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Article

A Langevin-Type q-Variant System of Nonlinear Fractional Integro-Difference Equations with Nonlocal Boundary Conditions

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Abstract: We introduce a new class of boundary value problems consisting of a q-variant system of Langevin-type nonlinear coupled fractional integro-difference equations and nonlocal multipoint boundary conditions. We make use of standard fixed-point theorems to derive the existence and uniqueness results for the given problem. Illustrative examples for the obtained results are also presented.

Keywords: fractional q-difference equations; Riemann–Liouville integral; nonlocal boundary conditions; existence; fixed point

MSC: 34A08; 34B10; 34B15; 39A13

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1. Introduction

The Langevin equation provides a decent approach to describe the evolution of fluctuating physical phenomena. Examples include anomalous diffusion [1], time evolution of the velocity of the Brownian motion [2,3], diffusion with inertial effects [4], gait variability [5], harmonization of a many-body problem [6], financial aspects [7], etc. However, the failure of the ordinary Langevin equation for correct description of the dynamical systems in complex media led to its several generalizations. One such example is that of the Langevin equation, involving fractional-order derivative operators, which provides a more flexible model for fractal processes. For some recent results on Langevin equation, see ([8–12]) and the references therein.

The topic of q-difference equations has evolved into an important area of research, as such equations are always completely controllable and appear in the q-optimal control problem [13]. Furthermore, the variational q-calculus is regarded as a generalization of the continuous variational calculus due to the presence of an extra parameter q whose nature may be physical or economical. The variational calculus on the q-uniform lattice is concerned with the study of the q-Euler equation and its applications to commutation equations, and isoperimetric and Lagrange problems. In other words, the q-Euler–Lagrange equation is solved for finding the extremum of the functional involved instead of solving the Euler–Lagrange equation [14]. There do exist q-variants of certain significant concepts, such as q-analogues of fractional operators, q-Laplace transform, q-Taylor's formula, etc.

Fractional-order operators are found to be of great utility in improving the mathematical modeling of several real-world problems. The variational principles based on fractional derivative operators lead to the class of fractional Euler–Lagrange equations [15]. In addition, one can find some interesting results on optimal control theories for fractional differential systems in the articles [16–21].

The popularity of fractional calculus in the recent years led to the birth of the fractional analogue of q-difference equations (fractional q-difference equations), for instance, see [22,23].

One can find interesting results on nonlinear boundary value problems involving fractional q-derivative and q-integral operators, and different kinds of boundary conditions in the articles [24–37]. In a recent work [38], the authors studied the existence of solutions for a nonlinear fractional q-integro-difference equation equipped with q-integral boundary conditions. However, it is observed that there are a few results for coupled systems of fractional q-integro-difference equations [39]. More recently, a coupled system of nonlinear fractional q-integro-difference equations with q-integral coupled boundary conditions was studied in [40].

The objective of the present work is to enrich the literature on boundary value problems of coupled systems of fractional q-integro-difference equations. Keeping in mind the importance of the fractional Langevin equation, we introduce and study a new problem consisting of a coupled system of Langevin-type nonlinear fractional q-integro-difference equations complemented with nonlocal multipoint boundary conditions. The proposed problem is interesting in the sense that it enhances the literature on fractional q-variant of Langevin equations with mixed nonlinearities in terms of the parameter q. On the other hand, the consideration of multipoint non-separated boundary conditions involving the values of the unknown functions together with their q-derivatives at the end points as well as the interior nonlocal positions of given domain extends the scope of the present work to a more general situation (also see Section 5). For the motivation of nonlocal boundary conditions, we recall that nonlocal multipoint boundary conditions appear in feedback controls problems, optimal boundary control of (finite) string vibrations arising from interior arbitrary positions, etc. For more details, see [41–44]. In precise terms, we investigate the following boundary value problem:

$$\begin{cases} {}^cD_q^{p_1}({}^cD_q^{p_2}+\lambda_1)x(t)=\alpha_1 f_1(t,x(t),y(t))+\beta_1 I_q^{\xi_1}g_1(t,x(t),y(t)), & 0\le t\le 1, \\ {}^cD_q^{r_1}({}^cD_q^{r_2}+\lambda_2)y(t)=\alpha_2 f_2(t,x(t),y(t))+\beta_2 I_q^{\xi_2}g_2(t,x(t),y(t)), & 0\le t\le 1, \end{cases} \tag{1}$$

$$\begin{cases} \mu_1 x(0)-\mu_2\left(t^{(1-p_2)}D_q x(t)\right)\Big|_{t=0}=\displaystyle\sum_{j=1}^{n} a_j y(\eta_j), \\ \mu_3 y(0)-\mu_4\left(t^{(1-r_2)}D_q y(t)\right)\Big|_{t=0}=\displaystyle\sum_{j=1}^{n} b_j x(\eta_j), \\ \sigma_1 x(1)+\sigma_2 D_q x(1)=\displaystyle\sum_{j=1}^{n} k_j D_q y(\eta_j), \\ \sigma_3 y(1)+\sigma_4 D_q y(1)=\displaystyle\sum_{j=1}^{n} m_j D_q x(\eta_j), \end{cases} \tag{2}$$

where ${}^cD_q^{p_i}$ and ${}^cD_q^{r_i}$ denote the fractional q-derivative operators of the Caputo type, $0<p_i,r_i\le 1$, $0<q<1$, $I_q^{\xi_i}(.)$ denotes Riemann–Liouville integral of order $\xi_i>0$, f_i,g_i are given continuous functions, $\lambda_i\neq 0,\alpha_i,\beta_i, i=1,2,$ and $a_j,b_j,k_j,m_j, j=1,\ldots,n$ are real constants and $\mu_1,\mu_2,\mu_3,\mu_4,\sigma_1,\sigma_2,\sigma_3,\sigma_4\in\mathbb{R},\eta_j\in(0,1),\ j=1,\ldots,n$.

Here, one can notice that the right-hand sides of the fractional q-Langevin equations in the system (1) involve the usual as well as q-integral-type nonlinearities. These equations correspond to different combinations of nonlinearities, such as ordinary nonlinearities, $f_1(t,x(t),y(t))$ and $f_2(t,x(t),y(t))$ for $\beta_1=0=\beta_2$, purely q-integral-type nonlinearities, $I_q^{\xi_1}g_1(t,x(t),y(t))$ and $I_q^{\xi_2}g_2(t,x(t),y(t))$ for $\alpha_1=0=\alpha_2$, and so on.

The paper is organized as follows. In Section 2, we recall some general concepts and results on q-calculus and fractional calculus. We then solve a linear variant of the given problem that provides a platform to define the solution for the problem at hand. Section 3 is devoted to the main existence results, which are established with the aid of some classical fixed-point theorems. The paper concludes with an illustrative example.

2. Preliminaries on Fractional q-Calculus

Here, we recall some basic definitions and known results on fractional q-calculus.

Definition 1. *Let $\beta \geq 0$, $0 < q < 1$, and f be a function defined on $[0,1]$. The fractional q-integral of the Riemann–Liouville type is $(I_q^0 f)(t) = f(t)$ and*

$$(I_q^\beta f)(t) = \int_0^t \frac{(t-qs)^{(\beta-1)}}{\Gamma_q(\beta)} f(s) d_q(s), \;\; \beta > 0, \; t \in [0,1],$$

where

$$\Gamma_q(\beta) = \frac{(1-q)^{(\beta-1)}}{(1-q)^{\beta-1}}, \; 0 < q < 1$$

and satisfies the relation:
$\Gamma_q(\beta+1) = [\beta]_q \Gamma_q(\beta)$, *with*

$$[\beta]_q = \frac{q^\beta - 1}{q-1}, \; (1-q)^{(0)} = 1, \;\; (1-q)^{(n)} = \prod_{k=0}^{n-1} (1-q^{k+1}), \; n \in \mathbb{N}.$$

More generally, if $\alpha \in \mathbb{R}$, then

$$(1-q)^{(\alpha)} = \prod_{i=0}^{\infty} \frac{(1-q^{i+1})}{(1-q^{1+\alpha+i})}.$$

For $0 < q < 1$, we define the q-derivative of a real valued function f as

$$D_q f(t) = \frac{f(t) - f(qt)}{(1-q)t}, \; t \neq 0, \;\; D_q f(0) = \lim_{n\to\infty} \frac{f(sq^n) - f(0)}{sq^n}, \; s \neq 0.$$

For more details, see [22].

Definition 2 ([45]). *The fractional q-derivative of the Riemann–Liouville type of order $\beta \geq 0$ is defined by $(D_q^0 f)(t) = f(t)$ and*

$$(D_q^\beta f)(t) = (D_q^{[\beta]} I_q^{[\beta]-\beta} f)(t), \;\; \beta > 0,$$

where $[\beta]$ is the smallest integer greater than or equal to β.

Definition 3 ([45]). *The fractional q-derivative of the Caputo type of order $\beta \geq 0$ is defined by*

$$({}^c D_q^\beta f)(t) = (I_q^{[\beta]-\beta} D_q^{[\beta]} f)(t), \;\; \beta > 0,$$

where $[\beta]$ is the smallest integer greater than or equal to β.

Definition 4. *(q-Beta function) For any $x, y > 0$,*

$$B_q(x,y) = \int_0^1 t^{(x-1)} (1-qt)^{(y-1)} d_q t$$

is called the q-beta function.

Recall that

$$B_q(x,y) = \frac{\Gamma_q(x)\Gamma_q(y)}{\Gamma_q(x+y)}. \tag{3}$$

Lemma 1 ([45]). *Let $\beta, \gamma \geq 0$ and let f be a function defined on $[0,1]$. Then*
(i) $(I_q^\gamma I_q^\beta f)(t) = (I_q^{\beta+\gamma} f)(t)$,
(ii) $(D_q^\beta I_q^\beta f)(t) = f(t)$.

Lemma 2 ([45])**.** *Let $\beta > 0$. Then the following equality holds:*

$$(I_q^{\beta}\, {}^c D_q^{\beta} f)(t) = f(t) - \sum_{k=0}^{[\beta]-1} \frac{t^k}{\Gamma_q(k+1)} (D_q^k f)(0).$$

Lemma 3 ([25])**.** *Let $\beta \geq 0$ and $n \in \mathbb{N}$. Then the following equality holds:*

$$(I_q^{\beta} D_q^n f)(t) = D_q^n I_q^{\beta} f(t) - \sum_{k=0}^{[\beta]-1} \frac{t^{\beta-n+k}}{\Gamma_q(\beta-n+k)} (D_q^k f)(0).$$

Lemma 4 ([46])**.** *For $\beta \in \mathbb{R}^+$, $\lambda \in (-1, \infty)$, the following is valid*

$$I_q^{\beta}\Big((x-a)^{(\lambda)}\Big) = \frac{\Gamma_q(\lambda+1)}{\Gamma_q(\beta+\lambda+1)} (x-a)^{(\beta+\lambda)}, \quad 0 < a < x < b.$$

In particular, for $\lambda = 0, a = 0$, using q-integration by parts, we have

$$\begin{aligned}(I_q^{\beta} 1)(x) &= \frac{1}{\Gamma_q(\beta)} \int_0^x (x-qt)^{(\beta-1)} d_q t = -\frac{1}{\Gamma_q(\beta)} \int_0^x \frac{D_q((x-t)^{(\beta)})}{[\beta]_q} d_q t \\ &= -\frac{1}{\Gamma_q(\beta+1)} \int_0^x D_q((x-t)^{(\beta)}) d_q t = \frac{1}{\Gamma_q(\beta+1)} x^{(\beta)}.\end{aligned}$$

In order to define the solution for the problem (1) and (2), we need the following lemma.

Lemma 5. *Let $\Lambda \neq 0$ and $h_1, h_2 \in C([0,1], \mathbb{R})$. Then the unique solution of the following linear system of equations:*

$$\begin{cases} {}^c D_q^{p_1}({}^c D_q^{p_2} + \lambda_1) x(t) = h_1(t), \ \ 0 \leq t \leq 1, \\ {}^c D_q^{r_1}({}^c D_q^{r_2} + \lambda_2) y(t) = h_2(t), \ \ 0 \leq t \leq 1, \end{cases} \tag{4}$$

subject to the boundary conditions (2) *is given by*

$$\begin{aligned} x(t) &= \int_0^t \tfrac{(t-qu)^{(p_1+p_2-1)}}{\Gamma_q(p_1+p_2)} h_1(u) d_q u - \lambda_1 \int_0^t \tfrac{(t-qu)^{(p_2-1)}}{\Gamma_q(p_2)} x(u) d_q u \\ &- \delta_1\Big(\tfrac{\rho_1 t^{p_2}+\rho_5}{\Lambda}\Big)\Bigg(\int_0^{\eta_j} \tfrac{(\eta_j-qu)^{(r_1+r_2-1)}}{\Gamma_q(r_1+r_2)} h_2(u) d_q u - \lambda_2 \int_0^{\eta_j} \tfrac{(\eta_j-qu)^{(r_2-1)}}{\Gamma_q(r_2)} y(u) d_q u \Bigg) \\ &- \delta_2\Big(\tfrac{\rho_2 t^{p_2}+\rho_6}{\Lambda}\Big)\Bigg(\int_0^{\eta_j} \tfrac{(\eta_j-qu)^{(p_1+p_2-1)}}{\Gamma_q(p_1+p_2)} h_1(u) d_q u - \lambda_1 \int_0^{\eta_j} \tfrac{(\eta_j-qu)^{(p_2-1)}}{\Gamma_q(p_2)} x(u) d_q u \Bigg) \\ &- \delta_3\Big(\tfrac{\rho_3 t^{p_2}+\rho_7}{\Lambda}\Big)\Bigg(\int_0^{\eta_j} \tfrac{(\eta_j-qu)^{(r_1+r_2-2)}}{\Gamma_q(r_1+r_2-1)} h_2(u) d_q u - \lambda_2 \int_0^{\eta_j} \tfrac{(\eta_j-qu)^{(r_2-2)}}{\Gamma_q(r_2-1)} y(u) d_q u \Bigg) \\ &+ \sigma_1\Big(\tfrac{\rho_3 t^{p_2}+\rho_7}{\Lambda}\Big)\Bigg(\int_0^1 \tfrac{(1-qu)^{(p_1+p_2-1)}}{\Gamma_q(p_1+p_2)} h_1(u) d_q u - \lambda_1 \int_0^1 \tfrac{(1-qu)^{(p_2-1)}}{\Gamma_q(p_2)} x(u) d_q u \Bigg) \\ &+ \sigma_2\Big(\tfrac{\rho_3 t^{p_2}+\rho_7}{\Lambda}\Big)\Bigg(\int_0^1 \tfrac{(1-qu)^{(p_1+p_2-2)}}{\Gamma_q(p_1+p_2-1)} h_1(u) d_q u - \lambda_1 \int_0^1 \tfrac{(1-qu)^{(p_2-2)}}{\Gamma_q(p_2-1)} x(u) d_q u \Bigg) \\ &- \delta_4\Big(\tfrac{\rho_4 t^{p_2}+\rho_8}{\Lambda}\Big)\Bigg(\int_0^{\eta_j} \tfrac{(\eta_j-qu)^{(p_1+p_2-2)}}{\Gamma_q(p_1+p_2-1)} h_1(u) d_q u - \lambda_1 \int_0^{\eta_j} \tfrac{(\eta_j-qu)^{(p_2-2)}}{\Gamma_q(p_2-1)} x(u) d_q u \Bigg) \\ &+ \sigma_3\Big(\tfrac{\rho_4 t^{p_2}+\rho_8}{\Lambda}\Big)\Bigg(\int_0^1 \tfrac{(1-qu)^{(r_1+r_2-1)}}{\Gamma_q(r_1+r_2)} h_2(u) d_q u - \lambda_2 \int_0^1 \tfrac{(1-qu)^{(r_2-1)}}{\Gamma_q(r_2)} y(u) d_q u \Bigg) \\ &+ \sigma_4\Big(\tfrac{\rho_4 t^{p_2}+\rho_8}{\Lambda}\Big)\Bigg(\int_0^1 \tfrac{(1-qu)^{(r_1+r_2-2)}}{\Gamma_q(r_1+r_2-1)} h_2(u) d_q u - \lambda_2 \int_0^1 \tfrac{(1-qu)^{(r_2-2)}}{\Gamma_q(r_2-1)} y(u) d_q u \Bigg), \end{aligned} \tag{5}$$

and

$$
\begin{aligned}
y(t) \;=\;& \int_0^t \frac{(t-qu)^{(r_1+r_2-1)}}{\Gamma_q(r_1+r_2)} h_2(u)d_qu - \lambda_2 \int_0^t \frac{(t-qu)^{(r_2-1)}}{\Gamma_q(r_2)} y(u)d_qu \\
-\;& \delta_1\left(\frac{\rho_9 t^{r_2}+\rho_{13}}{\Lambda}\right)\left(\int_0^{\eta_j} \frac{(\eta_j-qu)^{(r_1+r_2-1)}}{\Gamma_q(r_1+r_2)} h_2(u)d_qu - \lambda_2 \int_0^{\eta_j} \frac{(\eta_j-qu)^{(r_2-1)}}{\Gamma_q(r_2)} y(u)d_qu\right) \\
-\;& \delta_2\left(\frac{\rho_{10} t^{r_2}+\rho_{14}}{\Lambda}\right)\left(\int_0^{\eta_j} \frac{(\eta_j-qu)^{(p_1+p_2-1)}}{\Gamma_q(p_1+p_2)} h_1(u)d_qu - \lambda_1 \int_0^{\eta_j} \frac{(\eta_j-qu)^{(p_2-1)}}{\Gamma_q(p_2)} x(u)d_qu\right) \\
-\;& \delta_3\left(\frac{\rho_{11} t^{r_2}+\rho_{15}}{\Lambda}\right)\left(\int_0^{\eta_j} \frac{(\eta_j-qu)^{(r_1+r_2-2)}}{\Gamma_q(r_1+r_2-1)} h_2(u)d_qu - \lambda_2 \int_0^{\eta_j} \frac{(\eta_j-qu)^{(r_2-2)}}{\Gamma_q(r_2-1)} y(u)d_qu\right) \\
+\;& \sigma_1\left(\frac{\rho_{11} t^{r_2}+\rho_{15}}{\Lambda}\right)\left(\int_0^{1} \frac{(1-qu)^{(p_1+p_2-1)}}{\Gamma_q(p_1+p_2)} h_1(u)d_qu - \lambda_1 \int_0^{1} \frac{(1-qu)^{(p_2-1)}}{\Gamma_q(p_2)} x(u)d_qu\right) \\
+\;& \sigma_2\left(\frac{\rho_{11} t^{r_2}+\rho_{15}}{\Lambda}\right)\left(\int_0^{1} \frac{(1-qu)^{(p_1+p_2-2)}}{\Gamma_q(p_1+p_2-1)} h_1(u)d_qu - \lambda_1 \int_0^{1} \frac{(1-qu)^{(p_2-2)}}{\Gamma_q(p_2-1)} x(u)d_qu\right) \\
-\;& \delta_4\left(\frac{\rho_{12} t^{r_2}+\rho_{16}}{\Lambda}\right)\left(\int_0^{\eta_j} \frac{(\eta_j-qu)^{(p_1+p_2-2)}}{\Gamma_q(p_1+p_2-1)} h_1(u)d_qu - \lambda_1 \int_0^{\eta_j} \frac{(\eta_j-qu)^{(p_2-2)}}{\Gamma_q(p_2-1)} x(u)d_qu\right) \\
+\;& \sigma_3\left(\frac{\rho_{12} t^{r_2}+\rho_{16}}{\Lambda}\right)\left(\int_0^{1} \frac{(1-qu)^{(r_1+r_2-1)}}{\Gamma_q(r_1+r_2)} h_2(u)d_qu - \lambda_2 \int_0^{1} \frac{(1-qu)^{(r_2-1)}}{\Gamma_q(r_2)} y(u)d_qu\right) \\
+\;& \sigma_4\left(\frac{\rho_{12} t^{r_2}+\rho_{16}}{\Lambda}\right)\left(\int_0^{1} \frac{(1-qu)^{(r_1+r_2-2)}}{\Gamma_q(r_1+r_2-1)} h_2(u)d_qu - \lambda_2 \int_0^{1} \frac{(1-qu)^{(r_2-2)}}{\Gamma_q(r_2-1)} y(u)d_qu\right),
\end{aligned}
\tag{6}
$$

where

$$
\begin{aligned}
\Lambda \;=\;& (\delta_1\delta_2-\mu_1\mu_3)(\sigma_1+\sigma_2[p_2]_q)(\sigma_3+\sigma_4[r_2]_q) - \sigma_1(\delta_1\delta_6+\mu_2\mu_3[p_2]_q)(\sigma_3+\sigma_4[r_2]_q) \\
& +(\mu_1\mu_3\delta_7\delta_8-\mu_2\delta_2\delta_7\sigma_3-\mu_2\mu_4\sigma_1\sigma_3-\delta_1\delta_2\delta_7\delta_8-\mu_4\delta_1\delta_8\sigma_1)[p_2]_q[r_2]_q \\
& -\sigma_3(\delta_2\delta_5+\mu_1\mu_4[r_2]_q)(\sigma_1+\sigma_2[p_2]_q) - \mu_3\delta_5\delta_8\sigma_1[p_2]_q - \mu_1\delta_6\delta_7\sigma_3[r_2]_q + \delta_5\delta_6\sigma_1\sigma_3,
\end{aligned}
\tag{7}
$$

