\zeta \\
&\le \Xi_2^{\varrho_1-1}M_1^{\varrho_1-1}\bigg\{\frac{1}{\Gamma(\delta_1-1)}\int_0^\eta [\eta^{\delta_1-2}(1-\zeta)^{\delta_1-\alpha_0-1} + (\eta-\zeta)^{\delta_1-2}]\,d\zeta \\
&\quad + \frac{1}{\Gamma(\delta_1-1)}\int_\eta^1 \eta^{\delta_1-2}(1-\zeta)^{\delta_1-\alpha_0-1}d\zeta \\
&\quad + \frac{(\delta_1-1)\eta^{\delta_1-2}}{\mathfrak{a}_2}\int_0^1 \sum_{i=1}^n \left(\int_0^1 \mathfrak{g}_{2i}(\tau,\zeta)\,d\mathcal{H}_i(\tau)\right)d\zeta\bigg\}.
\end{aligned}
$$

Therefore, for $\eta \in (0,1)$ we obtain

$$
\begin{aligned}
&|(\mathcal{A}_1(u,v))'(\eta)| \le \Xi_2^{\varrho_1-1}M_1^{\varrho_1-1}\bigg[\frac{1}{\Gamma(\delta_1-1)}\left(\frac{\eta^{\delta_1-2}}{\delta_1-\alpha_0}+\frac{\eta^{\delta_1-1}}{\delta_1-1}\right) \\
&\quad + \frac{(\delta_1-1)\eta^{\delta_1-2}}{\mathfrak{a}_2}\int_0^1 \sum_{i=1}^n \left(\int_0^1 \frac{1}{\Gamma(\delta_1-\alpha_i)}(1-\zeta)^{\delta_1-\alpha_0-1}\,d\zeta\right)\tau^{\delta_1-\alpha_i-1}\,d\mathcal{H}_i(\tau)\bigg] \\
&= \Xi_2^{\varrho_1-1}M_1^{\varrho_1-1}\bigg[\frac{1}{\Gamma(\delta_1-1)}\left(\frac{\eta^{\delta_1-2}}{\delta_1-\alpha_0}+\frac{\eta^{\delta_1-1}}{\delta_1-1}\right) + \frac{(\delta_1-1)\eta^{\delta_1-2}}{\mathfrak{a}_2(\delta_1-\alpha_0)}\sum_{i=1}^n \frac{1}{\Gamma(\delta_1-\alpha_i)} \\
&\quad \times \int_0^1 \tau^{\delta_1-\alpha_i-1}\,d\mathcal{H}_i(\tau)\bigg].
\end{aligned} \tag{23}
$$

We denote by

$$
\begin{aligned}
\Theta_0(\eta) &= \frac{1}{\Gamma(\delta_1-1)}\left(\frac{\eta^{\delta_1-2}}{\delta_1-\alpha_0}+\frac{\eta^{\delta_1-1}}{\delta_1-1}\right) \\
&\quad + \frac{(\delta_1-1)\eta^{\delta_1-2}}{\mathfrak{a}_2(\delta_1-\alpha_0)}\sum_{i=1}^n \frac{1}{\Gamma(\delta_1-\alpha_i)}\int_0^1 \tau^{\delta_1-\alpha_i-1}\,d\mathcal{H}_i(\tau), \quad \eta \in (0,1).
\end{aligned}
$$

This function $\Theta_0 \in L^1(0,1)$, because

$$\int_0^1 \Theta_0(\eta)\, d\eta = \frac{1}{\Gamma(\delta_1)}\left(\frac{1}{\delta_1-\alpha_0}+\frac{1}{\delta_1}\right)+\frac{1}{\mathfrak{a}_2(\delta_1-\alpha_0)} \times \sum_{i=1}^{n}\frac{1}{\Gamma(\delta_1-\alpha_i)}\int_0^1 \tau^{\delta_1-\alpha_i-1}\, d\mathcal{H}_i(\tau) < \infty. \tag{24}$$

Then for any $s_1, s_2 \in [0,1]$ with $s_1 < s_2$ and $(u,v) \in \mathcal{E}$, by (23) and (24) we conclude

$$|\mathcal{A}_1(u,v)(s_1) - \mathcal{A}_1(u,v)(s_2)| = \left|\int_{s_1}^{s_2} (\mathcal{A}_1(u,v))'(\tau)\, d\tau\right| \le \Xi_2^{\varrho_1-1} M_1^{\varrho_1-1}\int_{s_1}^{s_2}\Theta_0(\tau)\, d\tau. \tag{25}$$

By (24) and (25), we deduce that $\mathcal{A}_1(\mathcal{E})$ is equicontinuous. By a similar method, we find that $\mathcal{A}_2(\mathcal{E})$ is also equicontinuous, and then $\mathcal{A}(\mathcal{E})$ is equicontinuous too. Using the Arzela–Ascoli theorem, we conclude that $\mathcal{A}_1(\mathcal{E})$ and $\mathcal{A}_2(\mathcal{E})$ are relatively compact sets, and so $\mathcal{A}(\mathcal{E})$ is also relatively compact. In addition, we can show that $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{A}$ are continuous on $\mathcal{P}$ (see Lemma 1.4.1 from [14]). Hence, $\mathcal{A}$ is a completely continuous operator on $\mathcal{P}$. □

We define now the cone

$$\mathcal{P}_0 = \{(u,v) \in \mathcal{P},\ u(\eta) \ge \eta^{\delta_1-1}\|u\|,\ v(\eta) \ge \eta^{\delta_2-1}\|v\|,\ \eta \in [0,1]\}.$$

Under the assumptions $(I1)$ and $(I2)$, by using Lemma 4, we deduce that $\mathcal{A}(\mathcal{P}) \subset \mathcal{P}_0$, and so $\mathcal{A}|_{\mathcal{P}_0} : \mathcal{P}_0 \to \mathcal{P}_0$ (denoted again by $\mathcal{A}$) is also a completely continuous operator. For $\theta > 0$ we denote by B_θ the open ball centered at zero of radius θ, and by $\overline{B}_\theta$ and ∂B_θ its closure and its boundary, respectively.

We also denote by $M_1 = \int_0^1 \mathcal{J}_1(\tau)\psi_1(\tau)d\tau$, $M_2 = \int_0^1 \mathcal{J}_3(\tau)\psi_2(\tau)d\tau$, $M_3 = \int_0^1 \mathcal{J}_2(\tau)d\tau$, $M_4 = \int_0^1 \mathcal{J}_4(\tau)d\tau$, and for $\theta_1, \theta_2 \in (0,1)$, $\theta_1 < \theta_2$, $M_5 = \int_{\theta_1}^{\theta_2} \mathcal{J}_2(\zeta)\left(\int_{\theta_1}^{\zeta} \mathcal{G}_1(\zeta,\tau)\, d\tau\right)^{\varrho_1-1} d\zeta$, $M_6 = \int_{\theta_1}^{\theta_2} \mathcal{J}_4(\zeta)\left(\int_{\theta_1}^{\zeta} \mathcal{G}_3(\zeta,\tau)\, d\tau\right)^{\varrho_2-1} d\zeta$.

Theorem 2. *We suppose that assumptions $(I1)$, $(I2)$,*

$(I3)$ *There exist $c_i \ge 0,\ i = 1,\dots,4$ with $\sum_{i=1}^4 c_i > 0$, $d_i \ge 0,\ i = 1,\dots,4$ with $\sum_{i=1}^4 d_i > 0$, and $\mu_1 \ge 1,\ \mu_2 \ge 1$ such that*

$$\chi_{10} = \limsup_{\sum_{i=1}^4 c_i z_i \to 0}\ \max_{\eta\in[0,1]} \frac{\chi_1(\eta,z_1,z_2,z_3,z_4)}{\varphi_{r_1}((c_1z_1+c_2z_2+c_3z_3+c_4z_4)^{\mu_1})} < l_1,$$

and

$$\chi_{20} = \limsup_{\sum_{i=1}^4 d_i z_i \to 0}\ \max_{\eta\in[0,1]} \frac{\chi_2(\eta,z_1,z_2,z_3,z_4)}{\varphi_{r_2}((d_1z_1+d_2z_2+d_3z_3+d_4z_4)^{\mu_2})} < l_2,$$

where $l_1 = (2^{r_1-1}M_1M_3^{r_1-1}\rho_1^{\mu_1(r_1-1)})^{-1}$, $l_2 = (2^{r_2-1}M_2M_4^{r_2-1}\rho_2^{\mu_2(r_2-1)})^{-1}$, with $\rho_1 = 2\max\left\{c_1, c_2, \frac{c_3}{\Gamma(\sigma_1+1)}, \frac{c_4}{\Gamma(\sigma_2+1)}\right\}$, $\rho_2 = 2\max\left\{d_1, d_2, \frac{d_3}{\Gamma(\varsigma_1+1)}, \frac{d_4}{\Gamma(\varsigma_2+1)}\right\}$;

