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Third Hankel Determinant for a Subclass of Univalent Functions Associated with Lemniscate of Bernoulli

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Abstract: This paper deals with a new subclass of univalent function associated with the right half of the lemniscate of Bernoulli. We find the upper bound of the Hankel determinant $H_3(1)$ for this subclass by applying the Carlson–Shaffer operator to it. The present work also deals with certain properties of this newly defined subclass, such as the upper bound of the Hankel determinant of order 3, coefficient estimates, etc.

Keywords: Hankel determinant; Carlson–Shaffer operator; analytic functions; lemniscate of Bernoulli; starlike functions

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1. Introduction

Suppose that $\mathcal{H}(E)$ represents the class of those functions that are analytic in any open unit disk, i.e.,

$$E = \{z : z \in \mathbb{C} \quad \text{such that} \quad |z| < 1\}.$$

Here, $\mathbb{C}$ denotes the set of complex numbers.

In a similar way, we denote the class $\mathcal{A}$ of those analytic functions, which satisfies

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (\text{for all } \ z \in E). \tag{1}$$

The class $\mathcal{A}$ is normalized by

$$f(0) = 0 = f'(0) - 1.$$

Let us consider the analytic functions with the form

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n, \tag{2}$$

are denoted by the class $\mathcal{P}$, such that

$$\Re(p(z)) > 0 \qquad (\text{for all } \ z \in E).$$

Moreover, here $\mathcal{S}$ represents the class of univalent function in E. We represent by $\mathcal{S}^*$, the class of starlike function in E, which satisfies

$$\frac{zf'(z)}{f(z)} \in \mathcal{P} \qquad (\text{for all } \ z \in E).$$

Furthermore, $\mathcal{SL}^*$ represents the class of those functions that satisfying

$$\left| \left(\frac{zf'(z)}{f(z)} \right)^2 - 1 \right| < 1 \qquad (\text{for all } \ z \in E).$$

Hence, $f \in \mathcal{SL}^*$, iff, $\frac{zf'(z)}{f(z)}$ is the inside region that is bounded by the right half of the lemniscate of Bernoulli, it can be expressed by

$$\left| \omega^2 - 1 \right| < 1.$$

Sokól [1], and Sokól and Stankiewicz (see [2]) have introduced this class. One may represent subordination between any two analytic functions; f and g in E as

$$f(z) \prec g(z) \quad \text{or} \quad f \prec g\,.$$

If we have a Schwarz function w in E, which is analytic and satisfying the following conditions

$$|w(z)| < 1 \qquad \& \qquad w(0) = 0,$$

implies

$$f(z) = g(w(z)).$$

Furthermore, if g satisfies the condition of univalent function in E, then the equivalence becomes

$$f(z) \prec g(z) \quad (z \in E) \Rightarrow f(0) = g(0) \ \ \& \ \ f(E) \subset g(E).$$

Definition 1. *Suppose that $SL^*(\alpha, \beta)$ is the subclass of analytic functions given by*

$$SL^*(\alpha,\beta) = \left\{ f \in A : \left| \left(\frac{z[L(\alpha,\beta)f(z)]'}{L(\alpha,\beta)f(z)} \right)^2 - 1 \right| < 1 \right\}, \tag{3}$$

or

$$\frac{z[L(\alpha,\beta)f(z)]'}{L(\alpha,\beta)f(z)} \prec \sqrt{1+z} \qquad (z \in E), \tag{4}$$

where

$$L(\alpha,\beta)f(z) = z + \sum_{n=2}^{\infty} \frac{(\alpha)_{n-1}}{(\beta)_{n-1}} a_n z^n, \tag{5}$$

and

$$(x)_n = x(x+1)(x+2)\cdots(x+n-1)$$

with

$$(\alpha)_1 = \alpha, \quad (\beta)_1 = \beta,$$

where

$$(\alpha)_2 = \alpha(\alpha+1), \quad (\beta)_2 = \beta(\beta+1).$$

Suppose that $q \geq 1$ and $n \geq 0$. The definition of qth Hankel determinant is given by

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdot & \cdot & \cdot & a_{n+q-1} \\ a_{n+1} & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ a_{n+q-1} & \cdot & \cdot & \cdot & \cdot & a_{n+2(q-1)} \end{vmatrix}.$$