$$
\begin{cases}
\delta_1 = \sum_{j=1}^{n} a_j, \quad \delta_2 = \sum_{j=1}^{n} b_j, \quad \delta_3 = \sum_{j=1}^{n} k_j, \quad \delta_4 = \sum_{j=1}^{n} m_j, \\
\delta_5 = \sum_{j=1}^{n} a_j\eta_j^{r_2}, \quad \delta_6 = \sum_{j=1}^{n} b_j\eta_j^{p_2}, \quad \delta_7 = \sum_{j=1}^{n} k_j\eta_j^{r_2-1}, \quad \delta_8 = \sum_{j=1}^{n} m_j\eta_j^{p_2-1}, \\
\rho_1 = -\mu_3\sigma_1(\sigma_3+\sigma_4[r_2]_q) - (\delta_2\delta_7\sigma_3+\mu_4\sigma_1\sigma_3)[r_2]_q, \\
\rho_2 = -\delta_1\sigma_1(\sigma_3+\sigma_4[r_2]_q) - \mu_1\delta_7\sigma_3[r_2]_q + \delta_5\sigma_1\sigma_3, \\
\rho_3 = (\mu_1\mu_3-\delta_1\delta_2)(\sigma_3+\sigma_4[r_2]_q) + \mu_1\mu_4\sigma_3[r_2]_q + \delta_2\delta_5\sigma_3, \\
\rho_4 = (\mu_1\mu_3\delta_7-\delta_1\delta_2\delta_7-\mu_4\delta_1\sigma_1)[r_2]_q - \mu_3\delta_5\sigma_1, \\
\rho_5 = \mu_3(\sigma_1+\sigma_2[p_2]_q)(\sigma_3+\sigma_4[r_2]_q) + \mu_4\sigma_3[r_2]_q(\sigma_1+\sigma_2[p_2]_q) - \mu_3\delta_7\delta_8[p_2]_q[r_2]_q + \delta_6\delta_7\sigma_3[r_2]_q, \\
\rho_6 = \delta_1(\sigma_1+\sigma_2[p_2]_q)(\sigma_3+\sigma_4[r_2]_q) - \delta_5\sigma_3(\sigma_1+\sigma_2[p_2]_q) - (\delta_1\delta_7\delta_8+\mu_2\delta_7\sigma_3)[p_2]_q[r_2]_q, \\
\rho_7 = (\delta_1\delta_6+\mu_2\mu_3[p_2]_q)(\sigma_3+\sigma_4[r_2]_q) + (\mu_4\delta_1\delta_8+\mu_2\mu_4\sigma_3)[p_2]_q[r_2]_q + \mu_3\delta_5\delta_8[p_2]_q - \delta_5\delta_6\sigma_3, \\
\rho_8 = (\mu_3\delta_5+\mu_4\delta_1[r_2]_q)(\sigma_1+\sigma_2[p_2]_q) + \mu_2\mu_3\delta_7[p_2]_q[r_2]_q + \delta_1\delta_6\delta_7[r_2]_q, \\
\rho_9 = -\delta_2\sigma_3(\sigma_1+\sigma_2[p_2]_q) - \mu_3\delta_8\sigma_1[p_2]_q + \delta_6\sigma_1\sigma_3, \\
\rho_{10} = -\mu_1\sigma_3(\sigma_1+\sigma_2[p_2]_q) - (\mu_2\sigma_1\sigma_3+\delta_1\delta_8\sigma_1)[p_2]_q, \\
\rho_{11} = (\mu_1\mu_3\delta_8-\delta_1\delta_2\delta_8-\mu_2\delta_2\sigma_3)[p_2]_q - \mu_1\delta_6\sigma_3, \\
\rho_{12} = (\mu_1\mu_3-\delta_1\delta_2)(\sigma_1+\sigma_2[p_2]_q) + \mu_2\mu_3\sigma_1[p_2]_q + \delta_1\delta_6\sigma_1, \\
\rho_{13} = \delta_2(\sigma_1+\sigma_2[p_2]_q)(\sigma_3+\sigma_4[r_2]_q) - \delta_6\sigma_1(\sigma_3+\sigma_4[r_2]_q) - (\delta_2\delta_7\delta_8+\mu_4\delta_8\sigma_1)[p_2]_q[r_2]_q, \\
\rho_{14} = \mu_1(\sigma_1+\sigma_2[p_2]_q)(\sigma_3+\sigma_4[r_2]_q) + \mu_2\sigma_1[p_2]_q(\sigma_3+\sigma_4[r_2]_q) - \mu_1\delta_7\delta_8[p_2]_q[r_2]_q + \delta_5\delta_8\sigma_1[p_2]_q, \\
\rho_{15} = (\mu_1\delta_6+\mu_2\delta_2[p_2]_q)(\sigma_3+\sigma_4[r_2]_q) + \mu_1\mu_4\delta_8[p_2]_q[r_2]_q + \delta_2\delta_5\delta_8[p_2]_q, \\
\rho_{16} = (\delta_2\delta_5+\mu_1\mu_4[r_2]_q)(\sigma_1+\sigma_2[p_2]_q) + (\mu_2\delta_2\delta_7+\mu_2\mu_4\sigma_1)[p_2]_q[r_2]_q + \mu_1\delta_6\delta_7[r_2]_q - \delta_5\delta_6\sigma_1.
\end{cases}
\tag{8}
$$

Proof. Applying the q-integral operators $I_q^{p_1}$ and $I_q^{r_1}$, respectively, on the first and second equations of (4), we obtain

$$
({}^cD_q^{p_2}+\lambda_1)x(t) = I_q^{p_1}h_1(t) - c_0, \;\; ({}^cD_q^{r_2}+\lambda_2)y(t) = I_q^{r_1}h_2(t) - d_0,
$$

where c_0 and d_0 are arbitrary real constants. Now, applying the q-integral operators $I_q^{p_2}$ and $I_q^{r_2}$, respectively, to both sides of the above equations, we obtain

$$x(t) = \int_0^t \frac{(t-qu)^{(p_1+p_2-1)}}{\Gamma_q(p_1+p_2)} h_1(u) d_q u - \lambda_1 \int_0^t \frac{(t-qu)^{(p_2-1)}}{\Gamma_q(p_2)} x(u) d_q u - c_0 \frac{t^{p_2}}{\Gamma_q(p_2+1)} - c_1, \tag{9}$$

$$y(t) = \int_0^t \frac{(t-qu)^{(r_1+r_2-1)}}{\Gamma_q(r_1+r_2)} h_2(u) d_q u - \lambda_2 \int_0^t \frac{(t-qu)^{(r_2-1)}}{\Gamma_q(r_2)} y(u) d_q u - d_0 \frac{t^{r_2}}{\Gamma_q(r_2+1)} - d_1, \tag{10}$$

where $c_1, d_1 \in \mathbb{R}$ are arbitrary constants. By using the conditions (2), we obtain a system of equations in the unknown constants c_0, c_1, d_0 and d_1 given by

$$\begin{cases} \dfrac{\mu_2 [p_2]_q}{\Gamma_q(p_2+1)} c_0 - \mu_1 c_1 + \dfrac{\delta_5}{\Gamma_q(r_2+1)} d_0 + \delta_1 d_1 = F_1, \\ \dfrac{\delta_6}{\Gamma_q(p_2+1)} c_0 + \delta_2 c_1 + \dfrac{\mu_4 [r_2]_q}{\Gamma_q(r_2+1)} d_0 - \mu_3 d_1 = F_2, \\ -\dfrac{(\sigma_1 + \sigma_2 [p_2]_q)}{\Gamma_q(p_2+1)} c_0 - \sigma_1 c_1 + \dfrac{\delta_7 [r_2]_q}{\Gamma_q(r_2+1)} d_0 = F_3, \\ \dfrac{\delta_8 [p_2]_q}{\Gamma_q(p_2+1)} c_0 - \dfrac{(\sigma_3 + \sigma_4 [r_2]_q)}{\Gamma_q(r_2+1)} d_0 - \sigma_3 d_1 = F_4, \end{cases} \tag{11}$$

where $\delta_1, \delta_2, \delta_5, \delta_6, \delta_7, \delta_8$ are given in (8), and

$$\begin{aligned} F_1 &= \delta_1 \Big(\int_0^{\eta_j} \frac{(\eta_j - qu)^{(r_1+r_2-1)}}{\Gamma_q(r_1+r_2)} h_2(u) d_q u - \lambda_2 \int_0^{\eta_j} \frac{(\eta_j - qu)^{(r_2-1)}}{\Gamma_q(r_2)} y(u) d_q u \Big), \\ F_2 &= \delta_2 \Big(\int_0^{\eta_j} \frac{(\eta_j - qu)^{(p_1+p_2-1)}}{\Gamma_q(p_1+p_2)} h_1(u) d_q u - \lambda_1 \int_0^{\eta_j} \frac{(\eta_j - qu)^{(p_2-1)}}{\Gamma_q(p_2)} x(u) d_q u \Big), \end{aligned}$$

$$\begin{aligned} F_3 &= \delta_3 \Big(\int_0^{\eta_j} \frac{(\eta_j - qu)^{(r_1+r_2-2)}}{\Gamma_q(r_1+r_2-1)} h_2(u) d_q u - \lambda_2 \int_0^{\eta_j} \frac{(\eta_j - qu)^{(r_2-2)}}{\Gamma_q(r_2-1)} y(u) d_q u \Big) \\ &\quad - \sigma_1 \Big(\int_0^1 \frac{(1-qu)^{(p_1+p_2-1)}}{\Gamma_q(p_1+p_2)} h_1(u) d_q u - \lambda_1 \int_0^1 \frac{(1-qu)^{(p_2-1)}}{\Gamma_q(p_2)} x(u) d_q u \Big) \\ &\quad - \sigma_2 \Big(\int_0^1 \frac{(1-qu)^{(p_1+p_2-2)}}{\Gamma_q(p_1+p_2-1)} h_1(u) d_q u - \lambda_1 \int_0^1 \frac{(1-qu)^{(p_2-2)}}{\Gamma_q(p_2-1)} x(u) d_q u \Big), \end{aligned}$$

$$\begin{aligned} F_4 &= \delta_4 \Big(\int_0^{\eta_j} \frac{(\eta_j - qu)^{(p_1+p_2-2)}}{\Gamma_q(p_1+p_2-1)} h_1(u) d_q u - \lambda_1 \int_0^{\eta_j} \frac{(\eta_j - qu)^{(p_2-2)}}{\Gamma_q(p_2-1)} x(u) d_q u \Big) \\ &\quad - \sigma_3 \Big(\int_0^1 \frac{(1-qu)^{(r_1+r_2-1)}}{\Gamma_q(r_1+r_2)} h_2(u) d_q u - \lambda_2 \int_0^1 \frac{(1-qu)^{(r_2-1)}}{\Gamma_q(r_2)} y(u) d_q u \Big) \\ &\quad - \sigma_4 \Big(\int_0^1 \frac{(1-qu)^{(r_1+r_2-2)}}{\Gamma_q(r_1+r_2-1)} h_2(u) d_q u - \lambda_2 \int_0^1 \frac{(1-qu)^{(r_2-2)}}{\Gamma_q(r_2-1)} y(u) d_q u \Big). \end{aligned}$$

Solving the system (11) for c_0, c_1, d_0 and d_1, we find that

$$\begin{aligned} c_0 &= \frac{\Gamma_q(p_2+1)}{\Lambda} \Big(\rho_1 F_1 + \rho_2 F_2 + \rho_3 F_3 + \rho_4 F_4 \Big), \; c_1 = \frac{1}{\Lambda} \Big(\rho_5 F_1 + \rho_6 F_2 + \rho_7 F_3 + \rho_8 F_4 \Big), \\ d_0 &= \frac{\Gamma_q(r_2+1)}{\Lambda} \Big(\rho_9 F_1 + \rho_{10} F_2 + \rho_{11} F_3 + \rho_{12} F_4 \Big), \; d_1 = \frac{1}{\Lambda} \Big(\rho_{13} F_1 + \rho_{14} F_2 + \rho_{15} F_3 + \rho_{16} F_4 \Big), \end{aligned}$$

where Λ is given by (7). Substituting the values of c_0, c_1, d_0 and d_1 in (9) and (10) yields the solution (5) and (6). By direct computation, one can obtain the converse of the lemma. This completes the proof. □

Let $\mathcal{C} = \{x | x \in C([0,1], \mathbb{R})\}$ be the space equipped with the norm $\|x\| = \sup_{t\in[0,1]} |x(t)|$. Obviously, $(\mathcal{C}, \|.\|)$ is a Banach space. Then, the product space $(\mathcal{C} \times \mathcal{C}, \|.\|)$ is also a Banach space with the norm $\|(x,y)\| = \|x\| + \|y\|$ for $(x,y) \in \mathcal{C} \times \mathcal{C}$.

In view of Lemma 5, we define an operator $\mathcal{G} : \mathcal{C} \times \mathcal{C} \to \mathcal{C} \times \mathcal{C}$ by

$$\mathcal{G}(x,y)(t) = \begin{pmatrix} \mathcal{G}_1(x,y)(t) \\ \mathcal{G}_2(x,y)(t) \end{pmatrix}, \tag{12}$$

where

$$\begin{aligned}
\mathcal{G}_1(x,y)(t) &= \alpha_1 \int_0^t \frac{(t-qu)^{(p_1+p_2-1)}}{\Gamma_q(p_1+p_2)} f_1(u,x(u),y(u))d_qu \\
&+ \beta_1 \int_0^t \frac{(t-qu)^{(p_1+p_2+\xi_1-1)}}{\Gamma_q(p_1+p_2+\xi)} g_1(u,x(u),y(u))d_qu - \lambda_1 \int_0^t \frac{(t-qu)^{(p_2-1)}}{\Gamma_q(p_2)} x(u)d_qu \\
&- \delta_1 \Big(\frac{\rho_1 t^{p_2}+\rho_5}{\Lambda}\Big)\Bigg(\alpha_2 \int_0^{\eta_j} \frac{(\eta_j-qu)^{(r_1+r_2-1)}}{\Gamma_q(r_1+r_2)} f_2(u,x(u),y(u))d_qu \\
&+ \beta_2 \int_0^{\eta_j} \frac{(\eta_j-qu)^{(r_1+r_2+\xi_2-1)}}{\Gamma_q(r_1+r_2+\xi_2)} g_2(u,x(u),y(u))d_qu - \lambda_2 \int_0^{\eta_j} \frac{(\eta_j-qu)^{(r_2-1)}}{\Gamma_q(r_2)} y(u)d_qu\Bigg) \\
&- \delta_2 \Big(\frac{\rho_2 t^{p_2}+\rho_6}{\Lambda}\Big)\Bigg(\alpha_1 \int_0^{\eta_j} \frac{(\eta_j-qu)^{(p_1+p_2-1)}}{\Gamma_q(p_1+p_2)} f_1(u,x(u),y(u))d_qu \\
&+ \beta_1 \int_0^{\eta_j} \frac{(\eta_j-qu)^{(p_1+p_2+\xi_1-1)}}{\Gamma_q(p_1+p_2+\xi_1)} g_1(u,x(u),y(u))d_qu - \lambda_1 \int_0^{\eta_j} \frac{(\eta_j-qu)^{(p_2-1)}}{\Gamma_q(p_2)} x(u)d_qu\Bigg) \\
&- \delta_3 \Big(\frac{\rho_3 t^{p_2}+\rho_7}{\Lambda}\Big)\Bigg(\alpha_2 \int_0^{\eta_j} \frac{(\eta_j-qu)^{(r_1+r_2-2)}}{\Gamma_q(r_1+r_2-1)} f_2(u,x(u),y(u))d_qu \\
&+ \beta_2 \int_0^{\eta_j} \frac{(\eta_j-qu)^{(r_1+r_2+\xi_2-2)}}{\Gamma_q(r_1+r_2+\xi_2-1)} g_2(u,x(u),y(u))d_qu - \lambda_2 \int_0^{\eta_j} \frac{(\eta_j-qu)^{(r_2-2)}}{\Gamma_q(r_2-1)} y(u)d_qu\Bigg) \\
&+ \sigma_1 \Big(\frac{\rho_3 t^{p_2}+\rho_7}{\Lambda}\Big)\Bigg(\alpha_1 \int_0^{1} \frac{(1-qu)^{(p_1+p_2-1)}}{\Gamma_q(p_1+p_2)} f_1(u,x(u),y(u))d_qu \\
&+ \beta_1 \int_0^{1} \frac{(1-qu)^{(p_1+p_2+\xi_1-1)}}{\Gamma_q(p_1+p_2+\xi_1)} g_1(u,x(u),y(u))d_qu - \lambda_1 \int_0^{1} \frac{(1-qu)^{(p_2-1)}}{\Gamma_q(p_2)} x(u)d_qu\Bigg) \\
&+ \sigma_2 \Big(\frac{\rho_3 t^{p_2}+\rho_7}{\Lambda}\Big)\Bigg(\alpha_1 \int_0^{1} \frac{(1-qu)^{(p_1+p_2-2)}}{\Gamma_q(p_1+p_2-1)} f_1(u,x(u),y(u))d_qu \\
&+ \beta_1 \int_0^{1} \frac{(1-qu)^{(p_1+p_2+\xi_1-2)}}{\Gamma_q(p_1+p_2+\xi_1-1)} g_1(u,x(u),y(u))d_qu - \lambda_1 \int_0^{1} \frac{(1-qu)^{(p_2-2)}}{\Gamma_q(p_2-1)} x(u)d_qu\Bigg) \\
&- \delta_4 \Big(\frac{\rho_4 t^{p_2}+\rho_8}{\Lambda}\Big)\Bigg(\alpha_1 \int_0^{\eta_j} \frac{(\eta_j-qu)^{(p_1+p_2-2)}}{\Gamma_q(p_1+p_2-1)} f_1(u,x(u),y(u))d_qu \\
&+ \beta_1 \int_0^{\eta_j} \frac{(\eta_j-qu)^{(p_1+p_2+\xi_1-2)}}{\Gamma_q(p_1+p_2+\xi_1-1)} g_1(u,x(u),y(u))d_qu - \lambda_1 \int_0^{\eta_j} \frac{(\eta_j-qu)^{(p_2-2)}}{\Gamma_q(p_2-1)} x(u)d_qu\Bigg) \\
&+ \sigma_3 \Big(\frac{\rho_4 t^{p_2}+\rho_8}{\Lambda}\Big)\Bigg(\alpha_2 \int_0^{1} \frac{(1-qu)^{(r_1+r_2-1)}}{\Gamma_q(r_1+r_2)} f_2(u,x(u),y(u))d_qu \\
&+ \beta_2 \int_0^{1} \frac{(1-qu)^{(r_1+r_2+\xi_2-1)}}{\Gamma_q(r_1+r_2+\xi_2)} g_2(u,x(u),y(u))d_qu - \lambda_2 \int_0^{1} \frac{(1-qu)^{(r_2-1)}}{\Gamma_q(r_2)} y(u)d_qu\Bigg) \\
&+ \sigma_4 \Big(\frac{\rho_4 t^{p_2}+\rho_8}{\Lambda}\Big)\Bigg(\alpha_2 \int_0^{1} \frac{(1-qu)^{(r_1+r_2-2)}}{\Gamma_q(r_1+r_2-1)} f_2(u,x(u),y(u))d_qu \\
&+ \beta_2 \int_0^{1} \frac{(1-qu)^{(r_1+r_2+\xi_2-2)}}{\Gamma_q(r_1+r_2+\xi_2-1)} g_2(u,x(u),y(u))d_qu - \lambda_2 \int_0^{1} \frac{(1-qu)^{(r_2-2)}}{\Gamma_q(r_2-1)} y(u)d_qu\Bigg),
\end{aligned}$$

$$
\begin{aligned}
\mathcal{G}_2(x,y)(t) &= \alpha_2\int_0^t \frac{(t-qu)^{(r_1+r_2-1)}}{\Gamma_q(r_1+r_2)}f_2(u,x(u),y(u))d_qu \\
&+ \beta_2\int_0^t \frac{(t-qu)^{(r_1+r_2+\xi_2-1)}}{\Gamma_q(r_1+r_2+\xi_2)}g_2(u,x(u),y(u))d_qu - \lambda_2\int_0^t \frac{(t-qu)^{(r_2-1)}}{\Gamma_q(r_2)}y(u)d_qu \\
&- \delta_1\Big(\frac{\rho_9 t^{r_2}+\rho_{13}}{\Lambda}\Big)\bigg(\alpha_2\int_0^{\eta_j}\frac{(\eta_j-qu)^{(r_1+r_2-1)}}{\Gamma_q(r_1+r_2)}f_2(u,x(u),y(u))d_qu \\
&+ \beta_2\int_0^{\eta_j}\frac{(\eta_j-qu)^{(r_1+r_2+\xi_2-1)}}{\Gamma_q(r_1+r_2+\xi_2)}g_2(u,x(u),y(u))d_qu - \lambda_2\int_0^{\eta_j}\frac{(\eta_j-qu)^{(r_2-1)}}{\Gamma_q(r_2)}y(u)d_qu\bigg) \\
&- \delta_2\Big(\frac{\rho_{10} t^{r_2}+\rho_{14}}{\Lambda}\Big)\bigg(\alpha_1\int_0^{\eta_j}\frac{(\eta_j-qu)^{(p_1+p_2-1)}}{\Gamma_q(p_1+p_2)}f_1(u,x(u),y(u))d_qu \\
&+ \beta_1\int_0^{\eta_j}\frac{(\eta_j-qu)^{(p_1+p_2+\xi_1-1)}}{\Gamma_q(p_1+p_2+\xi_1)}g_1(u,x(u),y(u))d_qu - \lambda_1\int_0^{\eta_j}\frac{(\eta_j-qu)^{(p_2-1)}}{\Gamma_q(p_2)}x(u)d_qu\bigg) \\
&- \delta_3\Big(\frac{\rho_{11} t^{r_2}+\rho_{15}}{\Lambda}\Big)\bigg(\alpha_2\int_0^{\eta_j}\frac{(\eta_j-qu)^{(r_1+r_2-2)}}{\Gamma_q(r_1+r_2-1)}f_2(u,x(u),y(u))d_qu \\
&+ \beta_2\int_0^{\eta_j}\frac{(\eta_j-qu)^{(r_1+r_2+\xi_2-2)}}{\Gamma_q(r_1+r_2+\xi_2-1)}g_2(u,x(u),y(u))d_qu - \lambda_2\int_0^{\eta_j}\frac{(\eta_j-qu)^{(r_2-2)}}{\Gamma_q(r_2-1)}y(u)d_qu\bigg) \\
&+ \sigma_1\Big(\frac{\rho_{11} t^{r_2}+\rho_{15}}{\Lambda}\Big)\bigg(\alpha_1\int_0^{1}\frac{(1-qu)^{(p_1+p_2-1)}}{\Gamma_q(p_1+p_2)}f_1(u,x(u),y(u))d_qu \\
&+ \beta_1\int_0^{1}\frac{(1-qu)^{(p_1+p_2+\xi_1-1)}}{\Gamma_q(p_1+p_2+\xi_1)}g_1(u,x(u),y(u))d_qu - \lambda_1\int_0^{1}\frac{(1-qu)^{(p_2-1)}}{\Gamma_q(p_2)}x(u)d_qu\bigg) \\
&+ \sigma_2\Big(\frac{\rho_{11} t^{r_2}+\rho_{15}}{\Lambda}\Big)\bigg(\alpha_1\int_0^{1}\frac{(1-qu)^{(p_1+p_2-2)}}{\Gamma_q(p_1+p_2-1)}f_1(u,x(u),y(u))d_qu \\
&+ \beta_1\int_0^{1}\frac{(1-qu)^{(p_1+p_2+\xi_1-2)}}{\Gamma_q(p_1+p_2+\xi_1-1)}g_1(u,x(u),y(u))d_qu - \lambda_1\int_0^{1}\frac{(1-qu)^{(p_2-2)}}{\Gamma_q(p_2-1)}x(u)d_qu\bigg) \\
&- \delta_4\Big(\frac{\rho_{12} t^{r_2}+\rho_{16}}{\Lambda}\Big)\bigg(\alpha_1\int_0^{\eta_j}\frac{(\eta_j-qu)^{(p_1+p_2-2)}}{\Gamma_q(p_1+p_2-1)}f_1(u,x(u),y(u))d_qu \\
&+ \beta_1\int_0^{\eta_j}\frac{(\eta_j-qu)^{(p_1+p_2+\xi_1-2)}}{\Gamma_q(p_1+p_2+\xi_1-1)}g_1(u,x(u),y(u))d_qu - \lambda_1\int_0^{\eta_j}\frac{(\eta_j-qu)^{(p_2-2)}}{\Gamma_q(p_2-1)}x(u)d_qu\bigg) \\
&+ \sigma_3\Big(\frac{\rho_{12} t^{r_2}+\rho_{16}}{\Lambda}\Big)\bigg(\alpha_2\int_0^{1}\frac{(1-qu)^{(r_1+r_2-1)}}{\Gamma_q(r_1+r_2)}f_2(u,x(u),y(u))d_qu \\
&+ \beta_2\int_0^{1}\frac{(1-qu)^{(r_1+r_2+\xi_2-1)}}{\Gamma_q(r_1+r_2+\xi_2)}g_2(u,x(u),y(u))d_qu - \lambda_2\int_0^{1}\frac{(1-qu)^{(r_2-1)}}{\Gamma_q(r_2)}y(u)d_qu\bigg) \\
&+ \sigma_4\Big(\frac{\rho_{12} t^{r_2}+\rho_{16}}{\Lambda}\Big)\bigg(\alpha_2\int_0^{1}\frac{(1-qu)^{(r_1+r_2-2)}}{\Gamma_q(r_1+r_2-1)}f_2(u,x(u),y(u))d_qu \\
&+ \beta_2\int_0^{1}\frac{(1-qu)^{(r_1+r_2+\xi_2-2)}}{\Gamma_q(r_1+r_2+\xi_2-1)}g_2(u,x(u),y(u))d_qu - \lambda_2\int_0^{1}\frac{(1-qu)^{(r_2-2)}}{\Gamma_q(r_2-1)}y(u)d_qu\bigg).
\end{aligned}
$$

