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Article

A Novel Implementation of Mönch's Fixed Point Theorem to a System of Nonlinear Hadamard Fractional Differential Equations

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Abstract: In this article, we employed Mönch's fixed point theorem to investigate the existence of solutions for a system of nonlinear Hadamard fractional differential equations and nonlocal non-conserved boundary conditions in terms of Hadamard integral. Followed by a study of the stability of this solution using the Ulam-Hyres technique. This study concludes with an applied numerical example that helps in understanding the theoretical results obtained.

Keywords: Mönch's fixed point; existence; Hadamard derivatives; stability

MSC: 26A33; 34B15; 34B18.

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1. Introduction

Fractional Calculus (FCs): Some consider this science a part of mathematical analysis and deal with the applications of integration and derivation in the case of non-integer order. As for this field (fractional differentiation), it helps us to find the derivative with order half or 0.3 or 0.7…etc. see [1,2]. The origins of this trend lie in the seventeenth century when Newton and Leibniz laid the foundations of calculus. Leibniz put the famous $\frac{d^n y}{dx^n}$ to denote the nth derivative of the function f, so Leibniz sent a message to L'Hopital telling him this new symbol, but L'Hopital responded to the message with a confusing question: "What if $n=\frac{1}{2}$?" The letter was written in 1695 and is today the first appearance of the fractional derivative (FDs). The mathematician Liouville began investigating and researching the subject and issued a series of papers in the period 1832–1837, where he knew the first operator of fractional integration, and after Riemann considered this subject and developed on it, what is known today as the definition of Riemann appeared. An unprecedented interest and development in this field followed. To learn more about the history of the emergence of this branch of mathematics, we direct the reader to look at [3,4]. Whereas the study of BVPs for equations with nonlinear fractional differentials has a prominent and important role in the theory of fractional Calculus, and in the study of physical phenomena through the physical interpretation of boundary conditions. To pass quickly on the practical applications of FDs in various applied sciences, Refs. [5–14] present some valuable works in applications of fractional calculus. .

Through the in-depth and comprehensive study of FDEs, the existence and uniqueness of solutions to FDEs are proven using a set of fixed point theories such as Banach's, Darbo's, Leary-Schuader alternative, and also Mönch's.

Recently, many researchers have given a lot of attention to investigating the existence of solutions to fractional differential equations, and most of these works were focused on the Caputo fractional derivative, so the FDE involving derivatives of the Hadamard type did not gain that much attention, add to this, that a very large percentage used Banach Contraction Mapping and Leray-Schauder's alternative, whereas the Mönch's fixed point theorem is not mentioned much. In this work, we shed light on the differential equations that combine the derivative and integral of the Hadamard fractional type and study the existence of a solution to this system through the rarely used Mönch's fixed point theorem. Herein lies the originality and distinction of this work. Just as ordinary differential equations have applications in various sciences, fractional differential equations have found a place in these applications, and were even distinguished in some cases over ordinary differential equations as they reduce the percentage error in estimation of the variable of interest; see [15–20].

This fractional derivative is invariant with respect to dilation on the whole axis, Hadamard fractional derivatives are nonlocal fractional derivatives with a singular logarithmic kernel with memory, and hence they are suitable to describe complex systems, keeping in mind that, just like Riemann-Liouville, the Hadamard derivative has its own disadvantages as well, one of which is the fact that the derivative of a constant is not equal to zero; see [21].

The following fractional differential equation (FDE) comprises a Hadamard fractional derivative (H-FD) of variable order. In [22], the authors employed Darbo's fixed point theorem to investigate the existence and stability of the solution.

$$\begin{cases} {}^{\mathcal{H}}\mathcal{D}_{1+}^{\mathcal{P}_1}\psi(\tau) = \mathcal{F}_1(\tau, \psi(\tau)), \ \ \tau \in [1, \mathcal{T}], \\ \psi(1) = \psi(\mathcal{T}) = 0, \end{cases}$$

where $1 < \mathcal{P}_1 \leq 2, \mathcal{F}_1 : [1, \mathcal{T}] \times \mathcal{R}_e \to \mathcal{R}_e$ is a continuous function and ${}^{\mathcal{H}}\mathcal{D}_{1+}^{\mathcal{P}_1}, {}^{\mathcal{H}}\mathcal{I}_{1+}^{\mathcal{P}_1}$ are the Hadamard fractional derivative (H-FD) and integral of variable-order $\psi(\tau)$, respectively.

