fractal and fractional

Article

Analytic Resolving Families for Equations with the Dzhrbashyan–Nersesyan Fractional Derivative

Vladimir E. Fedorov [1,*], Marina V. Plekhanova [1,2] and Elizaveta M. Izhberdeeva [1]

1 Mathematical Analysis Department, Chelyabinsk State University, 129, Kashirin Brothers St., Chelyabinsk 454001, Russia
2 Computational Mechanics Department, South Ural State University, 76, Lenin Av., Chelyabinsk 454080, Russia
* Correspondence: kar@csu.ru; Tel.: +7-351-799-7106

Abstract: In this paper, a criterion for generating an analytic family of operators, which resolves a linear equation solved with respect to the Dzhrbashyan–Nersesyan fractional derivative, via a linear closed operator is obtained. The properties of the resolving families are investigated and applied to prove the existence of a unique solution for the corresponding initial value problem of the inhomogeneous equation with the Dzhrbashyan–Nersesyan fractional derivative. A solution is presented explicitly using resolving families of operators. A theorem on perturbations of operators from the found class of generators of resolving families is proved. The obtained results are used for a study of an initial-boundary value problem to a model of the viscoelastic Oldroyd fluid dynamics. Thus, the Dzhrbashyan–Nersesyan initial value problem is investigated in the essentially infinite-dimensional case. The use of the proved abstract results to study initial-boundary value problems for a system of partial differential equations is demonstrated.

Keywords: fractional Dzhrbashyan–Nersesyan derivative; differential equation with fractional derivatives; resolving family of operators; perturbation theorem; initial value problem; initial-boundary value problem; viscoelastic Oldroyd fluid

MSC: 34G10; 35R11; 34A08

Citation: Fedorov, V.E.; Plekhanova, M.V.; Izhberdeeva, E.M. Analytic Resolving Families for Equations with the Dzhrbashyan–Nersesyan Fractional Derivative. *Fractal Fract.* **2022**, *6*, 541. https://doi.org/10.3390/fractalfract6100541

Academic Editor: Carlo Cattani

Received: 2 August 2022
Accepted: 22 September 2022
Published: 25 September 2022

1. Introduction

Consider the differential equation

$$D^{\sigma_n}z(t) = Az(t) + f(t), \quad t \in (0, T], \tag{1}$$

where A is a linear closed operator, which has a dense domain D_A in a Banach space $\mathcal{Z}$, $T > 0$, $f : [0, T] \to \mathcal{Z}$ is a given function. Let D_t^β be the Riemann–Liouville integral for $\beta \leq 0$ and the Riemann–Liouville derivative for $\beta > 0$. Here $D^{\sigma_n}z := D_t^{\alpha_n - 1}D_t^{\alpha_{n-1}}D_t^{\alpha_{n-2}} \dots D_t^{\alpha_0}z(t)$, where $\alpha_k \in (0, 1]$, is the Dzhrbashyan–Nersesyan fractional derivative [1]. Note that this derivative includes as partial cases the Gerasimov–Caputo ($\alpha_k = 1$, $k = 0, 1, \dots, n-1$, $\alpha_n = \alpha - n + 1$) and the Riemann–Liouville ($\alpha_0 = \alpha - n + 1$, $\alpha_k = 1$, $k = 1, 2, \dots, n$) fractional derivatives of an order α from $(n-1, n]$.

In recent decades, fractional-order equations have been actively used in modeling various complex systems and processes in physics, chemistry, social sciences, and humanities [2–6]. We note recent works [7–12], combining theoretical studies in various fields of fractional integro-differential calculus and their use in real-world modeling problems, particularly when modeling biological processes in virology, which is especially important at present. Readers should also note the works [13,14], which consider some applied problems with the Dzhrbashyan–Nersesyan fractional derivative.

The initial value problem

$$D^{\sigma_k} z(0) = z_k, \quad k = 0, 1, \ldots, n-1, \tag{2}$$

with

$$D^{\sigma_0} z(t) := D_t^{\alpha_0 - 1} z(t), \quad D^{\sigma_k} z(t) := D_t^{\alpha_k - 1} D_t^{\alpha_{k-1}} D_t^{\alpha_{k-2}} \ldots D_t^{\alpha_0} z(t), \quad k = 1, 2, \ldots, n,$$

for Equation (1) in the scalar case ($\mathcal{Z} = \mathbb{R}$, $A \in \mathbb{R}$) is studied by M.M. Dzrbashyan, A.B. Nersesyan in [1]. The unique solvability theorem for such a problem with $\mathcal{Z} = \mathbb{R}^n$ and a matrix A was obtained in [15]. Various equations with partial derivatives of Dzhrbashyan and Nersesyan were studied in papers [16–21]. Problem (1), (2) with a linear continuous operator $A \in \mathcal{L}(\mathcal{Z})$ in an arbitrary Banach space $\mathcal{Z}$ was researched in [22] considering the methods used to resolve families of operators; see [23].

The results obtained in this work generalize the corresponding results of the theory of analytic semigroups of operators solving first-order equations in Banach spaces [24,25]. We also note the works in which the theory of analytical resolving families is constructed for evolutionary integral equations [26], equations with a Gerasimov–Caputo [27] or Riemann–Liouville [28] derivative, fractional multi-term linear differential equations in Banach spaces [29], and equations with various distributed fractional derivatives [30–34].

After the Introduction and Preliminaries, in the second section of the present work, the notion of a k-resolving family for homogeneous Equation (1), i.e., with $f \equiv 0$, $k = 0, 1, \ldots, n-1$, is introduced. In the third section, it is shown that the existence of k-resolving families, $k = 1, 2, \ldots, n-1$, follows from the existence of a zero-resolving family. In the fourth section, a criterion of the existence of a zero-resolving family of operators to the homogeneous Equation (1) is found in terms of conditions for a linear closed operator A. The class of operators which satisfy these conditions is denoted as $\mathcal{A}_{\{\alpha_k\}}(\theta_0, a_0)$. Various properties of the resolving families are investigated, and a perturbation theorem for operators from $\mathcal{A}_{\{\alpha_k\}}(\theta_0, a_0)$ is proved in the fifth section. For problem (1), (2) with a function f, which is continuous in the graph norm of A or Hölderian, the existence of a unique solution is obtained in the sixth section. In the last section, this result is used to prove the theorem on a unique solution existence for an initial-boundary value problem to a fractional linearized model of the viscoelastic Oldroyd fluid dynamics.

The theoretical significance of the obtained results lies in the fact that they give a correct statement of an initial problem and conditions for its unique solvability for equations with the Dzhrbashyan–Nersesian fractional derivative and with an unbounded linear operator at the unknown function. The unboundedness of the operator in the equation makes it possible to reduce initial-boundary value problems to various equations and systems of partial differential equations in problems of this type.

2. Preliminaries

Let $\mathcal{Z}$ be a Banach space. For the function $z : \mathbb{R}_+ \to \mathcal{Z}$, the Riemann–Liouville fractional integral of an order $\beta > 0$ has the form

$$J_t^\beta z(t) := \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} z(s) ds, \quad t > 0.$$

For the function z, the Riemann–Liouville fractional derivative of an order $\alpha \in (m-1, m]$, where $m \in \mathbb{N}$ is defined as $D_t^\alpha z(t) := D_t^m J_t^{m-\alpha} z(t)$, $D_t^m := \frac{d^m}{dt^m}$. Further, we use the notation $D_t^{-\alpha} := J_t^\alpha$ for $\alpha > 0$; $D_t^0 = J_t^0$ is the identical operator.

Let $\{\alpha_k\}_0^n$ be a set of numbers $\alpha_k \in (0, 1]$, $k = 0, 1, \ldots, n \in \mathbb{N}$. We will use the denotations $\sigma_k := \sum_{j=0}^{k} \alpha_j - 1$, $k = 0, 1, \ldots, n$, hence $\sigma_k \in (-1, k-1]$. Further, we will assume that $\sigma_n > 0$. Define the Dzhrbashyan–Nersesyan fractional derivatives, which correspond to the sequence $\{\alpha_k\}_0^n$, by relations

$$D^{\sigma_0}z(t) := D_t^{\alpha_0-1}z(t), \tag{3}$$

$$D^{\sigma_k}z(t) := D_t^{\alpha_k-1}D_t^{\alpha_{k-1}}D_t^{\alpha_{k-2}}\dots D_t^{\alpha_0}z(t), \quad k = 1,2,\dots,n. \tag{4}$$

Example 1. *Take* $\alpha \in (n-1,n]$, $\alpha_0 = \alpha - n + 1 \in (0,1]$, $\alpha_k = 1$, $k = 1,2,\dots,n$, *then* $D^{\sigma_0}z(t) := D_t^{\alpha-n}z(t) := J_t^{n-\alpha}z(t)$, $D^{\sigma_k}z(t) := D_t^{k-1}D_t^{\alpha-n+1}z(t) = D_t^kJ_t^{n-\alpha}z(t) := D_t^{k-n+\alpha}z(t)$, $k = 1,2,\dots,n$, *are the Riemann–Liouville fractional derivatives. In particular,* $D^{\sigma_n}z(t) = D_t^nJ_t^{n-\alpha}z(t) := D_t^\alpha z(t)$.

Example 2. *If* $\alpha \in (n-1,n]$, $\alpha_k = 1$, $k = 0,1,\dots,n-1$, $\alpha_n = \alpha - n + 1$, *then* $D^{\sigma_k}z(t) := D_t^kz(t)$, $k = 0,1,\dots,n-1$, $D^{\sigma_n}z(t) := D_t^{\alpha-n}D_t^nz(t) := J_t^{n-\alpha}D_t^nz(t) :={}^CD_t^\alpha$ *is the Gerasimov–Caputo fractional derivative.*

Example 3. *In [23], it is shown that the compositions of the Gerasimov–Caputo and the Riemann–Liouville fractional derivatives* $D_t^\alpha D_t^\beta$, $D_t^\alpha{}^CD_t^\beta$, ${}^CD_t^\alpha D_t^\beta$, ${}^CD_t^\alpha{}^CD_t^\beta$ *may be presented as Dzhrbashyan–Nersesyan fractional derivatives* D^{σ_n} *for some sequences* $\{\sigma_0,\sigma_1,\dots,\sigma_n\}$.