3. Existence and Uniqueness Results

In the sequel, we set the notation

$$
\begin{cases}
\Psi_1 = \frac{|\alpha_1|}{\Gamma_q(p_1+p_2+1)} + \frac{|\alpha_1|(\gamma_1+\gamma_2)}{|\Lambda|},\ \Psi_2 = \frac{|\alpha_2|(\gamma_3+\gamma_4)}{|\Lambda|},\ \Psi_3 = \frac{|\alpha_1|(\gamma_{13}+\gamma_{14})}{|\Lambda|}, \\
\Psi_4 = \frac{|\alpha_2|}{\Gamma_q(r_1+r_2+1)} + \frac{|\alpha_2|(\gamma_{15}+\gamma_{16})}{|\Lambda|},\ \Phi_1 = \frac{|\beta_1|}{\Gamma_q(p_1+p_2+\xi_1+1)} + \frac{|\beta_1|(\gamma_5+\gamma_6)}{|\Lambda|}, \\
\Phi_2 = \frac{|\beta_2|(\gamma_7+\gamma_8)}{|\Lambda|},\ \Phi_3 = \frac{|\beta_1|(\gamma_{17}+\gamma_{18})}{|\Lambda|},\ \Phi_4 = \frac{|\beta_2|}{\Gamma_q(r_1+r_2+\xi_2+1)} + \frac{|\beta_2|(\gamma_{19}+\gamma_{20})}{|\Lambda|}, \\
\Theta_1 = \frac{|\lambda_1|}{\Gamma_q(p_2+1)} + \frac{|\lambda_1|(\gamma_9+\gamma_{10})}{|\Lambda|} + \frac{|\lambda_2|(\gamma_{11}+\gamma_{12})}{|\Lambda|}, \\
\Theta_2 = \frac{|\lambda_1|(\gamma_{21}+\gamma_{22})}{|\Lambda|} + \frac{|\lambda_2|}{\Gamma_q(r_2+1)} + \frac{|\lambda_2|(\gamma_{23}+\gamma_{24})}{|\Lambda|},
\end{cases}
\tag{13}
$$

$$\begin{cases} \gamma_1 = \dfrac{|\delta_2||\rho_2+\rho_6|\eta_j^{p_1+p_2} + |\sigma_1||\rho_3+\rho_7|}{\Gamma_q(p_1+p_2+1)}, & \gamma_2 = \dfrac{|\delta_4||\rho_4+\rho_8|\eta_j^{p_1+p_2-1} + |\sigma_2||\rho_3+\rho_7|}{\Gamma_q(p_1+p_2)}, \\ \gamma_3 = \dfrac{|\delta_1||\rho_1+\rho_5|\eta_j^{r_1+r_2} + |\sigma_3||\rho_4+\rho_8|}{\Gamma_q(r_1+r_2+1)}, & \gamma_4 = \dfrac{|\delta_3||\rho_3+\rho_7|\eta_j^{r_1+r_2-1} + |\sigma_4||\rho_4+\rho_8|}{\Gamma_q(r_1+r_2)}, \\ \gamma_5 = \dfrac{|\delta_2||\rho_2+\rho_6|\eta_j^{p_1+p_2+\xi_1} + |\sigma_1||\rho_3+\rho_7|}{\Gamma_q(p_1+p_2+\xi_1+1)}, & \gamma_6 = \dfrac{|\delta_4||\rho_4+\rho_8|\eta_j^{p_1+p_2+\xi_1-1} + |\sigma_2||\rho_3+\rho_7|}{\Gamma_q(p_1+p_2+\xi_1)}, \\ \gamma_7 = \dfrac{|\delta_1||\rho_1+\rho_5|\eta_j^{r_1+r_2+\xi_2} + |\sigma_3||\rho_4+\rho_8|}{\Gamma_q(r_1+r_2+\xi_2+1)}, & \gamma_8 = \dfrac{|\delta_3||\rho_3+\rho_7|\eta_j^{r_1+r_2+\xi_2-1} + |\sigma_4||\rho_4+\rho_8|}{\Gamma_q(r_1+r_2+\xi_2)}, \\ \gamma_9 = \dfrac{|\delta_2||\rho_2+\rho_6|\eta_j^{p_2} + |\sigma_1||\rho_3+\rho_7|}{\Gamma_q(p_2+1)}, & \gamma_{10} = \dfrac{|\delta_4||\rho_4+\rho_8|\eta_j^{p_2-1} + |\sigma_2||\rho_3+\rho_7|}{\Gamma_q(p_2)}, \\ \gamma_{11} = \dfrac{|\delta_1||\rho_1+\rho_5|\eta_j^{r_2} + |\sigma_3||\rho_4+\rho_8|}{\Gamma_q(r_2+1)}, & \gamma_{12} = \dfrac{|\delta_3||\rho_3+\rho_7|\eta_j^{r_2-1} + |\sigma_4||\rho_4+\rho_8|}{\Gamma_q(r_2)}, \\ \gamma_{13} = \dfrac{|\delta_2||\rho_{10}+\rho_{14}|\eta_j^{p_1+p_2} + |\sigma_1||\rho_{11}+\rho_{15}|}{\Gamma_q(p_1+p_2+1)}, & \gamma_{14} = \dfrac{|\delta_4||\rho_{12}+\rho_{16}|\eta_j^{p_1+p_2-1} + |\sigma_2||\rho_{11}+\rho_{15}|}{\Gamma_q(p_1+p_2)}, \\ \gamma_{15} = \dfrac{|\delta_1||\rho_9+\rho_{13}|\eta_j^{r_1+r_2} + |\sigma_3||\rho_{12}+\rho_{16}|}{\Gamma_q(r_1+r_2+1)}, & \gamma_{16} = \dfrac{|\delta_3||\rho_{11}+\rho_{15}|\eta_j^{r_1+r_2-1} + |\sigma_4||\rho_{12}+\rho_{16}|}{\Gamma_q(r_1+r_2)}, \\ \gamma_{17} = \dfrac{|\delta_2||\rho_{10}+\rho_{14}|\eta_j^{p_1+p_2+\xi_1} + |\sigma_1||\rho_{11}+\rho_{15}|}{\Gamma_q(p_1+p_2+\xi_1+1)}, & \gamma_{18} = \dfrac{|\delta_4||\rho_{12}+\rho_{16}|\eta_j^{p_1+p_2+\xi_1-1} + |\sigma_2||\rho_{11}+\rho_{15}|}{\Gamma_q(p_1+p_2+\xi_1)}, \\ \gamma_{19} = \dfrac{|\delta_1||\rho_9+\rho_{13}|\eta_j^{r_1+r_2+\xi_2} + |\sigma_3||\rho_{12}+\rho_{16}|}{\Gamma_q(r_1+r_2+\xi_2+1)}, & \gamma_{20} = \dfrac{|\delta_3||\rho_{11}+\rho_{15}|\eta_j^{r_1+r_2+\xi_2-1} + |\sigma_4||\rho_{12}+\rho_{16}|}{\Gamma_q(r_1+r_2+\xi_2)}, \\ \gamma_{21} = \dfrac{|\delta_2||\rho_{10}+\rho_{14}|\eta_j^{p_2} + |\sigma_1||\rho_{11}+\rho_{15}|}{\Gamma_q(p_2+1)}, & \gamma_{22} = \dfrac{|\delta_4||\rho_{12}+\rho_{16}|\eta_j^{p_2-1} + |\sigma_2||\rho_{11}+\rho_{15}|}{\Gamma_q(p_2)}, \\ \gamma_{23} = \dfrac{|\delta_1||\rho_9+\rho_{13}|\eta_j^{r_2} + |\sigma_3||\rho_{12}+\rho_{16}|}{\Gamma_q(r_2+1)}, & \gamma_{24} = \dfrac{|\delta_3||\rho_{11}+\rho_{15}|\eta_j^{r_2-1} + |\sigma_4||\rho_{12}+\rho_{16}|}{\Gamma_q(r_2)}. \end{cases} \tag{14}$$

In the following theorem, we prove the existence of a unique solution to the system (1) and (2) by applying the Banach contraction mapping principle [47].

Theorem 1. *Let $f_1, f_2 : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $g_1, g_2 : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be continuous functions satisfying the following conditions:*

($\mathbf{A}_1$) *There exist positive constants ι_1, ι_2 such that for each $t \in [0,1]$ and $x_i, y_i \in \mathbb{R}$, $i = 1,2$,*

$$|f_1(t,x_1,y_1) - f_1(t,x_2,y_2)| \le \iota_1(|x_1 - x_2| + |y_1 - y_2|),$$

$$|f_2(t,x_1,y_1) - f_2(t,x_2,y_2)| \le \iota_2(|x_1 - x_2| + |y_1 - y_2|).$$

($\mathbf{A}_2$) *There exist positive constants κ_1, κ_2 such that for each $t \in [0,1]$ and $x_i, y_i \in \mathbb{R}$, $i = 1,2$,*

$$|g_1(t,x_1,y_1) - g_1(t,x_2,y_2)| \le \kappa_1(|x_1 - x_2| + |y_1 - y_2|),$$

$$|g_2(t,x_1,y_1) - g_2(t,x_2,y_2)| \le \kappa_2(|x_1 - x_2| + |y_1 - y_2|).$$

Then the system (1) and (2) has a unique solution on $[0,1]$, provided that

$$\Upsilon = (\Psi_1 + \Psi_3)\iota_1 + (\Psi_2 + \Psi_4)\iota_2 + (\Phi_1 + \Phi_3)\kappa_1 + (\Phi_2 + \Phi_4)\kappa_2 + \Theta_1 + \Theta_2 < 1. \tag{15}$$

where $\Psi_1, \Psi_2, \Psi_3, \Psi_4, \Phi_1, \Phi_2, \Phi_3, \Phi_4, \Theta_1, \Theta_2$ are given in (13).

Proof. Let N_1, N_2, M_1, M_2 be finite numbers such that

$$\begin{aligned} N_1 &= \sup_{t\in[0,1]} |f_1(t,0,0)|, \quad N_2 = \sup_{t\in[0,1]} |f_2(t,0,0)|, \\ M_1 &= \sup_{t\in[0,1]} |g_1(t,0,0)|, \quad M_2 = \sup_{t\in[0,1]} |g_2(t,0,0)|. \end{aligned} \tag{16}$$

Now we show that $\mathcal{G}B_\varrho \subset B_\varrho$, where $B_\varrho = \{(x,y) \in \mathcal{G} \times \mathcal{G} : \|(x,y)\| \le \varrho\}$ with

$$\varrho \geq \frac{(\Psi_1+\Psi_3)N_1 + (\Psi_2+\Psi_4)N_2 + (\Phi_1+\Phi_3)M_1 + (\Phi_2+\Phi_4)M_2}{1-\Upsilon},$$

where Υ is given in (15). For any $(x,y) \in B_\varrho, t \in [0,1]$, using $(\mathbf{A}_1)$, we have

$$\begin{aligned}
|f_1(t,x(t),y(t))| &\leq |f_1(t,x(t),y(t)) - f_1(t,0,0)| + |f_1(t,0,0)| \\
&\leq \iota_1(|x(t)| + |y(t)|) + |f_1(t,0,0)| \\
&\leq \iota_1(\|x\| + \|y\|) + N_1 \\
&\leq \iota_1\varrho + N_1.
\end{aligned}$$

Similarly, we can find that

$$|f_2(t,x(t),y(t))| \leq \iota_2\varrho + N_2, \quad |g_1(t,x(t),y(t))| \leq \kappa_1\varrho + M_1, \quad |g_2(t,x(t),y(t))| \leq \kappa_2\varrho + M_2.$$

Then we have

$$\begin{aligned}
&\|\mathcal{G}_1(x,y)\| \\
&\leq \sup_{t\in[0,1]} \Bigg\{ |\alpha_1| \int_0^t \frac{(t-qu)^{(p_1+p_2-1)}}{\Gamma_q(p_1+p_2)} |f_1(u,x(u),y(u))| d_q u \\
&+|\beta_1| \int_0^t \frac{(t-qu)^{(p_1+p_2+\xi_1-1)}}{\Gamma_q(p_1+p_2+\xi)} |g_1(u,x(u),y(u))| d_q u + |\lambda_1| \int_0^t \frac{(t-qu)^{(p_2-1)}}{\Gamma_q(p_2)} |x(u)| d_q u \\
&+\frac{|\delta_1||\rho_1 t^{p_2}+\rho_5|}{|\Lambda|} \Bigg(|\alpha_2| \int_0^{\eta_j} \frac{(\eta_j-qu)^{(r_1+r_2-1)}}{\Gamma_q(r_1+r_2)} |f_2(u,x(u),y(u))| d_q u \\
&+|\beta_2| \int_0^{\eta_j} \frac{(\eta_j-qu)^{(r_1+r_2+\xi_2-1)}}{\Gamma_q(r_1+r_2+\xi_2)} |g_2(u,x(u),y(u))| d_q u + |\lambda_2| \int_0^{\eta_j} \frac{(\eta_j-qu)^{(r_2-1)}}{\Gamma_q(r_2)} |y(u)| d_q u \Bigg) \\
&+\frac{|\delta_2||\rho_2 t^{p_2}+\rho_6|}{|\Lambda|} \Bigg(|\alpha_1| \int_0^{\eta_j} \frac{(\eta_j-qu)^{(p_1+p_2-1)}}{\Gamma_q(p_1+p_2)} |f_1(u,x(u),y(u))| d_q u \\
&+|\beta_1| \int_0^{\eta_j} \frac{(\eta_j-qu)^{(p_1+p_2+\xi_1-1)}}{\Gamma_q(p_1+p_2+\xi_1)} |g_1(u,x(u),y(u))| d_q u + |\lambda_1| \int_0^{\eta_j} \frac{(\eta_j-qu)^{(p_2-1)}}{\Gamma_q(p_2)} |x(u)| d_q u \Bigg) \\
&+\frac{|\delta_3||\rho_3 t^{p_2}+\rho_7|}{|\Lambda|} \Bigg(|\alpha_2| \int_0^{\eta_j} \frac{(\eta_j-qu)^{(r_1+r_2-2)}}{\Gamma_q(r_1+r_2-1)} |f_2(u,x(u),y(u))| d_q u \\
&+|\beta_2| \int_0^{\eta_j} \frac{(\eta_j-qu)^{(r_1+r_2+\xi_2-2)}}{\Gamma_q(r_1+r_2+\xi_2-1)} |g_2(u,x(u),y(u))| d_q u + |\lambda_2| \int_0^{\eta_j} \frac{(\eta_j-qu)^{(r_2-2)}}{\Gamma_q(r_2-1)} |y(u)| d_q u \Bigg) \\
&+\frac{|\sigma_1||\rho_3 t^{p_2}+\rho_7|}{|\Lambda|} \Bigg(|\alpha_1| \int_0^1 \frac{(1-qu)^{(p_1+p_2-1)}}{\Gamma_q(p_1+p_2)} |f_1(u,x(u),y(u))| d_q u \\
&+|\beta_1| \int_0^1 \frac{(1-qu)^{(p_1+p_2+\xi_1-1)}}{\Gamma_q(p_1+p_2+\xi_1)} |g_1(u,x(u),y(u))| d_q u + |\lambda_1| \int_0^1 \frac{(1-qu)^{(p_2-1)}}{\Gamma_q(p_2)} |x(u)| d_q u \Bigg) \\
&+\frac{|\sigma_2||\rho_3 t^{p_2}+\rho_7|}{|\Lambda|} \Bigg(|\alpha_1| \int_0^1 \frac{(1-qu)^{(p_1+p_2-2)}}{\Gamma_q(p_1+p_2-1)} |f_1(u,x(u),y(u))| d_q u \\
&+|\beta_1| \int_0^1 \frac{(1-qu)^{(p_1+p_2+\xi_1-2)}}{\Gamma_q(p_1+p_2+\xi_1-1)} |g_1(u,x(u),y(u))| d_q u + |\lambda_1| \int_0^1 \frac{(1-qu)^{(p_2-2)}}{\Gamma_q(p_2-1)} |x(u)| d_q u \Bigg) \\
&+\frac{|\delta_4||\rho_4 t^{p_2}+\rho_8|}{|\Lambda|} \Bigg(|\alpha_1| \int_0^{\eta_j} \frac{(\eta_j-qu)^{(p_1+p_2-2)}}{\Gamma_q(p_1+p_2-1)} |f_1(u,x(u),y(u))| d_q u \\
&+|\beta_1| \int_0^{\eta_j} \frac{(\eta_j-qu)^{(p_1+p_2+\xi_1-2)}}{\Gamma_q(p_1+p_2+\xi_1-1)} |g_1(u,x(u),y(u))| d_q u + |\lambda_1| \int_0^{\eta_j} \frac{(\eta_j-qu)^{(p_2-2)}}{\Gamma_q(p_2-1)} |x(u)| d_q u \Bigg) \\
&+\frac{|\sigma_3||\rho_4 t^{p_2}+\rho_8|}{|\Lambda|} \Bigg(|\alpha_2| \int_0^1 \frac{(1-qu)^{(r_1+r_2-1)}}{\Gamma_q(r_1+r_2)} |f_2(u,x(u),y(u))| d_q u
\end{aligned}$$