Several authors worked on this determinant. Different authors [3–8] worked on $H_2(2)$ for various classes of functions and find its sharp upper bound. The functional $|a_3 - a_2^2| = H_2(1)$ is known as a Fekete–Szegö functional. For any real and complex values of μ, this functional was generalized as $|a_3 - \mu a_2^2|$. For a class of univalent functions $f \in \mathcal{S}$ and some real values of μ, the sharp estimates of $|a_3 - \mu a_2^2|$ were evaluated by Fekete and Szegö, which is also known as functional $|a_2 a_4 - a_3^2|$ equivalent to $H_2(2)$. Similarly, for a subclass of analytic functions, the Hankel determinant of $H_3(1)$ was studied by Babalola [9]. Several authors (Refs. [10–12]) also studied the Hankel determinant $H_3(1)$. Our main focus in this work is for the class $SL^*(\alpha, \beta)$ on the Hankel determinant $H_3(1)$.

2. Set of Lemmas

Lemma 1 ([13])**.** *Assuming that $p \in \mathcal{P}$ be the form of Equation (2), we may write*

$$|p_2 - vp_1^2| \leq \begin{cases} -2 + 4v & , \quad v > 1 \\ 2 & , \quad 0 \leq v \leq 1 \\ 2 - 4v & , \quad v < 0 \end{cases}.$$

For $0 < v < 1$, the sharpness of the upper bound stated above may be enhanced by

$$(1 - v)|p_1|^2 + |p_2 - vp_1^2| \leq 2 \qquad (\frac{1}{2} < v \leq 1)$$

&

$$v|p_1|^2 + |p_2 - vp_1^2| \leq 2 \qquad (0 < v \leq \frac{1}{2}).$$

Lemma 2 ([13])**.** *Let us assume that $p \in \mathcal{P}$ be the form Equation (2), and for any complex number v, we have*

$$|p_2 - vp_1^2| \leq 2\max(1, |1 - 2v|).$$

Sharp results can be obtained by following

$$p(z) = \frac{1+z}{1-z},$$

and

$$p(z) = \frac{1+z^2}{1-z^2}.$$

Lemma 3 ([14])**.** *Let us assume that $p \in \mathcal{P}$ be the form Equation (2); then, we have*

$$p_2 = 4x + (1 - x)p_1^2,$$

for any x, such that $|x| \leq 1$

$$p_3 = \frac{p_1^3}{4} + \left[\frac{p_1}{2}x - \frac{p_1}{4}x^2 + \frac{1}{2}(1 - |x|^2)z\right](4 - p_1^2),$$

for any z, if $|z| \leq 1$.

3. Main Results

This section will provide proofs of the main results.

Theorem 1. *Assuming that $L(\alpha,\beta)f(z) \in SL^*(\alpha,\beta)$ and is of the form (5). Then*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{1}{16}\left(\frac{\beta(\beta+1)}{\alpha(\alpha+1)} - 4\mu\frac{\beta^2}{\alpha^2}\right), & \mu < -\frac{3}{4}, \\ \frac{1}{4}, & -\frac{3}{4} \leq \mu \leq \frac{5}{4}, \\ \frac{1}{16}\left(4\mu\frac{\beta^2}{\alpha^2} - \frac{\beta(\beta+1)}{\alpha(\alpha+1)}\right), & \mu > \frac{5}{4}. \end{cases}$$

Proof. If $L(\alpha,\beta)f(z) \in SL^*(\alpha,\beta)$, then it follows from Equation (4) that

$$\frac{z[L(\alpha,\beta)f(z)]'}{L(\alpha,\beta)f(z)} \prec \Phi(z).$$

Let us define the function,

$$p(z) = 1 + \sum p_n z^n = \frac{1+w(z)}{1-w(z)}.$$

As $p \in \mathcal{P}$, so

$$\frac{p(z)-1}{p(z)+1} = w(z).$$

Using Equation (4), we have

$$\frac{z[L(\alpha,\beta)f(z)]'}{L(\alpha,\beta)f(z)} = \Phi(w(z)).$$

Now as

$$\left[\frac{2p(z)}{1+p(z)}\right]^{\frac{1}{2}} = \left[2 - \frac{2}{1+p(z)}\right]^{\frac{1}{2}},$$

so, we have

$$\begin{aligned} \left[\frac{2p(z)}{1+p(z)}\right]^{\frac{1}{2}} &= 1 + \frac{1}{4}p_1 z + \left(\frac{1}{4}p_2 - \frac{5}{32}p_1^2\right)z^2 \\ &\quad + \left(\frac{1}{4}p_3 - \frac{5}{16}p_1p_2 + \frac{13}{128}p_1^3\right)z^3 + \cdots. \end{aligned}$$