In 2021, Ref. [23] Bashir Ahmad, et al. investigated the existence and uniqueness of the following system of FDE involving H-FD

$$\begin{cases} {}^{\mathcal{H}}\mathcal{D}^{\mathcal{P}_1}\psi(\tau) = \mathcal{F}_1(\tau, \psi(\tau), \varpi(\tau)), \ \ \tau \in [1, \mathcal{T}], \ \ 0 < \mathcal{P}_1 \leq 1, \\ {}^{\mathcal{H}}\mathcal{D}^{\mathcal{P}_2}\varpi(\tau) = \mathcal{F}_2(\tau, \psi(\tau), \varpi(\tau)), \ \ \tau \in [1, \mathcal{T}], \ \ 0 < \mathcal{P}_2 \leq 1, \end{cases}$$

with the following coupled BCs:

$$\begin{cases} \psi(1) + \sum_{\nu=1}^{m} \delta_{1\nu}\varpi(\tau_\nu) = 0, \\ \varpi(1) + \sum_{\nu=1}^{m} \delta_{2\nu}\psi(\tau_\nu) = 0, \end{cases}$$

where $\mathcal{D}^\theta$ is the (H-FD) of order $\theta \in \{\mathcal{P}_1, \mathcal{P}_2\}$, respectively $\mathcal{F}_1, \mathcal{F}_2 : [1, \mathcal{T}] \times \mathcal{R}_e^2 \to \mathcal{R}_e$ are Carathéodory functions, τ_ν are given points with $1 \leq \tau_1 \leq \cdots \leq \tau_m < \mathcal{T}$ and δ_1, δ_2 are real number such that $1 - \sum_{\nu=1}^{m} \delta_{1\nu} \sum_{\nu=1}^{m} \delta_{2\nu} \neq 0$.

In [24], the authors studied the existence and uniqueness of a multipoint BVP with H-FD (sequential type):

$$\begin{cases} ({}^{\mathcal{H}}\mathcal{D}^{\mathcal{P}_1} + \lambda\, {}^{\mathcal{H}}\mathcal{D}^{\mathcal{P}_1 - 1})\psi(\tau) = \mathcal{F}_1(\tau, \psi(\tau)), \ \ \tau \in [1, \mathcal{T}], \ \ 1 < \mathcal{P}_1 \leq 2, \\ \psi(1) = 0, \ \ \psi(\mathcal{T}) = \sum_{\nu=1}^{m} \delta_{1\nu}\varpi(\tau_\nu), \end{cases}$$

where ${}^{\mathcal{H}}\mathcal{D}^{\mathcal{P}_1}$ is the (H-FD) of order $\mathcal{P}_1$, $\mathcal{F}_1 : [1, \mathcal{T}] \times \mathcal{R}_e \to \mathcal{R}_e$ is a continuous function, $\lambda \in \mathcal{R}_e^+, \tau_\nu, \nu = 1, 2, \cdots m$, are given points with $1 \leq \tau_1 \leq \cdots \leq \tau_m < \mathcal{T}$, and $\delta_{1\nu}$ are appropriate real numbers.

The authors in [25] used the Banach and Schaefer theorems to establish the necessary conditions that ensured the stability and existence of the subsequent FDE with H-FD and solutions:

$$\begin{cases} \mathcal{D}^{\mathcal{P}_1}\psi(\kappa) = \mathcal{F}_1(\kappa, \psi(\kappa), \varpi(\kappa)), \ \ \kappa \in [1, \mathcal{T}], \ \ 0 < \mathcal{P}_1 \leq 1, \\ \mathcal{D}^{\mathcal{P}_1}\varpi(\kappa) = \mathcal{F}_2(\kappa, \psi(\kappa), \varpi(\kappa)), \ \ \kappa \in [1, \mathcal{T}], \ \ 0 < \mathcal{P}_1 \leq 1, \end{cases}$$

with the following coupled BCs:

$$\begin{cases} \psi(1) = \delta_1 \varpi(\mathcal{T}), \\ \varpi(1) = \delta_2 \psi(\mathcal{T}), \end{cases}$$

where $\mathcal{D}^{\theta}$ is the (H-FD) of order $\theta \in \{\mathcal{P}_1, \mathcal{P}_2\}, \mathcal{F}_1, \mathcal{F}_2 : [1, \mathcal{T}] \times \mathcal{R}_e^2 \to \mathcal{R}_e$ are appropriate functions, and δ_1, δ_2 are real number with $\delta_1\delta_2 \neq 1$.