$$
\begin{aligned}
&+|\beta_2|\int_0^1 \frac{(1-qu)^{(r_1+r_2+\xi_2-1)}}{\Gamma_q(r_1+r_2+\xi_2)}|g_2(u,x(u),y(u))|d_qu+|\lambda_2|\int_0^1 \frac{(1-qu)^{(r_2-1)}}{\Gamma_q(r_2)}|y(u)|d_qu\Bigg) \\
&+\frac{|\sigma_4||\rho_4t^{p_2}+\rho_8|}{|\Lambda|}\Bigg(|\alpha_2|\int_0^1 \frac{(1-qu)^{(r_1+r_2-2)}}{\Gamma_q(r_1+r_2-1)}|f_2(u,x(u),y(u))|d_qu \\
&+|\beta_2|\int_0^1 \frac{(1-qu)^{(r_1+r_2+\xi_2-2)}}{\Gamma_q(r_1+r_2+\xi_2-1)}|g_2(u,x(u),y(u))|d_qu+|\lambda_2|\int_0^1 \frac{(1-qu)^{(r_2-2)}}{\Gamma_q(r_2-1)}|y(u)|d_qu\Bigg)\Bigg\} \\
&\le (\iota_1 r+N_1)\sup_{t\in[0,1]}\Bigg\{|\alpha_1|\int_0^t \frac{(t-qu)^{(p_1+p_2-1)}}{\Gamma_q(p_1+p_2)}d_qu+\frac{|\alpha_1||\delta_2||\rho_2t^{p_2}+\rho_6|}{|\Lambda|} \\
&\times\int_0^{\eta_j} \frac{(\eta_j-qu)^{(p_1+p_2-1)}}{\Gamma_q(p_1+p_2)}d_qu+\frac{|\alpha_1||\sigma_1||\rho_3t^{p_2}+\rho_7|}{|\Lambda|}\int_0^1 \frac{(1-qu)^{(p_1+p_2-1)}}{\Gamma_q(p_1+p_2)}d_qu \\
&+\frac{|\alpha_1||\sigma_2||\rho_3t^{p_2}+\rho_7|}{|\Lambda|}\int_0^1 \frac{(1-qu)^{(p_1+p_2-2)}}{\Gamma_q(p_1+p_2-1)}d_qu+\frac{|\alpha_1||\delta_4||\rho_4t^{p_2}+\rho_8|}{|\Lambda|}\int_0^{\eta_j} \frac{(\eta_j-qu)^{(p_1+p_2-2)}}{\Gamma_q(p_1+p_2-1)}d_qu\Bigg\} \\
&+(\kappa_1 r+M_1)\sup_{t\in[0,1]}\Bigg\{|\beta_1|\int_0^t \frac{(t-qu)^{(p_1+p_2+\xi_1-1)}}{\Gamma_q(p_1+p_2+\xi_1)}d_qu+\frac{|\beta_1||\delta_2||\rho_2t^{p_2}+\rho_6|}{|\Lambda|} \\
&\times\int_0^{\eta_j} \frac{(\eta_j-qu)^{(p_1+p_2+\xi_1-1)}}{\Gamma_q(p_1+p_2+\xi_1)}d_qu+\frac{|\beta_1||\sigma_1||\rho_3t^{p_2}+\rho_7|}{|\Lambda|}\int_0^1 \frac{(1-qu)^{(p_1+p_2+\xi_1-1)}}{\Gamma_q(p_1+p_2+\xi_1)}d_qu \\
&+\frac{|\beta_1||\sigma_2||\rho_3t^{p_2}+\rho_7|}{|\Lambda|}\int_0^1 \frac{(1-qu)^{(p_1+p_2+\xi_1-2)}}{\Gamma_q(p_1+p_2+\xi_1-1)}d_qu+\frac{|\beta_1||\delta_4||\rho_4t^{p_2}+\rho_8|}{|\Lambda|} \\
&\times\int_0^{\eta_j} \frac{(\eta_j-qu)^{(p_1+p_2+\xi_1-2)}}{\Gamma_q(p_1+p_2+\xi_1-1)}d_qu\Bigg\}+(\iota_2 r+N_2)\sup_{t\in[0,1]}\Bigg\{\frac{|\alpha_2||\delta_1||\rho_1t^{p_2}+\rho_5|}{|\Lambda|} \\
&\times\int_0^{\eta_j} \frac{(\eta_j-qu)^{(r_1+r_2-1)}}{\Gamma_q(r_1+r_2)}d_qu+\frac{|\alpha_2||\delta_3||\rho_3t^{p_2}+\rho_7|}{|\Lambda|}\int_0^{\eta_j} \frac{(\eta_j-qu)^{(r_1+r_2-2)}}{\Gamma_q(r_1+r_2-1)}d_qu \\
&+\frac{|\alpha_2||\sigma_3||\rho_4t^{p_2}+\rho_8|}{|\Lambda|}\int_0^1 \frac{(1-qu)^{(r_1+r_2-1)}}{\Gamma_q(r_1+r_2)}d_qu+\frac{|\alpha_2||\sigma_4||\rho_4t^{p_2}+\rho_8|}{|\Lambda|}\int_0^1 \frac{(1-qu)^{(r_1+r_2-2)}}{\Gamma_q(r_1+r_2-1)}d_qu\Bigg\} \\
&+(\kappa_2 r+M_2)\sup_{t\in[0,1]}\Bigg\{\frac{|\beta_2||\delta_1||\rho_1t^{p_2}+\rho_5|}{|\Lambda|}\int_0^{\eta_j} \frac{(\eta_j-qu)^{(r_1+r_2+\xi_2-1)}}{\Gamma_q(r_1+r_2+\xi_2)}d_qu+\frac{|\beta_2||\delta_3||\rho_3t^{p_2}+\rho_7|}{|\Lambda|} \\
&\times\int_0^{\eta_j} \frac{(\eta_j-qu)^{(r_1+r_2+\xi_2-2)}}{\Gamma_q(r_1+r_2+\xi_2-1)}d_qu+\frac{|\beta_2||\sigma_3||\rho_4t^{p_2}+\rho_8|}{|\Lambda|}\int_0^1 \frac{(1-qu)^{(r_1+r_2+\xi_2-1)}}{\Gamma_q(r_1+r_2+\xi_2)}d_qu
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{|\beta_2||\sigma_4||\rho_4t^{p_2}+\rho_8|}{|\Lambda|}\int_0^1 \frac{(1-qu)^{(r_1+r_2+\xi_2-2)}}{\Gamma_q(r_1+r_2+\xi_2-1)}d_qu\Bigg\}+\varrho\sup_{t\in[0,1]}\Bigg\{|\lambda_1|\int_0^t \frac{(t-qu)^{(p_2-1)}}{\Gamma_q(p_2)}d_qu \\
&+\frac{|\lambda_2||\delta_1||\rho_1t^{p_2}+\rho_5|}{|\Lambda|}\int_0^{\eta_j} \frac{(\eta_j-qu)^{(r_2-1)}}{\Gamma_q(r_2)}d_qu+\frac{|\lambda_1||\delta_2||\rho_2t^{p_2}+\rho_6|}{|\Lambda|}\int_0^{\eta_j} \frac{(\eta_j-qu)^{(p_2-1)}}{\Gamma_q(p_2)}d_qu \\
&+\frac{|\lambda_2||\delta_3||\rho_3t^{p_2}+\rho_7|}{|\Lambda|}\int_0^{\eta_j} \frac{(\eta_j-qu)^{(r_2-2)}}{\Gamma_q(r_2-1)}d_qu+\frac{|\lambda_1||\sigma_1||\rho_3t^{p_2}+\rho_7|}{|\Lambda|}\int_0^1 \frac{(1-qu)^{(p_2-1)}}{\Gamma_q(p_2)}d_qu \\
&+\frac{|\lambda_1||\sigma_2||\rho_3t^{p_2}+\rho_7|}{|\Lambda|}\int_0^1 \frac{(1-qu)^{(p_2-2)}}{\Gamma_q(p_2-1)}d_qu+\frac{|\lambda_1||\delta_4||\rho_4t^{p_2}+\rho_8|}{|\Lambda|}\int_0^{\eta_j} \frac{(\eta_j-qu)^{(p_2-2)}}{\Gamma_q(p_2-1)}d_qu \\
&+\frac{|\lambda_2||\sigma_3||\rho_4t^{p_2}+\rho_8|}{|\Lambda|}\int_0^1 \frac{(1-qu)^{(r_2-1)}}{\Gamma_q(r_2)}d_qu+\frac{|\lambda_2||\sigma_4||\rho_4t^{p_2}+\rho_8|}{|\Lambda|}\int_0^1 \frac{(1-qu)^{(r_2-2)}}{\Gamma_q(r_2-1)}d_qu\Bigg\} \\
&\le (\iota_1\varrho+N_1)\Psi_1+(\iota_2\varrho+N_2)\Psi_2+(\kappa_1\varrho+M_1)\Phi_1+(\kappa_2\varrho+M_2)\Phi_2+\varrho\Theta_1,
\end{aligned}
$$

where $\Psi_1,\ \Psi_2,\ \Phi_1,\ \Phi_2,\ \Theta_1$ are given in (13).

Furthermore, we obtain

$$
\begin{aligned}
&\|\mathcal{G}_2(x,y)\| \\
&\le \sup_{t\in[0,1]}\Bigg\{|\alpha_2|\int_0^t \frac{(t-qu)^{(r_1+r_2-1)}}{\Gamma_q(r_1+r_2)}|f_2(u,x(u),y(u))|d_qu \\
&+|\beta_2|\int_0^t \frac{(t-qu)^{(r_1+r_2+\xi_2-1)}}{\Gamma_q(r_1+r_2+\xi_2)}|g_2(u,x(u),y(u))|d_qu+|\lambda_2|\int_0^t \frac{(t-qu)^{(r_2-1)}}{\Gamma_q(r_2)}|y(u)|d_qu
\end{aligned}
$$

$$+\frac{|\delta_1||\rho_9 t^{r_2}+\rho_{13}|}{|\Lambda|}\Bigg(|\alpha_2|\int_0^{\eta_j}\frac{(\eta_j-qu)^{(r_1+r_2-1)}}{\Gamma_q(r_1+r_2)}|f_2(u,x(u),y(u))|d_qu$$

$$+|\beta_2|\int_0^{\eta_j}\frac{(\eta_j-qu)^{(r_1+r_2+\xi_2-1)}}{\Gamma_q(r_1+r_2+\xi_2)}|g_2(u,x(u),y(u))|d_qu+|\lambda_2|\int_0^{\eta_j}\frac{(\eta_j-qu)^{(r_2-1)}}{\Gamma_q(r_2)}|y(u)|d_qu\Bigg)$$

$$+\frac{|\delta_2||\rho_{10} t^{r_2}+\rho_{14}|}{|\Lambda|}\Bigg(|\alpha_1|\int_0^{\eta_j}\frac{(\eta_j-qu)^{(p_1+p_2-1)}}{\Gamma_q(p_1+p_2)}|f_1(u,x(u),y(u))|d_qu$$

$$+|\beta_1|\int_0^{\eta_j}\frac{(\eta_j-qu)^{(p_1+p_2+\xi_1-1)}}{\Gamma_q(p_1+p_2+\xi_1)}|g_1(u,x(u),y(u))|d_qu+|\lambda_1|\int_0^{\eta_j}\frac{(\eta_j-qu)^{(p_2-1)}}{\Gamma_q(p_2)}|x(u)|d_qu\Bigg)$$

$$+\frac{|\delta_3||\rho_{11} t^{r_2}+\rho_{15}|}{|\Lambda|}\Bigg(|\alpha_2|\int_0^{\eta_j}\frac{(\eta_j-qu)^{(r_1+r_2-2)}}{\Gamma_q(r_1+r_2-1)}|f_2(u,x(u),y(u))|d_qu$$

$$+|\beta_2|\int_0^{\eta_j}\frac{(\eta_j-qu)^{(r_1+r_2+\xi_2-2)}}{\Gamma_q(r_1+r_2+\xi_2-1)}|g_2(u,x(u),y(u))|d_qu+|\lambda_2|\int_0^{\eta_j}\frac{(\eta_j-qu)^{(r_2-2)}}{\Gamma_q(r_2-1)}|y(u)|d_qu\Bigg)$$

$$+\frac{|\sigma_1||\rho_{11} t^{r_2}+\rho_{15}|}{|\Lambda|}\Bigg(|\alpha_1|\int_0^{1}\frac{(1-qu)^{(p_1+p_2-1)}}{\Gamma_q(p_1+p_2)}|f_1(u,x(u),y(u))|d_qu$$

$$+|\beta_1|\int_0^{1}\frac{(1-qu)^{(p_1+p_2+\xi_1-1)}}{\Gamma_q(p_1+p_2+\xi_1)}|g_1(u,x(u),y(u))|d_qu+|\lambda_1|\int_0^{1}\frac{(1-qu)^{(p_2-1)}}{\Gamma_q(p_2)}|x(u)|d_qu\Bigg)$$

$$+\frac{|\sigma_2||\rho_{11} t^{r_2}+\rho_{15}|}{|\Lambda|}\Bigg(|\alpha_1|\int_0^{1}\frac{(1-qu)^{(p_1+p_2-2)}}{\Gamma_q(p_1+p_2-1)}|f_1(u,x(u),y(u))|d_qu$$

$$+|\beta_1|\int_0^{1}\frac{(1-qu)^{(p_1+p_2+\xi_1-2)}}{\Gamma_q(p_1+p_2+\xi_1-1)}|g_1(u,x(u),y(u))|d_qu+|\lambda_1|\int_0^{1}\frac{(1-qu)^{(p_2-2)}}{\Gamma_q(p_2-1)}|x(u)|d_qu\Bigg)$$

$$+\frac{|\delta_4||\rho_{12} t^{r_2}+\rho_{16}|}{|\Lambda|}\Bigg(|\alpha_1|\int_0^{\eta_j}\frac{(\eta_j-qu)^{(p_1+p_2-2)}}{\Gamma_q(p_1+p_2-1)}|f_1(u,x(u),y(u))|d_qu$$

$$+|\beta_1|\int_0^{\eta_j}\frac{(\eta_j-qu)^{(p_1+p_2+\xi_1-2)}}{\Gamma_q(p_1+p_2+\xi_1-1)}|g_1(u,x(u),y(u))|d_qu+|\lambda_1|\int_0^{\eta_j}\frac{(\eta_j-qu)^{(p_2-2)}}{\Gamma_q(p_2-1)}|x(u)|d_qu\Bigg)$$

$$+\frac{|\sigma_3||\rho_{12} t^{r_2}+\rho_{16}|}{|\Lambda|}\Bigg(|\alpha_2|\int_0^{1}\frac{(1-qu)^{(r_1+r_2-1)}}{\Gamma_q(r_1+r_2)}|f_2(u,x(u),y(u))|d_qu$$

$$+|\beta_2|\int_0^{1}\frac{(1-qu)^{(r_1+r_2+\xi_2-1)}}{\Gamma_q(r_1+r_2+\xi_2)}|g_2(u,x(u),y(u))|d_qu+|\lambda_2|\int_0^{1}\frac{(1-qu)^{(r_2-1)}}{\Gamma_q(r_2)}|y(u)|d_qu\Bigg)$$

$$+\frac{|\sigma_4||\rho_{12} t^{r_2}+\rho_{16}|}{|\Lambda|}\Bigg(|\alpha_2|\int_0^{1}\frac{(1-qu)^{(r_1+r_2-2)}}{\Gamma_q(r_1+r_2-1)}|f_2(u,x(u),y(u))|d_qu$$

$$+|\beta_2|\int_0^{1}\frac{(1-qu)^{(r_1+r_2+\xi_2-2)}}{\Gamma_q(r_1+r_2+\xi_2-1)}|g_2(u,x(u),y(u))|d_qu+|\lambda_2|\int_0^{1}\frac{(1-qu)^{(r_2-2)}}{\Gamma_q(r_2-1)}|y(u)|d_qu\Bigg)\Bigg\}$$

$$\leq(\iota_1 r+N_1)\sup_{t\in[0,1]}\Bigg\{\frac{|\alpha_1||\delta_2||\rho_{10} t^{r_2}+\rho_{14}|}{|\Lambda|}\int_0^{\eta_j}\frac{(\eta_j-qu)^{(p_1+p_2-1)}}{\Gamma_q(p_1+p_2)}d_qu$$

$$+\frac{|\alpha_1||\sigma_1||\rho_{11} t^{r_2}+\rho_{15}|}{|\Lambda|}\int_0^{1}\frac{(1-qu)^{(p_1+p_2-1)}}{\Gamma_q(p_1+p_2)}d_qu+\frac{|\alpha_1||\sigma_2||\rho_{11} t^{r_2}+\rho_{15}|}{|\Lambda|}\int_0^{1}\frac{(1-qu)^{(p_1+p_2-2)}}{\Gamma_q(p_1+p_2-1)}d_qu$$

$$+\frac{|\alpha_1||\delta_4||\rho_{12} t^{r_2}+\rho_{16}|}{|\Lambda|}\int_0^{\eta_j}\frac{(\eta_j-qu)^{(p_1+p_2-2)}}{\Gamma_q(p_1+p_2-1)}d_qu\Bigg\}+(\kappa_1 r+M_1)\sup_{t\in[0,1]}\Bigg\{\frac{|\beta_1||\delta_2||\rho_{10} t^{r_2}+\rho_{14}|}{|\Lambda|}$$

$$\times\int_0^{\eta_j}\frac{(\eta_j-qu)^{(p_1+p_2+\xi_1-1)}}{\Gamma_q(p_1+p_2+\xi_1)}d_qu+\frac{|\beta_1||\sigma_1||\rho_{11} t^{r_2}+\rho_{15}|}{|\Lambda|}\int_0^{1}\frac{(1-qu)^{(p_1+p_2+\xi_1-1)}}{\Gamma_q(p_1+p_2+\xi_1)}d_qu$$

$$+\frac{|\beta_1||\sigma_2||\rho_{11} t^{r_2}+\rho_{15}|}{|\Lambda|}\int_0^{1}\frac{(1-qu)^{(p_1+p_2+\xi_1-2)}}{\Gamma_q(p_1+p_2+\xi_1-1)}d_qu+\frac{|\beta_1||\delta_4||\rho_{12} t^{r_2}+\rho_{16}|}{|\Lambda|}$$

$$\times\int_0^{\eta_j}\frac{(\eta_j-qu)^{(p_1+p_2+\xi_1-2)}}{\Gamma_q(p_1+p_2+\xi_1-1)}d_qu\Bigg\}+(\iota_2 r+N_2)\sup_{t\in[0,1]}\Bigg\{|\alpha_2|\int_0^{t}\frac{(t-qu)^{(r_1+r_2-1)}}{\Gamma_q(r_1+r_2)}d_qu$$

$$+\frac{|\alpha_2||\delta_1||\rho_9 t^{r_2}+\rho_{13}|}{|\Lambda|}\int_0^{\eta_j}\frac{(\eta_j-qu)^{(r_1+r_2-1)}}{\Gamma_q(r_1+r_2)}d_qu+\frac{|\alpha_2||\delta_3||\rho_{11}t^{r_2}+\rho_{15}|}{|\Lambda|}\int_0^{\eta_j}\frac{(\eta_j-qu)^{(r_1+r_2-2)}}{\Gamma_q(r_1+r_2-1)}d_qu$$
$$+\frac{|\alpha_2||\sigma_3||\rho_{12}t^{r_2}+\rho_{16}|}{|\Lambda|}\int_0^{1}\frac{(1-qu)^{(r_1+r_2-1)}}{\Gamma_q(r_1+r_2)}d_qu+\frac{|\alpha_2||\sigma_4||\rho_{12}t^{r_2}+\rho_{16}|}{|\Lambda|}\int_0^{1}\frac{(1-qu)^{(r_1+r_2-2)}}{\Gamma_q(r_1+r_2-1)}d_qu\Bigg\}$$
$$+(\kappa_2 r+M_2)\sup_{t\in[0,1]}\Bigg\{|\beta_2|\int_0^t\frac{(t-qu)^{(r_1+r_2+\xi_2-1)}}{\Gamma_q(r_1+r_2+\xi_2)}d_qu+\frac{|\beta_2||\delta_1||\rho_9t^{r_2}+\rho_{13}|}{|\Lambda|}$$
$$\times\int_0^{\eta_j}\frac{(\eta_j-qu)^{(r_1+r_2+\xi_2-1)}}{\Gamma_q(r_1+r_2+\xi_2)}d_qu+\frac{|\beta_2||\delta_3||\rho_{11}t^{r_2}+\rho_{15}|}{|\Lambda|}\int_0^{\eta_j}\frac{(\eta_j-qu)^{(r_1+r_2+\xi_2-2)}}{\Gamma_q(r_1+r_2+\xi_2-1)}d_qu$$
$$+\frac{|\beta_2||\sigma_3||\rho_{12}t^{r_2}+\rho_{16}|}{|\Lambda|}\int_0^{1}\frac{(1-qu)^{(r_1+r_2+\xi_2-1)}}{\Gamma_q(r_1+r_2+\xi_2)}d_qu+\frac{|\beta_2||\sigma_4||\rho_{12}t^{r_2}+\rho_{16}|}{|\Lambda|}$$
$$\times\int_0^{1}\frac{(1-qu)^{(r_1+r_2+\xi_2-2)}}{\Gamma_q(r_1+r_2+\xi_2-1)}d_qu\Bigg\}+\varrho\sup_{t\in[0,1]}\Bigg\{|\lambda_2|\int_0^t\frac{(t-qu)^{(r_2-1)}}{\Gamma_q(r_2)}d_qu$$
$$+\frac{|\lambda_2||\delta_1||\rho_9t^{r_2}+\rho_{13}|}{|\Lambda|}\int_0^{\eta_j}\frac{(\eta_j-qu)^{(r_2-1)}}{\Gamma_q(r_2)}d_qu+\frac{|\lambda_1||\delta_2||\rho_{10}t^{r_2}+\rho_{14}|}{|\Lambda|}\int_0^{\eta_j}\frac{(\eta_j-qu)^{(p_2-1)}}{\Gamma_q(p_2)}d_qu$$
$$+\frac{|\lambda_2||\delta_3||\rho_{11}t^{r_2}+\rho_{15}|}{|\Lambda|}\int_0^{\eta_j}\frac{(\eta_j-qu)^{(r_2-2)}}{\Gamma_q(r_2-1)}d_qu+\frac{|\lambda_1||\sigma_1||\rho_{11}t^{r_2}+\rho_{15}|}{|\Lambda|}\int_0^{1}\frac{(1-qu)^{(p_2-1)}}{\Gamma_q(p_2)}d_qu$$
$$+\frac{|\lambda_1||\sigma_2||\rho_{11}t^{r_2}+\rho_{15}|}{|\Lambda|}\int_0^{1}\frac{(1-qu)^{(p_2-2)}}{\Gamma_q(p_2-1)}d_qu+\frac{|\lambda_1||\delta_4||\rho_{12}t^{r_2}+\rho_{16}|}{|\Lambda|}\int_0^{\eta_j}\frac{(\eta_j-qu)^{(p_2-2)}}{\Gamma_q(p_2-1)}d_qu$$
$$+\frac{|\lambda_2||\sigma_3||\rho_{12}t^{r_2}+\rho_{16}|}{|\Lambda|}\int_0^{1}\frac{(1-qu)^{(r_2-1)}}{\Gamma_q(r_2)}d_qu+\frac{|\lambda_2||\sigma_4||\rho_{12}t^{r_2}+\rho_{16}|}{|\Lambda|}\int_0^{1}\frac{(1-qu)^{(r_2-2)}}{\Gamma_q(r_2-1)}d_qu\Bigg\}$$
$$\le(\iota_1\varrho+N_1)\Psi_3+(\iota_2\varrho+N_2)\Psi_4+(\kappa_1\varrho+M_1)\Phi_3+(\kappa_2\varrho+M_2)\Phi_4+\varrho\Theta_2,$$

where $\Psi_3,\ \Psi_4,\ \Phi_3,\ \Phi_4,\ \Theta_2$ are given in (13).