Similarly,

$$\begin{aligned} \frac{z[L(\alpha,\beta)f(z)]'}{L(\alpha,\beta)f(z)} &= 1 + \frac{\alpha}{\beta}a_2 z + \left[\frac{\alpha(\alpha+1)}{\beta(\beta+1)}2a_3 - \frac{\alpha^2}{\beta^2}a_2^2\right]z^2 \\ &\quad + \left[3a_4\left(\frac{\alpha(\alpha+1)(\alpha+2)}{\beta(\beta+1)(\beta+2)}\right) - 3a_2a_3\frac{\alpha^2(\alpha+1)}{\beta^2(\beta+1)} + a_2^3\frac{\alpha^3}{\beta^3}\right]z^3 + \cdots. \end{aligned}$$

Thus,

$$a_2 = \frac{1}{4}\frac{\beta}{\alpha}p_1, \tag{6}$$

$$a_3 = \frac{\beta(\beta+1)}{\alpha(\alpha+1)}\left[\frac{1}{8}p_2 - \frac{3}{64}p_1^2\right] \tag{7}$$

and

$$a_4 = \frac{\beta(\beta+1)(\beta+2)}{\alpha(\alpha+1)(\alpha+2)}\left[\frac{1}{12}p_3 - \frac{7}{96}p_1p_2 + \frac{13}{768}p_1^3\right]. \tag{8}$$

Now, making use of Equations (6) and (7), we have

$$\left|a_3-\mu a_2^2\right|=\frac{1}{8}\frac{\beta(\beta+1)}{\alpha(\alpha+1)}\left|p_2-\frac{1}{8}\left[4\mu\frac{\beta(\alpha+1)}{\alpha(\beta+1)}+3\right]p_1^2\right|. \tag{9}$$

Using Lemma 1 in conjunction with Equation (9), we obtained the require result. □

Theorem 2. *Let, for any complex number μ, $L(\alpha,\beta)f(z)\in SL^*(\alpha,\beta)$ having the form Equation (5). Then*

$$\left|\frac{\alpha(\alpha+1)}{\beta(\beta+1)}a_3-\mu a^2\frac{\alpha^2}{\beta^2}\right|\leq\frac{1}{4}\max\left(1;\left|\mu\frac{\beta(\beta+1)}{\alpha(\alpha+1)}-\frac{1}{4}\right|\right).$$

Proof. The proof of this theorem is simple, so we omit the proof. □

Special Cases:

1. For $L(\alpha,\alpha)$ we get;
$$\left|a_3-\mu a_2^2\right|\leq\frac{1}{4}\max\left(1;\left|\mu-\frac{1}{4}\right|\right),$$
which is proved by Raza and Malik [15].
2. For $L(\alpha,\alpha)$ and $\mu=1$, we can get $H_2(1)$.

Theorem 3. *Assume that $L(\alpha,\beta)f\in SL^*(\alpha,\beta)$ is in the form Equation (5). Then*

$$\left|a_2a_4-a_3^2\right|\leq\frac{1}{16}\left(\frac{\beta(\beta+1)}{\alpha(\alpha+1)}\right)^2.$$

Proof. By make use of Equations (6)–(8), we have

$$\begin{aligned}a_2a_4-a_3^2=&\left(\frac{\beta}{4\alpha}p_1\right)\left(\frac{\beta(\beta+1)(\beta+2)}{\alpha(\alpha+1)(\alpha+2)}\right)\left(\frac{1}{12}p_3-\frac{7}{96}p_1p_2+\frac{13}{768}p_1^3\right)\\&-\left[\frac{\beta(\beta+1)}{\alpha(\alpha+1)}\left(\frac{1}{8}p_2-\frac{3}{64}p_1^2\right)\right]^2.\end{aligned}$$