Due to the importance of the subject and the possibility of employing it in various scientific fields, many researchers in the field of fractional differential have studied the systems of FDEs with a variety of serious conditions accompanying them. For more information on these scientific papers, the reader can see [26–33], A large group of researchers interested in FCs studies the stability of solutions for FDEs after studying the existence of their solutions. To enrich the reader, it is possible to see [34–36].

In this study, in Section 3 we will employ Mönch's theorem to prove the existence of a solution to the system of FDEs mentioned below

$$\begin{cases} {}^{\mathcal{H}}\mathcal{D}^{\mathcal{P}_1}\psi(\kappa) = \mathcal{G}_1(\kappa, \psi(\kappa), \varpi(\kappa)), \quad \kappa \in (1, e), \ \ \mathcal{P}_1 \in (1, 2], \\ {}^{\mathcal{H}}\mathcal{D}^{\mathcal{P}_2}\varpi(\kappa) = \mathcal{G}_2(\kappa, \psi(\kappa), \varpi(\kappa)), \\ \psi(1) = 0, \ \ \varepsilon_1 {}^{\mathcal{H}}\mathcal{I}^{\mathcal{Q}_1}\psi(\zeta) + \varepsilon_2\psi(e) = \varepsilon_3, \\ \varpi(1) = 0, \ \ \delta_1 {}^{\mathcal{H}}\mathcal{I}^{\mathcal{Q}_2}\varpi(\xi) + \delta_2\varpi(e) = \delta_3, \end{cases} \tag{1}$$

where ${}^{\mathcal{H}}\mathcal{D}^{\mathcal{P}_i}$ is the Hadamard fractional derivative of order $\mathcal{P}_i, i = 1, 2, \mathcal{G}_i : [1, e] \times \mathcal{R}_e{}^2 \to \mathcal{R}_e$ are given continuous functions, $\mathcal{Q}_i > 0, i = 1, 2$. $\varepsilon_\nu, \delta_\nu \in \mathcal{R}_e, \nu = 1, 2, 3.1 < \zeta < \xi < e$. ${}^{\mathcal{H}}\mathcal{I}^{(\cdot)}$ represent the Hadamard fractional integral.

In Section 2 preliminaries for this study are mentioned. In the Section 4, which looks at the stability of this solution using the Ulam-Hyres stability technique, Section 5 will represent an applied numerical example of the system of equations mentioned above. Finally, a conclusion is obtained in the Section 6.

2. Preliminaries

This section introduces fundamental FCs concepts, principles, and initial results [1–3].

Definition 1 ([37]). *The H-D of fractional order ψ for a function k: $[1,\infty) \to \mathcal{R}_e$ is defined as*

$$\mathcal{D}^{\psi}k(\kappa) = \frac{1}{\Gamma(n-\psi)} \left(\kappa \frac{d}{d\kappa}\right)^n \int_1^{\kappa} \left(\log \frac{\kappa}{s}\right)^{n-\psi-1} \frac{k(s)}{s} ds, \ \ n-1 < q < n, \ \ n = [\psi] + 1,$$

where $[\psi]$ denotes the integer part of the real number ψ and $\log(\cdot) = \log_e(\cdot)$.

Definition 2 ([37]). *The Hadamard fractional integral of order ψ for a function k is defined as*

$$\mathcal{I}^{\psi}k(\kappa) = \frac{1}{\Gamma(\psi)} \int_1^{\kappa} \left(\log \frac{\kappa}{s}\right)^{\psi-1} \frac{k(s)}{s} ds, \ \ \psi > 0.$$

Remark 1. *If $k \in \mathcal{C}^n[0,\infty)$, then*

$$^{c}D^{\psi}_{0+}k(\kappa) = \frac{1}{\Gamma(n-\psi)}\int_0^{\kappa}\frac{k^n(\rho)}{(\kappa-\rho)^{\psi+1-n}}ds = I^{n-\psi}k(n)(\kappa), \kappa > 0, n-1 < \psi < n,$$

Definition 3. *The Kuratowski measure of non compactness k defined on bounded set ψ of Banach space $\hat{\mathcal{M}}^*$ is :*

$$k(\psi) := \inf\{r > 0 : \psi = \psi_i \ \textit{and diam} \ \ (\psi_i) \le r \ \textit{for} \ 1 \le i \le m\}.$$

To discuss the problem in this paper, we need the following lemmas.