$(I4)$ *There exist $p_i \ge 0,\ i = 1,\dots,4$ with $\sum_{i=1}^4 p_i > 0$, $q_i \ge 0,\ i = 1,\dots,4$ with $\sum_{i=1}^4 q_i > 0$, $\theta_1, \theta_2 \in (0,1)$, $\theta_1 < \theta_2$ and $\lambda_1 > 1$, $\lambda_2 > 1$ such that*

$$f_\infty = \liminf_{\sum_{i=1}^4 p_i z_i \to \infty}\ \min_{\eta\in[\theta_1,\theta_2]} \frac{f(\eta,z_1,z_2,z_3,z_4)}{\varphi_{r_1}(p_1z_1+p_2z_2+p_3z_3+p_4z_4)} > l_3,$$

or

$$g_\infty = \liminf_{\sum_{i=1}^4 q_i z_i \to \infty}\ \min_{\eta\in[\theta_1,\theta_2]} \frac{g(\eta,z_1,z_2,z_3,z_4)}{\varphi_{r_2}(q_1z_1+q_2z_2+q_3z_3+q_4z_4)} > l_4,$$

where $l_3 = \lambda_1(2\rho_3 M_5 \theta_1^{\delta_1-1})^{1-r_1}$, $l_4 = \lambda_2(2\rho_4 M_6 \theta_1^{\delta_2-1})^{1-r_2}$ *with* $\rho_3 = \min\Big\{p_1\theta_1^{\delta_1-1}, p_2\theta_1^{\delta_2-1}, \frac{p_3\theta_1^{\sigma_1+\delta_1-1}\Gamma(\delta_1)}{\Gamma(\delta_1+\sigma_1)}, \frac{p_4\theta_1^{\sigma_2+\delta_2-1}\Gamma(\delta_2)}{\Gamma(\delta_2+\sigma_2)}\Big\}$, $\rho_4 = \min\Big\{q_1\theta_1^{\delta_1-1}, q_2\theta_1^{\delta_2-1}, \frac{q_3\theta_1^{\varsigma_1+\delta_1-1}\Gamma(\delta_1)}{\Gamma(\delta_1+\varsigma_1)}, \frac{q_4\theta_1^{\varsigma_2+\delta_2-1}\Gamma(\delta_2)}{\Gamma(\delta_2+\varsigma_2)}\Big\}$,

hold. Then there exists a positive solution $(u(\tau), v(\tau))$, $\tau \in [0,1]$ *of problems (1) and (2).*

Proof. By $(I3)$ there exists $R \in (0,1)$ such that

$$\begin{aligned} \chi_1(\eta, z_1, z_2, z_3, z_4) &\le l_1\varphi_{r_1}((c_1z_1 + c_2z_2 + c_3z_3 + c_4z_4)^{\mu_1}),\\ \chi_2(\eta, z_1, z_2, z_3, z_4) &\le l_2\varphi_{r_2}((d_1z_1 + d_2z_2 + d_3z_3 + d_4z_4)^{\mu_2}), \end{aligned} \tag{26}$$

for all $\eta \in [0,1]$, $z_i \ge 0$, $i = 1, \ldots, 4$ with $\sum_{i=1}^4 c_iz_i \le R$ and $\sum_{i=1}^4 d_iz_i \le R$. We define $R_1 \le \min\{R/\rho_1, R/\rho_2, R\}$. For any $(u,v) \in \overline{B}_{R_1} \cap \mathcal{P}$ and $\zeta \in [0,1]$ we have

$$\begin{aligned} &c_1u(\zeta) + c_2v(\zeta) + c_3I_{0+}^{\sigma_1}u(\zeta) + c_4I_{0+}^{\sigma_2}v(\zeta)\\ &\quad\le 2\max\Big\{c_1, c_2, \tfrac{c_3}{\Gamma(\sigma_1+1)}, \tfrac{c_4}{\Gamma(\sigma_2+1)}\Big\}\|(u,v)\|_{\mathcal{Y}} = \rho_1\|(u,v)\|_{\mathcal{Y}} \le \rho_1R_1 \le R,\\ &d_1u(\zeta) + d_2v(\zeta) + d_3I_{0+}^{\varsigma_1}u(\zeta) + d_4I_{0+}^{\varsigma_2}v(\zeta)\\ &\quad\le 2\max\Big\{d_1, d_2, \tfrac{d_3}{\Gamma(\varsigma_1+1)}, \tfrac{d_4}{\Gamma(\varsigma_2+1)}\Big\}\|(u,v)\|_{\mathcal{Y}} = \rho_2\|(u,v)\|_{\mathcal{Y}} \le \rho_2R_1 \le R. \end{aligned}$$

Then by (26) and Lemma 3, for any $(u,v) \in \partial B_{R_1} \cap \mathcal{P}_0$ and $\eta \in [0,1]$, we deduce

$$\begin{aligned} &(\mathcal{A}_1(u,v))(\eta) \le \int_0^1 \mathcal{J}_2(\zeta)\varphi_{\varrho_1}\Big(\int_0^1 \mathcal{J}_1(\vartheta)f(\vartheta, u(\vartheta), v(\vartheta), I_{0+}^{\sigma_1}u(\vartheta), I_{0+}^{\sigma_2}v(\vartheta))\,d\vartheta\Big)d\zeta\\ &= M_3\varphi_{\varrho_1}\Big(\int_0^1 \mathcal{J}_1(\vartheta)f(\vartheta, u(\vartheta), v(\vartheta), I_{0+}^{\sigma_1}u(\vartheta), I_{0+}^{\sigma_2}v(\vartheta))\,d\vartheta\Big)\\ &\le M_3\varphi_{\varrho_1}\Big(\int_0^1 \mathcal{J}_1(\vartheta)\psi_1(\vartheta)\chi_1(\vartheta, u(\vartheta), v(\vartheta), I_{0+}^{\sigma_1}u(\vartheta), I_{0+}^{\sigma_2}v(\vartheta))\,d\vartheta\Big)\\ &\le M_3\varphi_{\varrho_1}\Big(\int_0^1 \mathcal{J}_1(\vartheta)\psi_1(\vartheta)l_1\varphi_{r_1}((c_1u(\vartheta) + c_2v(\vartheta) + c_3I_{0+}^{\sigma_1}u(\vartheta) + c_4I_{0+}^{\sigma_2}v(\vartheta))^{\mu_1})d\vartheta\Big)\\ &\le M_3\varphi_{\varrho_1}(\varphi_{r_1}((\rho_1\|(u,v)\|_{\mathcal{Y}})^{\mu_1}))\varphi_{\varrho_1}(l_1)\varphi_{\varrho_1}(M_1)\\ &= M_3M_1^{\varrho_1-1}l_1^{\varrho_1-1}\rho_1^{\mu_1}\|(u,v)\|_{\mathcal{Y}}^{\mu_1} \le M_3M_1^{\varrho_1-1}l_1^{\varrho_1-1}\rho_1^{\mu_1}\|(u,v)\|_{\mathcal{Y}} = \tfrac{1}{2}\|(u,v)\|_{\mathcal{Y}},\\ &(\mathcal{A}_2(u,v))(\eta) \le \int_0^1 \mathcal{J}_4(\zeta)\varphi_{\varrho_2}\Big(\int_0^1 \mathcal{J}_3(\vartheta)g(\vartheta, u(\vartheta), v(\vartheta), I_{0+}^{\varsigma_1}u(\vartheta), I_{0+}^{\varsigma_2}v(\vartheta))\,d\vartheta\Big)d\zeta\\ &= M_4\varphi_{\varrho_2}\Big(\int_0^1 \mathcal{J}_3(\vartheta)g(\vartheta, u(\vartheta), v(\vartheta), I_{0+}^{\varsigma_1}u(\vartheta), I_{0+}^{\varsigma_2}v(\vartheta))\,d\vartheta\Big)\\ &\le M_4\varphi_{\varrho_2}\Big(\int_0^1 \mathcal{J}_3(\vartheta)\psi_2(\vartheta)\chi_2(\vartheta, u(\vartheta), v(\vartheta), I_{0+}^{\varsigma_1}u(\vartheta), I_{0+}^{\varsigma_2}v(\vartheta))\,d\vartheta\Big)\\ &\le M_4\varphi_{\varrho_2}\Big(\int_0^1 \mathcal{J}_3(\vartheta)\psi_2(\vartheta)l_2\varphi_{r_2}((d_1u(\vartheta) + d_2v(\vartheta) + d_3I_{0+}^{\varsigma_1}u(\vartheta) + d_4I_{0+}^{\varsigma_2}v(\vartheta))^{\mu_2})d\vartheta\Big)\\ &\le M_4\varphi_{\varrho_2}(\varphi_{r_2}((\rho_2\|(u,v)\|_{\mathcal{Y}})^{\mu_2}))\varphi_{\varrho_2}(l_2)\varphi_{\varrho_2}(M_2)\\ &= M_4M_2^{\varrho_2-1}l_2^{\varrho_2-1}\rho_2^{\mu_2}\|(u,v)\|_{\mathcal{Y}}^{\mu_2} \le M_4M_2^{\varrho_2-1}l_2^{\varrho_2-1}\rho_2^{\mu_2}\|(u,v)\|_{\mathcal{Y}} = \tfrac{1}{2}\|(u,v)\|_{\mathcal{Y}}. \end{aligned}$$

Then we conclude that

$$\|\mathcal{A}(u,v)\|_{\mathcal{Y}} = \|\mathcal{A}_1(u,v)\| + \|\mathcal{A}_2(u,v)\| \le \|(u,v)\|_{\mathcal{Y}}, \ \forall\, (u,v) \in \partial B_{R_1} \cap \mathcal{P}_0. \tag{27}$$