Let $\alpha \in (m-1,m]$, $m \in \mathbb{N}$. Then, for a function $z : \mathbb{R}_+ \to \mathcal{Z}$, we use $\widehat{z}$ to denote the Laplace transform, and for too-large expressions for z as $\mathrm{Lap}[z]$. In [22], it is proved that

$$\widehat{D^{\sigma_n}z}(\lambda) = \lambda^{\sigma_n}\widehat{z}(\lambda) - \sum_{k=0}^{n-1}\lambda^{\sigma_n-\sigma_k-1}D^{\sigma_k}z(0). \tag{5}$$

$\mathcal{L}(\mathcal{Z})$ denotes the Banach space of all linear continuous operators on a Banach space $\mathcal{Z}$; $\mathcal{C}l(\mathcal{Z})$ denotes the set of all linear closed operators, which are densely defined in $\mathcal{Z}$ and act into $\mathcal{Z}$. For an operator $A \in \mathcal{C}l(\mathcal{Z})$, its domain D_A is endowed by the norm $\|\cdot\|_{D_A} := \|\cdot\|_{\mathcal{Z}} + \|A\cdot\|_{\mathcal{Z}}$, which is a Banach space due to the closedness of A.

Consider the initial value problem

$$D^{\sigma_k}z(0) = z_k, \quad k = 0,1,\dots,n-1. \tag{6}$$

to the linear homogeneous equation

$$D^{\sigma_n}z(t) = Az(t), \quad t > 0, \tag{7}$$

where $A \in \mathcal{C}l(\mathcal{Z})$, D^{σ_n} is the Dzhrbashyan–Nersesyan fractional derivative, associated with a set of real numbers $\{\alpha_k\}_0^n$, $0 < \alpha_k \le 1$, $k = 0,1,\dots,n \in \mathbb{N}$, by (3), (4), $\sigma_n > 0$.

A solution to problem (6), (7) is a function $z \in C(\mathbb{R}_+; D_A)$, such that $D_t^{\sigma_k}z \in C(\overline{\mathbb{R}}_+;\mathcal{Z})$, $k = 0,1,\dots,n-1$, $D_t^{\sigma_n}z \in C(\mathbb{R}_+;\mathcal{Z})$, (7) holds for all $t \in \mathbb{R}_+$ and conditions (6) are valid. Hereafter, $\overline{\mathbb{R}}_+ := \mathbb{R}_+ \cup \{0\}$.

Denote $S_{\theta,a} := \{\lambda \in \mathbb{C} : |\arg(\lambda - a)| < \theta, \lambda \ne a\}$, $\theta \in [\pi/2,\pi]$, $a \in \mathbb{R}$, $\Sigma_\psi := \{t \in \mathbb{C} : |\arg t| < \psi,\ t \ne 0\}$ for $\psi \in (0,\pi/2]$ and formulate an assertion that is important for further considerations.

Theorem 1 ([34])**.** *Let* $\theta_0 \in (\pi/2,\pi]$, $a \in \mathbb{R}$, $\beta \in [0,1)$, $\mathcal{X}$ *be a Banach space,* $H : (a,\infty) \to \mathcal{X}$. *Then, the next statements are equivalent.*

(i) *There exists an analytic function* $F : \Sigma_{\theta_0-\pi/2} \to \mathcal{X}$. *For every* $\theta \in (\pi/2,\theta_0)$, *there exists such a* $C(\theta) > 0$ *that the inequality* $\|F(t)\|_{\mathcal{X}} \le C(\theta)|t|^{-\beta}e^{a\mathrm{Re}\,t}$ *is satisfied for all* $t \in \Sigma_{\theta-\pi/2}$; *for* $\lambda > a$ $\widehat{F}(\lambda) = H(\lambda)$.

(ii) *H is analytically extendable on $S_{\theta_0,a}$; for every $\theta \in (\pi/2, \theta_0)$ there exists $K(\theta) > 0$, such that for all $\lambda \in S_{\theta,a}$*

$$\|H(\lambda)\|_{\mathcal{X}} \leq \frac{K(\theta)}{|\lambda - a|^{1-\beta}}.$$

3. k-Resolving Families of Operators

Definition 1. *A set of linear bounded operators $\{S_l(t) \in \mathcal{L}(\mathcal{Z}) : t > 0\}$ is called k-resolving family, $k \in \{0, 1, \dots, n-1\}$, for Equation (7), if it satisfies the next conditions:*
(i) *$S_k(t)$ is a strongly continuous family at $t > 0$;*
(ii) *$S_k(t)[D_A] \subset D_A$, for all $x \in D_A$, $t > 0$ $S_k(t)Ax = AS_k(t)x$;*
(iii) *For every $z_k \in D_A$ $S_k(t)z_k$ is a solution of initial value problem $D^{\sigma_k}z(0) = z_k$, $D^{\sigma_l}z(0) = 0$, $l \in \{0, \dots, n-1\} \setminus \{k\}$ to Equation (7).*

Let $\rho(A) := \{\lambda \in \mathbb{C} : R_\lambda(A) := (\lambda I - A)^{-1} \in \mathcal{L}(\mathcal{Z})\}$ be the resolvent set of operator A.

Proposition 1. *Let $\alpha_l \in (0, 1]$, $l = 0, 1, \dots, n$, $\sigma_n > 0$. For $k \in \{0, 1, \dots, m-1\}$ there exists a k-resolving family of operators $\{S_k(t) \in \mathcal{L}(\mathcal{Z}) : t > 0\}$ for Equation (7), such that at some $K > 0$, $a \in \mathbb{R}$, $\beta \in [0, 1)$ $\|S_k(t)\|_{\mathcal{L}(\mathcal{Z})} \leq Ke^{at}t^{-\beta}$ for all $t > 0$. Then, $\lambda^{\sigma_n} \in \rho(A)$ for $\mathrm{Re}\lambda > a$,*

$$\widehat{S}_k(\lambda) = \lambda^{\sigma_n - \sigma_k - 1} R_{\lambda^{\sigma_n}}(A) \tag{8}$$

and a k-resolving family of operators for Equation (7) is unique.

Proof. Due to identity (5) and Definition 1 for arbitrary $z_k \in D_A$, $\mathrm{Re}\lambda > a$ $\lambda^{\sigma_n}\widehat{S}_k(\lambda)z_k - \lambda^{\sigma_n - \sigma_k - 1}z_k = A\widehat{S}_k(\lambda)z_k = \widehat{S}_k(\lambda)Az_k$. Therefore, the operator $\lambda^{\sigma_n}I - A : D_A \to \mathcal{Z}$ is invertible and equality (8) holds. Since $\widehat{S}_k(\lambda) \in \mathcal{L}(\mathcal{Z})$ for $\mathrm{Re}\lambda > a$, we have $\lambda^{\sigma_n} \in \rho(A)$. Due to equality (8) from the uniqueness of the inverse Laplace transform, we see the uniqueness of a k-resolving family for Equation (7). □

Proposition 2. *Let $\alpha_k \in (0, 1]$, $k = 0, 1, \dots, n$, $\sigma_n > 0$. There exists a 0-resolving family $\{S_0(t) \in \mathcal{L}(\mathcal{Z}) : t > 0\}$ for (7), such that at some $K > 0$, $a \in \mathbb{R}$ $\|S_0(t)\|_{\mathcal{L}(\mathcal{Z})} \leq Ke^{at}t^{\sigma_0}$ for all $t > 0$. Then, for every $k = 0, 1, \dots, n-1$, there exists a unique k-resolving family $\{S_k(t) \in \mathcal{L}(\mathcal{Z}) : t > 0\}$. Moreover, $S_k(t) \equiv J_t^{\sigma_k - \sigma_0}S_0(t)$ and $\|S_k(t)\|_{\mathcal{L}(\mathcal{Z})} \leq K_1e^{at}t^{\sigma_k}$ at some $K_1 > 0$ for all $t > 0$, $k = 1, 2, \dots, n-1$.*

Proof. Since every $z_0 \in D_A \setminus \{0\}$ $J^{1-\alpha_0}S_0(t)z_0$ has a nonzero limit z_0 as $t \to 0+$, due to ([29], Lemma 1) $S_0(t)z_0 = t^{\alpha_0 - 1}z_0/\Gamma(\alpha_0) + o(t^{\alpha_0 - 1})$ as $t \to 0+$. Therefore, for every $z_0 \in \mathcal{Z}$, $T > 0$ $S_0(t)z_0 \in L_1(0, T; \mathcal{Z})$ and there are Riemann–Liouville fractional integrals for this function.

Define for $k = 1, 2, \dots, n-1$ the families $\{S_k(t) := J_t^{\sigma_k - \sigma_0}S_0(t) \in \mathcal{L}(\mathcal{Z}) : t > 0\}$. By this construction, it satisfies condition (i) in the Definition 1. For $x \in D_A$, $t > 0$

$$J_t^{\sigma_k - \sigma_0}S_0(t)Ax = \int_0^t \frac{(t-s)^{\sigma_k - \sigma_0 - 1}}{\Gamma(\sigma_k - \sigma_0)} S_0(s)Axds = AJ_t^{\sigma_k - \sigma_0}S_0(t)x,$$

since $\{S_0(t) \in \mathcal{L}(\mathcal{Z}) : t > 0\}$ satisfies condition (ii) in Definition 1 and the operator A is closed. So, condition (ii) holds for $\{S_k(t) \in \mathcal{L}(\mathcal{Z}) : t > 0\}$, where $k = 1, 2, \dots, n-1$.

Further, we have

$$\|S_k(t)\|_{\mathcal{L}(\mathcal{Z})} \leq K\int_0^t \frac{(t-s)^{\sigma_k - \sigma_0 - 1}}{\Gamma(\sigma_k - \sigma_0)} s^{\sigma_0}e^{as}ds \leq \frac{Ke^{at}t^{\sigma_k}\Gamma(\sigma_0 + 1)}{\Gamma(\sigma_k + 1)} = K_1e^{at}t^{\sigma_k}, \quad t > 0.$$

For $z_k \in D_A$, multiply the equality $\lambda^{\sigma_n}\widehat{S}_0(\lambda)z_k - \lambda^{\sigma_n-\sigma_0-1}z_k = A\widehat{S}_0(\lambda)z_k$, which follows from point (iii) of Definition 1 for $k=0$ after the Laplace transform action, by $\lambda^{\sigma_0-\sigma_k}$ and obtain the equality $\lambda^{\sigma_n}\widehat{J_t^{\sigma_k-\sigma_0}S_0}(\lambda)z_k - \lambda^{\sigma_n-\sigma_k-1}z_k = A\widehat{J_t^{\sigma_k-\sigma_0}S_0}(\lambda)z_k$, i.e., $\lambda^{\sigma_n}\widehat{S}_k(\lambda)z_k - \lambda^{\sigma_n-\sigma_k-1}z_k = A\widehat{S}_k(\lambda)z_k$, which means that $\{S_k(t) \in \mathcal{L}(\mathcal{Z}) : t > 0\}$ is a k-resolving family for Equation (7) due to the uniqueness of the inverse Laplace transform. Hence equality (8) is valid and a k-resolving family of Equation (7) is unique by Proposition 1. □