Lemma 1. *Given the Banach space $\hat{\mathcal{M}}^*$ with $\psi, \mathcal{V}$ are two bounded proper subsets of $\hat{\mathcal{M}}^*$, then the ensuing characteristics are true.*

(1) *If $\psi \subset \mathcal{V}$, then $k(\psi) \le k(\mathcal{V})$;*
(2) *$k(\psi) = k(\bar{\psi}) = k(\overline{conv}\psi)$;*
(3) *ψ is relatively compact $k(\psi) = 0$;*
(4) *$k(\delta\psi) = |\delta|k(\psi), \delta \in \mathcal{R}_e$;*
(5) *$k(\psi \cup \mathcal{V}) = \max\{k(\psi), k(\mathcal{V})\}$;*
(6) *$k(\psi + \mathcal{V}) = k(\psi) + k(\mathcal{V}), \psi + \mathcal{V} = \{x | x = \mathfrak{u} + \mathfrak{v}, \mathfrak{u} \in \psi, \mathfrak{v} \in \mathcal{V}\}$;*
(7) *$k(\psi + y) = k(\psi), \forall y \in \hat{\mathcal{M}}^*$.*

For more details and the proof of these properties, see [38].

Lemma 2. *Given an equicontinuous and bounded set $\mathcal{W}^* \subset \mathcal{C}([1,e], \hat{\mathcal{M}}^*)$, then the function $\varpi \rightarrowtail k(\mathcal{W}^*(\varpi))$ is continuous on $[1,e], k_{\mathcal{C}}(\mathcal{W}^*) = \max_{\varpi\in[1,e]} k(\mathcal{W}^*(\varpi))$, and*

$$k\left(\int_{\mathfrak{a}}^{\mathcal{T}} x(\kappa)d\kappa\right) \le \left(\int_{\mathfrak{a}}^{\mathcal{T}} (x(\kappa))d\kappa\right), \mathcal{W}^*(\kappa) = \{x(\kappa) : x \in \mathcal{W}^*\}. \tag{2}$$

Definition 4. *Given the function $\Psi : [1,e] \times \hat{\mathcal{M}}^* \to \hat{\mathcal{M}}^*, \Psi$ satisfy Carathéodory's conditions, if the following conditions applies:*

$\Psi(\varpi, z)$ is measurable in ϖ for $z \in \hat{\mathcal{M}}^*$;

$\Psi(\varpi, z)$ is continuous in $z \in \hat{\mathcal{M}}^*$ for $\varpi \in [1,e]$.

Theorem 1. *Given a bounded, closed, and convex subset $\Omega \subset \hat{\mathcal{M}}^*$, such that $0 \in \Omega$, let also $\mathcal{T}$ be a continuous mapping of Ω into itself. (Mönch's fixed point theorem).*

If $\mathcal{W}^* = \overline{conv}\mathcal{T}(\mathcal{W}^*)$, or $\mathcal{W}^* = \mathcal{T}(\mathcal{W}^*) \cup \{0\}$, then $k(\mathcal{W}^*) = 0$,

satisfied $\forall \ \mathcal{W}^* \subset \Omega$, then $\mathcal{T}$ has a fixed point.