Now we suppose in $(I4)$ that $f_\infty > l_3$ (in a similar manner we study the case $g_\infty > l_4$). Then there exists $C_1 > 0$ such that

$$f(\eta, z_1, z_2, z_3, z_4) \ge l_3\varphi_{r_1}(p_1z_1 + p_2z_2 + p_3z_3 + p_4z_4) - C_1, \tag{28}$$

for all $\eta \in [\theta_1, \theta_2]$ and $z_i \geq 0,\ i = 1, \ldots, 4$. By definition of $I_{0+}^{\sigma_1}$, for any $(u, v) \in \mathcal{P}_0$ and $\eta \in [0, 1]$ we have

$$\begin{aligned}
I_{0+}^{\sigma_1} u(\eta) &= \frac{1}{\Gamma(\sigma_1)} \int_0^{\eta} (\eta - \zeta)^{\sigma_1 - 1} u(\zeta)\, d\zeta \geq \frac{1}{\Gamma(\sigma_1)} \int_0^{\eta} (\eta - \zeta)^{\sigma_1 - 1} \zeta^{\delta_1 - 1} \|u\|\, d\zeta \\
&\overset{\zeta = \eta y}{=} \frac{\|u\|}{\Gamma(\sigma_1)} \int_0^1 (\eta - \eta y)^{\sigma_1 - 1} \eta^{\delta_1 - 1} y^{\delta_1 - 1} \eta\, dy = \frac{\|u\|}{\Gamma(\sigma_1)} \eta^{\sigma_1 + \delta_1 - 1} \int_0^1 y^{\delta_1 - 1} (1 - y)^{\sigma_1 - 1}\, dy \\
&= \frac{\|u\|}{\Gamma(\sigma_1)} \eta^{\sigma_1 + \delta_1 - 1} B(\delta_1, \sigma_1) = \frac{\|u\| \eta^{\sigma_1 + \delta_1 - 1} \Gamma(\delta_1)}{\Gamma(\delta_1 + \sigma_1)},
\end{aligned} \tag{29}$$

and in a similar way

$$I_{0+}^{\sigma_2} v(\eta) \geq \frac{\|v\| \eta^{\sigma_2 + \delta_2 - 1} \Gamma(\delta_2)}{\Gamma(\delta_2 + \sigma_2)},$$

where $B(p, q)$ is the first Euler function. Then by using (28) and (29), for any $(u, v) \in \mathcal{P}_0$ and $\eta \in [\theta_1, \theta_2]$ we obtain

$$\begin{aligned}
&(\mathcal{A}_1(u, v))(\eta) \geq \int_{\theta_1}^{\theta_2} \mathcal{G}_2(\eta, \zeta) \varphi_{\varrho_1} \left(\int_{\theta_1}^{\zeta} \mathcal{G}_1(\zeta, \vartheta) f(\vartheta, u(\vartheta), v(\vartheta), I_{0+}^{\sigma_1} u(\vartheta), I_{0+}^{\sigma_2} v(\vartheta)) d\vartheta \right) d\zeta \\
&\geq \theta_1^{\delta_1 - 1} \int_{\theta_1}^{\theta_2} \mathcal{J}_2(\zeta) \left(\int_{\theta_1}^{\zeta} \mathcal{G}_1(\zeta, \vartheta) \left[l_3 (p_1 u(\vartheta) + p_2 v(\vartheta) + p_3 I_{0+}^{\sigma_1} u(\vartheta) + p_4 I_{0+}^{\sigma_2} v(\vartheta))^{r_1 - 1} \right.\right. \\
&\quad \left. - C_1] d\vartheta \right)^{\varrho_1 - 1} d\zeta \\
&\geq \theta_1^{\delta_1 - 1} \int_{\theta_1}^{\theta_2} \mathcal{J}_2(\zeta) \left(\int_{\theta_1}^{\zeta} \mathcal{G}_1(\zeta, \vartheta) \left[l_3 \left(p_1 \theta_1^{\delta_1 - 1} \|u\| + p_2 \theta_1^{\delta_2 - 1} \|v\| \right.\right.\right. \\
&\quad \left.\left.\left. + p_3 \frac{\theta_1^{\sigma_1 + \delta_1 - 1} \Gamma(\delta_1)}{\Gamma(\delta_1 + \sigma_1)} \|u\| + p_4 \frac{\theta_1^{\sigma_2 + \delta_2 - 1} \Gamma(\delta_2)}{\Gamma(\delta_2 + \sigma_2)} \|v\| \right)^{r_1 - 1} - C_1 \right] d\vartheta \right)^{\varrho_1 - 1} d\zeta \\
&\geq \theta_1^{\delta_1 - 1} \int_{\theta_1}^{\theta_2} \mathcal{J}_2(\zeta) \left(\int_{\theta_1}^{\zeta} \mathcal{G}_1(\zeta, \vartheta) \left[l_3 \left(\min\left\{ p_1 \theta_1^{\delta_1 - 1}, p_2 \theta_1^{\delta_2 - 1}, p_3 \frac{\theta_1^{\sigma_1 + \delta_1 - 1} \Gamma(\delta_1)}{\Gamma(\delta_1 + \sigma_1)}, \right.\right.\right.\right. \\
&\quad \left.\left.\left.\left. \frac{p_4 \theta_1^{\sigma_2 + \delta_2 - 1} \Gamma(\delta_2)}{\Gamma(\delta_2 + \sigma_2)} \right\} 2 \|(u, v)\|_{\mathcal{Y}} \right)^{r_1 - 1} - C_1 \right] d\vartheta \right)^{\varrho_1 - 1} d\zeta \\
&= \theta_1^{\delta_1 - 1} \int_{\theta_1}^{\theta_2} \mathcal{J}_2(\zeta) \left(\int_{\theta_1}^{\zeta} \mathcal{G}_1(\zeta, \vartheta) \left[l_3 (2\rho_3 \|(u, v)\|_{\mathcal{Y}})^{r_1 - 1} - C_1 \right] d\vartheta \right)^{\varrho_1 - 1} d\zeta \\
&= M_5 \theta_1^{\delta_1 - 1} \left[l_3 (2\rho_3 \|(u, v)\|_{\mathcal{Y}})^{r_1 - 1} - C_1 \right]^{\varrho_1 - 1} \\
&= \left(M_5^{r_1 - 1} \theta_1^{(\delta_1 - 1)(r_1 - 1)} l_3 2^{r_1 - 1} \rho_3^{r_1 - 1} \|(u, v)\|_{\mathcal{Y}}^{r_1 - 1} - M_5^{r_1 - 1} \theta_1^{(\delta_1 - 1)(r_1 - 1)} C_1 \right)^{\varrho_1 - 1} \\
&= \left(\lambda_1 \|(u, v)\|_{\mathcal{Y}}^{r_1 - 1} - C_2 \right)^{\varrho_1 - 1}, \quad C_2 = M_5^{r_1 - 1} \theta_1^{(\delta_1 - 1)(r_1 - 1)} C_1.
\end{aligned}$$

Then we deduce

$$\|\mathcal{A}(u, v)\|_{\mathcal{Y}} \geq \|\mathcal{A}_1(u, v)\| \geq |\mathcal{A}_1(u, v)(\theta_1)| \geq \left(\lambda_1 \|(u, v)\|_{\mathcal{Y}}^{r_1 - 1} - C_2 \right)^{\varrho_1 - 1}, \quad \forall\, (u, v) \in \mathcal{P}_0.$$

We choose $R_2 \geq \max\left\{ 1, C_2^{\varrho_1 - 1} / (\lambda_1 - 1)^{\varrho_1 - 1} \right\}$ and we obtain

$$\|\mathcal{A}(u, v)\|_{\mathcal{Y}} \geq \|(u, v)\|_{\mathcal{Y}}, \quad \forall\, (u, v) \in \partial B_{R_2} \cap \mathcal{P}_0. \tag{30}$$

By Lemma 5, (27), (30) and Theorem 1 (i), we conclude that $\mathcal{A}$ has a fixed point $(u, v) \in (\overline{B}_{R_2} \setminus B_{R_1}) \cap \mathcal{P}_0$, so $R_1 \leq \|(u, v)\|_{\mathcal{Y}} \leq R_2$, and $u(\tau) \geq \tau^{\delta_1 - 1} \|u\|$ and $v(\tau) \geq \tau^{\delta_2 - 1} \|v\|$ for all $\tau \in [0, 1]$. Then $\|u\| > 0$ or $\|v\| > 0$, that is $u(\tau) > 0$ for all $\tau \in (0, 1]$ or $v(\tau) > 0$ for all $\tau \in (0, 1]$. Hence $(u(\tau), v(\tau)),\ \tau \in [0, 1]$ is a positive solution of problem (1) and (2). □

Theorem 3. *We suppose that assumptions* $(I1)$, $(I2)$,

$(I5)$ *There exist* $e_i \geq 0,\ i = 1, \ldots, 4$ *with* $\sum_{i=1}^4 e_i > 0$, $k_i \geq 0,\ i = 1, \ldots, 4$ *with* $\sum_{i=1}^4 k_i > 0$ *such that*

$$\chi_{1\infty} = \limsup_{\sum_{i=1}^4 e_i z_i \to \infty} \max_{\eta \in [0,1]} \frac{\chi_1(\eta, z_1, z_2, z_3, z_4)}{\varphi_{r_1}(e_1 z_1 + e_2 z_2 + e_3 z_3 + e_4 z_4)} < m_1,$$

and

$$\chi_{2\infty} = \limsup_{\sum_{i=1}^4 k_i z_i \to \infty} \max_{\eta \in [0,1]} \frac{\chi_2(\eta, z_1, z_2, z_3, z_4)}{\varphi_{r_2}(k_1 z_1 + k_2 z_2 + k_3 z_3 + k_4 z_4)} < m_2,$$

where $m_1 < \min\{1/(2M_1(\xi_1 M_3)^{r_1-1}), 1/(M_1(2\xi_1 M_3)^{r_1-1})\}$, $m_2 < \min\{1/(2M_2(\xi_2 M_4)^{r_2-1}), 1/(M_2(2\xi_2 M_4)^{r_2-1})\}$ *with* $\xi_1 = 2\max\left\{e_1, e_2, \frac{e_3}{\Gamma(\sigma_1+1)}, \frac{e_4}{\Gamma(\sigma_2+1)}\right\}$, $\xi_2 = 2\max\left\{k_1, k_2, \frac{k_3}{\Gamma(\varsigma_1+1)}, \frac{k_4}{\Gamma(\varsigma_2+1)}\right\}$;