Remark 1. *The parameter σ_0 in the formulation of Proposition 2 defines the power singularity of the family $\{S_0(t) \in \mathcal{L}(\mathcal{Z}) : t > 0\}$ at zero. At the beginning of the proof of Proposition 2, it was shown that we have two possibilities only: the singularity at zero has a power of $\sigma_0 := \alpha_0 - 1 < 0$, or a singularity is absent in the case $\alpha_0 = 1$. Due to Proposition 2, the k-resolving family $\{S_k(t) \in \mathcal{L}(\mathcal{Z}) : t > 0\}$ has the singularity of the power $\sigma_k < 0$, or it is absent at zero, if $\sigma_k \geq 0$.*

Theorem 2. *Let $\alpha_l \in (0,1]$, $l = 0,1,\ldots,n$, $\sigma_n > 0$, there exist a k-resolving family of operators $\{S_k(t) \in \mathcal{L}(\mathcal{Z}) : t > 0\}$ of (7) for some $k \in \{0,1,\ldots,n-1\}$, such that $\|S_k(t)\|_{\mathcal{L}(\mathcal{Z})} \leq Ke^{at}t^{\sigma_k}$ at some $K > 0$, $a \in \mathbb{R}$ for all $t > 0$. Then, there exists a limit $\lim\limits_{t\to 0+} D^{\sigma_k}S_k(t) = I$ in the norm of the space $\mathcal{L}(\mathcal{Z})$, if and only if $A \in \mathcal{L}(\mathcal{Z})$.*

Proof. Note that $\widehat{D^{\sigma_k}S_k} = \lambda^{\sigma_k}\widehat{S}_k = \lambda^{\sigma_n-1}(\lambda^{\sigma_n}I - A)^{-1}$ due to (5), Definition 1 and Proposition 1. Hence for $z_k \in D_A$, $b > a$

$$D^{\sigma_k}S_k(t)z_k = \int\limits_{b-i\infty}^{b+i\infty} \lambda^{\sigma_n-1}R_{\lambda^{\sigma_n}}(A)e^{\lambda t}z_k d\lambda = z_k + \int\limits_{b-i\infty}^{b+i\infty} \lambda^{-1}R_{\lambda^{\sigma_n}}(A)e^{\lambda t}Az_k d\lambda. \quad (9)$$

Since, for large enough $|\lambda|$

$$\|\lambda^{-1}R_{\lambda^{\sigma_n}}(A)\|_{\mathcal{L}(\mathcal{Z})} \leq \frac{C_1}{|\lambda|^{\sigma_n-\sigma_0+\alpha_0}} = \frac{C_1}{|\lambda|^{\sigma_n+1}},$$

we have $\|D^{\sigma_k}S_k(t)z_k\|_{\mathcal{Z}} \leq K_1e^{bt}$.

For $\mathrm{Re}\lambda > b$

$$\int\limits_0^\infty e^{-\lambda t}(D^{\sigma_k}S_k(t) - I)dt = \lambda^{\sigma_n-1}R_{\lambda^{\sigma_n}}(A) - \lambda^{-1}I.$$

Assume that $\eta(t) := \|D^{\sigma_k}S_k(t) - I\|_{\mathcal{L}(\mathcal{Z})}$ is a continuous function on $[0,1]$ and $\eta(0) = 0$. For arbitrary $\varepsilon > 0$, take $\delta > 0$, such that for all $t \in [0,\delta]$ $\eta(t) \leq \varepsilon$; therefore, due to the inequality $\eta(t) \leq K_1e^{bt} + 1$ for $t \geq 0$, we have

$$\left\|\lambda^{\sigma_n-1}R_{\lambda^{\sigma_n}}(A) - \lambda^{-1}I\right\|_{\mathcal{L}(\mathcal{Z})} \leq \int\limits_0^\delta e^{-\lambda t}\eta(t)dt + \int\limits_\delta^\infty e^{-\lambda t}\eta(t)dt \leq \frac{\varepsilon}{\lambda} + o\left(\frac{1}{\lambda}\right)$$

as $\mathrm{Re}\lambda \to +\infty$. Hence, for large enough $\mathrm{Re}\lambda > 0$ $\left\|\lambda^{\sigma_n-1}R_{\lambda^{\sigma_n}}(A) - \lambda^{-1}I\right\|_{\mathcal{L}(\mathcal{Z})} < 1$. Consequently, $R_{\lambda^{\sigma_n}}(A)$ is a continuously invertible operator, so $A \in \mathcal{L}(\mathcal{Z})$.

Let $A \in \mathcal{L}(\mathcal{Z})$, $R > \|A\|_{\mathcal{L}(\mathcal{Z})}^{1/\sigma_n}$, $\Gamma_{1,R} := \{Re^{i\varphi} : \varphi \in (-\pi,\pi)\}$, $\Gamma_{2,R} := \{re^{i\pi} : r \in [R,\infty)\}$, $\Gamma_{3,R} := \{re^{-i\pi} : r \in [R,\infty)\}$, $\Gamma_R := \Gamma_{1,R} \cup \Gamma_{2,R} \cup \Gamma_{3,R}$. Due to equality (9), we obtain for $t > 0$

$$D^{\sigma_k}S_k(t) = I + \frac{1}{2\pi i}\int\limits_{\Gamma_R} \lambda^{-1}R_{\lambda^{\sigma_n}}(A)Ae^{\lambda t}d\lambda = I + \frac{1}{2\pi i}\int\limits_{\Gamma_R} \frac{1}{\lambda}\sum_{l=1}^\infty \frac{A^l e^{\lambda t}d\lambda}{\lambda^{l\sigma_n}}.$$

Take $R = 1/t$ for small $t > 0$; then,

$$\|D^{\sigma_k} S_k(t) - I\|_{\mathcal{L}(\mathcal{Z})} \leq C_1 \sum_{k=1}^{3} \sum_{l=1}^{\infty} \int_{\Gamma_{k,R}} \frac{\|A\|^l_{\mathcal{L}(\mathcal{Z})} |d\lambda|}{|\lambda|^{l\sigma_n+1}} \leq \frac{C_2 t^{\sigma_n} \|A\|_{\mathcal{L}(\mathcal{Z})}}{1 - t^{\sigma_n} \|A\|_{\mathcal{L}(\mathcal{Z})}} \to 0$$

as $t \to 0+$. □

Remark 2. *An analogous result of Theorem 2 is well-known for resolving semigroups of operators for first-order equations (see, e.g., [35]). On resolving families of operators for equations, which are solved with respect to a Gerasimov–Caputo derivative, a similar theorem was obtained in work [27].*

4. Generation of Analytic *k*-Resolving Families

Let $k \in \{0, 1, \dots, n-1\}$. A k-resolving family of operators is called *analytic*, if at some $\psi_0 \in (0, \pi/2]$ it has an analytic continuation to Σ_{ψ_0}. An analytic k-resolving family of operators $\{S_k(t) \in \mathcal{L}(\mathcal{Z}) : t > 0\}$ has a type (ψ_0, a_0, β) at some $\psi_0 \in (0, \pi/2]$, $a_0 \in \mathbb{R}$, $\beta \geq 0$, if, for arbitrary $\psi \in (0, \psi_0)$, $a > a_0$, there exists $C(\psi, a)$, such that the inequality $\|S_k(t)\|_{\mathcal{L}(\mathcal{Z})} \leq C(\psi, a) e^{a \mathrm{Re}\, t} |t|^{-\beta}$ is satisfied for all $t \in \Sigma_\psi$.

Remark 3. *From Proposition 2 and Remark 1 it follows that for a k-resolving family of operators $\{S_k(t) \in \mathcal{L}(\mathcal{Z}) : t > 0\}$, we may have $\beta = -\sigma_k$, or $\beta = 0$.*

Definition 2. *An operator $A \in \mathcal{C}l(\mathcal{Z})$ belongs to the class $\mathcal{A}_{\{\alpha_k\}}(\theta_0, a_0)$, $\theta_0 \in (\pi/2, \pi)$, $a_0 \geq 0$, $\alpha_k \in (0, 1]$, $k = 0, 1, \dots, n$, $\sigma_n > 0$, if:*

(i) *For all $\lambda \in S_{\theta_0, a_0}$ we have $\lambda^{\sigma_n} \in \rho(A)$;*

(ii) *For arbitrary $\theta \in (\pi/2, \theta_0)$, $a > a_0$, there exists a constant $K(\theta, a) > 0$, such that for every $\lambda \in S_{\theta,a}$*

$$\|R_{\lambda^{\sigma_n}}(A)\|_{\mathcal{L}(\mathcal{Z})} \leq \frac{K(\theta, a)}{|\lambda - a|^{\alpha_0} |\lambda|^{\sigma_n - \sigma_0 - 1}}.$$

If $A \in \mathcal{A}_{\{\alpha_k\}}(\theta_0, a_0)$, the operators

$$Z_k(t) = \frac{1}{2\pi i} \int_\gamma \lambda^{\sigma_n - \sigma_k - 1} R_{\lambda^{\sigma_n}}(A) e^{\lambda t} d\lambda, \quad t > 0, \; k = 0, 1, \dots, n-1,$$

are defined, where $\Gamma := \Gamma_+ \cup \Gamma_- \cup \Gamma_0$, $\Gamma_\pm := \{\lambda \in \mathbb{C} : \lambda = a + re^{\pm i\theta}, r \in (\delta, \infty)\}$, $\Gamma_0 := \{\lambda \in \mathbb{C} : \lambda = a + \delta e^{i\varphi}, \varphi \in (-\theta, \theta)\}$, $\theta \in (\pi/2, \theta_0)$, $a > a_0$, $\delta > 0$.

Theorem 3. *Let $\alpha_k \in (0, 1]$, $k = 0, 1, \dots, n$, $\alpha_0 + \alpha_n > 0$, $\theta_0 \in (\pi/2, \pi]$, $a_0 \geq 0$.*

(i) *If there exists an analytic 0-resolving family of operators of the type $(\theta_0 - \pi/2, a_0, -\sigma_0)$ for (7), then $A \in \mathcal{A}_{\{\alpha_k\}}(\theta_0, a_0)$.*

(ii) *If $A \in \mathcal{A}_{\{\alpha_k\}}(\theta_0, a_0)$, then for every $k = 0, 1, \dots, n-1$ there exists a unique analytic k-resolving family of operators $\{S_k(t) \in \mathcal{L}(\mathcal{Z}) : t > 0\}$ of the type $(\theta_0 - \pi/2, a_0, \max\{-\sigma_k, 0\})$ for (7). Moreover, for $t > 0$, $k = 0, 1, \dots, n-1$ $S_k(t) \equiv Z_k(t) \equiv J_t^{\sigma_k - \sigma_0} Z_0(t)$.*

Proof. Choose $R > \delta$,

$$\Gamma_R := \bigcup_{k=1}^{4} \Gamma_{k,R}, \quad \Gamma_{1,R} := \Gamma_0, \quad \Gamma_{2,R} := \{\lambda \in \mathbb{C} : \lambda = a + Re^{i\varphi}, \varphi \in (-\theta, \theta)\},$$

$$\Gamma_{3,R} := \{\lambda \in \mathbb{C} : \lambda = a + re^{i\theta}, r \in [\delta, R]\}, \quad \Gamma_{4,R} := \{\lambda \in \mathbb{C} : \lambda = a + re^{-i\theta}, r \in [\delta, R]\},$$

Γ_R is the positively oriented closed loop,

$$\Gamma_{5,R} := \{\lambda \in \mathbb{C} : \lambda = a + re^{i\theta}, r \in [R, \infty)\}, \quad \Gamma_{6,R} := \{\lambda \in \mathbb{C} : \lambda = a + re^{-i\theta}, r \in [R, \infty)\},$$

then $\Gamma = \Gamma_{5,R} \cup \Gamma_{6,R} \cup \Gamma_R \setminus \Gamma_{2,R}$.