From the foregoing inequalities, it follows that

$$\|\mathcal{G}(x,y)\|\le\Upsilon\varrho+(\Psi_1+\Psi_3)N_1+(\Psi_2+\Psi_4)N_2+(\Phi_1+\Phi_3)M_1+(\Phi_2+\Phi_4)M_2,$$

which implies that $\mathcal{G}B_\varrho\subset B_\varrho$. Next we show that the operator $\mathcal{G}$ is a contraction. Using conditions $(\mathbf{A}_1)$ and $(\mathbf{A}_2)$, for any $(x_1,y_1),(x_2,y_2)\in\mathcal{C}\times\mathcal{C}, t\in[0,1]$, we obtain

$$\|\mathcal{G}_1(x_1,y_1)-\mathcal{G}_1(x_2,y_2)\|$$
$$\le\sup_{t\in[0,1]}\Bigg\{|\alpha_1|\int_0^t\frac{(t-qu)^{(p_1+p_2-1)}}{\Gamma_q(p_1+p_2)}|f_1(u,x_1(u),y_1(u))-f_1(u,x_2(u),y_2(u))|d_qu$$
$$+|\beta_1|\int_0^t\frac{(t-qu)^{(p_1+p_2+\xi_1-1)}}{\Gamma_q(p_1+p_2+\xi_1)}|g_1(u,x_1(u),y_1(u))-g_1(u,x_2(u),y_2(u))|d_qu$$
$$+|\lambda_1|\int_0^t\frac{(t-qu)^{(p_2-1)}}{\Gamma_q(p_2)}|x_1(u)-x_2(u)|d_qu+\frac{|\delta_1||\rho_1t^{p_2}+\rho_5|}{|\Lambda|}\Bigg(|\alpha_2|\int_0^{\eta_j}\frac{(\eta_j-qu)^{(r_1+r_2-1)}}{\Gamma_q(r_1+r_2)}$$
$$\times|f_2(u,x_1(u),y_1(u))-f_2(u,x_2(u),y_2(u))|d_qu+|\beta_2|\int_0^{\eta_j}\frac{(\eta_j-qu)^{(r_1+r_2+\xi_2-1)}}{\Gamma_q(r_1+r_2+\xi_2)}$$
$$\times|g_2(u,x_1(u),y_1(u))-g_2(u,x_2(u),y_2(u))|d_qu+|\lambda_2|\int_0^{\eta_j}\frac{(\eta_j-qu)^{(r_2-1)}}{\Gamma_q(r_2)}|y_1(u)-y_2(u)|d_qu\Bigg)$$
$$+\frac{|\delta_2||\rho_2t^{p_2}+\rho_6|}{|\Lambda|}\Bigg(|\alpha_1|\int_0^{\eta_j}\frac{(\eta_j-qu)^{(p_1+p_2-1)}}{\Gamma_q(p_1+p_2)}|f_1(u,x_1(u),y_1(u))-f_1(u,x_2(u),y_2(u))|d_qu$$
$$+|\beta_1|\int_0^{\eta_j}\frac{(\eta_j-qu)^{(p_1+p_2+\xi_1-1)}}{\Gamma_q(p_1+p_2+\xi_1)}|g_1(u,x_1(u),y_1(u))-g_1(u,x_2(u),y_2(u))|d_qu$$
$$+|\lambda_1|\int_0^{\eta_j}\frac{(\eta_j-qu)^{(p_2-1)}}{\Gamma_q(p_2)}|x_1(u)-x_2(u)|d_qu\Bigg)+\frac{|\delta_3||\rho_3t^{p_2}+\rho_7|}{|\Lambda|}\Bigg(|\alpha_2|\int_0^{\eta_j}\frac{(\eta_j-qu)^{(r_1+r_2-2)}}{\Gamma_q(r_1+r_2-1)}$$
$$\times|f_2(u,x_1(u),y_1(u))-f_2(u,x_2(u),y_2(u))|d_qu+|\beta_2|\int_0^{\eta_j}\frac{(\eta_j-qu)^{(r_1+r_2+\xi_2-2)}}{\Gamma_q(r_1+r_2+\xi_2-1)}$$

$$\times|g_2(u,x_1(u),y_1(u))-g_2(u,x_2(u),y_2(u))|d_qu+|\lambda_2|\int_0^{\eta_j}\frac{(\eta_j-qu)^{(r_2-2)}}{\Gamma_q(r_2-1)}|y_1(u)-y_2(u)|d_qu\Bigg)$$
$$+\frac{|\sigma_1||\rho_3t^{p_2}+\rho_7|}{|\Lambda|}\Bigg(|\alpha_1|\int_0^1\frac{(1-qu)^{(p_1+p_2-1)}}{\Gamma_q(p_1+p_2)}|f_1(u,x_1(u),y_1(u))-f_1(u,x_2(u),y_2(u))|d_qu$$
$$+|\beta_1|\int_0^1\frac{(1-qu)^{(p_1+p_2+\xi_1-1)}}{\Gamma_q(p_1+p_2+\xi_1)}|g_1(u,x_1(u),y_1(u))-g_1(u,x_2(u),y_2(u))|d_qu$$
$$+|\lambda_1|\int_0^1\frac{(1-qu)^{(p_2-1)}}{\Gamma_q(p_2)}|x_1(u)-x_2(u)|d_qu\Bigg)+\frac{|\sigma_2||\rho_3t^{p_2}+\rho_7|}{|\Lambda|}\Bigg(|\alpha_1|\int_0^1\frac{(1-qu)^{(p_1+p_2-2)}}{\Gamma_q(p_1+p_2-1)}$$
$$\times|f_1(u,x_1(u),y_1(u))-f_1(u,x_2(u),y_2(u))|d_qu+|\beta_1|\int_0^1\frac{(1-qu)^{(p_1+p_2+\xi_1-2)}}{\Gamma_q(p_1+p_2+\xi_1-1)}$$
$$\times|g_1(u,x_1(u),y_1(u))-g_1(u,x_2(u),y_2(u))|d_qu+|\lambda_1|\int_0^1\frac{(1-qu)^{(p_2-2)}}{\Gamma_q(p_2-1)}|x_1(u)-x_2(u)|d_qu\Bigg)$$
$$+\frac{|\delta_4||\rho_4t^{p_2}+\rho_8|}{|\Lambda|}\Bigg(|\alpha_1|\int_0^{\eta_j}\frac{(\eta_j-qu)^{(p_1+p_2-2)}}{\Gamma_q(p_1+p_2-1)}|f_1(u,x_1(u),y_1(u))-f_1(u,x_2(u),y_2(u))|d_qu$$
$$+|\beta_1|\int_0^{\eta_j}\frac{(\eta_j-qu)^{(p_1+p_2+\xi_1-2)}}{\Gamma_q(p_1+p_2+\xi_1-1)}|g_1(u,x_1(u),y_1(u))-g_1(u,x_2(u),y_2(u))|d_qu$$
$$+|\lambda_1|\int_0^{\eta_j}\frac{(\eta_j-qu)^{(p_2-2)}}{\Gamma_q(p_2-1)}|x_1(u)-x_2(u)|d_qu\Bigg)+\frac{|\sigma_3||\rho_4t^{p_2}+\rho_8|}{|\Lambda|}\Bigg(|\alpha_2|\int_0^1\frac{(1-qu)^{(r_1+r_2-1)}}{\Gamma_q(r_1+r_2)}$$
$$\times|f_2(u,x_1(u),y_1(u))-f_2(u,x_2(u),y_2(u))|d_qu+|\beta_2|\int_0^1\frac{(1-qu)^{(r_1+r_2+\xi_2-1)}}{\Gamma_q(r_1+r_2+\xi_2)}$$
$$\times|g_2(u,x_1(u),y_1(u))-g_2(u,x_2(u),y_2(u))|d_qu+|\lambda_2|\int_0^1\frac{(1-qu)^{(r_2-1)}}{\Gamma_q(r_2)}|y_1(u)-y_2(u)|d_qu\Bigg)$$
$$+\frac{|\sigma_4||\rho_4t^{p_2}+\rho_8|}{|\Lambda|}\Bigg(|\alpha_2|\int_0^1\frac{(1-qu)^{(r_1+r_2-2)}}{\Gamma_q(r_1+r_2-1)}|f_2(u,x_1(u),y_1(u))-f_2(u,x_2(u),y_2(u))|d_qu$$
$$+|\beta_2|\int_0^1\frac{(1-qu)^{(r_1+r_2+\xi_2-2)}}{\Gamma_q(r_1+r_2+\xi_2-1)}|g_2(u,x_1(u),y_1(u))-g_2(u,x_2(u),y_2(u))|d_qu$$
$$+|\lambda_2|\int_0^1\frac{(1-qu)^{(r_2-2)}}{\Gamma_q(r_2-1)}|y_1(u)-y_2(u)|d_qu\Bigg)\Bigg\}$$
$$\leq\iota_1(\|x_1-x_2\|+\|y_1-y_2\|)\sup_{t\in[0,1]}\Bigg\{|\alpha_1|\int_0^t\frac{(t-qu)^{(p_1+p_2-1)}}{\Gamma_q(p_1+p_2)}d_qu+\frac{|\alpha_1||\delta_2||\rho_2t^{p_2}+\rho_6|}{|\Lambda|}$$
$$\times\int_0^{\eta_j}\frac{(\eta_j-qu)^{(p_1+p_2-1)}}{\Gamma_q(p_1+p_2)}d_qu+\frac{|\alpha_1||\sigma_1||\rho_3t^{p_2}+\rho_7|}{|\Lambda|}\int_0^1\frac{(1-qu)^{(p_1+p_2-1)}}{\Gamma_q(p_1+p_2)}d_qu$$
$$+\frac{|\alpha_1||\sigma_2||\rho_3t^{p_2}+\rho_7|}{|\Lambda|}\int_0^1\frac{(1-qu)^{(p_1+p_2-2)}}{\Gamma_q(p_1+p_2-1)}d_qu+\frac{|\alpha_1||\delta_4||\rho_4t^{p_2}+\rho_8|}{|\Lambda|}\int_0^{\eta_j}\frac{(\eta_j-qu)^{(p_1+p_2-2)}}{\Gamma_q(p_1+p_2-1)}d_qu\Bigg\}$$
$$+\kappa_1(\|x_1-x_2\|+\|y_1-y_2\|)\sup_{t\in[0,1]}\Bigg\{|\beta_1|\int_0^t\frac{(t-qu)^{(p_1+p_2+\xi_1-1)}}{\Gamma_q(p_1+p_2+\xi_1)}d_qu+\frac{|\beta_1||\delta_2||\rho_2t^{p_2}+\rho_6|}{|\Lambda|}$$
$$\times\int_0^{\eta_j}\frac{(\eta_j-qu)^{(p_1+p_2+\xi_1-1)}}{\Gamma_q(p_1+p_2+\xi_1)}d_qu+\frac{|\beta_1||\sigma_1||\rho_3t^{p_2}+\rho_7|}{|\Lambda|}\int_0^1\frac{(1-qu)^{(p_1+p_2+\xi_1-1)}}{\Gamma_q(p_1+p_2+\xi_1)}d_qu$$
$$+\frac{|\beta_1||\sigma_2||\rho_3t^{p_2}+\rho_7|}{|\Lambda|}\int_0^1\frac{(1-qu)^{(p_1+p_2+\xi_1-2)}}{\Gamma_q(p_1+p_2+\xi_1-1)}d_qu+\frac{|\beta_1||\delta_4||\rho_4t^{p_2}+\rho_8|}{|\Lambda|}\int_0^{\eta_j}\frac{(\eta_j-qu)^{(p_1+p_2+\xi_1-2)}}{\Gamma_q(p_1+p_2+\xi_1-1)}d_qu\Bigg\}$$
$$+\iota_2(\|x_1-x_2\|+\|y_1-y_2\|)\sup_{t\in[0,1]}\Bigg\{\frac{|\alpha_2||\delta_1||\rho_1t^{p_2}+\rho_5|}{|\Lambda|}\int_0^{\eta_j}\frac{(\eta_j-qu)^{(r_1+r_2-1)}}{\Gamma_q(r_1+r_2)}d_qu$$
$$+\frac{|\alpha_2||\delta_3||\rho_3t^{p_2}+\rho_7|}{|\Lambda|}\int_0^{\eta_j}\frac{(\eta_j-qu)^{(r_1+r_2-2)}}{\Gamma_q(r_1+r_2-1)}d_qu+\frac{|\alpha_2||\sigma_3||\rho_4t^{p_2}+\rho_8|}{|\Lambda|}\int_0^1\frac{(1-qu)^{(r_1+r_2-1)}}{\Gamma_q(r_1+r_2)}d_qu$$
$$+\frac{|\alpha_2||\sigma_4||\rho_4t^{p_2}+\rho_8|}{|\Lambda|}\int_0^1\frac{(1-qu)^{(r_1+r_2-2)}}{\Gamma_q(r_1+r_2-1)}d_qu\Bigg\}+\kappa_2(\|x_1-x_2\|+\|y_1-y_2\|)$$

$$\times \sup_{t\in[0,1]}\left\{\frac{|\beta_2||\delta_1||\rho_1 t^{p_2}+\rho_5|}{|\Lambda|}\int_0^{\eta_j}\frac{(\eta_j-qu)^{(r_1+r_2+\xi_2-1)}}{\Gamma_q(r_1+r_2+\xi_2)}d_qu+\frac{|\beta_2||\delta_3||\rho_3 t^{p_2}+\rho_7|}{|\Lambda|}\right.$$
$$\times\int_0^{\eta_j}\frac{(\eta_j-qu)^{(r_1+r_2+\xi_2-2)}}{\Gamma_q(r_1+r_2+\xi_2-1)}d_qu+\frac{|\beta_2||\sigma_3||\rho_4 t^{p_2}+\rho_8|}{|\Lambda|}\int_0^1\frac{(1-qu)^{(r_1+r_2+\xi_2-1)}}{\Gamma_q(r_1+r_2+\xi_2)}d_qu$$
$$\left.+\frac{|\beta_2||\sigma_4||\rho_4 t^{p_2}+\rho_8|}{|\Lambda|}\int_0^1\frac{(1-qu)^{(r_1+r_2+\xi_2-2)}}{\Gamma_q(r_1+r_2+\xi_2-1)}d_qu\right\}+(\|x_1-x_2\|+\|y_1-y_2\|)$$
$$\times \sup_{t\in[0,1]}\left\{|\lambda_1|\int_0^t\frac{(t-qu)^{(p_2-1)}}{\Gamma_q(p_2)}d_qu+\frac{|\lambda_2||\delta_1||\rho_1 t^{p_2}+\rho_5|}{|\Lambda|}\int_0^{\eta_j}\frac{(\eta_j-qu)^{(r_2-1)}}{\Gamma_q(r_2)}d_qu\right.$$
$$+\frac{|\lambda_1||\delta_2||\rho_2 t^{p_2}+\rho_6|}{|\Lambda|}\int_0^{\eta_j}\frac{(\eta_j-qu)^{(p_2-1)}}{\Gamma_q(p_2)}d_qu+\frac{|\lambda_2||\delta_3||\rho_3 t^{p_2}+\rho_7|}{|\Lambda|}\int_0^{\eta_j}\frac{(\eta_j-qu)^{(r_2-2)}}{\Gamma_q(r_2-1)}d_qu$$
$$+\frac{|\lambda_1||\sigma_1||\rho_3 t^{p_2}+\rho_7|}{|\Lambda|}\int_0^1\frac{(1-qu)^{(p_2-1)}}{\Gamma_q(p_2)}d_qu+\frac{|\lambda_1||\sigma_2||\rho_3 t^{p_2}+\rho_7|}{|\Lambda|}\int_0^1\frac{(1-qu)^{(p_2-2)}}{\Gamma_q(p_2-1)}d_qu$$
$$+\frac{|\lambda_1||\delta_4||\rho_4 t^{p_2}+\rho_8|}{|\Lambda|}\int_0^{\eta_j}\frac{(\eta_j-qu)^{(p_2-2)}}{\Gamma_q(p_2-1)}d_qu+\frac{|\lambda_2||\sigma_3||\rho_4 t^{p_2}+\rho_8|}{|\Lambda|}\int_0^1\frac{(1-qu)^{(r_2-1)}}{\Gamma_q(r_2)}d_qu$$
$$\left.+\frac{|\lambda_2||\sigma_4||\rho_4 t^{p_2}+\rho_8|}{|\Lambda|}\int_0^1\frac{(1-qu)^{(r_2-2)}}{\Gamma_q(r_2-1)}d_qu\right\}$$
$$\le(\iota_1\Psi_1+\iota_2\Psi_2+\kappa_1\Phi_1+\kappa_2\Phi_2+\Theta_1)(\|x_1-x_2\|+\|y_1-y_2\|),$$

where $\Psi_1,\ \Psi_2,\ \Phi_1,\ \Phi_2,\ \Theta_1$ are given in (13).

Similarly, one can obtain

$$\|\mathcal{G}_2(x_1,y_1)-\mathcal{G}_2(x_2,y_2)\|$$
$$\le\sup_{t\in[0,1]}\left\{|\alpha_2|\int_0^t\frac{(t-qu)^{(r_1+r_2-1)}}{\Gamma_q(r_1+r_2)}|f_2(u,x_1(u),y_1(u))-f_2(u,x_2(u),y_2(u))|d_qu\right.$$
$$+|\beta_2|\int_0^t\frac{(t-qu)^{(r_1+r_2+\xi_2-1)}}{\Gamma_q(r_1+r_2+\xi_2)}|g_2(u,x_1(u),y_1(u))-g_2(u,x_2(u),y_2(u))|d_qu$$
$$+|\lambda_2|\int_0^t\frac{(t-qu)^{(r_2-1)}}{\Gamma_q(r_2)}|y_1(u)-y_2(u)|d_qu+\frac{|\delta_1||\rho_9 t^{r_2}+\rho_{13}|}{|\Lambda|}\left(|\alpha_2|\int_0^{\eta_j}\frac{(\eta_j-qu)^{(r_1+r_2-1)}}{\Gamma_q(r_1+r_2)}\right.$$
$$\times|f_2(u,x_1(u),y_1(u))-f_2(u,x_2(u),y_2(u))|d_qu+|\beta_2|\int_0^{\eta_j}\frac{(\eta_j-qu)^{(r_1+r_2+\xi_2-1)}}{\Gamma_q(r_1+r_2+\xi_2)}$$
$$\left.\times|g_2(u,x_1(u),y_1(u))-g_2(u,x_2(u),y_2(u))|d_qu+|\lambda_2|\int_0^{\eta_j}\frac{(\eta_j-qu)^{(r_2-1)}}{\Gamma_q(r_2)}|y_1(u)-y_2(u)|d_qu\right)$$
$$+\frac{|\delta_2||\rho_{10} t^{r_2}+\rho_{14}|}{|\Lambda|}\left(|\alpha_1|\int_0^{\eta_j}\frac{(\eta_j-qu)^{(p_1+p_2-1)}}{\Gamma_q(p_1+p_2)}|f_1(u,x_1(u),y_1(u))-f_1(u,x_2(u),y_2(u))|d_qu\right.$$
$$+|\beta_1|\int_0^{\eta_j}\frac{(\eta_j-qu)^{(p_1+p_2+\xi_1-1)}}{\Gamma_q(p_1+p_2+\xi_1)}|g_1(u,x_1(u),y_1(u))-g_1(u,x_2(u),y_2(u))|d_qu$$
$$\left.+|\lambda_1|\int_0^{\eta_j}\frac{(\eta_j-qu)^{(p_2-1)}}{\Gamma_q(p_2)}|x_1(u)-x_2(u)|d_qu\right)+\frac{|\delta_3||\rho_{11} t^{r_2}+\rho_{15}|}{|\Lambda|}\left(|\alpha_2|\int_0^{\eta_j}\frac{(\eta_j-qu)^{(r_1+r_2-2)}}{\Gamma_q(r_1+r_2-1)}\right.$$
$$\times|f_2(u,x_1(u),y_1(u))-f_2(u,x_2(u),y_2(u))|d_qu+|\beta_2|\int_0^{\eta_j}\frac{(\eta_j-qu)^{(r_1+r_2+\xi_2-2)}}{\Gamma_q(r_1+r_2+\xi_2-1)}$$
$$\left.\times|g_2(u,x_1(u),y_1(u))-g_2(u,x_2(u),y_2(u))|d_qu+|\lambda_2|\int_0^{\eta_j}\frac{(\eta_j-qu)^{(r_2-2)}}{\Gamma_q(r_2-1)}|y_1(u)-y_2(u)|d_qu\right)$$
$$+\frac{|\sigma_1||\rho_{11} t^{r_2}+\rho_{15}|}{|\Lambda|}\left(|\alpha_1|\int_0^1\frac{(1-qu)^{(p_1+p_2-1)}}{\Gamma_q(p_1+p_2)}|f_1(u,x_1(u),y_1(u))-f_1(u,x_2(u),y_2(u))|d_qu\right.$$
$$+|\beta_1|\int_0^1\frac{(1-qu)^{(p_1+p_2+\xi_1-1)}}{\Gamma_q(p_1+p_2+\xi_1)}|g_1(u,x_1(u),y_1(u))-g_1(u,x_2(u),y_2(u))|d_qu$$
$$\left.+|\lambda_1|\int_0^1\frac{(1-qu)^{(p_2-1)}}{\Gamma_q(p_2)}|x_1(u)-x_2(u)|d_qu\right)+\frac{|\sigma_2||\rho_{11} t^{r_2}+\rho_{15}|}{|\Lambda|}\left(|\alpha_1|\int_0^1\frac{(1-qu)^{(p_1+p_2-2)}}{\Gamma_q(p_1+p_2-1)}\right.$$