$(I6)$ *There exist* $s_i \geq 0,\ i = 1, \ldots, 4$ *with* $\sum_{i=1}^4 s_i > 0$, $t_i \geq 0,\ i = 1, \ldots, 4$ *with* $\sum_{i=1}^4 t_i > 0$, $\theta_1, \theta_2 \in (0,1)$, $\theta_1 < \theta_2$ *and* $\nu_1 \in (0,1]$, $\nu_2 \in (0,1]$, $\lambda_3 \geq 1$, $\lambda_4 \geq 1$ *such that*

$$f_0 = \liminf_{\sum_{i=1}^4 s_i z_i \to 0} \min_{\eta \in [\theta_1, \theta_2]} \frac{f(\eta, z_1, z_2, z_3, z_4)}{\varphi_{r_1}((s_1 z_1 + s_2 z_2 + s_3 z_3 + s_4 z_4)^{\nu_1})} > m_3,$$

or

$$g_0 = \liminf_{\sum_{i=1}^4 t_i z_i \to 0} \min_{\eta \in [\theta_1, \theta_2]} \frac{g(\eta, z_1, z_2, z_3, z_4)}{\varphi_{r_2}((t_1 z_1 + t_2 z_2 + t_3 z_3 + t_4 z_4)^{\nu_2})} > m_4,$$

where $m_3 = \lambda_3^{r_1-1}(M_5 2^{\nu_1}\xi_3^{\nu_1}\theta_1^{\delta_1-1})^{1-r_1}$, $m_4 = \lambda_4^{r_2-1}(M_6 2^{\nu_2}\xi_4^{\nu_2}\theta_1^{\delta_2-1})^{1-r_2}$, *with* $\xi_3 = \min\left\{s_1\theta_1^{\delta_1-1}, s_2\theta_1^{\delta_2-1}, \frac{s_3\theta_1^{\sigma_1+\delta_1-1}\Gamma(\delta_1)}{\Gamma(\delta_1+\sigma_1)}, \frac{s_4\theta_1^{\sigma_2+\delta_2-1}\Gamma(\delta_2)}{\Gamma(\delta_2+\sigma_2)}\right\}$, $\xi_4 = \min\left\{t_1\theta_1^{\delta_1-1}, t_2\theta_1^{\delta_2-1}, \frac{t_3\theta_1^{\varsigma_1+\delta_1-1}\Gamma(\delta_1)}{\Gamma(\delta_1+\varsigma_1)}, \frac{t_4\theta_1^{\varsigma_2+\delta_2-1}\Gamma(\delta_2)}{\Gamma(\delta_2+\varsigma_2)}\right\}$,

hold. Then there exists a positive solution $(u(\tau), v(\tau))$, $\tau \in [0,1]$ *of problem (1) and (2).*

Proof. From $(I5)$ there exist $C_3 > 0$, $C_4 > 0$ such that

$$\begin{aligned} &\chi_1(\eta, z_1, z_2, z_3, z_4) \leq m_1 \varphi_{r_1}(e_1 z_1 + e_2 z_2 + e_3 z_3 + e_4 z_4) + C_3, \\ &\chi_2(\eta, z_1, z_2, z_3, z_4) \leq m_2 \varphi_{r_2}(k_1 z_1 + k_2 z_2 + k_3 z_3 + k_4 z_4) + C_4, \end{aligned} \tag{31}$$

for any $\eta \in [0,1]$ and $z_i \geq 0,\ i = 1, \ldots, 4$. By using $(I2)$ and (31) for any $(u,v) \in \mathcal{P}_0$ and $\eta \in [0,1]$ we find

$$\begin{aligned}
\mathcal{A}_1(u,v)(\eta) &\leq \int_0^1 \mathcal{J}_2(\zeta)\varphi_{\varrho_1}\left(\int_0^1 \mathcal{J}_1(\vartheta) f(\vartheta, u(\vartheta), v(\vartheta), I_{0+}^{\sigma_1}u(\vartheta), I_{0+}^{\sigma_2}v(\vartheta))\, d\vartheta\right) d\zeta \\
&\leq M_3 \varphi_{\varrho_1}\left(\int_0^1 \mathcal{J}_1(\vartheta)\psi_1(\vartheta)\chi_1(\vartheta, u(\vartheta), v(\vartheta), I_{0+}^{\sigma_1}u(\vartheta), I_{0+}^{\sigma_2}v(\vartheta))\, d\vartheta\right) \\
&\leq M_3 \varphi_{\varrho_1}\left(\int_0^1 \mathcal{J}_1(\vartheta)\psi_1(\vartheta)\left[m_1 \varphi_{r_1}\left(e_1 u(\vartheta) + e_2 v(\vartheta) + e_3 I_{0+}^{\sigma_1}u(\vartheta) + e_4 I_{0+}^{\sigma_2}v(\vartheta)\right) + C_3\right] d\vartheta\right) \\
&\leq M_3 \varphi_{\varrho_1}\left(\int_0^1 \mathcal{J}_1(\vartheta)\psi_1(\vartheta)\left[m_1\left(e_1\|u\| + e_2\|v\| + \frac{e_3\|u\|}{\Gamma(\sigma_1+1)} + \frac{e_4\|v\|}{\Gamma(\sigma_2+1)}\right)^{r_1-1} + C_3\right] d\vartheta\right) \\
&\leq M_3 \varphi_{\varrho_1}\left[m_1\left(\max\left\{e_1, e_2, \tfrac{e_3}{\Gamma(\sigma_1+1)}, \tfrac{e_4}{\Gamma(\sigma_2+1)}\right\} 2\|(u,v)\|_{\mathcal{Y}}\right)^{r_1-1} + C_3\right] \\
&\quad \times \left(\int_0^1 \mathcal{J}_1(\vartheta)\psi_1(\vartheta)\, d\vartheta\right)^{\varrho_1-1} \\
&= M_1^{\varrho_1-1} M_3 \left(m_1 \xi_1^{r_1-1}\|(u,v)\|_{\mathcal{Y}}^{r_1-1} + C_3\right)^{\varrho_1-1},
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{A}_2(u,v)(\eta) &\le \int_0^1 \mathcal{J}_4(\zeta)\varphi_{\varrho_2}\left(\int_0^1 \mathcal{J}_3(\vartheta)g(\vartheta,u(\vartheta),v(\vartheta),I_{0+}^{\varsigma_1}u(\vartheta),I_{0+}^{\varsigma_2}v(\vartheta))\,d\vartheta\right)d\zeta\\
&\le M_4\varphi_{\varrho_2}\left(\int_0^1 \mathcal{J}_3(\vartheta)\psi_2(\vartheta)\chi_2(\vartheta,u(\vartheta),v(\vartheta),I_{0+}^{\varsigma_1}u(\vartheta),I_{0+}^{\varsigma_2}v(\vartheta))\,d\vartheta\right)\\
&\le M_4\varphi_{\varrho_2}\left(\int_0^1 \mathcal{J}_3(\vartheta)\psi_2(\vartheta)\left[m_2\varphi_{r_2}\left(k_1u(\vartheta)+k_2v(\vartheta)+k_3I_{0+}^{\varsigma_1}u(\vartheta)+k_4I_{0+}^{\varsigma_2}v(\vartheta)\right)+C_4\right]d\vartheta\right)\\
&\le M_4\varphi_{\varrho_2}\left(\int_0^1 \mathcal{J}_3(\vartheta)\psi_2(\vartheta)\left[m_2\left(k_1\|u\|+k_2\|v\|+\frac{k_3\|u\|}{\Gamma(\varsigma_1+1)}+\frac{k_4\|v\|}{\Gamma(\varsigma_2+1)}\right)^{r_2-1}+C_4\right]d\vartheta\right)\\
&\le M_4\varphi_{\varrho_2}\left[m_2\left(\max\left\{k_1,k_2,\tfrac{k_3}{\Gamma(\varsigma_1+1)},\tfrac{k_4}{\Gamma(\varsigma_2+1)}\right\}2\|(u,v)\|_{\mathcal{Y}}\right)^{r_2-1}+C_4\right]\\
&\quad\times\left(\int_0^1 \mathcal{J}_3(\vartheta)\psi_2(\vartheta)\,d\vartheta\right)^{\varrho_2-1}\\
&= M_2^{\varrho_2-1}M_4\left(m_2\xi_2^{r_2-1}\|(u,v)\|_{\mathcal{Y}}^{r_2-1}+C_4\right)^{\varrho_2-1}.
\end{aligned}$$