If $A \in \mathcal{A}_{\{\alpha_k\}}(\theta_0, a_0)$, then following Theorem 1 with $\mathcal{X} = \mathcal{L}(\mathcal{Z})$, the operators family $\{Z_0(t) \in \mathcal{L}(\mathcal{Z}) : t > 0\}$ is analytic of the type $(\theta_0 - \pi/2, a_0, -\sigma_0)$, it implies point (i) of Definition 1, and point (ii) of this definition is evidently fulfilled.

For any $\theta \in (\pi/2, \theta_0)$, $a > a_0$, we have such a $K(\theta, a) > 0$, that for every $\lambda \in S_{\theta,a}$

$$\left\|\lambda^{\sigma_n-\sigma_k-1}R_{\lambda^{\sigma_n}}(A)\right\|_{\mathcal{L}(\mathcal{Z})} \le C\left\|\lambda^{\sigma_n-\sigma_0-1}R_{\lambda^{\sigma_n}}(A)\right\|_{\mathcal{L}(\mathcal{Z})} \le \frac{CK(\theta,a)}{|\lambda-a|^{\alpha_0}}.$$

So, for $k = 0, 1, \ldots, n-1$, $\mathrm{Re}\lambda > a_0$ there exists the Laplace transforms $\widehat{Z}_k(\lambda) = \lambda^{\sigma_n-\sigma_k-1}R_{\lambda^{\sigma_n}}(A)$, $\widehat{J_t^{\beta} Z_k}(\lambda) = \lambda^{\sigma_n-\sigma_k-1-\beta}R_{\lambda^{\sigma_n}}(A)$, $\beta > 0$, therefore, $Z_k(t) = J_t^{\sigma_k-\sigma_0}Z_0(t)$.

For $z_0 \in D_A$

$$D_t^{\alpha_0-1}Z_0(t)z_0 = \frac{1}{2\pi i}\int_\Gamma \lambda^{\sigma_n-1}R_{\lambda^{\sigma_n}}(A)e^{\lambda t}z_0 d\lambda =$$

$$= \frac{1}{2\pi i}\int_\Gamma \frac{e^{\lambda t}}{\lambda}z_0 d\lambda + \frac{1}{2\pi i}\int_\Gamma \lambda^{-1}R_{\lambda^{\sigma_n}}(A)e^{\lambda t}Az_0 d\lambda = z_0 + \frac{1}{2\pi i}\int_\Gamma \lambda^{-1}R_{\lambda^{\sigma_n}}(A)e^{\lambda t}Az_0 d\lambda.$$

If $t \in [0,1]$, $\lambda \in \Gamma \setminus \{\mu \in \mathbb{C} : |\mu| \le 2a\}$, then

$$\left\|\lambda^{-1}R_{\lambda^{\sigma_n}}(A)e^{\lambda t}Az_0\right\|_{\mathcal{Z}} \le \frac{e^{a+\delta}K(\theta,a)\|Az_0\|_{\mathcal{Z}}}{|\lambda-a|^{\alpha_0}|\lambda|^{\sigma_n-\sigma_0}} \le \frac{C_1}{|\lambda|^{\sigma_n+1}}.$$

Hence,

$$\frac{1}{2\pi i}\int_\Gamma \lambda^{-1}R_{\lambda^{\sigma_n}}(A)e^{\lambda t}Az_0 d\lambda =$$

$$= \lim_{R\to\infty}\frac{1}{2\pi i}\left(\int_{\Gamma_R} - \int_{\Gamma_{2,R}} + \int_{\Gamma_{5,R}} + \int_{\Gamma_{6,R}}\right)\lambda^{-1}R_{\lambda^{\sigma_n}}(A)e^{\lambda t}Az_0 d\lambda = 0,$$

since by the Cauchy theorem

$$\int_{\Gamma_R}\lambda^{-1}R_{\lambda^{\sigma_n}}(A)e^{\lambda t}Az_0 d\lambda = 0, \quad \left\|\int_{\Gamma_{s,R}}\lambda^{-1}R_{\lambda^{\sigma_n}}(A)e^{\lambda t}Az_0 d\lambda\right\|_{\mathcal{Z}} \le \frac{C_2}{R^{\sigma_n}} \to 0$$

as $R \to \infty$ for $s = 2, 5, 6$.

At the same time, due equality (5)

$$\mathrm{Lap}[D^{\sigma_1}Z_0(\cdot)z_0](\lambda) = \lambda^{\sigma_n-\sigma_0-1+\sigma_1}R_{\lambda^{\sigma_n}}(A)z_0 - \lambda^{\sigma_1-\sigma_0-1}z_0 = \lambda^{\alpha_1-1}R_{\lambda^{\sigma_n}}(A)Az_0,$$

for $\lambda \in \Gamma \setminus \{\mu \in \mathbb{C} : |\mu| \le 2a\}$

$$\|\lambda^{\alpha_1-1}R_{\lambda^{\sigma_n}}(A)Az_0\|_{\mathcal{Z}} \le \frac{C_3}{|\lambda|^{\sigma_n-\sigma_0+\alpha_0-\alpha_1}} = \frac{C_3}{|\lambda|^{\alpha_0+\alpha_2+\alpha_3+\cdots+\alpha_n}},$$

$\alpha_0 + \alpha_2 + \alpha_3 + \cdots + \alpha_n > \alpha_0 + \alpha_n > 1$, hence $D^{\sigma_1}Z_0(0)z_0 = 0$. Further, for every $k = 2, 3, \ldots, n-1$

$$\mathrm{Lap}[D^{\sigma_k}Z_0(\cdot)z_0](\lambda) = \lambda^{\sigma_n-\sigma_0-1+\sigma_k}R_{\lambda^{\sigma_n}}(A)z_0 - \lambda^{\sigma_k-\sigma_0-1}z_0 = \lambda^{\sigma_k-\sigma_0-1}R_{\lambda^{\sigma_n}}(A)Az_0,$$

for $\lambda \in \Gamma \setminus \{\mu \in \mathbb{C} : |\mu| \le 2a\}$

$$\|\lambda^{\sigma_k-\sigma_0-1}R_{\lambda^{\sigma_n}}(A)Az_0\|_{\mathcal{Z}} \le \frac{C_3}{|\lambda|^{\sigma_n-\sigma_k+\alpha_0}} = \frac{C_3}{|\lambda|^{\alpha_0+\alpha_{k+1}+\alpha_{k+2}+\cdots+\alpha_n}},$$

thus, $D^{\sigma_k} Z_0(0) z_0 = 0$. Finally,

$$\mathrm{Lap}[D^{\sigma_n} Z_0(\cdot) z_0](\lambda) = \lambda^{\sigma_n - \sigma_0 - 1 + \sigma_n} R_{\lambda^{\sigma_n}}(A) z_0 - \lambda^{\sigma_n - \sigma_0 - 1} z_0 = A \lambda^{\sigma_n - \sigma_0 - 1} R_{\lambda^{\sigma_n}}(A) z_0.$$

Acting on the inverse Laplace transform, we get the equality $D^{\sigma_n} Z_0(t) z_0 = A Z_0(t) z_0$, so $\{Z_0(t) \in \mathcal{L}(\mathcal{Z}) : t > 0\}$ is a zero-resolving family of operators for Equation (7). Then, by Proposition 2 for every $k = 1, 2, \ldots, n-1$, there exists a k-resolving family of operators, which coincide with operators $J_t^{\sigma_k - \sigma_0} Z_0(t) = Z_k(t)$. Every such family is analytic with the type $(\theta_0 - \pi/2, a_0, \max\{-\sigma_k, 0\})$; see the proof of Proposition 2 and Remark 3.

If there exists a zero-resolving family with the type $(\theta_0 - \pi/2, a_0, -\sigma_0)$, equality (8) at $k = 0$ and Theorem 1 with $\mathcal{X} = \mathcal{L}(\mathcal{Z})$ implies that $A \in \mathcal{A}_{\{\alpha_k\}}(\theta_0, a_0)$. □

Remark 4. *Note that $\sigma_n > 0$, if $\alpha_0 + \alpha_n > 1$.*

Remark 5. *An analogous for Theorem 3 result on the first-order equations is called the Solomyak–Yosida theorem on generation of analytic semigroups of operators [24,25]. Previously, similar results were obtained for evolutionary integral equations [26], differential equations with a Gerasimov–Caputo fractional derivative [27], with a Riemann–Liouville derivative [28], for multi-term linear fractional differential equations in Banach spaces [29], and equations with distributed fractional derivatives [30,31,33,34].*

Corollary 1. *Let $A \in \mathcal{A}_{\{\alpha_k\}}(\theta_0, a_0)$, $\alpha_k \in (0,1]$, $k = 0, 1, \ldots, n$, $\alpha_0 + \alpha_n > 0$, $\theta_0 \in (\pi/2, \pi]$, $a_0 \geq 0$. Then, for any $z_0, z_1, \ldots, z_{n-1} \in D_A$ problem (6), (7) has a unique solution, and it has the form*

$$z(t) = \sum_{k=0}^{n-1} Z_k(t) z_k.$$

The solution is analytic in $\Sigma_{\theta_0 - \pi/2}$.