$$
\begin{aligned}
&\times|f_1(u,x_1(u),y_1(u))-f_1(u,x_2(u),y_2(u))|d_qu+|\beta_1|\int_0^1\frac{(1-qu)^{(p_1+p_2+\xi_1-2)}}{\Gamma_q(p_1+p_2+\xi_1-1)}\\
&\times|g_1(u,x_1(u),y_1(u))-g_1(u,x_2(u),y_2(u))|d_qu+|\lambda_1|\int_0^1\frac{(1-qu)^{(p_2-2)}}{\Gamma_q(p_2-1)}|x_1(u)-x_2(u)|d_qu\Bigg)\\
&+\frac{|\delta_4||\rho_{12}t^{r_2}+\rho_{16}|}{|\Lambda|}\Bigg(|\alpha_1|\int_0^{\eta_j}\frac{(\eta_j-qu)^{(p_1+p_2-2)}}{\Gamma_q(p_1+p_2-1)}|f_1(u,x_1(u),y_1(u))-f_1(u,x_2(u),y_2(u))|d_qu\\
&+|\beta_1|\int_0^{\eta_j}\frac{(\eta_j-qu)^{(p_1+p_2+\xi_1-2)}}{\Gamma_q(p_1+p_2+\xi_1-1)}|g_1(u,x_1(u),y_1(u))-g_1(u,x_2(u),y_2(u))|d_qu\\
&+|\lambda_1|\int_0^{\eta_j}\frac{(\eta_j-qu)^{(p_2-2)}}{\Gamma_q(p_2-1)}|x_1(u)-x_2(u)|d_qu\Bigg)+\frac{|\sigma_3||\rho_{12}t^{r_2}+\rho_{16}|}{|\Lambda|}\Bigg(|\alpha_2|\int_0^1\frac{(1-qu)^{(r_1+r_2-1)}}{\Gamma_q(r_1+r_2)}\\
&\times|f_2(u,x_1(u),y_1(u))-f_2(u,x_2(u),y_2(u))|d_qu+|\beta_2|\int_0^1\frac{(1-qu)^{(r_1+r_2+\xi_2-1)}}{\Gamma_q(r_1+r_2+\xi_2)}\\
&\times|g_2(u,x_1(u),y_1(u))-g_2(u,x_2(u),y_2(u))|d_qu+|\lambda_2|\int_0^1\frac{(1-qu)^{(r_2-1)}}{\Gamma_q(r_2)}|y_1(u)-y_2(u)|d_qu\Bigg)\\
&+\frac{|\sigma_4||\rho_{12}t^{r_2}+\rho_{16}|}{|\Lambda|}\Bigg(|\alpha_2|\int_0^1\frac{(1-qu)^{(r_1+r_2-2)}}{\Gamma_q(r_1+r_2-1)}|f_2(u,x_1(u),y_1(u))-f_2(u,x_2(u),y_2(u))|d_qu\\
&+|\beta_2|\int_0^1\frac{(1-qu)^{(r_1+r_2+\xi_2-2)}}{\Gamma_q(r_1+r_2+\xi_2-1)}|g_2(u,x_1(u),y_1(u))-g_2(u,x_2(u),y_2(u))|d_qu\\
&+|\lambda_2|\int_0^1\frac{(1-qu)^{(r_2-2)}}{\Gamma_q(r_2-1)}|y_1(u)-y_2(u)|d_qu\Bigg)\Bigg\}\\
&\leq\iota_1(\|x_1-x_2\|+\|y_1-y_2\|)\sup_{t\in[0,1]}\Bigg\{\frac{|\alpha_1||\delta_2||\rho_{10}t^{r_2}+\rho_{14}|}{|\Lambda|}\int_0^{\eta_j}\frac{(\eta_j-qu)^{(p_1+p_2-1)}}{\Gamma_q(p_1+p_2)}d_qu\\
&+\frac{|\alpha_1||\sigma_1||\rho_{11}t^{r_2}+\rho_{15}|}{|\Lambda|}\int_0^1\frac{(1-qu)^{(p_1+p_2-1)}}{\Gamma_q(p_1+p_2)}d_qu+\frac{|\alpha_1||\sigma_2||\rho_{11}t^{r_2}+\rho_{15}|}{|\Lambda|}\int_0^1\frac{(1-qu)^{(p_1+p_2-2)}}{\Gamma_q(p_1+p_2-1)}d_qu\\
&+\frac{|\alpha_1||\delta_4||\rho_{12}t^{r_2}+\rho_{16}|}{|\Lambda|}\int_0^{\eta_j}\frac{(\eta_j-qu)^{(p_1+p_2-2)}}{\Gamma_q(p_1+p_2-1)}d_qu\Bigg\}+\kappa_1(\|x_1-x_2\|+\|y_1-y_2\|)\\
&\times\sup_{t\in[0,1]}\Bigg\{\frac{|\beta_1||\delta_2||\rho_{10}t^{r_2}+\rho_{14}|}{|\Lambda|}\int_0^{\eta_j}\frac{(\eta_j-qu)^{(p_1+p_2+\xi_1-1)}}{\Gamma_q(p_1+p_2+\xi_1)}d_qu+\frac{|\beta_1||\sigma_1||\rho_{11}t^{r_2}+\rho_{15}|}{|\Lambda|}\\
&\times\int_0^1\frac{(1-qu)^{(p_1+p_2+\xi_1-1)}}{\Gamma_q(p_1+p_2+\xi_1)}d_qu+\frac{|\beta_1||\sigma_2||\rho_{11}t^{r_2}+\rho_{15}|}{|\Lambda|}\int_0^1\frac{(1-qu)^{(p_1+p_2+\xi_1-2)}}{\Gamma_q(p_1+p_2+\xi_1-1)}d_qu\\
&+\frac{|\beta_1||\delta_4||\rho_{12}t^{r_2}+\rho_{16}|}{|\Lambda|}\int_0^{\eta_j}\frac{(\eta_j-qu)^{(p_1+p_2+\xi_1-2)}}{\Gamma_q(p_1+p_2+\xi_1-1)}d_qu\Bigg\}+\iota_2(\|x_1-x_2\|+\|y_1-y_2\|)\\
&\times\sup_{t\in[0,1]}\Bigg\{|\alpha_2|\int_0^t\frac{(t-qu)^{(r_1+r_2-1)}}{\Gamma_q(r_1+r_2)}d_qu+\frac{|\alpha_2||\delta_1||\rho_9t^{r_2}+\rho_{13}|}{|\Lambda|}\int_0^{\eta_j}\frac{(\eta_j-qu)^{(r_1+r_2-1)}}{\Gamma_q(r_1+r_2)}d_qu\\
&+\frac{|\alpha_2||\delta_3||\rho_{11}t^{r_2}+\rho_{15}|}{|\Lambda|}\int_0^{\eta_j}\frac{(\eta_j-qu)^{(r_1+r_2-2)}}{\Gamma_q(r_1+r_2-1)}d_qu+\frac{|\alpha_2||\sigma_3||\rho_{12}t^{r_2}+\rho_{16}|}{|\Lambda|}\int_0^1\frac{(1-qu)^{(r_1+r_2-1)}}{\Gamma_q(r_1+r_2)}d_qu\\
&+\frac{|\alpha_2||\sigma_4||\rho_{12}t^{r_2}+\rho_{16}|}{|\Lambda|}\int_0^1\frac{(1-qu)^{(r_1+r_2-2)}}{\Gamma_q(r_1+r_2-1)}d_qu\Bigg\}+\kappa_2(\|x_1-x_2\|+\|y_1-y_2\|)\\
&\times\sup_{t\in[0,1]}\Bigg\{|\beta_2|\int_0^t\frac{(t-qu)^{(r_1+r_2+\xi_2-1)}}{\Gamma_q(r_1+r_2+\xi_2)}d_qu+\frac{|\beta_2||\delta_1||\rho_9t^{r_2}+\rho_{13}|}{|\Lambda|}\int_0^{\eta_j}\frac{(\eta_j-qu)^{(r_1+r_2+\xi_2-1)}}{\Gamma_q(r_1+r_2+\xi_2)}d_qu\\
&+\frac{|\beta_2||\delta_3||\rho_{11}t^{r_2}+\rho_{15}|}{|\Lambda|}\int_0^{\eta_j}\frac{(\eta_j-qu)^{(r_1+r_2+\xi_2-2)}}{\Gamma_q(r_1+r_2+\xi_2-1)}d_qu+\frac{|\beta_2||\sigma_3||\rho_{12}t^{r_2}+\rho_{16}|}{|\Lambda|}\\
&\times\int_0^1\frac{(1-qu)^{(r_1+r_2+\xi_2-1)}}{\Gamma_q(r_1+r_2+\xi_2)}d_qu+\frac{|\beta_2||\sigma_4||\rho_{12}t^{r_2}+\rho_{16}|}{|\Lambda|}\int_0^1\frac{(1-qu)^{(r_1+r_2+\xi_2-2)}}{\Gamma_q(r_1+r_2+\xi_2-1)}d_qu\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
&+(\|x_1-x_2\|+\|y_1-y_2\|)\sup_{t\in[0,1]}\bigg\{|\lambda_2|\int_0^t\frac{(t-qu)^{(r_2-1)}}{\Gamma_q(r_2)}d_qu\\
&+\frac{|\lambda_2||\delta_1||\rho_9t^{r_2}+\rho_{13}|}{|\Lambda|}\int_0^{\eta_j}\frac{(\eta_j-qu)^{(r_2-1)}}{\Gamma_q(r_2)}d_qu+\frac{|\lambda_1||\delta_2||\rho_{10}t^{r_2}+\rho_{14}|}{|\Lambda|}\int_0^{\eta_j}\frac{(\eta_j-qu)^{(p_2-1)}}{\Gamma_q(p_2)}d_qu\\
&+\frac{|\lambda_2||\delta_3||\rho_{11}t^{r_2}+\rho_{15}|}{|\Lambda|}\int_0^{\eta_j}\frac{(\eta_j-qu)^{(r_2-2)}}{\Gamma_q(r_2-1)}d_qu+\frac{|\lambda_1||\sigma_1||\rho_{11}t^{r_2}+\rho_{15}|}{|\Lambda|}\int_0^{1}\frac{(1-qu)^{(p_2-1)}}{\Gamma_q(p_2)}d_qu\\
&+\frac{|\lambda_1||\sigma_2||\rho_{11}t^{r_2}+\rho_{15}|}{|\Lambda|}\int_0^{1}\frac{(1-qu)^{(p_2-2)}}{\Gamma_q(p_2-1)}d_qu+\frac{|\lambda_1||\delta_4||\rho_{12}t^{r_2}+\rho_{16}|}{|\Lambda|}\int_0^{\eta_j}\frac{(\eta_j-qu)^{(p_2-2)}}{\Gamma_q(p_2-1)}d_qu\\
&+\frac{|\lambda_2||\sigma_3||\rho_{12}t^{r_2}+\rho_{16}|}{|\Lambda|}\int_0^{1}\frac{(1-qu)^{(r_2-1)}}{\Gamma_q(r_2)}d_qu+\frac{|\lambda_2||\sigma_4||\rho_{12}t^{r_2}+\rho_{16}|}{|\Lambda|}\int_0^{1}\frac{(1-qu)^{(r_2-2)}}{\Gamma_q(r_2-1)}d_qu\bigg\}\\
&\le(\iota_1\Psi_3+\iota_2\Psi_4+\kappa_1\Phi_3+\kappa_2\Phi_4+\Theta_2)(\|x_1-x_2\|+\|y_1-y_2\|),
\end{aligned}
$$

where $\Psi_3,\ \Psi_4,\ \Phi_3,\ \Phi_4,\ \Theta_2$ are given in (13). Consequently, we obtain

$$\|\mathcal{G}(x_1,y_1)-\mathcal{G}(x_2,y_2)\|\le\Upsilon\big(\|x_1-x_2\|+\|y_1-y_2\|\big).$$

As $\Upsilon<1$ by (15), therefore $\mathcal{G}$ is a contraction. Hence, we deduce by the conclusion of the Banach contraction mapping principle that the operator $\mathcal{G}$ has a unique fixed point, which is indeed the unique solution of the problem (1) and (2). The proof is completed. □

Next, we present an existence result for the problem (1) and (2) which is proved by means of the Leray–Schauder nonlinear alternative [48].

Theorem 2. *Assume that*

($\mathbf{A}_3$) $f_1,f_2,g_1,g_2:[0,1]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ *are continuous functions and that there exist real constants* $\tau_i,\widetilde{\tau}_i,\epsilon_i,\widetilde{\epsilon}_i\ge0,\ (i=1,2)$ *and* $\tau_0,\widetilde{\tau}_0,\epsilon_0,\widetilde{\epsilon}_0>0$ *such that,* $\forall x,y\in\mathbb{R}$,

$$|f_1(t,x,y)|\le\tau_0+\tau_1|x|+\tau_2|y|,\quad |f_2(t,x,y)|\le\widetilde{\tau}_0+\widetilde{\tau}_1|x|+\widetilde{\tau}_2|y|,$$

$$|g_1(t,x,y)|\le\epsilon_0+\epsilon_1|x|+\epsilon_2|y|,\quad |g_2(t,x,y)|\le\widetilde{\epsilon}_0+\widetilde{\epsilon}_1|x|+\widetilde{\epsilon}_2|y|,$$

Then the system (1) and (2) has at least one solution on $[0,1]$ *provided that*

$$(\Psi_1+\Psi_3)\tau_1+(\Psi_2+\Psi_4)\widetilde{\tau}_1+(\Phi_1+\Phi_3)\epsilon_1+(\Phi_2+\Phi_4)\widetilde{\epsilon}_1+v_1<1,$$

$$(\Psi_1+\Psi_3)\tau_2+(\Psi_2+\Psi_4)\widetilde{\tau}_2+(\Phi_1+\Phi_3)\epsilon_2+(\Phi_2+\Phi_4)\widetilde{\epsilon}_2+v_2<1,$$

where $\Psi_i,\ \Phi_i,\ i=1,2,3,4$ *are given by (13) and*

$$v_1=|\lambda_1|\left(\frac{1}{\Gamma_q(p_2+1)}+\frac{|\gamma_9+\gamma_{10}|}{|\Lambda|}+\frac{|\gamma_{21}+\gamma_{22}|}{|\Lambda|}\right),$$

$$v_2=|\lambda_2|\left(\frac{1}{\Gamma_q(r_2+1)}+\frac{|\gamma_{11}+\gamma_{12}|}{|\Lambda|}+\frac{|\gamma_{23}+\gamma_{24}|}{|\Lambda|}\right).$$

Proof. In the first step, it will be shown that the operator $\mathcal{G}:\mathcal{C}\times\mathcal{C}\to\mathcal{C}\times\mathcal{C}$ is completely continuous. Notice that the operator $\mathcal{G}$ is continuous in view of the continuity of the functions f_1,f_2,g_1,g_2. Let $\mathcal{Y}\subset\mathcal{C}\times\mathcal{C}$ be bounded. Then, for all $(x,y)\in\mathcal{Y}$, there exist constants $\hbar_1,\hbar_2,\varpi_1,\varpi_2$ such that $|f_1(t,x(t),y(t))|\le\hbar_1,|f_2(t,x(t),y(t))|\le$

$\hbar_2, |g_1(t,x(t),y(t))| \leq \varpi_1$, $|g_2(t,x(t),y(t))| \leq \varpi_2$. Let $(x,y) \in \mathcal{Y}$. Then there exists φ such that $\|(x,y)\| = \|x\| + \|y\| \leq \varphi$, and for any $(x,y) \in \mathcal{Y}$, we have

$$
\begin{aligned}
&\|\mathcal{G}_1(x,y)\| \\
&\leq \sup_{t\in[0,1]} \Bigg\{ |\alpha_1| \int_0^t \frac{(t-qu)^{(p_1+p_2-1)}}{\Gamma_q(p_1+p_2)} |f_1(u,x(u),y(u))| d_q u \\
&+|\beta_1| \int_0^t \frac{(t-qu)^{(p_1+p_2+\xi_1-1)}}{\Gamma_q(p_1+p_2+\xi)} |g_1(u,x(u),y(u))| d_q u + |\lambda_1| \int_0^t \frac{(t-qu)^{(p_2-1)}}{\Gamma_q(p_2)} |x(u)| d_q u \\
&+\frac{|\delta_1||\rho_1 t^{p_2}+\rho_5|}{|\Lambda|} \Bigg(|\alpha_2| \int_0^{\eta_j} \frac{(\eta_j-qu)^{(r_1+r_2-1)}}{\Gamma_q(r_1+r_2)} |f_2(u,x(u),y(u))| d_q u \\
&+|\beta_2| \int_0^{\eta_j} \frac{(\eta_j-qu)^{(r_1+r_2+\xi_2-1)}}{\Gamma_q(r_1+r_2+\xi_2)} |g_2(u,x(u),y(u))| d_q u + |\lambda_2| \int_0^{\eta_j} \frac{(\eta_j-qu)^{(r_2-1)}}{\Gamma_q(r_2)} |y(u)| d_q u \Bigg) \\
&+\frac{|\delta_2||\rho_2 t^{p_2}+\rho_6|}{|\Lambda|} \Bigg(|\alpha_1| \int_0^{\eta_j} \frac{(\eta_j-qu)^{(p_1+p_2-1)}}{\Gamma_q(p_1+p_2)} |f_1(u,x(u),y(u))| d_q u \\
&+|\beta_1| \int_0^{\eta_j} \frac{(\eta_j-qu)^{(p_1+p_2+\xi_1-1)}}{\Gamma_q(p_1+p_2+\xi_1)} |g_1(u,x(u),y(u))| d_q u + |\lambda_1| \int_0^{\eta_j} \frac{(\eta_j-qu)^{(p_2-1)}}{\Gamma_q(p_2)} |x(u)| d_q u \Bigg) \\
&+\frac{|\delta_3||\rho_3 t^{p_2}+\rho_7|}{|\Lambda|} \Bigg(|\alpha_2| \int_0^{\eta_j} \frac{(\eta_j-qu)^{(r_1+r_2-2)}}{\Gamma_q(r_1+r_2-1)} |f_2(u,x(u),y(u))| d_q u \\
&+|\beta_2| \int_0^{\eta_j} \frac{(\eta_j-qu)^{(r_1+r_2+\xi_2-2)}}{\Gamma_q(r_1+r_2+\xi_2-1)} |g_2(u,x(u),y(u))| d_q u + |\lambda_2| \int_0^{\eta_j} \frac{(\eta_j-qu)^{(r_2-2)}}{\Gamma_q(r_2-1)} |y(u)| d_q u \Bigg) \\
&+\frac{|\sigma_1||\rho_3 t^{p_2}+\rho_7|}{|\Lambda|} \Bigg(|\alpha_1| \int_0^{1} \frac{(1-qu)^{(p_1+p_2-1)}}{\Gamma_q(p_1+p_2)} |f_1(u,x(u),y(u))| d_q u \\
&+|\beta_1| \int_0^{1} \frac{(1-qu)^{(p_1+p_2+\xi_1-1)}}{\Gamma_q(p_1+p_2+\xi_1)} |g_1(u,x(u),y(u))| d_q u + |\lambda_1| \int_0^{1} \frac{(1-qu)^{(p_2-1)}}{\Gamma_q(p_2)} |x(u)| d_q u \Bigg) \\
&+\frac{|\sigma_2||\rho_3 t^{p_2}+\rho_7|}{|\Lambda|} \Bigg(|\alpha_1| \int_0^{1} \frac{(1-qu)^{(p_1+p_2-2)}}{\Gamma_q(p_1+p_2-1)} |f_1(u,x(u),y(u))| d_q u \\
&+|\beta_1| \int_0^{1} \frac{(1-qu)^{(p_1+p_2+\xi_1-2)}}{\Gamma_q(p_1+p_2+\xi_1-1)} |g_1(u,x(u),y(u))| d_q u + |\lambda_1| \int_0^{1} \frac{(1-qu)^{(p_2-2)}}{\Gamma_q(p_2-1)} |x(u)| d_q u \Bigg) \\
&+\frac{|\delta_4||\rho_4 t^{p_2}+\rho_8|}{|\Lambda|} \Bigg(|\alpha_1| \int_0^{\eta_j} \frac{(\eta_j-qu)^{(p_1+p_2-2)}}{\Gamma_q(p_1+p_2-1)} |f_1(u,x(u),y(u))| d_q u \\
&+|\beta_1| \int_0^{\eta_j} \frac{(\eta_j-qu)^{(p_1+p_2+\xi_1-2)}}{\Gamma_q(p_1+p_2+\xi_1-1)} |g_1(u,x(u),y(u))| d_q u + |\lambda_1| \int_0^{\eta_j} \frac{(\eta_j-qu)^{(p_2-2)}}{\Gamma_q(p_2-1)} |x(u)| d_q u \Bigg) \\
&+\frac{|\sigma_3||\rho_4 t^{p_2}+\rho_8|}{|\Lambda|} \Bigg(|\alpha_2| \int_0^{1} \frac{(1-qu)^{(r_1+r_2-1)}}{\Gamma_q(r_1+r_2)} |f_2(u,x(u),y(u))| d_q u \\
&+|\beta_2| \int_0^{1} \frac{(1-qu)^{(r_1+r_2+\xi_2-1)}}{\Gamma_q(r_1+r_2+\xi_2)} |g_2(u,x(u),y(u))| d_q u + |\lambda_2| \int_0^{1} \frac{(1-qu)^{(r_2-1)}}{\Gamma_q(r_2)} |y(u)| d_q u \Bigg) \\
&+\frac{|\sigma_4||\rho_4 t^{p_2}+\rho_8|}{|\Lambda|} \Bigg(|\alpha_2| \int_0^{1} \frac{(1-qu)^{(r_1+r_2-2)}}{\Gamma_q(r_1+r_2-1)} |f_2(u,x(u),y(u))| d_q u \\
&+|\beta_2| \int_0^{1} \frac{(1-qu)^{(r_1+r_2+\xi_2-2)}}{\Gamma_q(r_1+r_2+\xi_2-1)} |g_2(u,x(u),y(u))| d_q u + |\lambda_2| \int_0^{1} \frac{(1-qu)^{(r_2-2)}}{\Gamma_q(r_2-1)} |y(u)| d_q u \Bigg) \Bigg\} \\
&\leq \hbar_1 \Psi_1 + \hbar_2 \Psi_2 + \varpi_1 \Phi_1 + \varpi_2 \Phi_2 + \varphi \Theta_1
\end{aligned}
$$

where $\Psi_1,\ \Psi_2,\ \Phi_1,\ \Phi_2,\ \Theta_1$ are given in (13). Similarly, we can find that

$$
\begin{aligned}
&\|\mathcal{G}_2(x,y)\| \\
&\le \sup_{t\in[0,1]}\Bigg\{|\alpha_2|\int_0^t \frac{(t-qu)^{(r_1+r_2-1)}}{\Gamma_q(r_1+r_2)}|f_2(u,x(u),y(u))|d_qu \\
&+|\beta_2|\int_0^t \frac{(t-qu)^{(r_1+r_2+\xi_2-1)}}{\Gamma_q(r_1+r_2+\xi_2)}|g_2(u,x(u),y(u))|d_qu+|\lambda_2|\int_0^t \frac{(t-qu)^{(r_2-1)}}{\Gamma_q(r_2)}|y(u)|d_qu \\
&+\frac{|\delta_1||\rho_9 t^{r_2}+\rho_{13}|}{|\Lambda|}\Bigg(|\alpha_2|\int_0^{\eta_j} \frac{(\eta_j-qu)^{(r_1+r_2-1)}}{\Gamma_q(r_1+r_2)}|f_2(u,x(u),y(u))|d_qu \\
&+|\beta_2|\int_0^{\eta_j} \frac{(\eta_j-qu)^{(r_1+r_2+\xi_2-1)}}{\Gamma_q(r_1+r_2+\xi_2)}|g_2(u,x(u),y(u))|d_qu+|\lambda_2|\int_0^{\eta_j} \frac{(\eta_j-qu)^{(r_2-1)}}{\Gamma_q(r_2)}|y(u)|d_qu\Bigg) \\
&+\frac{|\delta_2||\rho_{10} t^{r_2}+\rho_{14}|}{|\Lambda|}\Bigg(|\alpha_1|\int_0^{\eta_j} \frac{(\eta_j-qu)^{(p_1+p_2-1)}}{\Gamma_q(p_1+p_2)}|f_1(u,x(u),y(u))|d_qu \\
&+|\beta_1|\int_0^{\eta_j} \frac{(\eta_j-qu)^{(p_1+p_2+\xi_1-1)}}{\Gamma_q(p_1+p_2+\xi_1)}|g_1(u,x(u),y(u))|d_qu+|\lambda_1|\int_0^{\eta_j} \frac{(\eta_j-qu)^{(p_2-1)}}{\Gamma_q(p_2)}|x(u)|d_qu\Bigg) \\
&+\frac{|\delta_3||\rho_{11} t^{r_2}+\rho_{15}|}{|\Lambda|}\Bigg(|\alpha_2|\int_0^{\eta_j} \frac{(\eta_j-qu)^{(r_1+r_2-2)}}{\Gamma_q(r_1+r_2-1)}|f_2(u,x(u),y(u))|d_qu \\
&+|\beta_2|\int_0^{\eta_j} \frac{(\eta_j-qu)^{(r_1+r_2+\xi_2-2)}}{\Gamma_q(r_1+r_2+\xi_2-1)}|g_2(u,x(u),y(u))|d_qu+|\lambda_2|\int_0^{\eta_j} \frac{(\eta_j-qu)^{(r_2-2)}}{\Gamma_q(r_2-1)}|y(u)|d_qu\Bigg) \\
&+\frac{|\sigma_1||\rho_{11} t^{r_2}+\rho_{15}|}{|\Lambda|}\Bigg(|\alpha_1|\int_0^{1} \frac{(1-qu)^{(p_1+p_2-1)}}{\Gamma_q(p_1+p_2)}|f_1(u,x(u),y(u))|d_qu \\
&+|\beta_1|\int_0^{1} \frac{(1-qu)^{(p_1+p_2+\xi_1-1)}}{\Gamma_q(p_1+p_2+\xi_1)}|g_1(u,x(u),y(u))|d_qu+|\lambda_1|\int_0^{1} \frac{(1-qu)^{(p_2-1)}}{\Gamma_q(p_2)}|x(u)|d_qu\Bigg) \\
&+\frac{|\sigma_2||\rho_{11} t^{r_2}+\rho_{15}|}{|\Lambda|}\Bigg(|\alpha_1|\int_0^{1} \frac{(1-qu)^{(p_1+p_2-2)}}{\Gamma_q(p_1+p_2-1)}|f_1(u,x(u),y(u))|d_qu \\
&+|\beta_1|\int_0^{1} \frac{(1-qu)^{(p_1+p_2+\xi_1-2)}}{\Gamma_q(p_1+p_2+\xi_1-1)}|g_1(u,x(u),y(u))|d_qu+|\lambda_1|\int_0^{1} \frac{(1-qu)^{(p_2-2)}}{\Gamma_q(p_2-1)}|x(u)|d_qu\Bigg) \\
&+\frac{|\delta_4||\rho_{12} t^{r_2}+\rho_{16}|}{|\Lambda|}\Bigg(|\alpha_1|\int_0^{\eta_j} \frac{(\eta_j-qu)^{(p_1+p_2-2)}}{\Gamma_q(p_1+p_2-1)}|f_1(u,x(u),y(u))|d_qu \\
&+|\beta_1|\int_0^{\eta_j} \frac{(\eta_j-qu)^{(p_1+p_2+\xi_1-2)}}{\Gamma_q(p_1+p_2+\xi_1-1)}|g_1(u,x(u),y(u))|d_qu+|\lambda_1|\int_0^{\eta_j} \frac{(\eta_j-qu)^{(p_2-2)}}{\Gamma_q(p_2-1)}|x(u)|d_qu\Bigg) \\
&+\frac{|\sigma_3||\rho_{12} t^{r_2}+\rho_{16}|}{|\Lambda|}\Bigg(|\alpha_2|\int_0^{1} \frac{(1-qu)^{(r_1+r_2-1)}}{\Gamma_q(r_1+r_2)}|f_2(u,x(u),y(u))|d_qu \\
&+|\beta_2|\int_0^{1} \frac{(1-qu)^{(r_1+r_2+\xi_2-1)}}{\Gamma_q(r_1+r_2+\xi_2)}|g_2(u,x(u),y(u))|d_qu+|\lambda_2|\int_0^{1} \frac{(1-qu)^{(r_2-1)}}{\Gamma_q(r_2)}|y(u)|d_qu\Bigg) \\
&+\frac{|\sigma_4||\rho_{12} t^{r_2}+\rho_{16}|}{|\Lambda|}\Bigg(|\alpha_2|\int_0^{1} \frac{(1-qu)^{(r_1+r_2-2)}}{\Gamma_q(r_1+r_2-1)}|f_2(u,x(u),y(u))|d_qu \\
&+|\beta_2|\int_0^{1} \frac{(1-qu)^{(r_1+r_2+\xi_2-2)}}{\Gamma_q(r_1+r_2+\xi_2-1)}|g_2(u,x(u),y(u))|d_qu+|\lambda_2|\int_0^{1} \frac{(1-qu)^{(r_2-2)}}{\Gamma_q(r_2-1)}|y(u)|d_qu\Bigg)\Bigg\} \\
&\le \hbar_1\Psi_3+\hbar_2\Psi_4+\varpi_1\Phi_3+\varpi_2\Phi_4+\varphi\Theta_2,
\end{aligned}
$$

where $\Psi_3,\ \Psi_4,\ \Phi_3,\ \Phi_4,\ \Theta_2$ are given in (13).