Then we obtain

$$\begin{aligned}
\|\mathcal{A}_1(u,v)\| &\le M_1^{\varrho_1-1}M_3\left(m_1\xi_1^{r_1-1}\|(u,v)\|_{\mathcal{Y}}^{r_1-1}+C_3\right)^{\varrho_1-1},\\
\|\mathcal{A}_2(u,v)\| &\le M_2^{\varrho_2-1}M_4\left(m_2\xi_2^{r_2-1}\|(u,v)\|_{\mathcal{Y}}^{r_2-1}+C_4\right)^{\varrho_2-1},
\end{aligned}$$

and so

$$\begin{aligned}
\|\mathcal{A}(u,v)\|_{\mathcal{Y}} \le{}& M_1^{\varrho_1-1}M_3\left(m_1\xi_1^{r_1-1}\|(u,v)\|_{\mathcal{Y}}^{r_1-1}+C_3\right)^{\varrho_1-1}\\
&+M_2^{\varrho_2-1}M_4\left(m_2\xi_2^{r_2-1}\|(u,v)\|_{\mathcal{Y}}^{r_2-1}+C_4\right)^{\varrho_2-1},
\end{aligned}$$

for all $(u,v)\in\mathcal{P}_0$. We choose

$$\begin{aligned}
R_3 \ge \max\Bigg\{1,\ &\frac{M_1^{\varrho_1-1}M_32^{\varrho_1-2}C_3^{\varrho_1-1}+M_2^{\varrho_2-1}M_42^{\varrho_2-2}C_4^{\varrho_2-1}}{1-(M_1^{\varrho_1-1}M_32^{\varrho_1-2}m_1^{\varrho_1-1}\xi_1+M_2^{\varrho_2-1}M_42^{\varrho_2-2}m_2^{\varrho_2-1}\xi_2)},\\
&\frac{M_1^{\varrho_1-1}M_3C_3^{\varrho_1-1}+M_2^{\varrho_2-1}M_4C_4^{\varrho_2-1}}{1-(M_1^{\varrho_1-1}M_3m_1^{\varrho_1-1}\xi_1+M_2^{\varrho_2-1}M_4m_2^{\varrho_2-1}\xi_2)},\\
&\frac{M_1^{\varrho_1-1}M_3C_3^{\varrho_1-1}+M_2^{\varrho_2-1}M_42^{\varrho_2-2}C_4^{\varrho_2-1}}{1-(M_1^{\varrho_1-1}M_3m_1^{\varrho_1-1}\xi_1+M_2^{\varrho_2-1}M_42^{\varrho_2-2}m_2^{\varrho_2-1}\xi_2)},\\
&\frac{M_1^{\varrho_1-1}M_32^{\varrho_1-2}C_3^{\varrho_1-1}+M_2^{\varrho_2-1}M_4C_4^{\varrho_2-1}}{1-(M_1^{\varrho_1-1}M_32^{\varrho_1-2}m_1^{\varrho_1-1}\xi_1+M_2^{\varrho_2-1}M_4m_2^{\varrho_2-1}\xi_2)}\Bigg\},
\end{aligned} \tag{32}$$

and then we conclude

$$\|\mathcal{A}(u,v)\|_{\mathcal{Y}} \le \|(u,v)\|_{\mathcal{Y}},\ \ \forall\,(u,v)\in\partial B_{R_3}\cap\mathcal{P}_0. \tag{33}$$

The above number R_3 was chosen based on the inequalities $(x+y)^{\varpi}\le 2^{\varpi-1}(x^{\varpi}+y^{\varpi})$ for $\varpi\ge 1$ and $x,y\ge 0$, and $(x+y)^{\varpi}\le x^{\varpi}+y^{\varpi}$ for $\varpi\in(0,1]$ and $x,y\ge 0$. Here $\varpi=\varrho_1-1$ or ϱ_2-1. We prove the inequality (33) in one case, namely $\varrho_1\in[2,\infty)$ and $\varrho_2\in[2,\infty)$. In this case, by using (32) and the relations $M_1^{\varrho_1-1}M_32^{\varrho_1-2}m_1^{\varrho_1-1}\xi_1<1/2$ and

$M_2^{\varrho_2-1}M_4 2^{\varrho_2-2}m_2^{\varrho_2-1}\xi_2 < 1/2$ (from the inequalities for m_1 and m_2 in $(I5)$) we have the inequalities

$$\begin{aligned}&M_1^{\varrho_1-1}M_3(m_1\xi_1^{r_1-1}R_3^{r_1-1}+C_3)^{\varrho_1-1}+M_2^{\varrho_2-1}M_4(m_2\xi_2^{r_2-1}R_3^{r_2-1}+C_4)^{\varrho_2-1}\\&\le M_1^{\varrho_1-1}M_3 2^{\varrho_1-2}(m_1^{\varrho_1-1}\xi_1R_3+C_3^{\varrho_1-1})+M_2^{\varrho_2-1}M_4 2^{\varrho_2-2}(m_2^{\varrho_2-1}\xi_2R_3+C_4^{\varrho_2-1})\\&=(M_1^{\varrho_1-1}M_3 2^{\varrho_1-2}m_1^{\varrho_1-1}\xi_1+M_2^{\varrho_2-1}M_4 2^{\varrho_2-2}m_2^{\varrho_2-1}\xi_2)R_3\\&\quad+(M_1^{\varrho_1-1}M_3 2^{\varrho_1-2}C_3^{\varrho_1-1}+M_2^{\varrho_2-1}M_4 2^{\varrho_2-2}C_4^{\varrho_2-1})\le R_3.\end{aligned}$$

In a similar manner we consider the cases $\varrho_1\in(1,2]$ and $\varrho_2\in(1,2]$; $\varrho_1\in[2,\infty)$ and $\varrho_2\in(1,2]$; $\varrho_1\in(1,2]$ and $\varrho_2\in[2,\infty)$.

In $(I6)$, we suppose that $g_0 > m_4$ (in a similar manner we can study the case $f_0 > m_3$). We deduce that there exists $\widetilde{R}_4\in(0,1]$ such that

$$g(\eta,z_1,z_2,z_3,z_4)\ge m_4\varphi_{r_2}((t_1z_1+t_2z_2+t_3z_3+t_4z_4)^{\nu_2}), \tag{34}$$

for all $\eta\in[\theta_1,\theta_2]$, $z_i\ge 0$, $i=1,\dots,4$, $\sum_{i=1}^4 t_iz_i\le\widetilde{R}_4$. We take $R_4\le\min\{\widetilde{R}_4/\widetilde{\xi}_4,\widetilde{R}_4\}$, where $\widetilde{\xi}_4=2\max\left\{t_1,t_2,\frac{t_3}{\Gamma(\varsigma_1+1)},\frac{t_4}{\Gamma(\varsigma_2+1)}\right\}$. Then for any $(u,v)\in\overline{B}_{R_4}\cap\mathcal{P}$ and $\eta\in[0,1]$ we have

$$\begin{aligned}&t_1u(\zeta)+t_2v(\zeta)+t_3I_{0+}^{\varsigma}u(\zeta)+t_4I_{0+}^{\varsigma}v(\zeta)\le t_1\|u\|+t_2\|v\|+\frac{t_3\|u\|}{\Gamma(\varsigma_1+1)}+\frac{t_4\|v\|}{\Gamma(\varsigma_2+1)}\\&\le\max\left\{t_1,t_2,\frac{t_3}{\Gamma(\varsigma_1+1)},\frac{t_4}{\Gamma(\varsigma_2+1)}\right\}2\|(u,v)\|_{\mathcal{Y}}=\widetilde{\xi}_4\|(u,v)\|_{\mathcal{Y}}\le\widetilde{\xi}_4R_4\le\widetilde{R}_4.\end{aligned}$$

Therefore by using (34) and (29), we obtain for any $(u,v)\in\partial B_{R_4}\cap\mathcal{P}_0$ and $\eta\in[\theta_1,\theta_2]$

$$\begin{aligned}&\mathcal{A}_2(u,v)(\eta)\ge\int_{\theta_1}^{\theta_2}\mathcal{G}_4(\eta,\zeta)\varphi_{\varrho_2}\left(\int_{\theta_1}^{\zeta}\mathcal{G}_3(\zeta,\vartheta)g(\vartheta,u(\vartheta),v(\vartheta),I_{0+}^{\varsigma_1}u(\vartheta),I_{0+}^{\varsigma_2}v(\vartheta))d\vartheta\right)d\zeta\\&\ge\theta_1^{\delta_2-1}\int_{\theta_1}^{\theta_2}\mathcal{J}_4(\zeta)\varphi_{\varrho_2}\left(\int_{\theta_1}^{\zeta}\mathcal{G}_3(\zeta,\vartheta)\left[m_4\varphi_{r_2}\left((t_1u(\vartheta)+t_2v(\vartheta)+t_3I_{0+}^{\varsigma_1}u(\vartheta)\right.\right.\right.\\&\quad\left.\left.\left.+t_4I_{0+}^{\varsigma_2}v(\vartheta))^{\nu_2}\right)\right]d\vartheta\right)d\zeta\\&\ge\theta_1^{\delta_2-1}\int_{\theta_1}^{\theta_2}\mathcal{J}_4(\zeta)\varphi_{\varrho_2}\left(\int_{\theta_1}^{\zeta}\mathcal{G}_3(\zeta,\vartheta)\left[m_4\left(t_1\theta_1^{\delta_1-1}\|u\|+t_2\theta_1^{\delta_2-1}\|v\|\right.\right.\right.\\&\quad\left.\left.\left.+\,+t_3\frac{\theta_1^{\varsigma_1+\delta_1-1}\Gamma(\delta_1)}{\Gamma(\delta_1+\varsigma_1)}\|u\|+t_4\frac{\theta_1^{\varsigma_2+\delta_2-1}\Gamma(\delta_2)}{\Gamma(\delta_2+\varsigma_2)}\|v\|\right)^{\nu_2(r_2-1)}\right]d\vartheta\right)d\zeta\\&\ge\theta_1^{\delta_2-1}\int_{\theta_1}^{\theta_2}\mathcal{J}_4(\zeta)\left(\int_{\theta_1}^{\zeta}\mathcal{G}_3(\zeta,\vartheta)m_4(2\xi_4\|(u,v)\|_{\mathcal{Y}})^{\nu_2(r_2-1)}d\vartheta\right)^{\varrho_2-1}d\zeta\\&=\theta_1^{\delta_2-1}m_4^{\varrho_2-1}(2\xi_4)^{\nu_2(\varrho_2-1)(r_2-1)}\|(u,v)\|_{\mathcal{Y}}^{\nu_2}\left(\int_{\theta_1}^{\theta_2}\mathcal{J}_4(\zeta)\left(\int_{\theta_1}^{\zeta}\mathcal{G}_3(\zeta,\vartheta)\,d\vartheta\right)^{\varrho_2-1}d\zeta\right)\\&=M_6\theta_1^{\delta_2-1}m_4^{\varrho_2-1}2^{\nu_2}\xi_4^{\nu_2}\|(u,v)\|_{\mathcal{Y}}^{\nu_2}=\lambda_4\|(u,v)\|_{\mathcal{Y}}^{\nu_2}\ge\|(u,v)\|_{\mathcal{Y}}^{\nu_2}\ge\|(u,v)\|_{\mathcal{Y}}.\end{aligned}$$