Proof. After Theorem 3, we need to prove the uniqueness of a solution only. If problem (6), (7) has two solutions y_1, y_2, then the difference $y = y_1 - y_2$ is a solution of (7) with the initial conditions $D^{\sigma_k} y(0) = 0$, $k = 0, 1, \ldots, n-1$. Redefine y on (T, ∞) for any $T > 0$ as a zero function. The got function y_T satisfies equality (7) at $t > 0$ without the point T. Using the Laplace transform obtained from Equation (7) and zero initial conditions, the equality $\lambda^{\sigma_n} \widehat{y}_T(\lambda) = A \widehat{y}_T(\lambda)$. Since $A \in \mathcal{A}_{\{\alpha_k\}}(\theta_0, a_0)$, we have $\widehat{y}_T(\lambda) \equiv 0$ for $\lambda \in S_{\theta_0, a_0}$. Therefore, $y_T \equiv 0$ for arbitrary $T > 0$, hence $y \equiv 0$ on $\mathbb{R}_+$ and a solution of problem (6), (7) is unique. □

Remark 6. *For $A \in \mathcal{L}(\mathcal{Z})$ the k-resolving operators of Equation (7) have the form (see [22])*

$$Z_k(t) = t^{\sigma_k} E_{\sigma_n, \sigma_k + 1}(t^{\sigma_n} A), \quad t \in S_{\pi,0}, \quad k = 0, 1, \ldots, n-1.$$

Here, according to $E_{\beta,\gamma}$ the Mittag–Leffler function is denoted. Indeed, decomposing the resolvent $R_{\sigma_n}(A)$ in the series for large enough $|\lambda|$ and using the Hankel integral, we obtain these equalities.

Theorem 4. *Let $A \in \mathcal{A}_{\{\alpha_k\}}(\theta_0, a_0)$, $\alpha_k \in (0,1]$, $k = 0, 1, \ldots, n$, $\alpha_0 + \alpha_n > 0$, $\sigma_n \geq 2$, $\theta_0 \in (\pi/2, \pi]$, $a_0 \geq 0$. Then $A \in \mathcal{L}(\mathcal{Z})$.*

Proof. For some $\nu_0 \in \mathbb{C}$, such that $|\nu_0| \geq R^{\sigma_n}$, take $\lambda_0 = \nu_0^{1/\sigma_n}$, hence $|\lambda_0| \geq R$, $\arg \lambda_0 = \arg \nu_0 / \sigma_n \in [-\pi/2, \pi/2]$, since $\sigma_n \geq 2$. Then, $\lambda_0 \in S_{\theta_0, a_0}$ for sufficiently large $R > 0$. Therefore, $\{\nu \in \mathbb{C} : |\nu| \geq R^{\sigma_n}\} \subset [S_{\theta_0, a_0}]^{\sigma_n} \subset \rho(A)$, since $A \in \mathcal{A}_{\{\alpha_k\}}(\theta_0, a_0)$. Here, we use the principal branch of the power function.

So, for $|\nu| \geq R^{\sigma_n}$, where $\nu = \lambda^{\sigma_n}$,

$$\|\nu R_\nu(A)\|_{\mathcal{L}(\mathcal{Z})} \leq \frac{K(\theta, a)|\lambda|^{\sigma_0+1}}{|\lambda - a|^{\alpha_0}} \leq C$$

and by Lemma 5.2 [36] the operator A is bounded. $\square$

Remark 7. *For strongly continuous resolving families of the equation with a Gerasimov–Caputo derivative, such a result was proved in [27].*

5. Inhomogeneous Equation

Let $f \in C([0,T];\mathcal{Z})$. Consider the equation

$$D^{\sigma_n} z(t) = Az(t) + f(t), \quad t \in (0,T]. \tag{10}$$

A solution of the initial value problem

$$D^{\sigma_k} z(0) = z_k, \quad k = 0, 1, \ldots, n-1, \tag{11}$$

to Equation (10) is a function $z \in C((0,T]; D_A)$, such that $D^{\sigma_k} z \in C([0,T];\mathcal{Z})$, $k = 0, 1, \ldots, n-1$, $D^{\sigma_n} z \in C((0,T];\mathcal{Z})$, for all $t \in (0,T]$ equality (10) is fulfilled and conditions (11) are valid.

Denote

$$Z(t) = \frac{1}{2\pi i}\int_\Gamma R_{\lambda^{\sigma_n}}(A) e^{\lambda t} d\lambda, \quad Y_\beta(t) = \frac{1}{2\pi i}\int_\Gamma \lambda^\beta R_{\lambda^{\sigma_n}}(A) e^{\lambda t} d\lambda, \quad \beta \in \mathbb{R}.$$

Lemma 1. *Let $A \in \mathcal{A}_{\{\alpha_k\}}(\theta_0, a_0)$, $\alpha_k \in (0,1]$, $k = 0, 1, \ldots, n$, $\alpha_0 + \alpha_n > 0$, $\theta_0 \in (\pi/2, \pi]$, $a_0 \geq 0$, $f \in C([0,T]; D_A)$. Then,*

$$z_f(t) = \int_0^t Z(t-s) f(s) ds \tag{12}$$

is a unique solution for the initial value problem

$$D^{\sigma_k} z(0) = 0, \quad k = 0, 1, \ldots, n-1, \tag{13}$$

to (10).

Proof. Since $A \in \mathcal{A}_{\{\alpha_k\}}(\theta_0, a_0)$, for sufficiently large $|\lambda|$ $\|R_{\lambda^{\sigma_n}}(A)\|_{\mathcal{L}(\mathcal{Z})} \leq C|\lambda|^{-\sigma_n}$, hence for $\mathrm{Re}\lambda > a_0$ $\widehat{Z}(\lambda) = R_{\lambda^{\sigma_n}}(A)$, $\widehat{D^{\sigma_0} Z}(\lambda) = \lambda^{\sigma_0} R_{\lambda^{\sigma_n}}(A)$, $\|Z(t)\|_{\mathcal{L}(\mathcal{Z})} \leq Ct^{\sigma_n - 1}$, $\|D^{\sigma_0} Z(t)\|_{\mathcal{L}(\mathcal{Z})} \leq Ct^{\sigma_n - \sigma_0 - 1}$ for $t \in (0,T]$. Analogously, $\|Y_\beta(t)\|_{\mathcal{L}(\mathcal{Z})} \leq Ct^{\sigma_n - \beta - 1}$ for $t \in (0,T]$, $\beta \in \mathbb{R}$.

Further,

$$\|D^{\sigma_0} z_f(t)\|_{\mathcal{Z}} = \left\| \int_0^t Y_{\sigma_0}(t-s) f(s) ds \right\|_{\mathcal{Z}} \leq C \max_{s\in[0,T]} \|f(s)\|_{\mathcal{Z}} t^{\sigma_n - \sigma_0},$$

hence $D^{\sigma_0} z_f(0) = 0$. Define f by zero outside the segment $[0,T]$; then, $z_f = Z * f$, $\widehat{z}_f(\lambda) = \widehat{Z}(\lambda)\widehat{f}(\lambda) = R_{\lambda^{\sigma_n}}(A)\widehat{f}(\lambda)$, $\widehat{D^{\sigma_1} z_f}(\lambda) = \lambda^{\sigma_1} R_{\lambda^{\sigma_n}}(A)\widehat{f}(\lambda)$,

$$\|D^{\sigma_1} z_f(t)\|_{\mathcal{Z}} = \left\| \int_0^t Y_{\sigma_1}(t-s) f(s) ds \right\|_{\mathcal{Z}} \leq C \max_{s\in[0,T]} \|f(s)\|_{\mathcal{Z}} t^{\sigma_n - \sigma_1}, \quad t \in (0,T],$$

$D^{\sigma_1} z_f(0) = 0$. Repeating the analogous reasoning sequentially, we get $k = 2, 3, \dots, n-1$ $\widehat{D^{\sigma_k} z_f}(\lambda) = \lambda^{\sigma_k} R_{\lambda^{\sigma_n}}(A) \widehat{f}(\lambda)$, $\|D^{\sigma_k} z_f(t)\|_{\mathcal{L}(\mathcal{Z})} \leq C \max_{s \in [0,T]} \|f(s)\|_{\mathcal{Z}} t^{\sigma_n - \sigma_k}$ for $t \in (0, T]$, $D^{\sigma_k} z_f(0) = 0$, $\widehat{D^{\sigma_n} z_f}(\lambda) = \lambda^{\sigma_n} R_{\lambda^{\sigma_n}}(A) \widehat{f}(\lambda)$.

Since $f \in C([0, T]; D_A)$, we have

$$\widehat{Az_f}(\lambda) = \widehat{z}_{Af}(\lambda) = AR_{\lambda^{\sigma_n}}(A)\widehat{f}(\lambda) = \lambda^{\sigma_n} R_{\lambda^{\sigma_n}}(A)\widehat{f}(\lambda) - \widehat{f}(\lambda),$$

so, $Az_f(t) = D^{\sigma_n} z_f(t) - f(t)$ for all $t > 0$. Thus, the function z_f satisfies equality (10). The proof of a solution's uniqueness is the same as for the homogeneous equation. □

Let $C^\gamma([0, T]; \mathcal{Z})$ for some $\gamma \in (0, 1]$ be the set of all functions $f : [0, T] \to \mathcal{Z}$, satisfying the Hölder condition:

$$\exists C > 0 \quad \forall s, t \in [0, T] \quad \|f(s) - f(t)\|_{\mathcal{Z}} \leq C|s - t|^\gamma.$$

Lemma 2. *Let* $A \in \mathcal{A}_{\{\alpha_k\}}(\theta_0, a_0)$, $\alpha_k \in (0, 1]$, $k = 0, 1, \dots, n$, $\alpha_0 + \alpha_n > 0$, $\theta_0 \in (\pi/2, \pi]$, $a_0 \geq 0$, $f \in C^\gamma([0, T]; \mathcal{Z})$, $\gamma \in (0, 1]$. *Then, problem* (10), (13) *has a unique solution; it has form* (12).