Consequently, we obtain

$$\|\mathcal{G}(x,y)\| \le (\Psi_1+\Psi_3)\hbar_1+(\Psi_2+\Psi_4)\hbar_2+(\Phi_1+\Phi_3)\varpi_1+(\Phi_2+\Phi_4)\varpi_2+(\Theta_1+\Theta_2)\varphi.$$

Therefore, the operator $\mathcal{G}$ is uniformly bounded. Next, we show that the operator $\mathcal{G}$ is equicontinuous. Let $t_1, t_2 \in [0,1]$ with $t_1 < t_2$. Then we have

$$\begin{aligned}
&|\mathcal{G}_1(x(t_2), y(t_2)) - \mathcal{G}_1(x(t_1), y(t_1))| \\
&\leq \frac{|\alpha_1|\hbar_1}{\Gamma_q(p_1+p_2)}\left|\int_0^{t_1}[(t_2-qu)^{(p_1+p_2-1)} - (t_1-qu)^{(p_1+p_2-1)}]d_qu + \int_{t_1}^{t_2}(t_2-qu)^{(p_1+p_2-1)}d_qu\right| \\
&+\frac{|\beta_1|\varpi_1}{\Gamma_q(p_1+p_2+\xi_1)}\left|\int_0^{t_1}[(t_2-qu)^{(p_1+p_2+\xi_1-1)} - (t_1-qu)^{(p_1+p_2+\xi_1-1)}]d_qu\right. \\
&\left.+\int_{t_1}^{t_2}(t_2-qu)^{(p_1+p_2+\xi_1-1)}d_qu\right| + \frac{|\lambda_1|\varphi}{\Gamma_q(p_2)}\left|\int_0^{t_1}[(t_2-qu)^{(p_2-1)} - (t_1-qu)^{(p_2-1)}]d_qu\right. \\
&\left.+\int_{t_1}^{t_2}(t_2-qu)^{(p_2-1)}d_qu\right| + \frac{|t_2^{p_2} - t_1^{p_2}|}{|\Lambda|}\Bigg[|\delta_1||\rho_1|\Big(\frac{|\alpha_2|\eta_j^{r_1+r_2}\hbar_2}{\Gamma_q(r_1+r_2+1)} + \frac{|\beta_2|\eta_j^{r_1+r_2+\xi_2}\varpi_2}{\Gamma_q(r_1+r_2+\xi_2+1)} \\
&+\frac{|\lambda_2|\eta_j^{r_2}\varphi}{\Gamma_q(r_2+1)}\Big) + |\delta_2||\rho_2|\Big(\frac{|\alpha_1|\eta_j^{p_1+p_2}\hbar_1}{\Gamma_q(p_1+p_2+1)} + \frac{|\beta_1|\eta_j^{p_1+p_2+\xi_1}\varpi_1}{\Gamma_q(p_1+p_2+\xi_1+1)} + \frac{|\lambda_1|\eta_j^{p_2}\varphi}{\Gamma_q(p_2+1)}\Big) \\
&+|\delta_3||\rho_3|\Big(\frac{|\alpha_2|\eta_j^{r_1+r_2-1}\hbar_2}{\Gamma_q(r_1+r_2)} + \frac{|\beta_2|\eta_j^{r_1+r_2+\xi_2-1}\varpi_2}{\Gamma_q(r_1+r_2+\xi_2)} + \frac{|\lambda_2|\eta_j^{r_2-1}\varphi}{\Gamma_q(r_2)}\Big) + |\sigma_1||\rho_3|\Big(\frac{|\alpha_1|\hbar_1}{\Gamma_q(p_1+p_2+1)} \\
&+\frac{|\beta_1|\varpi_1}{\Gamma_q(p_1+p_2+\xi_1+1)} + \frac{|\lambda_1|\varphi}{\Gamma_q(p_2+1)}\Big) + |\sigma_2||\rho_3|\Big(\frac{|\alpha_1|\hbar_1}{\Gamma_q(p_1+p_2)} + \frac{|\beta_1|\varpi_1}{\Gamma_q(p_1+p_2+\xi_1)} + \frac{|\lambda_1|\varphi}{\Gamma_q(p_2)}\Big) \\
&+|\delta_4||\rho_4|\Big(\frac{|\alpha_1|\eta_j^{p_1+p_2-1}\hbar_1}{\Gamma_q(p_1+p_2)} + \frac{|\beta_1|\eta_j^{p_1+p_2+\xi_1-1}\varpi_1}{\Gamma_q(p_1+p_2+\xi_1)} + \frac{|\lambda_1|\eta_j^{p_2-1}\varphi}{\Gamma_q(p_2)}\Big) + |\sigma_3||\rho_4|\Big(\frac{|\alpha_2|\hbar_2}{\Gamma_q(r_1+r_2+1)} \\
&+\frac{|\beta_2|\varpi_2}{\Gamma_q(r_1+r_2+\xi_2+1)} + \frac{|\lambda_2|\varphi}{\Gamma_q(r_2+1)}\Big) + |\sigma_4||\rho_4|\Big(\frac{|\alpha_2|\hbar_2}{\Gamma_q(r_1+r_2)} + \frac{|\beta_2|\varpi_2}{\Gamma_q(r_1+r_2+\xi_2)} + \frac{|\lambda_2|\varphi}{\Gamma_q(r_2)}\Big) \\
&\leq \frac{|\alpha_1|\hbar_1}{\Gamma_q(p_1+p_2+1)}\Big[2(t_2-t_1)^{p_1+p_2} + |t_2^{p_1+p_2} - t_1^{p_1+p_2}|\Big] + \frac{|\beta_1|\varpi_1}{\Gamma_q(p_1+p_2+\xi_1+1)} \\
&\times\Big[2(t_2-t_1)^{p_1+p_2+\xi_1} + |t_2^{p_1+p_2+\xi_1} - t_1^{p_1+p_2+\xi_1}|\Big] + \frac{|\lambda_1|\varphi}{\Gamma_q(p_2+1)}\Big[2(t_2-t_1)^{p_2} + |t_2^{p_2} - t_1^{p_2}|\Big] \\
&+\frac{|t_2^{p_2} - t_1^{p_2}|}{|\Lambda|}\Bigg[|\alpha_1|\hbar_1\left(\frac{|\delta_2||\rho_2|\eta_j^{p_1+p_2} + |\sigma_1||\rho_3|}{\Gamma_q(p_1+p_2+1)} + \frac{|\delta_4||\rho_4|\eta_j^{p_1+p_2-1} + |\sigma_2||\rho_3|}{\Gamma_q(p_1+p_2)}\right) \\
&+|\alpha_2|\hbar_2\left(\frac{|\delta_1||\rho_1|\eta_j^{r_1+r_2} + |\sigma_3||\rho_4|}{\Gamma_q(r_1+r_2+1)} + \frac{|\delta_3||\rho_3|\eta_j^{r_1+r_2-1} + |\sigma_4||\rho_4|}{\Gamma_q(r_1+r_2)}\right) \\
&+|\beta_1|\varpi_1\left(\frac{|\delta_2||\rho_2|\eta_j^{p_1+p_2+\xi_1} + |\sigma_1||\rho_3|}{\Gamma_q(p_1+p_2+\xi_1+1)} + \frac{|\delta_4||\rho_4|\eta_j^{p_1+p_2+\xi_1-1} + |\sigma_2||\rho_3|}{\Gamma_q(p_1+p_2+\xi_1)}\right) \\
&+|\beta_2|\varpi_2\left(\frac{|\delta_1||\rho_1|\eta_j^{r_1+r_2+\xi_2} + |\sigma_3||\rho_4|}{\Gamma_q(r_1+r_2+\xi_2+1)} + \frac{|\delta_3||\rho_3|\eta_j^{r_1+r_2+\xi_2-1} + |\sigma_4||\rho_4|}{\Gamma_q(r_1+r_2+\xi_2)}\right) \\
&+\varphi\Bigg(|\lambda_1|\Big(\frac{|\delta_2||\rho_2|\eta_j^{p_2} + |\sigma_1||\rho_3|}{\Gamma_q(p_2+1)} + \frac{|\delta_4||\rho_4|\eta_j^{p_2-1} + |\sigma_2||\rho_3|}{\Gamma_q(p_2)}\Big) + |\lambda_2|\Big(\frac{|\delta_1||\rho_1|\eta_j^{r_2} + |\sigma_3||\rho_4|}{\Gamma_q(r_2+1)} \\
&+\frac{|\delta_3||\rho_3|\eta_j^{r_2-1} + |\sigma_4||\rho_4|}{\Gamma_q(r_2)}\Big)\Bigg)\Bigg],
\end{aligned}$$

which tends to zero as $t_2 - t_1 \to 0$ independent of (x, y). Analogously, we can obtain

$$\begin{aligned}
&|\mathcal{G}_2(x(t_2), y(t_2)) - \mathcal{G}_2(x(t_1), y(t_1))| \\
&\leq \frac{|\alpha_2|\hbar_2}{\Gamma_q(r_1+r_2+1)}\Big[2(t_2-t_1)^{r_1+r_2} + |t_2^{r_1+r_2} - t_1^{r_1+r_2}|\Big] + \frac{|\beta_2|\varpi_2}{\Gamma_q(r_1+r_2+\xi_2+1)} \\
&\times\Big[2(t_2-t_1)^{r_1+r_2+\xi_2} + |t_2^{r_1+r_2+\xi_2} - t_1^{r_1+r_2+\xi_2}|\Big] + \frac{|\lambda_2|\varphi}{\Gamma_q(r_2+1)}\Big[2(t_2-t_1)^{r_2} + |t_2^{r_2} - t_1^{r_2}|\Big]
\end{aligned}$$

$$+\frac{|t_2^{r_2}-t_1^{r_2}|}{|\Lambda|}\Bigg[|\alpha_1|\hbar_1\Bigg(\frac{|\delta_2||\rho_{10}|\eta_j^{p_1+p_2}+|\sigma_1||\rho_{11}|}{\Gamma_q(p_1+p_2+1)}+\frac{|\delta_4||\rho_{12}|\eta_j^{p_1+p_2-1}+|\sigma_2||\rho_{11}|}{\Gamma_q(p_1+p_2)}\Bigg)$$
$$+|\alpha_2|\hbar_2\Bigg(\frac{|\delta_1||\rho_9|\eta_j^{r_1+r_2}+|\sigma_3||\rho_{12}|}{\Gamma_q(r_1+r_2+1)}+\frac{|\delta_3||\rho_{11}|\eta_j^{r_1+r_2-1}+|\sigma_4||\rho_{12}|}{\Gamma_q(r_1+r_2)}\Bigg)$$
$$+|\beta_1|\varpi_1\Bigg(\frac{|\delta_2||\rho_{10}|\eta_j^{p_1+p_2+\xi_1}+|\sigma_1||\rho_{11}|}{\Gamma_q(p_1+p_2+\xi_1+1)}+\frac{|\delta_4||\rho_{12}|\eta_j^{p_1+p_2+\xi_1-1}+|\sigma_2||\rho_{11}|}{\Gamma_q(p_1+p_2+\xi_1)}\Bigg)$$
$$+|\beta_2|\varpi_2\Bigg(\frac{|\delta_1||\rho_9|\eta_j^{r_1+r_2+\xi_2}+|\sigma_3||\rho_{12}|}{\Gamma_q(r_1+r_2+\xi_2+1)}+\frac{|\delta_3||\rho_{11}|\eta_j^{r_1+r_2+\xi_2-1}+|\sigma_4||\rho_{12}|}{\Gamma_q(r_1+r_2+\xi_2)}\Bigg)$$
$$+\varphi\Bigg(|\lambda_1|\Big(\frac{|\delta_2||\rho_{10}|\eta_j^{p_2}+|\sigma_1||\rho_{11}|}{\Gamma_q(p_2+1)}+\frac{|\delta_4||\rho_{12}|\eta_j^{p_2-1}+|\sigma_2||\rho_{11}|}{\Gamma_q(p_2)}\Big)+|\lambda_2|\Big(\frac{|\delta_1||\rho_9|\eta_j^{r_2}+|\sigma_3||\rho_{12}|}{\Gamma_q(r_2+1)}$$
$$+\frac{|\delta_3||\rho_{11}|\eta_j^{r_2-1}+|\sigma_4||\rho_{12}|}{\Gamma_q(r_2)}\Big)\Bigg)\Bigg].$$

Note that the right-hand side of the above inequality tends to zero as $t_2-t_1\to 0$ independent of (x,y). Thus the operator $\mathcal{G}(x,y)$ is equicontinuous. In view of the foregoing arguments, we deduce that the operator $\mathcal{G}(x,y)$ is completely continuous.

Finally, we show that $\Omega=\{(x,y)\in\mathcal{C}\times\mathcal{C}|(x,y)=\zeta\mathcal{G}(x,y),0<\zeta<1\}$ is bounded. Let $(x,y)\in\Omega$, with $(x,y)=\zeta\mathcal{G}(x,y)(t)$ and for any $t\in[0,1]$, we have

$$x(t)=\zeta\mathcal{G}_1(x,y)(t),\quad y(t)=\zeta\mathcal{G}_2(x,y)(t).$$

In view of condition $(\mathbf{A}_3)$, we can find that

$$|x(t)|$$
$$\le(\tau_0+\tau_1|x|+\tau_2|y|)\Bigg[|\alpha_1|\int_0^t\frac{(t-qu)^{(p_1+p_2-1)}}{\Gamma_q(p_1+p_2)}d_qu$$
$$+\frac{|\alpha_1||\delta_2||\rho_2t^{p_2}+\rho_6|}{|\Lambda|}\int_0^{\eta_j}\frac{(\eta_j-qu)^{(p_1+p_2-1)}}{\Gamma_q(p_1+p_2)}d_qu+\frac{|\alpha_1||\sigma_1||\rho_3t^{p_2}+\rho_7|}{|\Lambda|}\int_0^1\frac{(1-qu)^{(p_1+p_2-1)}}{\Gamma_q(p_1+p_2)}d_qu$$
$$+\frac{|\alpha_1||\sigma_2||\rho_3t^{p_2}+\rho_7|}{|\Lambda|}\int_0^1\frac{(1-qu)^{(p_1+p_2-2)}}{\Gamma_q(p_1+p_2-1)}d_qu+\frac{|\alpha_1||\delta_4||\rho_4t^{p_2}+\rho_8|}{|\Lambda|}\int_0^{\eta_j}\frac{(\eta_j-qu)^{(p_1+p_2-2)}}{\Gamma_q(p_1+p_2-1)}d_qu\Bigg]$$
$$+(\widetilde{\tau}_0+\widetilde{\tau}_1|x|+\widetilde{\tau}_2|y|)\Bigg[\frac{|\alpha_2||\delta_1||\rho_1t^{p_2}+\rho_5|}{|\Lambda|}\int_0^{\eta_j}\frac{(\eta_j-qu)^{(r_1+r_2-1)}}{\Gamma_q(r_1+r_2)}d_qu+\frac{|\alpha_2||\delta_3||\rho_3t^{p_2}+\rho_7|}{|\Lambda|}$$
$$\times\int_0^{\eta_j}\frac{(\eta_j-qu)^{(r_1+r_2-2)}}{\Gamma_q(r_1+r_2-1)}d_qu+\frac{|\alpha_2||\sigma_3||\rho_4t^{p_2}+\rho_8|}{|\Lambda|}\int_0^1\frac{(1-qu)^{(r_1+r_2-1)}}{\Gamma_q(r_1+r_2)}d_qu+\frac{|\alpha_2||\sigma_4||\rho_4t^{p_2}+\rho_8|}{|\Lambda|}$$
$$\times\int_0^1\frac{(1-qu)^{(r_1+r_2-2)}}{\Gamma_q(r_1+r_2-1)}d_qu\Bigg]+(\epsilon_0+\epsilon_1|x|+\epsilon_2|y|)\Bigg[|\beta_1|\int_0^t\frac{(t-qu)^{(p_1+p_2+\xi_1-1)}}{\Gamma_q(p_1+p_2+\xi_1)}d_qu$$
$$+\frac{|\beta_1||\delta_2||\rho_2t^{p_2}+\rho_6|}{|\Lambda|}\int_0^{\eta_j}\frac{(\eta_j-qu)^{(p_1+p_2+\xi_1-1)}}{\Gamma_q(p_1+p_2+\xi_1)}d_qu+\frac{|\beta_1||\sigma_1||\rho_3t^{p_2}+\rho_7|}{|\Lambda|}\int_0^1\frac{(1-qu)^{(p_1+p_2+\xi_1-1)}}{\Gamma_q(p_1+p_2+\xi_1)}d_qu$$
$$+\frac{|\beta_1||\sigma_2||\rho_3t^{p_2}+\rho_7|}{|\Lambda|}\int_0^1\frac{(1-qu)^{(p_1+p_2+\xi_1-2)}}{\Gamma_q(p_1+p_2+\xi_1-1)}d_qu+\frac{|\beta_1||\delta_4||\rho_4t^{p_2}+\rho_8|}{|\Lambda|}\int_0^{\eta_j}\frac{(\eta_j-qu)^{(p_1+p_2+\xi_1-2)}}{\Gamma_q(p_1+p_2+\xi_1-1)}d_qu\Bigg]$$
$$+(\widetilde{\epsilon}_0+\widetilde{\epsilon}_1|x|+\widetilde{\epsilon}_2|y|)\Bigg[\frac{|\beta_2||\delta_1||\rho_1t^{p_2}+\rho_5|}{|\Lambda|}\int_0^{\eta_j}\frac{(\eta_j-qu)^{(r_1+r_2+\xi_2-1)}}{\Gamma_q(r_1+r_2+\xi_2)}d_qu+\frac{|\beta_2||\delta_3||\rho_3t^{p_2}+\rho_7|}{|\Lambda|}$$
$$\times\int_0^{\eta_j}\frac{(\eta_j-qu)^{(r_1+r_2+\xi_2-2)}}{\Gamma_q(r_1+r_2+\xi_2-1)}d_qu+\frac{|\beta_2||\sigma_3||\rho_4t^{p_2}+\rho_8|}{|\Lambda|}\int_0^1\frac{(1-qu)^{(r_1+r_2+\xi_2-1)}}{\Gamma_q(r_1+r_2+\xi_2)}d_qu$$

$$
\begin{aligned}
&+\frac{|\beta_2||\sigma_4||\rho_4 t^{p_2}+\rho_8|}{|\Lambda|}\int_0^1 \frac{(1-qu)^{(r_1+r_2+\xi_2-2)}}{\Gamma_q(r_1+r_2+\xi_2-1)}d_qu\Bigg]+|\lambda_1|\Bigg[\int_0^t \frac{(t-qu)^{(p_2-1)}}{\Gamma_q(p_2)}d_qu \\
&+\frac{|\delta_2||\rho_2 t^{p_2}+\rho_6|}{|\Lambda|}\int_0^{\eta_j} \frac{(\eta_j-qu)^{(p_2-1)}}{\Gamma_q(p_2)}d_qu+\frac{|\sigma_1||\rho_3 t^{p_2}+\rho_7|}{|\Lambda|}\int_0^1 \frac{(1-qu)^{(p_2-1)}}{\Gamma_q(p_2)}d_qu \\
&+\frac{|\sigma_2||\rho_3 t^{p_2}+\rho_7|}{|\Lambda|}\int_0^1 \frac{(1-qu)^{(p_2-2)}}{\Gamma_q(p_2-1)}d_qu+\frac{|\delta_4||\rho_4 t^{p_2}+\rho_8|}{|\Lambda|}\int_0^{\eta_j} \frac{(\eta_j-qu)^{(p_2-2)}}{\Gamma_q(p_2-1)}d_qu\Bigg]|x| \\
&+|\lambda_2|\Bigg[\frac{|\delta_1||\rho_1 t^{p_2}+\rho_5|}{|\Lambda|}\int_0^{\eta_j} \frac{(\eta_j-qu)^{(r_2-1)}}{\Gamma_q(r_2)}d_qu+\frac{|\delta_3||\rho_3 t^{p_2}+\rho_7|}{|\Lambda|}\int_0^{\eta_j} \frac{(\eta_j-qu)^{(r_2-2)}}{\Gamma_q(r_2-1)}d_qu \\
&+\frac{|\sigma_3||\rho_4 t^{p_2}+\rho_8|}{|\Lambda|}\int_0^1 \frac{(1-qu)^{(r_2-1)}}{\Gamma_q(r_2)}d_qu+\frac{|\sigma_4||\rho_4 t^{p_2}+\rho_8|}{|\Lambda|}\int_0^1 \frac{(1-qu)^{(r_2-2)}}{\Gamma_q(r_2-1)}d_qu\Bigg]|y|,
\end{aligned}
$$