Then we deduce $\|\mathcal{A}_2(u,v)\|\ge\|(u,v)\|_{\mathcal{Y}}$ and then

$$\|\mathcal{A}(u,v)\|_{\mathcal{Y}}\ge\|(u,v)\|_{\mathcal{Y}},\ \ \forall\,(u,v)\in\partial B_{R_4}\cap\mathcal{P}_0. \tag{35}$$

From Lemma 5, (33), (35) and Theorem 1 (ii), we conclude that $\mathcal{A}$ has a fixed point $(u,v)\in(\overline{B}_{R_3}\setminus B_{R_4})\cap\mathcal{P}_0$, so $R_4\le\|(u,v)\|_{\mathcal{Y}}\le R_3$, which is a positive solution of problem (1) and (2). □

Theorem 4. *We suppose that assumptions $(I1)$, $(I2)$, $(I4)$ and $(I6)$ hold. In addition, the functions ψ_i and χ_i, $i=1,2$ satisfy the condition*

$(I7)$ $M_3 M_1^{\varrho_1 - 1} D_0^{\varrho_1 - 1} < \frac{1}{2}$, $M_4 M_2^{\varrho_2 - 1} D_0^{\varrho_2 - 1} < \frac{1}{2}$, *where*

$$D_0 = \max\left\{ \max_{\eta \in [0,1],\, z_i \in [0,\omega_0],\, i=1,\ldots,4} \chi_1(\eta, z_1, z_2, z_3, z_4), \max_{\eta \in [0,1],\, z_i \in [0,\omega_0],\, i=1,\ldots,4} \chi_2(\eta, z_1, z_2, z_3, z_4) \right\},$$

with $\omega_0 = \max\left\{1, \frac{1}{\Gamma(\sigma_1+1)}, \frac{1}{\Gamma(\sigma_2+1)}, \frac{1}{\Gamma(\varsigma_1+1)}, \frac{1}{\Gamma(\varsigma_2+1)}\right\}$.

Then there exist two positive solutions $(u_1(\tau), v_1(\tau))$, $(u_2(\tau), v_2(\tau))$, $\tau \in [0,1]$ *of problem (1) and (2).*

Proof. Under assumptions $(I1)$, $(I2)$ and $(I4)$, Theorem 2 gives us the existence of $R_2 > 1$ such that

$$\|\mathcal{A}(u,v)\|_{\mathcal{Y}} \geq \|(u,v)\|_{\mathcal{Y}}, \quad \forall\, (u,v) \in \partial B_{R_2} \cap \mathcal{P}_0. \tag{36}$$

Under assumptions $(I1)$, $(I2)$ and $(I6)$, Theorem 3 gives us the existence of $R_4 < 1$ such that

$$\|\mathcal{A}(u,v)\|_{\mathcal{Y}} \geq \|(u,v)\|_{\mathcal{Y}}, \quad \forall\, (u,v) \in \partial B_{R_4} \cap \mathcal{P}_0. \tag{37}$$

Now we consider the set $B_1 = \{(u,v) \in \mathcal{Y},\ \|(u,v)\|_{\mathcal{Y}} < 1\}$. By $(I7)$, for any $(u,v) \in \partial B_1 \cap \mathcal{P}_0$ and $\eta \in [0,1]$, we obtain

$$\begin{aligned}
&\mathcal{A}_1(u,v)(\eta) \leq \int_0^1 \mathcal{J}_2(\zeta) \varphi_{\varrho_1}\left(\int_0^1 \mathcal{J}_1(\vartheta) \psi_1(\vartheta) \chi_1(\vartheta, u(\vartheta), v(\vartheta), I_{0+}^{\sigma_1} u(\vartheta), I_{0+}^{\sigma_2} v(\vartheta))\, d\vartheta \right) d\zeta \\
&\leq D_0^{\varrho_1 - 1} \left(\int_0^1 \mathcal{J}_2(\zeta)\, d\zeta \right) \left(\int_0^1 \mathcal{J}_1(\vartheta) \psi_1(\vartheta)\, d\vartheta \right)^{\varrho_1 - 1} = M_3 D_0^{\varrho_1 - 1} M_1^{\varrho_1 - 1} < \frac{1}{2}, \\
&\mathcal{A}_2(u,v)(\eta) \leq \int_0^1 \mathcal{J}_4(\zeta) \varphi_{\varrho_2}\left(\int_0^1 \mathcal{J}_3(\vartheta) \psi_2(\vartheta) \chi_2(\vartheta, u(\vartheta), v(\vartheta), I_{0+}^{\varsigma_1} u(\vartheta), I_{0+}^{\varsigma_2} v(\vartheta))\, d\vartheta \right) d\zeta \\
&\leq D_0^{\varrho_2 - 1} \left(\int_0^1 \mathcal{J}_4(\zeta)\, d\zeta \right) \left(\int_0^1 \mathcal{J}_3(\vartheta) \psi_2(\vartheta)\, d\vartheta \right)^{\varrho_2 - 1} = M_4 D_0^{\varrho_2 - 1} M_2^{\varrho_2 - 1} < \frac{1}{2}.
\end{aligned}$$

Then $\|\mathcal{A}_i(u,v)\| < 1/2$ for all $(u,v) \in \partial B_1 \cap \mathcal{P}_0$, $i = 1,2$. Hence

$$\|\mathcal{A}(u,v)\|_{\mathcal{Y}} = \|\mathcal{A}_1(u,v)\| + \|\mathcal{A}_2(u,v)\| < 1 = \|(u,v)\|_{\mathcal{Y}}, \quad \forall\, (u,v) \in \partial B_1 \cap \mathcal{P}_0. \tag{38}$$

So from (36), (38) and Theorem 1, we deduce that problem (1) and (2) has one positive solution $(u_1, v_1) \in \mathcal{P}_0$ with $1 < \|(u_1, v_1)\|_{\mathcal{Y}} \leq R_2$. From (37) and (38) and the Guo–Krasnosel'skii fixed point theorem, we conclude that problem (1) and (2) have another positive solution $(u_2, v_2) \in \mathcal{P}_0$ with $R_4 \leq \|(u_2, v_2)\|_{\mathcal{Y}} < 1$. Then problem (1) and (2) have at least two positive solutions $(u_1(\tau), v_1(\tau))$, $(u_2(\tau), v_2(\tau))$, $\tau \in [0,1]$. □

4. Examples

Let $\gamma_1 = 3/2$, $\gamma_2 = 7/6$, $p = 4$, $q = 3$, $\delta_1 = 10/3$, $\delta_2 = 12/5$, $\sigma_1 = 2/5$, $\sigma_2 = 29/7$, $\varsigma_1 = 11/9$, $\varsigma_2 = 21/4$, $n = 2$, $m = 1$, $\alpha_0 = 13/8$, $\alpha_1 = 5/7$, $\alpha_2 = 3/4$, $\beta_0 = 10/9$, $\beta_1 = 7/8$, $r_1 = 17/4$, $r_2 = 25/8$, $\varrho_1 = 17/13$, $\varrho_2 = 25/17$, $\mathcal{H}_0(t) = \{2/7,\ t \in [0,3/4);\ 11/4,\ t \in [3/4,1]\}$, $\mathcal{H}_1(t) = t/2$, $t \in [0,1]$, $\mathcal{H}_2(t) = \{1/2,\ t \in [0,1/2);\ 13/10,\ t \in [1/2,1]\}$, $\mathcal{K}_0(t) = 4t/9$, $t \in [0,1]$, $\mathcal{K}_1(t) = \{1/4,\ t \in [0,1/3);\ 29/20,\ t \in [1/3,1]\}$.