Proof. Since A is closed,

$$AZ(t) = \frac{1}{2\pi i} \int_\Gamma AR_{\lambda^{\sigma_n}}(A) e^{\lambda t} d\lambda = \frac{1}{2\pi i} \int_\Gamma \lambda^{\sigma_n} R_{\lambda^{\sigma_n}}(A) e^{\lambda t} d\lambda = Y_{\sigma_n}(t), \quad t > 0,$$

therefore, $\operatorname{im} Z(t) \subset D_A$, as $t \to 0+$ $\|AZ(t)\|_{\mathcal{L}(\mathcal{Z})} = O(t^{-1})$ (see the previous proof). Therefore, for all $t, s \in (0, T]$

$$\|AZ(t - s)(f(s) - f(t))\|_{\mathcal{Z}} \leq C|t - s|^{\gamma - 1}.$$

Then

$$\int_0^t AZ(t - s)f(s)ds = \int_0^t AZ(t - s)(f(s) - f(t))ds + \int_0^t Y_{\sigma_n}(t - s)f(t)ds,$$

$$\int_0^t Y_{\sigma_n}(t - s)f(t)ds = -\int_0^t D_s^1 Y_{\sigma_n - 1}(t - s)f(t)ds = (Y_{\sigma_n - 1}(t) - Y_{\sigma_n - 1}(0))f(t).$$

Note that for any $x \in D_A$

$$Y_{\sigma_n - 1}(t)x = x + \frac{1}{2\pi i} \int_\Gamma \lambda^{-1} R_{\lambda^{\sigma_n}}(A) e^{\lambda t} Ax d\lambda \to x, \quad t \to 0+,$$

since for large enough $|\lambda|$ $\|\lambda^{-1} R_{\lambda^{\sigma_n}}(A)Ax\|_{\mathcal{Z}} \leq C\|Ax\|_{\mathcal{Z}}|\lambda|^{-\sigma_n - 1}$. At the same time, for sufficiently large $|\lambda|$ $\|\lambda^{\sigma_n - 1} R_{\lambda^{\sigma_n}}(A)\|_{\mathcal{L}(\mathcal{Z})} \leq C|\lambda|^{-1}$; therefore, the family $\{Y_{\sigma_n - 1}(t) \in \mathcal{L}(\mathcal{Z}) : t > 0\}$ is bounded uniformly. Since D_A is dense in $\mathcal{Z}$, for every $x \in \mathcal{Z}$ $\lim_{t \to 0+} Y_{\sigma_n - 1}(t)x = x$.

Thus,

$$\left\| \int_0^t AZ(t - s)f(s)ds \right\|_{\mathcal{Z}} \leq C_1 t^\gamma + \|Y_{\sigma_n - 1}(t) - Y_{\sigma_n - 1}(0)\|_{\mathcal{L}(\mathcal{Z})} \|f(t) - f(0)\|_{\mathcal{Z}} +$$

$$+ \|(Y_{\sigma_n - 1}(t) - Y_{\sigma_n - 1}(0))f(0)\|_{\mathcal{Z}} \leq$$

$$\leq C_1 t^\gamma + C_2 \|f(t) - f(0)\|_{\mathcal{Z}} + \|(Y_{\sigma_n - 1}(t) - Y_{\sigma_n - 1}(0))f(0)\|_{\mathcal{Z}} \to 0$$

as $t \to 0+$. Therefore, $z_f(t) \in D_A$, $z_f \in C([0,T];D_A)$.

Other arguing is the same as in the proof of the previous lemma. □

Corollary 1, Lemma 1 and Lemma 2 imply the following result.

Theorem 5. *Let $A \in \mathcal{A}_{\{\alpha_k\}}(\theta_0, a_0)$, $\alpha_k \in (0,1]$, $k = 0,1,\dots,n$, $\alpha_0 + \alpha_n > 0$, $\theta_0 \in (\pi/2, \pi]$, $a_0 \geq 0$, $\gamma \in (0,1]$, $f \in C([0,T];D_A) \cup C^\gamma([0,T];\mathcal{Z})$. Then problem (10), (11) has a unique solution, it has the form*

$$z(t) = \sum_{k=0}^{n-1} Z_k(t) z_k + \int_0^t Z(t-s) f(s) ds.$$

6. Perturbation Theorem

Theorem 6. *Let $A \in \mathcal{A}_{\{\alpha_k\}}(\theta_0, a_0)$, $\alpha_k \in (0,1]$, $k = 0,1,\dots,n$, $\alpha_0 + \alpha_n > 0$, $\theta_0 \in (\pi/2, \pi]$, $a_0 \geq 0$, $B \in \mathcal{C}l(\mathcal{Z})$, for some $\beta, \gamma \geq 0$*

$$\|Bx\|_{\mathcal{Z}} \leq \beta \|Ax\|_{\mathcal{Z}} + \gamma \|x\|_{\mathcal{Z}}, \quad x \in D_A \subset D_B, \tag{14}$$

there exists $q \in (0,1)$, such that $\beta(1 + K(\theta, a)) < q$ for all $\theta \in (\pi/2, \theta_0)$, $a > a_0$. Then, $A + B \in \mathcal{A}_{\{\alpha_k\}}(\theta_0, a_1)$ for sufficiently large $a_1 > a_0$.

Proof. Choose $l > \sin^{-1}\theta_0$, $\lambda \in S_{\theta, la} \subset S_{\theta, a}$ for some $\theta \in (\pi/2, \theta_0)$, $a > a_0$, then from (14), it follows that

$$\|BR_{\lambda^{\sigma_n}}(A)\|_{\mathcal{L}(\mathcal{Z})} \leq \beta \|AR_{\lambda^{\sigma_n}}(A)\|_{\mathcal{L}(\mathcal{Z})} + \gamma \|R_{\lambda^{\sigma_n}}(A)\|_{\mathcal{L}(\mathcal{Z})} \leq$$

$$\leq \beta\left(1 + \frac{|\lambda|^{\sigma_0+1} K_A(\theta, a)}{|\lambda - a|^{\alpha_0}}\right) + \frac{\gamma K_A(\theta, a)}{|\lambda - a|^{\alpha_0} |\lambda|^{\sigma_n - \sigma_0 - 1}},$$

where $K_A(\theta, a)$ is the constant from Definition 2. Note that the value

$$\frac{|\lambda|^{\alpha_0}}{|\lambda - a|^{\alpha_0}} \leq \frac{1}{\left(1 - \frac{a}{|\lambda|}\right)^{\alpha_0}} \leq \frac{1}{\left(1 - \frac{1}{l \sin\theta_0}\right)^{\alpha_0}}$$

is close to one, for a sufficiently large number l

$$\frac{|\lambda|^{\alpha_0}}{|\lambda - a|^{\alpha_0} |\lambda|^{\sigma_n}} \leq \frac{1}{\left(1 - \frac{1}{l \sin\theta_0}\right)^{\alpha_0} (l a_0 \sin\theta_0)^{\sigma_n}}$$

is close to zero. So, for such a l, we have

$$\|BR_{\lambda^{\sigma_n}}(A)\|_{\mathcal{L}(\mathcal{Z})} \leq \beta\left(1 + \frac{K_A(\theta, a)}{\left(1 - \frac{1}{l \sin\theta_0}\right)^{\alpha_0}}\right) + \frac{\gamma K_A(\theta, a)}{\left(1 - \frac{1}{l \sin\theta_0}\right)^{\alpha_0} (l a_0 \sin\theta_0)^{\sigma_n}} \leq$$

$$\leq \beta(1 + K(\theta, a)) + \varepsilon \leq q < 1.$$

Therefore,

$$R_{\lambda^{\sigma_n}}(A + B) = R_{\lambda^{\sigma_n}}(A)(I - BR_{\lambda^{\sigma_n}}(A))^{-1} = R_{\lambda^{\sigma_n}}(A) \sum_{k=0}^{\infty} [BR_{\lambda^{\sigma_n}}(A)]^k,$$

$$\frac{|\lambda - la|}{|\lambda - a|} = \left|1 - \frac{(l-1)a}{\lambda - a}\right| \leq 1 + \frac{(l-1)a}{|\lambda - a|} < 1 + \frac{1}{\sin\theta_0},$$

$$\|R_{\lambda^{\sigma_n}}(A+B)\|_{\mathcal{L}(\mathcal{Z})} \leq \frac{K_A(\theta,a)}{(1-q)|\lambda-a|^{\alpha_0}|\lambda|^{\sigma_n-\sigma_0-1}} \leq \frac{K_A(\theta,a)\left(1+\frac{1}{\sin\theta_0}\right)^{\alpha_0}}{(1-q)|\lambda-la|^{\alpha_0}|\lambda|^{\sigma_n-\sigma_0-1}}.$$

So, $A+B \in \mathcal{A}_{\{\alpha_k\}}(\theta_0,a_1)$ with $a_1 = la_0$, for all $\theta \in (\pi/2,\theta_0)$, $a > a_1$

$$K_{A+B}(\theta,a) = \frac{K_A(\theta,a/l)}{1-q}\left(1+\frac{1}{\sin\theta_0}\right)^{\alpha_0}.$$

□

Remark 8. *For every $B \in \mathcal{L}(\mathcal{Z})$ condition, (14) is satisfied with $\beta = 0$, $\gamma = \|B\|_{\mathcal{L}(\mathcal{Z})}$.*

Remark 9. *Theorem 6 generalizes the similar theorem for generators of analytic semigroups of operators [37]. Note that there are also analogous results for generators of resolving families for equations with distributed fractional derivatives in [30].*

7. Application to a Model of a Viscoelastic Oldroyd Fluid

Let $\alpha_k \in (0,1]$, $k = 0,1,\dots,n$, $\alpha_0+\alpha_n > 1$, $\sigma_n \in (0,2)$, $\Omega \subset \mathbb{R}^d$ be a bounded region, which has a smooth boundary $\partial\Omega$. We consider a fractional linearized model of the viscoelastic Oldroyd fluid dynamics with the order $N = 1$ (see [38])

$$D^{\sigma_k}v(x,0) = v_k(x),\ D^{\sigma_k}w(x,0) = w_k(x),\ x \in \Omega,\ k = 0,1,\dots,n-1, \quad (15)$$

$$v(x,t) = 0, \quad w(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T], \quad (16)$$

$$D^{\sigma_n}v = \mu\Delta v + \Delta w - \nabla p + g, \quad (x,t) \in \Omega \times (0,T], \quad (17)$$

$$D^{\sigma_n}w = bv + cw + h, \quad (x,t) \in \Omega \times (0,T], \quad (18)$$

$$\nabla\cdot v = 0, \quad \nabla\cdot w = 0, \quad (x,t) \in \Omega \times (0,T]. \quad (19)$$

Here, $T > 0$, D^{σ_k}, $k = 0,1,\dots,n$, are Dzhrbashyan–Nersesyan fractional derivatives with respect to time t, $x = (x_1,x_2,\dots,x_d)$ are spatial variables, $v = (v_1,v_2,\dots,v_d)$ is the fluid velocity vector, $w = (w_1,w_2,\dots,w_d)$ is a function of memory for the velocity, which is defined by a Volterra integral with respect to t for v, $\nabla p = (p_{x_1},p_{x_2},\dots,p_{x_d})$ is the pressure gradient of the fluid, Δ is the Laplace operator with respect to all the spatial variables, $\Delta v = (\Delta v_1,\Delta v_2,\dots,\Delta v_d)$, $\Delta w = (\Delta w_1,\Delta w_2,\dots,\Delta w_d)$, $\nabla\cdot v = v_{1x_1}+v_{2x_2}+\dots+v_{dx_d}$, $\nabla\cdot w = w_{1x_1}+w_{2x_2}+\dots+w_{dx_d}$. The constants $\mu,b,c \in \mathbb{R}$ and the functions $g,h : \Omega\times[0,T] \to \mathbb{R}^d$ are given.