Lemma 3. *Assume that $\mathcal{H}_1$ and $\mathcal{H}_2 \in \mathcal{C}([1,e], \mathcal{R}_e)$, the solution for the following system*

$$\begin{cases} {}^{\mathcal{H}}\mathcal{D}^{\mathcal{P}_1}\psi(\kappa) = \mathscr{H}_1, \\ {}^{\mathcal{H}}\mathcal{D}^{\mathcal{P}_2}\varpi(\kappa) = \mathscr{H}_2, \\ \psi(1) = 0, \ \ \varepsilon_1 {}^{\mathcal{H}}\mathcal{I}^{\mathcal{Q}_1}\psi(\zeta) + \varepsilon_2\psi(e) = \varepsilon_3, \\ \varpi(1) = 0, \ \ \delta_1 {}^{\mathcal{H}}\mathcal{I}^{\mathcal{Q}_2}\varpi(\xi) + \delta_2\varpi(e) = \delta_3. \end{cases} \tag{3}$$

is

$$\psi(\kappa) = {}^{\mathcal{H}}\mathcal{I}^{(\mathcal{P}_1)}\mathscr{H}_1(\kappa) + (In\ \kappa)^{\mathcal{P}_1-1}\frac{\varepsilon_3 - \varepsilon_1{}^{\mathcal{H}}\mathcal{I}^{(\mathcal{Q}_1+\mathcal{P}_1)}\mathscr{H}_1(\zeta) - \varepsilon_2{}^{\mathcal{H}}\mathcal{I}^{\mathcal{Q}_1}\mathscr{H}_1(e)}{\varepsilon_2 + \frac{\varepsilon_1\Gamma(\mathcal{P}_1)}{\Gamma(\mathcal{Q}_1+\mathcal{P}_1)}(In\ \zeta)^{\mathcal{Q}_1+\mathcal{P}_1-1}}. \tag{4}$$

$$\varpi(\kappa) = {}^{\mathcal{H}}\mathcal{I}^{(\mathcal{P}_2)}\mathscr{H}_2(\kappa) + (In\ \kappa)^{\mathcal{P}_2-1}\frac{\delta_3 - \delta_1{}^{\mathcal{H}}\mathcal{I}^{(\mathcal{Q}_2+\mathcal{P}_2)}\mathscr{H}_2(\xi) - \delta_2{}^{\mathcal{H}}\mathcal{I}^{\mathcal{Q}_2}\mathscr{H}_2(e)}{\delta_2 + \frac{\delta_1\Gamma(\mathcal{P}_2)}{\Gamma(\mathcal{Q}_2+\mathcal{P}_2)}(In\ \xi)^{\mathcal{Q}_2+\mathcal{P}_2-1}}. \tag{5}$$

Proof. Applying ${}^{\mathcal{H}}\mathcal{I}^{\mathcal{P}_1}$ to

$$^{\mathcal{H}}\mathcal{D}^{\mathcal{P}_1}\psi(\kappa) = \mathscr{H}_1,$$

gives

$$\psi(\kappa) = {}^{\mathcal{H}}\mathcal{I}^{(\mathcal{P}_1)}\mathscr{H}_1(\kappa) + \mathfrak{b}_1(In\ \kappa)^{\mathcal{P}_1-1} + \mathfrak{b}_2(In\ \kappa)^{\mathcal{P}_1-2}, \tag{6}$$

but $\psi(1) = 0$, yields $\mathfrak{b}_2 = 0$ observe that

$$\begin{aligned} {}^{\mathcal{H}}\mathcal{I}^{\mathcal{Q}_1}\psi(\zeta) =& {}^{\mathcal{H}}\mathcal{I}^{\mathcal{Q}_1+\mathcal{P}_1}\mathscr{H}_1(\zeta) + \frac{\mathfrak{b}_1}{\Gamma(\mathcal{Q}_1)}\int_1^{\zeta}\left(In\frac{\zeta}{r}\right)^{\mathcal{Q}_1-1}\left(In\frac{r}{r}\right)^{\mathcal{P}_1-1}dr \\ =& {}^{\mathcal{H}}\mathcal{I}^{\mathcal{Q}_1+\mathcal{P}_1}\mathscr{H}_1(\zeta) + \frac{\mathfrak{b}_1\Gamma(\mathcal{P}_1)}{\Gamma(\mathcal{Q}_1+\mathcal{P}_1)}(In\zeta)^{\mathcal{Q}_1+\mathcal{P}_1-1}. \end{aligned}$$