After simplification, we have

$$\begin{aligned}a_2a_4-a_3^2=&\frac{\beta^2(\beta+1)}{12288\alpha^2(\alpha+1)}\left[\frac{256(\beta+2)}{(\alpha+2)}p_1p_3-\frac{192(\beta+1)}{(\alpha+1)}p_2^2\right.\\&\left.+\left(\frac{224(\beta+2)}{(\alpha+2)}+\frac{144(\beta+1)}{(\alpha+1)}\right)p_1^2p_2+\left(\frac{52(\beta+2)}{(\alpha+2)}-\frac{27(\beta+1)}{(\alpha+1)}\right)p_1^4\right].\end{aligned}$$

By substituting values of p_2 and p_3 from Lemma 3, after some simplification, we have

$$\begin{aligned}a_2a_4-a_3^2\leq&\frac{\beta^2(\beta+1)}{12288\alpha^2(\alpha+1)}\left[\left(\frac{4(\beta+2)}{(\alpha+2)}-\frac{3(\beta+1)}{(\alpha+1)}\right)p_1^4\right.\\&+\frac{128(\beta+2)}{(\alpha+2)}(4-p_1^2)+\left(\frac{24(\beta+1)}{(\alpha+1)}-\frac{16(\beta+2)}{(\alpha+2)}\right)\\&(4-p_1^2)p_1^2\rho+\rho^2(4-p_1^2)\\&\left.\left\{\left(\frac{64(\beta+2)}{(\alpha+2)}-\frac{48(\beta+1)}{(\alpha+1)}\right)p_1^2+\frac{192(\beta+1)}{(\alpha+1)}\right\}\right],\end{aligned}$$

or by considering right-hand side as $F(p_1,\rho)$, we can write

$$a_2a_4-a_3^2=F(p_1,\rho).$$

Differentiating w.r.t. ρ, assuming $\rho > 0$ and taking $p_1 = p \in [0,2]$, we can obtain

$$\frac{\partial F(p,\rho)}{\partial p} = \frac{\beta^2(\beta+1)}{12288\alpha^2(\alpha+1)}\Bigg[\left(\frac{24(\beta+1)}{(\alpha+1)} - \frac{16(\beta+2)}{(\alpha+2)}\right) (4-p^2)p^2 + 2\rho(4-p^2) \left\{\left(\frac{64(\beta+2)}{(\alpha+2)} - \frac{48(\beta+1)}{(\alpha+1)}\right)p^2 + \frac{192(\beta+1)}{(\alpha+1)}\right\}\Bigg].$$

As $\frac{\partial F(p,\rho)}{\partial p} > 0$, we then find that $F(p,\rho)$ increases on $[0,1]$. Hence,

$$F(p)) = F(p,1) = \max F(p,\rho).$$

For $p = 0$, we can write

$$|a_2a_4 - a_3^2| \leq \frac{1}{16}\left(\frac{\beta(\beta+1)}{\alpha(\alpha+1)}\right)^2,$$

which is the desired result. □

Special Case:
If we put $\alpha = \beta$, then for $L(\alpha,\alpha)$, we can obtain

$$|a_2a_4 - a_3^2| = \frac{1}{16},$$

which is proved by Raza and Malik [15].

Theorem 4. *Let $L(\alpha,\beta)f \in SL^*(\alpha,\beta)$ is in the form Equation (5). Then*

$$|a_2a_4 - a_4| \leq \frac{1}{6}\left(\frac{\beta(\beta+1)(\beta+2)}{\alpha(\alpha+1)(\alpha+2)}\right)^2.$$

Proof. Using Lemma 3, we can write

$$a_2a_4 - a_4 = \frac{\beta(\beta+1)}{\alpha(\alpha+1)}\Bigg[\left(\frac{\beta}{32\alpha} - \frac{7(\beta+2)}{96(\alpha+2)}\right)p_1p_2 + \left(-\frac{3\beta}{256\alpha} + \frac{13(\beta+2)}{768(\alpha+2)}\right)p_1^3 - \frac{\beta+2}{12(\alpha+2)}p_3\Bigg].$$