Take $\mathbb{L}_2 := (L_2(\Omega))^d$, $\mathbb{H}^1 := (W_2^1(\Omega))^d$, $\mathbb{H}^2 := (W_2^2(\Omega))^d$. The closure of $\mathfrak{L} := \{u \in (C_0^\infty(\Omega))^d : \nabla\cdot u = 0\}$ in the norm of $\mathbb{L}_2$ will be denoted by $\mathbb{H}_\sigma$, and in the norm of the space $\mathbb{H}^1$ by $\mathbb{H}_\sigma^1$. We also denote $\mathbb{H}_\sigma^2 := \mathbb{H}_\sigma^1 \cap \mathbb{H}^2$, $\mathbb{H}_\pi$ is the orthogonal complement for $\mathbb{H}_\sigma$ in the Hilbert space $\mathbb{L}_2$, $\Sigma : \mathbb{L}_2 \to \mathbb{H}_\sigma$, $\Pi := I-\Sigma : \mathbb{L}_2 \to \mathbb{H}_\pi$ are the projectors.

The operator $B = \Sigma\Delta$, extended to a closed operator in the space $\mathbb{H}_\sigma$ with the domain $\mathbb{H}_\sigma^2$, has a real, negative, discrete spectrum with finite multiplicities of eigenvalues, condensed at $-\infty$ only [39]. Denote by $\{\lambda_k\}$ eigenvalues of B, numbered in non-increasing order, taking into account their multiplicities. Then, $\{\varphi_k\}$ will be used to denote the orthonormal system of eigenfunctions, which forms a basis in $\mathbb{H}_\sigma$ [39].

In order for Equation (19) to be fulfilled, take $\mathcal{Z} = \mathbb{H}_\sigma \times \mathbb{H}_\sigma$ and define in $\mathcal{Z}$ an operator

$$A = \begin{pmatrix} \mu B & B \\ bI & cI \end{pmatrix} \in \mathcal{C}l(\mathcal{Z}), \quad D_A = \mathbb{H}_\sigma^2 \times \mathbb{H}_\sigma^2. \quad (20)$$

Theorem 7. *Let $\alpha_k \in (0,1]$, $k = 0,1,\dots,n$, $\sigma_n \in [1,2)$, $\mu > 0$, $b,c \in \mathbb{R}$, $\mathcal{Z} = \mathbb{H}_\sigma \times \mathbb{H}_\sigma$, the operator A be defined by (20). Then, for some $\theta_0 \in (\pi/2,\pi)$, $a_0 > 0$ $A \in \mathcal{A}_{\{\alpha_k\}}(\theta_0,a_0)$.*

Proof. Let $\theta_0 \in (\pi/2, \pi)$, $\theta \in (\pi/2, \theta_0)$, $a_0 > 0$, $a > a_0$, then for $\lambda \in S_{\theta,a}$

$$\frac{|\lambda - a|}{|\lambda|} \le 1 + \frac{a}{|\lambda|} \le 1 + \frac{1}{\sin\theta_0},$$

so,

$$\frac{1}{|\lambda|^{\sigma_n}} = \frac{|\lambda - a|^{\alpha_0}}{|\lambda|^{\alpha_0}} \frac{1}{|\lambda - a|^{\alpha_0} |\lambda|^{\sigma_n - \alpha_0}} \le \frac{\left(1 + \frac{1}{\sin\theta_0}\right)^{\alpha_0}}{|\lambda - a|^{\alpha_0} |\lambda|^{\sigma_n - \sigma_0 - 1}}$$

and instead of estimates of the form $\|R_{\lambda^{\sigma_n}}(A)\|_{\mathcal{L}(\mathcal{Z})} \le \frac{K}{|\lambda - a|^{\alpha_0} |\lambda|^{\sigma_n - \sigma_0 - 1}}$, it will be enough to get inequalities $\|R_{\lambda^{\sigma_n}}(A)\|_{\mathcal{L}(\mathcal{Z})} \le \frac{K}{|\lambda|^{\sigma_n}}$.

Take $\theta_0 \in (\pi/2, \pi/\sigma_n)$, $a_0 = (l|c|)^{1/\sigma_n}$, where $l > 1$ is sufficiently large, then for $\lambda \in S_{\theta_0,a_0}$

$$\lambda^{\sigma_n} I - A = \sum_{k=1}^{\infty} \begin{pmatrix} \lambda^{\sigma_n} - \mu\lambda_k & -\lambda_k \\ -b & \lambda^{\sigma_n} - c \end{pmatrix} \langle \cdot, \varphi_k \rangle \varphi_k,$$

$$(\lambda^{\sigma_n} I - A)^{-1} = \sum_{k=1}^{\infty} \begin{pmatrix} \frac{\lambda^{\sigma_n} - c}{(\lambda^{\sigma_n} - c)(\lambda^{\sigma_n} - \mu\lambda_k) - b\lambda_k} & \frac{\lambda_k}{(\lambda^{\sigma_n} - c)(\lambda^{\sigma_n} - \mu\lambda_k) - b\lambda_k} \\ \frac{b}{(\lambda^{\sigma_n} - c)(\lambda^{\sigma_n} - \mu\lambda_k) - b\lambda_k} & \frac{\lambda^{\sigma_n} - \mu\lambda_k}{(\lambda^{\sigma_n} - c)(\lambda^{\sigma_n} - \mu\lambda_k) - b\lambda_k} \end{pmatrix} \langle \cdot, \varphi_k \rangle \varphi_k.$$

Since $\lambda^{\sigma_n} \in S_{\theta_0\sigma_n, l|c|}$ for $\lambda \in S_{\theta_0,a_0}$, we have $|\lambda^{\sigma_n} - c| \ge (l-1)|c| \sin(\pi - \theta_0\sigma_n)$, for sufficiently large l, the value $|b(\lambda^{\sigma_n} - c)^{-1}|$ is small enough and

$$\arg\left(\mu + \frac{b}{\lambda^{\sigma_n} - c}\right) < \frac{1}{2}(\pi - \theta_0\sigma_n).$$

Fix l, $a_0 = (l|c|)^{1/\sigma_n}$; then, for $\lambda \in S_{\theta_0,a_0}$, we have $\lambda^{\sigma_n} \in S_{\theta_0\sigma_n, l|c|} \subset S_{\theta_0\sigma_n, 0}$ and

$$\left|\frac{\lambda^{\sigma_n} - c}{(\lambda^{\sigma_n} - c)(\lambda^{\sigma_n} - \mu\lambda_k) - b\lambda_k}\right| = \frac{1}{\left|\lambda^{\sigma_n} - \lambda_k\left(\mu + \frac{b}{\lambda^{\sigma_n} - c}\right)\right|} \le \frac{1}{|\lambda|^{\sigma_n} \sin\frac{\pi - \theta_0\sigma_n}{2}},$$

$$\left|\frac{\lambda_k}{(\lambda^{\sigma_n} - c)(\lambda^{\sigma_n} - \mu\lambda_k) - b\lambda_k}\right| = \frac{1}{\left|(\lambda^{\sigma_n} - c)\left(\frac{\lambda^{\sigma_n}}{\lambda_k} - \mu\right) - b\right|} \le$$

$$\le \frac{1}{|\lambda|^{\sigma_n} \sin(\pi - \theta_0\sigma_n) \inf\limits_{k\in\mathbb{N}, \lambda\in S_{\theta_0,a_0}} \left|\frac{\lambda^{\sigma_n}}{\lambda_k} - \mu\right| - b} \le$$

$$\le \frac{2}{|\lambda|^{\sigma_n} \sin(\pi - \theta_0\sigma_n) \inf\limits_{k\in\mathbb{N}, \lambda\in S_{\theta_0,a_0}} \left|\frac{\lambda^{\sigma_n}}{\lambda_k} - \mu\right|},$$

if we take l, such that

$$|b| < \frac{l|c|}{2} \sin^{\sigma_n}\theta_0 \sin(\pi - \theta_0\sigma_n) \inf_{k\in\mathbb{N}, \lambda\in S_{\theta_0,a_0}} \left|\frac{\lambda^{\sigma_n}}{\lambda_k} - \mu\right| \le$$

$$\le \frac{|\lambda|^{\sigma_n}}{2} \sin(\pi - \theta_0\sigma_n) \inf_{k\in\mathbb{N}, \lambda\in S_{\theta_0,a_0}} \left|\frac{\lambda^{\sigma_n}}{\lambda_k} - \mu\right|.$$

Further, for large $k \in \mathbb{N}$

$$\left|\frac{b}{(\lambda^{\sigma_n} - c)(\lambda^{\sigma_n} - \mu\lambda_k) - b\lambda_k}\right| \le$$

$$\leq \frac{|b|}{\left|\left(\lambda^{\sigma_n} - \frac{c+\mu\lambda_k+\sqrt{\frac{(c-\mu\lambda_k)^2-4b\lambda_k}{2}}}{2}\right)\left(\lambda^{\sigma_n} - \frac{c+\mu\lambda_k-\sqrt{\frac{(c-\mu\lambda_k)^2-4b\lambda_k}{2}}}{2}\right)\right|} \leq$$

$$\leq \frac{|b|}{|\lambda|^{2\sigma_n}\sin^2(\pi-\theta_0\sigma_n)} \leq \frac{|b|(l|c|)^{-1}\sin^{-\sigma_n}\theta_0}{|\lambda|^{\sigma_n}\sin^2(\pi-\theta_0\sigma_n)},$$

$$\left|\frac{\lambda^{\sigma_n}-\mu\lambda_k}{(\lambda^{\sigma_n}-c)(\lambda^{\sigma_n}-\mu\lambda_k)-b\lambda_k}\right| \leq \frac{1}{\left|\lambda^{\sigma_n}-c-\frac{b\lambda_k}{\lambda^{\sigma_n}-\mu\lambda_k}\right|} \leq \frac{2}{|\lambda|^{\sigma_n}}$$

for sufficiently large l, since

$$\sup_{k\in\mathbb{N},\lambda\in S_{a_0,\theta_0}}\left|c+\frac{b\lambda_k}{\lambda^{\sigma_n}-\mu\lambda_k}\right| < \infty.$$

Thus, $A \in \mathcal{A}_{\{\alpha_k\}}(\theta_0, a_0)$ with $\theta_0 \in (\pi/2, \pi/\sigma_n)$, $a_0 = (l|c|)^{1/\sigma_n}$ with a chosen sufficiently large $l > 1$. □

Theorem 8. *Let* $\alpha_k \in (0,1]$, $k = 0,1,\ldots,n$, $\alpha_0+\alpha_n > 1$, $\sigma_n \in [1,2)$, $\Sigma g, h \in C([0,T];\mathbb{H}^2_\sigma) \cup C^\gamma([0,T];\mathbb{H}_\sigma)$, $\gamma \in (0,1]$. *Then, problem* (15)–(19) *has a unique solution.*

Proof. Problem (15)–(19) is represented as abstract problem (10), (11) due to the above choice of $\mathcal{Z}$ and A. Since we find the vector functions $v(\cdot,t)$ and $w(\cdot,t)$ with the values in $\mathbb{H}_\sigma$ for every $t \in (0,T]$, instead of Equation (17), we consider its projection on $\mathbb{H}_\sigma$

$$D^{\sigma_n}v = \mu Bv + Bw + \Sigma g, \quad (x,t) \in \Omega \times (0,T],$$

In this case, the projection of Equation (18) on $\mathbb{H}_\sigma$ has the form

$$D^{\sigma_n}w = bv + cw + \Sigma h, \quad (x,t) \in \Omega \times (0,T],$$

hence, $\Pi h \equiv 0$. Theorem 7 and Theorem 5 imply the required statement. □

Remark 10. *If we found* $v(x,t)$ *and* $w(x,t)$*, we obtain the pressure gradient using the formula* $\nabla p(\cdot,t) = \mu\Pi\Delta v(\cdot,t) + \Pi\Delta w(\cdot,t) + \Pi f(\cdot,t)$ *from the projection of Equation* (17) *on the subspace* $\mathbb{H}_\pi$.