The 2nd boundary condition gives

$$\varepsilon_1{}^{\mathcal{H}}\mathcal{I}^{\mathcal{Q}_1+\mathcal{P}_1}\mathscr{H}_1(\zeta) + \varepsilon_1\frac{\mathfrak{b}_1\Gamma(\mathcal{P}_1)}{\Gamma(\mathcal{Q}_1+\mathcal{P}_1)}(In\zeta)^{\mathcal{Q}_1+\mathcal{P}_1-1} + \varepsilon_2{}^{\mathcal{H}}\mathcal{I}^{\mathcal{P}_1}\mathscr{H}_1(e) + \varepsilon_2\mathfrak{b}_1 = \varepsilon_3,$$

implying that

$$\mathfrak{b}_1 = \frac{\varepsilon_3 - \varepsilon_1{}^{\mathcal{H}}\mathcal{I}^{(\mathcal{Q}_1+\mathcal{P}_1)}\mathscr{H}_1(\zeta) - \varepsilon_2{}^{\mathcal{H}}\mathcal{I}^{\mathcal{Q}_1}\mathscr{H}_1(e)}{\varepsilon_2 + \frac{\varepsilon_1\Gamma(\mathcal{P}_1)}{\Gamma(\mathcal{Q}_1+\mathcal{P}_1)}(In\ \zeta)^{\mathcal{Q}_1+\mathcal{P}_1-1}},$$

substitute the values of $\mathfrak{b}_1$, and $\mathfrak{b}_2$ in (6) yields (4). In a similar way, we can obtain (5). This completes the proof. □

In view of Lemma (3), Equations (4) and (5) can be rewritten as

$$\begin{aligned} \psi(\kappa) =& \frac{1}{\Gamma(\mathcal{P}_1)}\int_1^{\kappa}\left(In\frac{\kappa}{r}\right)^{\mathcal{P}_1-1}\frac{\mathscr{H}_1(r)}{r}dr \\ &+ \frac{(In\ \kappa)^{\mathcal{P}_1-1}}{\Delta_1}\left[\varepsilon_3 - \frac{\varepsilon_1}{\Gamma(\mathcal{Q}_1+\mathcal{P}_1)}\int_1^{\zeta}\left(In\frac{\zeta}{r}\right)^{\mathcal{Q}_1+\mathcal{P}_1-1}\frac{\mathscr{H}_1(r)}{r}dr \right.\\ &\left. - \frac{\varepsilon_2}{\Gamma(\mathcal{P}_1)}\int_1^{e}\left(In\frac{e}{r}\right)^{\mathcal{P}_1-1}\frac{\mathscr{H}_1(r)}{r}dr\right],\ \ \kappa \in [1, e], \end{aligned} \tag{7}$$

and

$$\begin{aligned}\varpi(\kappa) =&\frac{1}{\Gamma(\mathcal{P}_2)}\int_1^{\kappa}\Big(In\frac{\kappa}{r}\Big)^{\mathcal{P}_2-1}\frac{\mathscr{H}_2(r)}{r}dr \\ &+\frac{(In\,\kappa)^{\mathcal{P}_2-1}}{\Delta_2}\bigg[\delta_3-\frac{\delta_1}{\Gamma(\mathcal{Q}_2+\mathcal{P}_2)}\int_1^{\xi}\Big(In\frac{\xi}{r}\Big)^{\mathcal{Q}_2+\mathcal{P}_2-1}\frac{\mathscr{H}_2(r)}{r}dr \\ &-\frac{\delta_2}{\Gamma(\mathcal{P}_2)}\int_1^{e}\Big(In\frac{e}{r}\Big)^{\mathcal{P}_2-1}\frac{\mathscr{H}_2(r)}{r}dr\bigg],\ \ \kappa\in[1,e],\end{aligned} \tag{8}$$

with

$$\Delta_1=\varepsilon_2+\varepsilon_1\frac{\Gamma(\mathcal{P}_1)}{\Gamma(\mathcal{P}_1+\mathcal{Q}_1)}(In\zeta)^{\mathcal{P}_1+\mathcal{Q}_1-1},$$

$$\Delta_2=\delta_2+\delta_1\frac{\Gamma(\mathcal{P}_2)}{\Gamma(\mathcal{P}_2+\mathcal{Q}_2)}(In\xi)^{\mathcal{P}_2+\mathcal{Q}_2-1}.$$

3. Existence Results via Mönch's Fixed Point Theorem

Let $\widehat{\mathcal{E}}=\{(\psi(\kappa),\varpi(\kappa))|(\psi,\varpi)\in\mathcal{C}([1,e],\mathcal{R}_e)\times\mathcal{C}([1,e],\mathcal{R}_e)\}$. Clearly, the aforementioned set $\widehat{\mathcal{E}}$ is Banach space endowed with the norm

$$||(\psi,\varpi)||_{\widehat{\mathcal{E}}}=||\psi||_\infty+||\varpi||_\infty.$$

To show that our system (1) has a solution we set the following Assumptions,

($\mathcal{A}_1$) Suppose that $\psi,\varpi:[1,e]\times(\mathcal{R}_e)^2\to\mathcal{R}_e$ satisfy Carathéodory conditions.