We consider the system of fractional differential equations

$$\begin{cases} D_{0+}^{3/2}\left(\varphi_{17/4}\left(D_{0+}^{10/3} u(\tau)\right)\right) = f(\tau, u(\tau), v(\tau), I_{0+}^{2/5} u(\tau), I_{0+}^{29/7} v(\tau)), & \tau \in (0,1), \\ D_{0+}^{7/6}\left(\varphi_{25/8}\left(D_{0+}^{12/5} v(\tau)\right)\right) = g(\tau, u(\tau), v(\tau), I_{0+}^{11/9} u(\tau), I_{0+}^{21/4} v(\tau)), & \tau \in (0,1), \end{cases} \tag{39}$$

with the boundary conditions

$$\begin{cases} u(0)=u'(0)=u''(0)=0,\ D_{0+}^{10/3}u(0)=0,\ D_{0+}^{10/3}u(1)=\frac{1}{2^{4/13}}D_{0+}^{10/3}u\left(\frac{3}{4}\right), \\ D_{0+}^{13/8}u(1)=\frac{1}{2}\int_0^1 D_{0+}^{5/7}u(\eta)\,d\eta+\frac{4}{5}D_{0+}^{3/4}u\left(\frac{1}{2}\right), \\ v(0)=v'(0)=0,\ D_{0+}^{12/5}v(0)=0,\ \varphi_{25/8}\left(D_{0+}^{12/5}v(1)\right)=\frac{4}{9}\int_0^1 \varphi_{25/8}\left(D_{0+}^{12/5}v(\eta)\right)d\eta, \\ D_{0+}^{10/9}v(1)=\frac{6}{5}D_{0+}^{7/8}v\left(\frac{1}{3}\right). \end{cases} \tag{40}$$

We have here $\mathfrak{a}_1 \approx 0.56698729 > 0$, $\mathfrak{a}_2 \approx 2.16111947 > 0$, $\mathfrak{b}_1 \approx 0.61904762 > 0$, $\mathfrak{b}_2 \approx 0.43774133 > 0$. So, assumption $(I1)$ is satisfied. We also obtain

$$\begin{aligned}
\mathfrak{g}_1(\tau,\eta) &= \frac{1}{\Gamma(3/2)}\begin{cases} \tau^{1/2}(1-\eta)^{1/2}-(\tau-\eta)^{1/2},\ 0\le\eta\le\tau\le1, \\ \tau^{1/2}(1-\eta)^{1/2},\ 0\le\tau\le\eta\le1, \end{cases} \\
\mathfrak{g}_2(\tau,\eta) &= \frac{1}{\Gamma(10/3)}\begin{cases} \tau^{7/3}(1-\eta)^{17/24}-(\tau-\eta)^{7/3},\ 0\le\eta\le\tau\le1, \\ \tau^{7/3}(1-\eta)^{17/24},\ 0\le\tau\le\eta\le1, \end{cases} \\
\mathfrak{g}_{21}(\tau,\eta) &= \frac{1}{\Gamma(55/21)}\begin{cases} \tau^{34/21}(1-\eta)^{17/24}-(\tau-\eta)^{34/21},\ 0\le\eta\le\tau\le1, \\ \tau^{34/21}(1-\eta)^{17/24},\ 0\le\tau\le\eta\le1, \end{cases} \\
\mathfrak{g}_{22}(\tau,\eta) &= \frac{1}{\Gamma(31/12)}\begin{cases} \tau^{19/12}(1-\eta)^{17/24}-(\tau-\eta)^{19/12},\ 0\le\eta\le\tau\le1, \\ \tau^{19/12}(1-\eta)^{17/24},\ 0\le\tau\le\eta\le1, \end{cases} \\
\mathfrak{g}_3(\tau,\eta) &= \frac{1}{\Gamma(7/6)}\begin{cases} \tau^{1/6}(1-\eta)^{1/6}-(\tau-\eta)^{1/6},\ 0\le\eta\le\tau\le1, \\ \tau^{1/6}(1-\eta)^{1/6},\ 0\le\tau\le\eta\le1, \end{cases} \\
\mathfrak{g}_4(\tau,\eta) &= \frac{1}{\Gamma(12/5)}\begin{cases} \tau^{7/5}(1-\eta)^{13/45}-(\tau-\eta)^{7/5},\ 0\le\eta\le\tau\le1, \\ \tau^{7/5}(1-\eta)^{13/45},\ 0\le\tau\le\eta\le1, \end{cases} \\
\mathfrak{g}_{41}(\tau,\eta) &= \frac{1}{\Gamma(61/40)}\begin{cases} \tau^{21/40}(1-\eta)^{13/45}-(\tau-\eta)^{21/40},\ 0\le\eta\le\tau\le1, \\ \tau^{21/40}(1-\eta)^{13/45},\ 0\le\tau\le\eta\le1, \end{cases} \\
\mathcal{G}_1(\tau,\eta) &= \mathfrak{g}_1(\tau,\eta)+\frac{\tau^{1/2}}{2\mathfrak{a}_1}\mathfrak{g}_1\left(\frac{3}{4},\eta\right),\ (\tau,\eta)\in[0,1]\times[0,1], \\
\mathcal{G}_2(\tau,\eta) &= \mathfrak{g}_2(\tau,\eta)+\frac{\tau^{7/3}}{\mathfrak{a}_2}\left(\frac{1}{2}\int_0^1 \mathfrak{g}_{21}(\vartheta,\eta)\,d\vartheta+\frac{4}{5}\mathfrak{g}_{22}\left(\frac{1}{2},\eta\right)\right),\ (\tau,\eta)\in[0,1]\times[0,1], \\
\mathcal{G}_3(\tau,\eta) &= \mathfrak{g}_3(\tau,\eta)+\frac{4\tau^{1/6}}{9\mathfrak{b}_1}\int_0^1 \mathfrak{g}_3(\vartheta,\eta)\,d\vartheta,\ (\tau,\eta)\in[0,1]\times[0,1], \\
\mathcal{G}_4(\tau,\eta) &= \mathfrak{g}_4(\tau,\eta)+\frac{6\tau^{7/5}}{5\mathfrak{b}_2}\mathfrak{g}_{41}\left(\frac{1}{3},\eta\right),\ (\tau,\eta)\in[0,1]\times[0,1], \\
\mathfrak{h}_1(\eta) &= \frac{1}{\Gamma(3/2)}(1-\eta)^{1/2},\ \mathfrak{h}_2(\eta)=\frac{1}{\Gamma(10/3)}(1-\eta)^{17/24}(1-(1-\eta)^{13/8}),\ \eta\in[0,1], \\
\mathfrak{h}_3(\eta) &= \frac{1}{\Gamma(7/6)}(1-\eta)^{1/6},\ \mathfrak{h}_4(\eta)=\frac{1}{\Gamma(12/5)}(1-\eta)^{13/45}(1-(1-\eta)^{10/9}),\ \eta\in[0,1].
\end{aligned}$$

Besides we deduce

$$\begin{aligned}
\mathcal{J}_1(\zeta) &= \begin{cases} \mathfrak{h}_1(\zeta)+\frac{1}{2\mathfrak{a}_1\Gamma(3/2)}\left[\left(\frac{3}{4}\right)^{1/2}(1-\zeta)^{1/2}-\left(\frac{3}{4}-\zeta\right)^{1/2}\right],\ 0\le\zeta\le\frac{3}{4}, \\ \mathfrak{h}_1(\zeta)+\frac{1}{2\mathfrak{a}_1\Gamma(3/2)}\left(\frac{3}{4}\right)^{1/2}(1-\zeta)^{1/2},\ \frac{3}{4}<\zeta\le1, \end{cases} \\
\mathcal{J}_2(\zeta) &= \begin{cases} \mathfrak{h}_2(\zeta)+\frac{1}{\mathfrak{a}_2}\left\{\frac{1}{2\Gamma(76/21)}\left[(1-\zeta)^{17/24}-(1-\zeta)^{55/21}\right]\right. \\ \quad\left.+\frac{4}{5\Gamma(31/12)}\left[\left(\frac{1}{2}\right)^{19/12}(1-\zeta)^{17/24}-\left(\frac{1}{2}-\zeta\right)^{19/12}\right]\right\},\ 0\le\zeta\le\frac{1}{2}, \\ \mathfrak{h}_2(\zeta)+\frac{1}{\mathfrak{a}_2}\left\{\frac{1}{2\Gamma(76/21)}\left[(1-\zeta)^{17/24}-(1-\zeta)^{55/21}\right]\right. \\ \quad\left.+\frac{4}{5\Gamma(31/12)}\left(\frac{1}{2}\right)^{19/12}(1-\zeta)^{17/24}\right\},\ \frac{1}{2}<\zeta\le1, \end{cases} \\
\mathcal{J}_3(\zeta) &= \mathfrak{h}_3(\zeta)+\frac{4}{9\mathfrak{b}_1\Gamma(13/6)}\left[(1-\zeta)^{1/6}-(1-\zeta)^{7/6}\right],\ \zeta\in[0,1], \\
\mathcal{J}_4(\zeta) &= \begin{cases} \mathfrak{h}_4(\zeta)+\frac{6}{5\mathfrak{b}_2\Gamma(61/40)}\left[\left(\frac{1}{3}\right)^{21/40}(1-\zeta)^{13/45}-\left(\frac{1}{3}-\zeta\right)^{21/40}\right],\ 0\le\zeta\le\frac{1}{3}, \\ \mathfrak{h}_4(\zeta)+\frac{6}{5\mathfrak{b}_2\Gamma(61/40)}\left(\frac{1}{3}\right)^{21/40}(1-\zeta)^{13/45},\ \frac{1}{3}<\zeta\le1. \end{cases}
\end{aligned}$$