8. Conclusions

On the one hand, the results obtained will become the basis for the study of various classes of semilinear and quasilinear equations with the Dzhrbashyan–Nersesyan derivative. It is supposed to consider cases when the nonlinearity in the equation is continuous in the norm of the graph of the operator A and when it is Hölderian. In addition, there are plans to investigate similar equations with a degenerate linear operator at the Dzhrbashyan–Nersesyan derivative, linear, semi-linear and quasilinear. On the other hand, abstract results will be used to study various initial-boundary value problems for partial differential equations and their systems encountered in applications.

Author Contributions: Conceptualization, V.E.F. and M.V.P.; methodology, V.E.F.; software, E.M.I.; validation, E.M.I. and M.V.P.; formal analysis, E.M.I. and M.V.P.; investigation, E.M.I. and V.E.F.; resources, E.M.I.; data curation, E.M.I.; writing—original draft preparation, E.M.I. and M.V.P.; writing—review and editing, V.E.F. and M.V.P.; visualization, E.M.I.; supervision, V.E.F. and M.V.P.; project administration, V.E.F.; funding acquisition, V.E.F. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by the Russian Science Foundation, grant number 22-21-20095.

Data Availability Statement: Not applicable.

Conflicts of Interest: The funders had no role in the design of the study; in the collection, analyses, or interpretation of data; in the writing of the manuscript, or in the decision to publish the results.

References

1. Dzhrbashyan, M.M.; Nersesyan, A.B. Fractional derivatives and the Cauchy problem for differential equations of fractional order. *Izv. Akad. Nauk Armyanskoy SSR Mat.* **1968**, *3*, 3–28. (In Russian) [CrossRef]
2. Samko, S.G.; Kilbas, A.A.; Marichev, O.I. *Fractional Integrals and Derivatives. Theory and Applications*; Gordon and Breach Science Publishers: Philadelphia, PA, USA, 1993.
3. Podlubny, I. *Fractional Differential Equations*; Academic Press: Boston, MA, USA, 1999.
4. Nakhushev, A.M. *Fractional Calculus ant Its Applications*; Fizmatlit: Moscow, Russia, 2003. (In Russian)
5. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*; Elsevier Science Publishing: Amsterdam, The Netherlands; Boston, MA, USA; Heidelberg, Germany, 2006.
6. Tarasov, V.E. *Fractional Dynamics: Applications of Fractional Calculus to Dynamics of Particles, Fields and Media*; Springer: New York, NY, USA, 2011.
7. Khan, H.; Alzabut, J.; Shah, A.; Etemad S.; Rezapour, S.; Park, C. A study on the fractal-fractional tobacco smoking model. *AIMS Math.* **2022**, *7*, 13887–13909. [CrossRef]
8. Mohammadi, H.; Kumar, S.; Rezapour, S.; Etemad, S. A theoretical study of the Caputo—Fabrizio fractional modeling for hearing loss due to Mumps virus with optimal control. *Chaos Solitons Fractals* **2021**, *144*, 110668. [CrossRef]
9. Mohammadi, H.; Rezapour, S.; Etemad, S. On a hybrid fractional Caputo—Hadamard boundary value problem with hybrid Hadamard integral boundary value conditions. *Adv. Differ. Equ.* **2020**, *2020*, 455. [CrossRef]
10. Najafi, H.; Etemad, S.; Patanarapeelert, N.; Asamoah, J.K.K.; Rezapour, S.; Sitthiwirattham, T. A study on dynamics of $CD4^+$ T-cells under the effect of HIV-1 infection based on a mathematical fractal-fractional model via the Adams—Bashforth scheme and Newton polynomials. *Mathematics* **2022**, *10*, 1366. [CrossRef]
11. Rezapour, S.; Etemad, S.; Mohammadi, H. A mathematical analysis of a system of Caputo—Fabrizio fractional differential equations for the anthrax disease model in animals. *Adv. Differ. Equ.* **2020**, *2020*, 481. [CrossRef]
12. Fu, L.; Zhang, Z.; Yang, H. Fractional model of blood flow and rogue waves in arterial vessels. *Mathematical Methods in the Applied Sciences*; John Wiley and Sons, Inc.: Hoboken, NJ, USA, 2022. [CrossRef]
13. Losanova, F.M.; Kenetova, R.O. Nonlocal problem for generalized Mckendrick—Von Foerster equation with Caputo operator. *Nonlinear World* **2018**, *16*, 49–53. (In Russian)
14. Berezgova, R.Z. A priori estimate for the solution of a nonlocal boundary value problem for the Mckendrick—Von Foerster equation of fractional order. *Dokl. Adyg. (Cherkesskoy) Mezhdunarodnoy Akad.* **2020**, *20*, 9–14. (In Russian)
15. Mamchuev, M.O. Cauchy problem for a linear system of ordinary differential equations of the fractional order. *Mathematics* **2020**, *8*, 1475. [CrossRef]
16. Pskhu, A.V. The fundamental solution of a diffusion-wave equation of fractional order. *Izv. Math.* **2009**, *73*, 351–392. [CrossRef]
17. Pskhu, A.V. Fractional diffusion equation with discretely distributed differentiation operator. *Sib. Electron. Math. Rep.* **2016**, *13*, 1078–1098.
18. Pskhu, A.V. Boundary value problem for a first-order partial differential equation with a fractional discretely distributed differentiation operator. *Differ. Equ.* **2016**, *52*, 1610–1623. [CrossRef]
19. Pskhu, A.V. Stabilization of solutions to the Cauchy problem for fractional diffusion-wave equation. *J. Math. Sci.* **2020**, *250*, 800–810. [CrossRef]
20. Bogatyreva, F.T. On representation of a solution for first-order partial differential equation with Dzhrbashyan—Nersesyan operator of fractional differentiation. *Dokl. Adyg. (Cherkesskoy) Mezhdunarodnoy Akad. Nauk* **2020**, *20*, 6–11. (In Russian) [CrossRef]
21. Bogatyreva, F.T. Boundary value problems for first order partial differential equation with the Dzhrbashyan—Nersesyan operators. *Chelyabinsk Phys. Math. J.* **2021**, *6*, 403–416.
22. Fedorov, V.E.; Plekhanova, M.V.; Izhberdeeva, E.M. Initial value problems of linear equations with the Dzhrbashyan—Nersesyan derivative in Banach spaces. *Symmetry* **2021**, *13*, 1058. [CrossRef]
23. Volkova, A.R.; Izhberdeeva, E.M.; Fedorov, V.E. Initial value problems for equations with a composition of fractional derivatives. *Chelyabinsk Phys. Math.* **2021**, *6*, 269–277. [CrossRef]
24. Solomyak, M.Z. Application of semigroup theory to the study of differential equations in Banach spaces. *Dokl. Akad. Nauk SSSR* **1958**, *122*, 766–769.
25. Yosida, K. *Functional Analysis*; Springer: Berlin/Heidelberg, Germany; New York, NY, USA, 1965.
26. Prüss, J. *Evolutionary Integral Equations and Applications*; Springer: Basel, Switzerland, 1993.
27. Bajlekova, E.G. Fractional Evolution Equations in Banach Spaces. Ph.D. Thesis, Eindhoven University of Technology, Eindhoven, The Netherlands, 2001.
28. Fedorov, V.E.; Avilovich, A.S. A Cauchy type problem for a degenerate equation with the Riemann—Liouville derivative in the sectorial case. *Sib. Math. J.* **2019**, *60*, 359–372. [CrossRef]

29. Fedorov, V.E.; Du, W.-S.; Turov, M.M. On the unique solvability of incomplete Cauchy type problems for a class of multi-term equations with the Riemann—Liouville derivatives. *Symmetry* **2022**, *14*, 75. [CrossRef]
30. Fedorov, V.E. Generators of analytic resolving families for distributed order equations and perturbations. *Mathematics* **2020**, *8*, 1306. [CrossRef]
31. Fedorov, V.E.; Filin, N.V. Linear equations with discretely distributed fractional derivative in Banach spaces. *Tr. Inst. Mat. Mekhaniki UrO RAN* **2021**, *27*, 264–280.
32. Fedorov, V.E.; Filin, N.V. On strongly continuous resolving families of operators for fractional distributed order equations. *Fractal Fract.* **2021**, *5*, 20. [CrossRef]
33. Fedorov, V.E. On generation of family of resolving operators for a distributed order equation analytic in sector. *J. Math. Sci.* **2022**, *260*, 75–86. [CrossRef]
34. Fedorov, V.E.; Du, W.-S.; Kostic, M.; Abdrakhmanova, A.A. Analytic resolving families for equations with distributed Riemann—Liouville derivatives. *Mathematics* **2022**, *10*, 681. [CrossRef]
35. Pazy, A. *Semigroups and Linear Operators and Applications to Partial Differential Equations*; Springer: New York, NY, USA, 1983.
36. Goldstein, J.A. Semigroups and second-order differential equations. *J. Funct. Anal.* **1969**, *4*, 50–70. [CrossRef]
37. Kato, K. *Perturbation Theory for Linear Operators*; Springer: Berlin/Heidelberg, Germany, 1966.
38. Oskolkov, A.P. Initial-boundary value problems for equations of motion of Kelvin—Voight fluids and Oldroyd fluids. *Proc. Steklov Inst. Math.* **1989**, *179*, 137–182.
39. Ladyzhenskaya, O.A. *The Mathematical Theory of Viscous Incompressible Flow*; Gordon and Breach Science Publishers: New York, NY, USA, 1969.