($\mathcal{A}_2$) $\exists\ \mathcal{U}_\psi,\mathcal{U}_\varpi\in\mathcal{L}^1[1,e]\times(\mathcal{R}_e)_+$, and $\exists\ \mathfrak{H}_\psi,\mathfrak{H}_\varpi:(\mathcal{R}_e)_+\to(\mathcal{R}_e)_+$ such that $\forall\ \kappa\in[1,e],\forall(\psi,\varpi\in\widehat{\mathcal{E}})$ we have

$$\begin{aligned}||\psi(\kappa,\psi,\varpi)||_\infty&\le\mathcal{U}_\psi(\kappa)\mathfrak{H}_\psi(||\psi||_\infty+||\varpi||_\infty),\\ ||\varpi(\kappa,\psi,\varpi)||_\infty&\le\mathcal{U}_\varpi(\kappa)\mathfrak{H}_\varpi(||\psi||_\infty+||\varpi||_\infty),\end{aligned}$$

here $\mathfrak{H}_\psi$, $\mathfrak{H}_\varpi$ are non-decreasing continuous functions.

($\mathcal{A}_3$) Let $\mathcal{S}\subset\widehat{\mathcal{E}}\times\widehat{\mathcal{E}}$, assumed to be bounded, and

$$\begin{aligned}\mathcal{K}(\psi(\kappa,\mathcal{S}))&\le\mathcal{U}_\psi(\kappa)\mathcal{K}(\mathcal{S}),\\ \mathcal{K}(\varpi(\kappa,\mathcal{S}))&\le\mathcal{U}_\varpi(\kappa)\mathcal{K}(\mathcal{S}).\end{aligned}$$

For computational convenience, we set

$$\begin{aligned}\mathcal{O}_1=\sup_{1\le\kappa\le e}\bigg\{&\frac{1}{\Gamma(\mathcal{P}_1)}\int_1^{\kappa}\Big(In\frac{\kappa}{r}\Big)^{\mathcal{P}_1-1}\frac{1}{r}dr \\ &+\frac{(In\,\kappa)^{\mathcal{P}_1-1}}{\Delta_1}\bigg[\varepsilon_3-\frac{\varepsilon_1}{\Gamma(\mathcal{Q}_1+\mathcal{P}_1)}\int_1^{\zeta}\Big(In\frac{\zeta}{r}\Big)^{\mathcal{Q}_1+\mathcal{P}_1-1}\frac{1}{r}dr \\ &-\frac{\varepsilon_2}{\Gamma(\mathcal{P}_1)}\int_1^{e}\Big(In\frac{e}{r}\Big)^{\mathcal{P}_1-1}\frac{1}{r}dr\bigg]\bigg\},\end{aligned}$$

$$\mathcal{O}_1\le\frac{1}{\Gamma(\mathcal{P}_1+1)}+\frac{1}{|\Delta_1|}\bigg[\frac{|\varepsilon_1|(In\,\zeta)^{\mathcal{P}_1+\mathcal{Q}_1}}{\Gamma(\mathcal{P}_1+\mathcal{Q}_1+1)}+\frac{|\varepsilon_2|}{\Gamma(\mathcal{P}_1+1)}\bigg], \tag{9}$$

and

$$\mathcal{O}_2 \leq \frac{1}{\Gamma(\mathcal{P}_2+1)} + \frac{1}{|\Delta_2|}\left[\frac{|\delta_1|(In\ \xi)^{\mathcal{P}_2+\mathcal{Q}_2}}{\Gamma(\mathcal{P}_2+\mathcal{Q}_2+1)} + \frac{|\delta_2|}{\Gamma(\mathcal{P}_2+1)}\right]. \quad (10)$$

Theorem 2. *Assume that the Assumptions $(\mathcal{A}_1), (\mathcal{A}_2)$, and $(\mathcal{A}_3)$ are satisfied. If*

$$\max\{\mathcal{U}^*_\psi\mathcal{O}_1, \mathcal{U}^*_\varpi\mathcal{O}_2\} < 1, \quad (11)$$

then the system of fractional differential equations given by (1) has at least one solution on $[1, e]$.