By putting values of p_2 and p_3, we can obtain

$$a_2a_4 - a_4 = \frac{\beta(\beta+1)}{\alpha(\alpha+1)}\Bigg[\left(\frac{9\beta}{\alpha} - \frac{31(\beta+2)}{(\alpha+2)}\right)p1^3 + \left(\frac{9\beta}{\alpha} - \frac{60(\beta+2)}{(\alpha+2)}\right)(4-p_1^2)xp_1 + \frac{16(\beta+2)}{(\alpha+2)}(4-p_1^2)p_1x^2 - \frac{32(\beta+2)}{(\alpha+2)}(4-p_1^2)(1-|x|^2)z\Bigg].$$

Now, using triangular inequality, replacing $|x|$ with ρ, assuming $p_1 = p$ and differentiating w.r.t ρ after simplification, we obtain

$$F_1(p) = G_1(p) = \frac{\beta(\beta+1)}{768\alpha(\alpha+1)}\left[\left(\frac{9\beta}{\alpha} - \frac{31(\beta+2)}{(\alpha+2)}\right)p^3 + \frac{32(\beta+2)}{(\alpha+2)}(4-p_2)\right],$$

$$G_1'(p) = \frac{\beta(\beta+1)}{768\alpha(\alpha+1)}\left[\left(3\frac{9\beta}{\alpha} - \frac{31(\beta+2)}{(\alpha+2)}\right)p^2 - \left(\frac{64(\beta+2)}{(\alpha+2)}\right)p\right]$$

and

$$G_1''(p) = \frac{\beta(\beta+1)}{768\alpha(\alpha+1)}\left[\left(6\frac{9\beta}{\alpha} - \frac{31(\beta+2)}{(\alpha+2)}\right)p - \frac{64(\beta+2)}{(\alpha+2)}\right] < 0.$$

For $p = 0$, we can get

$$G_1(0) = \frac{128\beta(\beta+1)(\beta+2)}{768\alpha(\alpha+1)(\alpha+2)},$$

or

$$G_1(0) = \frac{\beta(\beta+1)(\beta+2)}{6\alpha(\alpha+1)(\alpha+2)}.$$

□

Theorem 5. *Let $L(\alpha,\beta)f \in SL^*(\alpha,\beta)$ be the form Equation (5). Then*

$$|H_3(1)| \leq \frac{1}{576}\left[\frac{9\beta^2(\beta+1)^2}{\alpha^2(\alpha+1)^2} + \frac{16\beta(\beta+1)(\beta+2)}{\alpha(\alpha+1)(\alpha+2)} + 18\right].$$

Proof. As,

$$a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_1a_3 - a_2^2) = H_3(1).$$

By applying triangular inequality; it gives

$$|a_2a_4 - a_3^2||a_3| + |a_2a_3 - a_4||a_4| + |a_1a_3 - a_2^2||a_5| = |H_3(1)|.$$

After simplification, we can write

$$|H_3(1)| \leq \frac{1}{4}\left(\frac{\beta^2(\beta+1)^2}{16\alpha^2(\alpha+1)^2}\right) + \frac{1}{6}\left(\frac{\beta(\beta+1)(\beta+2)}{6\alpha(\alpha+1)(\alpha+2)}\right) + \left(\frac{1}{8}\frac{1}{4}\right).$$

Hence,

$$|H_3(1)| \leq \frac{1}{576}\left[\frac{9\beta^2(\beta+1)^2}{\alpha^2(\alpha+1)^2} + \frac{16\beta(\beta+1)(\beta+2)}{\alpha(\alpha+1)(\alpha+2)} + 18\right].$$

□

4. Conclusions

In this work, we introduced a new subclass of univalent function associated with a Carlson–Shaffer operator, named as $SL^*(\alpha,\beta)$. By applying the Carlson–Shaffer operator, we derived an upper bound of $H_3(1)$ of the desired subclass associated to the right half of the lemniscate of Bernoulli. Certain properties such as: upper bound of $H_3(1)$, coefficient estimate, etc. for this newly defined subclass have also been discussed in detail. We also compare the obtained results with known results in special cases.

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