and

$$
\begin{aligned}
&|y(t)| \\
&\le(\tau_0+\tau_1|x|+\tau_2|y|)\Bigg[\frac{|\alpha_1||\delta_2||\rho_{10}t^{r_2}+\rho_{14}|}{|\Lambda|}\int_0^{\eta_j} \frac{(\eta_j-qu)^{(p_1+p_2-1)}}{\Gamma_q(p_1+p_2)}d_qu \\
&+\frac{|\alpha_1||\sigma_1||\rho_{11}t^{r_2}+\rho_{15}|}{|\Lambda|}\int_0^1 \frac{(1-qu)^{(p_1+p_2-1)}}{\Gamma_q(p_1+p_2)}d_qu+\frac{|\alpha_1||\sigma_2||\rho_{11}t^{r_2}+\rho_{15}|}{|\Lambda|}\int_0^1 \frac{(1-qu)^{(p_1+p_2-2)}}{\Gamma_q(p_1+p_2-1)}d_qu \\
&+\frac{|\alpha_1||\delta_4||\rho_{12}t^{r_2}+\rho_{16}|}{|\Lambda|}\int_0^{\eta_j} \frac{(\eta_j-qu)^{(p_1+p_2-2)}}{\Gamma_q(p_1+p_2-1)}d_qu\Bigg] \\
&+(\widetilde{\tau}_0+\widetilde{\tau}_1|x|+\widetilde{\tau}_2|y|)\Bigg[|\alpha_2|\int_0^t \frac{(t-qu)^{(r_1+r_2-1)}}{\Gamma_q(r_1+r_2)}d_qu+\frac{|\alpha_2||\delta_1||\rho_9 t^{r_2}+\rho_{13}|}{|\Lambda|}\int_0^{\eta_j} \frac{(\eta_j-qu)^{(r_1+r_2-1)}}{\Gamma_q(r_1+r_2)}d_qu \\
&+\frac{|\alpha_2||\delta_3||\rho_{11}t^{r_2}+\rho_{15}|}{|\Lambda|}\int_0^{\eta_j} \frac{(\eta_j-qu)^{(r_1+r_2-2)}}{\Gamma_q(r_1+r_2-1)}d_qu+\frac{|\alpha_2||\sigma_3||\rho_{12}t^{r_2}+\rho_{16}|}{|\Lambda|}\int_0^1 \frac{(1-qu)^{(r_1+r_2-1)}}{\Gamma_q(r_1+r_2)}d_qu \\
&+\frac{|\alpha_2||\sigma_4||\rho_{12}t^{r_2}+\rho_{16}|}{|\Lambda|}\int_0^1 \frac{(1-qu)^{(r_1+r_2-2)}}{\Gamma_q(r_1+r_2-1)}d_qu\Bigg] \\
&+(\epsilon_0+\epsilon_1|x|+\epsilon_2|y|)\Bigg[\frac{|\beta_1||\delta_2||\rho_{10}t^{r_2}+\rho_{14}|}{|\Lambda|}\int_0^{\eta_j} \frac{(\eta_j-qu)^{(p_1+p_2+\xi_1-1)}}{\Gamma_q(p_1+p_2+\xi_1)}d_qu \\
&+\frac{|\beta_1||\sigma_1||\rho_{11}t^{r_2}+\rho_{15}|}{|\Lambda|}\int_0^1 \frac{(1-qu)^{(p_1+p_2+\xi_1-1)}}{\Gamma_q(p_1+p_2+\xi_1)}d_qu+\frac{|\beta_1||\sigma_2||\rho_{11}t^{r_2}+\rho_{15}|}{|\Lambda|} \\
&\times\int_0^1 \frac{(1-qu)^{(p_1+p_2+\xi_1-2)}}{\Gamma_q(p_1+p_2+\xi_1-1)}d_qu+\frac{|\beta_1||\delta_4||\rho_{12}t^{r_2}+\rho_{16}|}{|\Lambda|}\int_0^{\eta_j} \frac{(\eta_j-qu)^{(p_1+p_2+\xi_1-2)}}{\Gamma_q(p_1+p_2+\xi_1-1)}d_qu\Bigg] \\
&+(\widetilde{\epsilon}_0+\widetilde{\epsilon}_1|x|+\widetilde{\epsilon}_2|y|)\Bigg[|\beta_2|\int_0^t \frac{(t-qu)^{(r_1+r_2+\xi_2-1)}}{\Gamma_q(r_1+r_2+\xi_2)}d_qu+\frac{|\beta_2||\delta_1||\rho_9 t^{r_2}+\rho_{13}|}{|\Lambda|} \\
&\times\int_0^{\eta_j} \frac{(\eta_j-qu)^{(r_1+r_2+\xi_2-1)}}{\Gamma_q(r_1+r_2+\xi_2)}d_qu+\frac{|\beta_2||\delta_3||\rho_{11}t^{r_2}+\rho_{15}|}{|\Lambda|}\int_0^{\eta_j} \frac{(\eta_j-qu)^{(r_1+r_2+\xi_2-2)}}{\Gamma_q(r_1+r_2+\xi_2-1)}d_qu \\
&+\frac{|\beta_2||\sigma_3||\rho_{12}t^{r_2}+\rho_{16}|}{|\Lambda|}\int_0^1 \frac{(1-qu)^{(r_1+r_2+\xi_2-1)}}{\Gamma_q(r_1+r_2+\xi_2)}d_qu+\frac{|\beta_2||\sigma_4||\rho_{12}t^{r_2}+\rho_{16}|}{|\Lambda|} \\
&\times\int_0^1 \frac{(1-qu)^{(r_1+r_2+\xi_2-2)}}{\Gamma_q(r_1+r_2+\xi_2-1)}d_qu\Bigg]+|\lambda_1|\Bigg[\frac{|\delta_2||\rho_{10}t^{r_2}+\rho_{14}|}{|\Lambda|}\int_0^{\eta_j} \frac{(\eta_j-qu)^{(p_2-1)}}{\Gamma_q(p_2)}d_qu \\
&+\frac{|\sigma_1||\rho_{11}t^{r_2}+\rho_{15}|}{|\Lambda|}\int_0^1 \frac{(1-qu)^{(p_2-1)}}{\Gamma_q(p_2)}d_qu+\frac{|\sigma_2||\rho_{11}t^{r_2}+\rho_{15}|}{|\Lambda|}\int_0^1 \frac{(1-qu)^{(p_2-2)}}{\Gamma_q(p_2-1)}d_qu \\
&+\frac{|\delta_4||\rho_{12}t^{r_2}+\rho_{16}|}{|\Lambda|}\int_0^{\eta_j} \frac{(\eta_j-qu)^{(p_2-2)}}{\Gamma_q(p_2-1)}d_qu\Bigg]|x|+|\lambda_2|\Bigg[\int_0^t \frac{(t-qu)^{(r_2-1)}}{\Gamma_q(r_2)}d_qu \\
&+\frac{|\delta_1||\rho_9 t^{r_2}+\rho_{13}|}{|\Lambda|}\int_0^{\eta_j} \frac{(\eta_j-qu)^{(r_2-1)}}{\Gamma_q(r_2)}d_qu+\frac{|\delta_3||\rho_{11}t^{r_2}+\rho_{15}|}{|\Lambda|}\int_0^{\eta_j} \frac{(\eta_j-qu)^{(r_2-2)}}{\Gamma_q(r_2-1)}d_qu \\
&+\frac{|\sigma_3||\rho_{12}t^{r_2}+\rho_{16}|}{|\Lambda|}\int_0^1 \frac{(1-qu)^{(r_2-1)}}{\Gamma_q(r_2)}d_qu+\frac{|\sigma_4||\rho_{12}t^{r_2}+\rho_{16}|}{|\Lambda|}\int_0^1 \frac{(1-qu)^{(r_2-2)}}{\Gamma_q(r_2-1)}d_qu\Bigg]|y|.
\end{aligned}
$$

In consequence, we obtain

$$
\begin{aligned}
\|x\| \quad & \leq (\tau_0 + \tau_1\|x\| + \tau_2\|y\|)\Psi_1 + (\widetilde{\tau}_0 + \widetilde{\tau}_1\|x\| + \widetilde{\tau}_2\|y\|)\Psi_2 \\
& + (\epsilon_0 + \epsilon_1\|x\| + \epsilon_2\|y\|)\Phi_1 + (\widetilde{\epsilon}_0 + \widetilde{\epsilon}_1\|x\| + \widetilde{\epsilon}_2\|y\|)\Phi_2 \\
& + |\lambda_1|\Big(\frac{1}{\Gamma_q(p_2+1)} + \frac{(\gamma_9+\gamma_{10})}{|\Lambda|}\Big)\|x\| + \frac{|\lambda_2|(\gamma_{11}+\gamma_{12})}{|\Lambda|}\|y\| \\
& = \tau_0\Psi_1 + \widetilde{\tau}_0\Psi_2 + \epsilon_0\Phi_1 + \widetilde{\epsilon}_0\Phi_2 \\
& + (\tau_1\Psi_1 + \widetilde{\tau}_1\Psi_2 + \epsilon_1\Phi_1 + \widetilde{\epsilon}_1\Phi_2)\|x\| \\
& + (\tau_2\Psi_1 + \widetilde{\tau}_2\Psi_2 + \epsilon_2\Phi_1 + \widetilde{\epsilon}_2\Phi_2)\|y\| \\
& + |\lambda_1|\Big(\frac{1}{\Gamma_q(p_2+1)} + \frac{(\gamma_9+\gamma_{10})}{|\Lambda|}\Big)\|x\| + \frac{|\lambda_2|(\gamma_{11}+\gamma_{12})}{|\Lambda|}\|y\|,
\end{aligned}
$$

and

$$
\begin{aligned}
\|y\| \quad & \leq (\tau_0 + \tau_1\|x\| + \tau_2\|y\|)\Psi_3 + (\widetilde{\tau}_0 + \widetilde{\tau}_1\|x\| + \widetilde{\tau}_2\|y\|)\Psi_4 \\
& + (\epsilon_0 + \epsilon_1\|x\| + \epsilon_2\|y\|)\Phi_3 + (\widetilde{\epsilon}_0 + \widetilde{\epsilon}_1\|x\| + \widetilde{\epsilon}_2\|y\|)\Phi_4 \\
& + \frac{|\lambda_1|(\gamma_{21}+\gamma_{22})}{|\Lambda|}\|x\| + |\lambda_2|\Big(\frac{1}{\Gamma_q(r_2+1)} + \frac{(\gamma_{23}+\gamma_{24})}{|\Lambda|}\Big)\|y\| \\
& = \tau_0\Psi_3 + \widetilde{\tau}_0\Psi_4 + \epsilon_0\Phi_3 + \widetilde{\epsilon}_0\Phi_4 \\
& + (\tau_1\Psi_3 + \widetilde{\tau}_1\Psi_4 + \epsilon_1\Phi_3 + \widetilde{\epsilon}_1\Phi_4)\|x\| \\
& + (\tau_2\Psi_3 + \widetilde{\tau}_2\Psi_4 + \epsilon_2\Phi_3 + \widetilde{\epsilon}_2\Phi_4)\|y\| \\
& + \frac{|\lambda_1|(\gamma_{21}+\gamma_{22})}{|\Lambda|}\|x\| + |\lambda_2|\Big(\frac{1}{\Gamma_q(r_2+1)} + \frac{(\gamma_{23}+\gamma_{24})}{|\Lambda|}\Big)\|y\|,
\end{aligned}
$$

which imply that

$$
\begin{aligned}
\|x\| + \|y\| \quad & \leq (\Psi_1+\Psi_3)\tau_0 + (\Psi_2+\Psi_4)\widetilde{\tau}_0 + (\Phi_1+\Phi_3)\epsilon_0 + (\Phi_2+\Phi_4)\widetilde{\epsilon}_0 \\
& + \Big[(\Psi_1+\Psi_3)\tau_1 + (\Psi_2+\Psi_4)\widetilde{\tau}_1 + (\Phi_1+\Phi_3)\epsilon_1 + (\Phi_2+\Phi_4)\widetilde{\epsilon}_1 + v_1\Big]\|x\| \\
& + \Big[(\Psi_1+\Psi_3)\tau_2 + (\Psi_2+\Psi_4)\widetilde{\tau}_2 + (\Phi_1+\Phi_3)\epsilon_2 + (\Phi_2+\Phi_4)\widetilde{\epsilon}_2 + v_2\Big]\|y\|.
\end{aligned}
$$

Thus we have

$$
\|(x,y)\| \quad \leq \frac{(\Psi_1+\Psi_3)\tau_0 + (\Psi_2+\Psi_4)\widetilde{\tau}_0 + (\Phi_1+\Phi_3)\epsilon_0 + (\Phi_2+\Phi_4)\widetilde{\epsilon}_0}{H_0},
$$

where

$$
\begin{aligned}
H_0 \quad = \quad & \min\{1 - [(\Psi_1+\Psi_3)\tau_1 + (\Psi_2+\Psi_4)\widetilde{\tau}_1 + (\Phi_1+\Phi_3)\epsilon_1 + (\Phi_2+\Phi_4)\widetilde{\epsilon}_1 + v_1], \\
& 1 - [(\Psi_1+\Psi_3)\tau_2 + (\Psi_2+\Psi_4)\widetilde{\tau}_2 + (\Phi_1+\Phi_3)\epsilon_2 + (\Phi_2+\Phi_4)\widetilde{\epsilon}_2 + v_2]\},
\end{aligned}
$$

which establishes that the set Ω is bounded. Thus, by Leray–Schauder nonlinear alternative [48], there exists a solution of the system (1)–(2) on $[0,1]$. The proof is complete. □

4. Examples

I. Illustration of Theorem 1

Example 1. *Let us consider a nonlinear system of coupled fractional q-integro-difference equations:*

$$
\begin{cases}
{}^cD_{0.5}^{0.05}({}^cD_{0.5}^{0.05} + 0.02)x(t) = 0.09f_1(t,x(t),y(t)) + 0.03I_{0.5}^{0.25}g_1(t,x(t),y(t)), & 0 \leq t \leq 1, \\
{}^cD_{0.5}^{0.35}({}^cD_{0.5}^{0.35} + 0.06)y(t) = 0.08f_2(t,x(t),y(t)) + 0.07I_{0.5}^{0.25}g_2(t,x(t),y(t)), & 0 \leq t \leq 1,
\end{cases} \tag{17}
$$

supplemented with four-point coupled boundary conditions

$$\begin{cases} 0.4x(0) - 0.2\Big(t^{(1-0.05)}D_q x(t)\Big)\Big|_{t=0} = \sum_{j=1}^{2} a_j y(\eta_j), \\ 0.4y(0) - 0.2\Big(t^{(1-0.35)}D_q y(t)\Big)\Big|_{t=0} = \sum_{j=1}^{2} b_j x(\eta_j), \\ 0.1x(1) + 0.2D_q x(1) = \sum_{j=1}^{2} k_j D_q y(\eta_j), \\ 0.1y(1) + 0.2D_q y(1) = \sum_{j=1}^{2} m_j D_q x(\eta_j), \end{cases} \tag{18}$$

where $p_1 = p_2 = 0.05$, $q = 0.5$, $r_1 = r_2 = 0.35$, $\alpha_1 = 0.09$, $\alpha_2 = 0.08$, $\beta_1 = 0.03$, $\beta_2 = 0.07$, $\xi_1 = \xi_2 = 0.25$, $\lambda_1 = 0.02, \lambda_2 = 0.06$, $\mu_1 = \mu_3 = 0.4$, $\mu_2 = \mu_4 = 0.2$, $\sigma_1 = \sigma_3 = 0.1$, $\sigma_2 = \sigma_4 = 0.2$, $a_1 = 0.35, a_2 = 0.3$, $b_1 = 0.2, b_2 = 0.25$, $k_1 = 0.7$, $k_2 = 0.1$, $m_1 = 0.6, m_2 = 0.8$, $\eta_1 = 0.45$, $\eta_2 = 0.65$, $t \in [0,1]$ *and*

$$\begin{aligned} f_1(t,x(t),y(t)) &= \tfrac{1}{196}\tfrac{|x(t)|}{1+|x(t)|} + \tfrac{\arctan y(t)}{(t^2+14)^2} - 10t, \\ f_2(t,x(t),y(t)) &= \tfrac{1}{\sqrt{225+105+t^3}}(x(t) + \cos y(t)), \\ g_1(t,x(t),y(t)) &= \tfrac{1}{16\sqrt{t^6+81}}(\sin x(t) + \arctan y(t) - \cos 2t), \\ g_2(t,x(t),y(t)) &= \tfrac{1}{20\sqrt{t+49}}(\sin x(t) + \tfrac{|y(t)|}{1+|y(t)|}) + 16e^{-t}. \end{aligned} \tag{19}$$

Then $\iota_1 = 1/196, \iota_2 = 1/120, \kappa_1 = 1/144, \kappa_2 = 1/140$ *as*

$$|f_1(t,x_1(t),y_1(t)) - f_1(t,x_2(t),y_2(t))| = \frac{1}{196}(|x_1 - x_2| + |y_1 - y_2|),$$

$$|f_2(t,x_1(t),y_1(t)) - f_2(t,x_2(t),y_2(t))| = \frac{1}{120}(|x_1 - x_2| + |y_1 - y_2|),$$

$$|g_1(t,x_1(t),y_1(t)) - g_1(t,x_2(t),y_2(t))| = \frac{1}{144}(|x_1 - x_2| + |y_1 - y_2|),$$

$$|g_2(t,x_1(t),y_1(t)) - g_2(t,x_2(t),y_2(t))| = \frac{1}{140}(|x_1 - x_2| + |y_1 - y_2|).$$

Using the given data, it is found that $\Psi_1 = 0.393067, \Psi_2 = 0.476841, \Psi_3 = 0.356139$, $\Psi_4 = 0.451896$, $\Phi_1 = 0.248188, \Phi_2 = 0.414528, \Phi_3 = 0.271996, \Phi_4 = 0.383275$, $\Theta_1 = 0.359396$, $\Theta_2 = 0.363914$, *and* $\Upsilon \approx 0.744182 < 1$. *Clearly the hypothesis of Theorem 1 holds true. So, by the conclusion of Theorem 1, the system (17) and (18) has a unique solution on* $[0,1]$.

II. Illustration of Theorem 2

Example 2. *Let us consider the system (17) and (18) with nonlinearities:*

$$\begin{aligned} f_1(t,x(t),y(t)) &= \tfrac{1}{(60+t)} + \tfrac{1}{263}x(t)\sin(t) + \tfrac{1}{170}\arctan y(t), \\ f_2(t,x(t),y(t)) &= \tfrac{1}{5\sqrt{1600t}} + \tfrac{1}{2(t^3+9)}\cos x(t) + \tfrac{1}{2(11+t^5)^2 y(t)}, \\ g_1(t,x(t),y(t)) &= \tfrac{1}{122\sqrt{t}} + \tfrac{2}{139}x(t) + \tfrac{1}{12\sqrt{256+t}}\cos y(t), \\ g_2(t,x(t),y(t)) &= \tfrac{5\sin t}{(t^3+126)} + \tfrac{1}{280}x(t) + \tfrac{2}{153}y(t). \end{aligned} \tag{20}$$

Notice that the condition $(\mathbf{A}_3)$ *holds true as*

$$\begin{aligned} |f_1(t,x(t),y(t))| &\leq \frac{1}{60} + \frac{1}{263}|x(t)| + \frac{1}{170}|y(t)|, \, |f_2(t,x(t),y(t))| \leq \frac{1}{20} + \frac{1}{162}|x(t)| + \frac{1}{242}|y(t)|, \\ |g_1(t,x(t),y(t))| &\leq \frac{1}{122} + \frac{2}{139}|x(t)| + \frac{1}{192}|y(t)|, |g_2(t,x(t),y(t))| \leq \frac{5}{126} + \frac{1}{280}|x(t)| + \frac{2}{153}|y(t)|, \end{aligned}$$

with $\tau_0 = 1/60, \tau_1 = 1/263, \tau_2 = 1/170, \widetilde{\tau}_0 = 1/200, \widetilde{\tau}_1 = 1/162, \widetilde{\tau}_2 = 1/242$, $\epsilon_0 = 1/122, \epsilon_1 = 2/139, \epsilon_2 = 1/192, \widetilde{\epsilon}_0 = 5/126, \widetilde{\epsilon}_1 = 1/280, \widetilde{\epsilon}_2 = 2/153$. *Moreover,*

$$(\Psi_1 + \Psi_3)\tau_1 + (\Psi_2 + \Psi_4)\widetilde{\tau}_1 + (\Phi_1 + \Phi_3)\epsilon_1 + (\Phi_2 + \Phi_4)\widetilde{\epsilon}_1 + v_1 \approx 0.138309 < 1,$$

$$(\Psi_1 + \Psi_3)\tau_2 + (\Psi_2 + \Psi_4)\widetilde{\tau}_2 + (\Phi_1 + \Phi_3)\epsilon_2 + (\Phi_2 + \Phi_4)\widetilde{\epsilon}_2 + v_2 \approx 0.625299 < 1.$$

Thus, all the assumptions of Theorem 2 are satisfied. Therefore, the conclusion of Theorem 2 applies and hence the system (17) and (18) with the nonlinearities (20) has at least one solution on $[0,1]$.

5. Conclusions

We have studied a new class of nonlocal multipoint boundary value problems of Langevin-type nonlinear coupled q-fractional integro-difference equations. First of all, the given problem was converted into an equivalent fixed-point problem. Then, we proved an existence and uniqueness result for the problem at hand by applying the Banach contraction mapping principle. In our second result, we presented the criteria ensuring the existence of a solution for the given problem. We also demonstrated the application of the obtained results by solving some particular problems. We emphasize that our results are new and contribute significantly to the literature on nonlocal multipoint boundary value problems of nonlinear coupled q-fractional integro-difference equations. It is imperative to note that our results correspond to the non-coupled separated boundary conditions for all $a_j = 0, b_j = 0, k_j = 0, m_j = 0, j = 1, \ldots, n$, which are indeed new in the given configuration.

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