Example 1. *We consider the functions*

$$f(\eta, z_1, z_2, z_3, z_4) = \frac{(2z_1+z_2+5z_3+7z_4)^{13a/4}}{\eta^{\kappa_1}(1-\eta)^{\kappa_2}}, \quad g(\eta, z_1, z_2, z_3, z_4) = \frac{(3z_1+8z_2+2z_3+9z_4)^{17b/8}}{\eta^{\kappa_3}(1-\eta)^{\kappa_4}}, \tag{41}$$

for $\eta \in (0,1)$, $z_i \geq 0$, $i = 1,\ldots,4$, *where* $a > 1$, $b > 1$, $\kappa_1 \in (0,1)$, $\kappa_2 \in (0,3/2)$, $\kappa_3 \in (0,1)$, $\kappa_4 \in (0,7/6)$. *Here* $\psi_1(\eta) = \frac{1}{\eta^{\kappa_1}(1-\eta)^{\kappa_2}}$, $\psi_2(\eta) = \frac{1}{\eta^{\kappa_3}(1-\eta)^{\kappa_4}}$ *for* $\eta \in (0,1)$, $\chi_1(\eta, z_1, z_2, z_3, z_4) = (2z_1 + z_2 + 5z_3 + 7z_4)^{13a/4}$ *and* $\chi_2(\eta, z_1, z_2, z_3, z_4) = (3z_1 + 8z_2 + 2z_3 + 9z_4)^{17b/8}$ *for* $\eta \in [0,1]$, $z_i \geq 0$, $i = 1,\ldots,4$. *We also find* $\Lambda_1 = \int_0^1 (1-\tau)^{1/2}\psi_1(\tau)\,d\tau = B(1-\kappa_1, \frac{3}{2}-\kappa_2) \in (0,\infty)$, $\Lambda_2 = \int_0^1 (1-\tau)^{1/6}\psi_2(\tau)\,d\tau = B(1-\kappa_3, \frac{7}{6}-\kappa_4) \in (0,\infty)$. *Then assumption* $(I2)$ *is also satisfied. Moreover, in* $(I3)$, *for* $c_1 = 2$, $c_2 = 1$, $c_3 = 5$, $c_4 = 7$, $\mu_1 = 1$, $d_1 = 3$, $d_2 = 8$, $d_3 = 2$, $d_4 = 9$, $\mu_2 = 1$, *we obtain* $\chi_{10} = 0$, $\chi_{20} = 0$. *In* $(I4)$, *for* $[\theta_1, \theta_2] \subset (0,1)$, $p_1 = 2$, $p_2 = 1$, $p_3 = 5$, $p_4 = 7$, *we have* $f_\infty = \infty$. *By Theorem 2, we deduce that there exists a positive solution* $(u(\tau), v(\tau))$, $\tau \in [0,1]$ *of problems (39) and (40) with the nonlinearities (41).*

Example 2. *We consider the functions*

$$\begin{aligned} f(\eta, z_1, z_2, z_3, z_4) &= \frac{s_0(\eta+2)}{(\eta^2+6)\sqrt[3]{\eta^2}}\left[\left(\frac{1}{4}z_1 + \frac{1}{3}z_2 + z_3 + \frac{1}{2}z_4\right)^{\omega_1} \right. \\ &\quad \left. + \left(\frac{1}{4}z_1 + \frac{1}{3}z_2 + z_3 + \frac{1}{2}z_4\right)^{\omega_2}\right], \ \eta \in (0,1], \ z_i \geq 0, \ i = 1,\ldots,4, \\ g(\eta, z_1, z_2, z_3, z_4) &= \frac{t_0(3+\sin\eta)}{(\eta+2)^4\sqrt[5]{(1-\eta)^3}}\left(e^{z_1} + \ln(z_2+z_3+1) + z_4^{\omega_3}\right), \\ &\quad \eta \in [0,1), \ z_i \geq 0, \ i = 1,\ldots,4, \end{aligned} \tag{42}$$

where $s_0 > 0$, $t_0 > 0$, $\omega_1 > \frac{13}{4}$, $\omega_2 \in \left(0, \frac{13}{4}\right)$, $\omega_3 > 0$. *Here, we have* $\psi_1(\eta) = \frac{1}{\sqrt[3]{\eta^2}}$, $\eta \in (0,1]$, $\chi_1(\eta, z_1, z_2, z_3, z_4) = \frac{s_0(\eta+2)}{(\eta^2+6)}\left[\left(\frac{1}{4}z_1 + \frac{1}{3}z_2 + z_3 + \frac{1}{2}z_4\right)^{\omega_1} + \left(\frac{1}{4}z_1 + \frac{1}{3}z_2 + z_3 + \frac{1}{2}z_4\right)^{\omega_2}\right]$, $\eta \in [0,1]$, $z_i \geq 0$, $i = 1,\ldots,4$, $\psi_2(\eta) = \frac{1}{\sqrt[5]{(1-\eta)^3}}$, $\eta \in [0,1)$, $\chi_2(\eta, z_1, z_2, z_3, z_4) = \frac{t_0(3+\sin\eta)}{(\eta+2)^4}\left(e^{z_1} + \ln(z_2+z_3+1) + z_4^{\omega_3}\right)$, $\eta \in [0,1]$, $z_i \geq 0$, $i = 1,\ldots,4$. *We find* $\Lambda_1 = \int_0^1 (1-\tau)^{1/2}\frac{1}{\sqrt[3]{\tau^2}}\,d\tau = B\left(\frac{1}{3}, \frac{3}{2}\right) \in (0,\infty)$, $\Lambda_2 = \int_0^1 (1-\tau)^{1/6}\frac{1}{\sqrt[5]{(1-\tau)^3}}\,d\tau = \frac{30}{17} \in (0,\infty)$. *Then assumption* $(I2)$ *is satisfied. For* $[\theta_1, \theta_2] \subset (0,1)$, $p_1 = 1/4$, $p_2 = 1/3$, $p_3 = 1$, $p_4 = 1/2$, *we obtain* $f_\infty = \infty$, *and for* $s_1 = 1/4$, $s_2 = 1/3$, $s_3 = 1$, $s_4 = 1/2$ *and* $\nu_1 \in \left(\frac{4\omega_2}{13}, 1\right]$, *we have* $f_0 = \infty$. *So assumptions* $(I4)$ *and* $(I6)$ *are satisfied. Then after some computations, we deduce* $M_1 = \int_0^1 \mathcal{J}_1(\tau)\psi_1(\tau)\,d\tau \approx 3.04682891$, $M_2 = \int_0^1 \mathcal{J}_3(\tau)\psi_2(\tau)\,d\tau \approx 2.64937892$, $M_3 = \int_0^1 \mathcal{J}_2(\tau)\,d\tau \approx 0.15582207$, $M_4 = \int_0^1 \mathcal{J}_4(\tau)\,d\tau \approx 1.25629509$. *In addition, we obtain that* $\omega_0 = \frac{1}{\Gamma(7/5)} \approx 1.12706049$, $D_0 = \max\left\{\frac{3s_0}{7}\left[\left(\frac{25}{12}\omega_0\right)^{\omega_1} + \left(\frac{25}{12}\omega_0\right)^{\omega_2}\right], t_0 m_0[e^{\omega_0} + \ln(2\omega_0+1) + \omega_0^{\omega_3}]\right\}$, *with* $m_0 = \max_{\eta\in[0,1]} \frac{3+\sin\eta}{(\eta+2)^4} \approx 3.0123699$. *If*

$$\begin{aligned} s_0 &< \min\left\{\frac{7}{3(2M_3)^{13/4}M_1[(25\omega_0/12)^{\omega_1} + (25\omega_0/12)^{\omega_2}]}, \frac{7}{3(2M_4)^{17/8}M_2[(25\omega_0/12)^{\omega_1} + (25\omega_0/12)^{\omega_2}]}\right\}, \\ t_0 &< \min\left\{\frac{1}{(2M_3)^{13/4}M_1 m_0\left[e^{\omega_0} + \ln(2\omega_0+1) + \omega_0^{\omega_3}\right]}, \frac{1}{(2M_4)^{17/8}M_2 m_0\left[e^{\omega_0} + \ln(2\omega_0+1) + \omega_0^{\omega_3}\right]}\right\}, \end{aligned}$$

then the inequalities $M_3 M_1^{4/13} D_0^{4/13} < \frac{1}{2}$, $M_4 M_2^{8/17} D_0^{8/17} < \frac{1}{2}$ *are satisfied (that is, assumption* $(I7)$ *is satisfied). For example, if* $\omega_1 = 4$, $\omega_2 = 2$, $\omega_3 = 3$, *and* $s_0 \leq 0.0034$ *and* $t_0 \leq 0.0031$, *then the above inequalities are satisfied. By Theorem 4, we conclude that problem (39) and (40) with the nonlinearities (42) has at least two positive solutions* $(u_1(\tau), v_1(\tau))$, $(u_2(\tau), v_2(\tau))$, $\tau \in [0,1]$.

5. Conclusions

In this paper we investigate the system of Riemann–Liouville fractional differential Equations (1) with r_1-Laplacian and r_2-Laplacian operators and fractional integral terms, subject to the uncoupled boundary conditions (2) which contain Riemann–Stieltjes integrals and fractional derivatives of various orders. The nonlinearities f and g from the system are nonnegative functions and they may be singular at $\tau = 0$ and/or $\tau = 1$. First we present the Green functions associated to our problem (1) and (2) and some of their properties. Then we give various conditions for the functions f and g such that (1) and (2) has at least one or two positive solutions. In the proof of our main results we use the Guo–Krasnosel'skii fixed point theorem of cone expansion and compression of norm type. We finally present two examples for illustrating the obtained existence theorems.

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