Proof. The continuous operator $\mathcal{T} : \widehat{\mathcal{E}} \to \widehat{\mathcal{E}}$ needs to be defined

$$\mathcal{T} = \mathcal{T}_1(\psi,\varpi)(\kappa), \mathcal{T}_2(\psi,\varpi)(\kappa),$$

where

$$\begin{aligned}\mathcal{T}_1 =& \Bigg\{\frac{1}{\Gamma(\mathcal{P}_1)}\int_1^\kappa \left(In\frac{\kappa}{r}\right)^{\mathcal{P}_1-1}\frac{\mathcal{G}_1(r,\psi(r),\varpi(r))}{r}dr \\ &+\frac{(In\ \kappa)^{\mathcal{P}_1-1}}{\Delta_1}\Bigg[\varepsilon_3 - \frac{\varepsilon_1}{\Gamma(\mathcal{Q}_1+\mathcal{P}_1)}\int_1^\zeta \left(In\frac{\zeta}{r}\right)^{\mathcal{Q}_1+\mathcal{P}_1-1}\frac{\mathcal{G}_1(r,\psi(r),\varpi(r))}{r}dr \\ &-\frac{\varepsilon_2}{\Gamma(\mathcal{P}_1)}\int_1^e \left(In\frac{e}{r}\right)^{\mathcal{P}_1-1}\frac{\mathcal{G}_1(r,\psi(r),\varpi(r))}{r}dr\Bigg]\Bigg\},\end{aligned} \quad (12)$$

and

$$\begin{aligned}\mathcal{T}_2 =& \Bigg\{\frac{1}{\Gamma(\mathcal{P}_2)}\int_1^\kappa \left(In\frac{\kappa}{r}\right)^{\mathcal{P}_2-1}\frac{\mathcal{G}_2(r,\psi(r),\varpi(r))}{r}dr \\ &+\frac{(In\ \kappa)^{\mathcal{P}_2-1}}{\Delta_2}\Bigg[\delta_3 - \frac{\delta_1}{\Gamma(\mathcal{Q}_2+\mathcal{P}_2)}\int_1^\zeta \left(In\frac{\zeta}{r}\right)^{\mathcal{Q}_2+\mathcal{P}_2-1}\frac{\mathcal{G}_2(r,\psi(r),\varpi(r))}{r}dr \\ &-\frac{\delta_2}{\Gamma(\mathcal{P}_2)}\int_1^e \left(In\frac{e}{r}\right)^{\mathcal{P}_2-1}\frac{\mathcal{G}_2(r,\psi(r),\varpi(r))}{r}dr\Bigg]\Bigg\}.\end{aligned} \quad (13)$$

Operator $\mathcal{T}$ equation

$$(\psi,\varpi) = \mathcal{T}(\psi,\varpi), \quad (14)$$

is equivalent to (7) and (8), keeping in mind that showing the existence of a solution for (14) is equivalent to showing the existence of solution for (1).

Next, we define $\mathcal{S}_\Theta = \{(\psi,\varpi) \in \widehat{\mathcal{E}} : ||(\psi,\varpi)||_{\widehat{\mathcal{E}}} \leq \Theta, \Theta > 0\}$ to be a closed bounded convex ball in $\widehat{\mathcal{E}}$ with

$$\Theta \geq \mathcal{U}^*_\psi\mathcal{O}_1\mathfrak{H}_\psi(\Theta) + \mathcal{U}^*_\varpi\mathcal{O}_2\mathfrak{H}_\varpi(\Theta).$$

satisfy Mönch's fixed point theorem condition we split our proof into four steps.

Step 1: We show that $\mathcal{T}\mathcal{S}_\Theta \subset \mathcal{S}_\Theta$, let $\mathcal{T} \in [1, e]$ and $\forall(\psi,\varpi) \in \mathcal{S}_\Theta$, we have

$$\begin{aligned}||\mathcal{T}_1(\psi,\varpi)||_\infty =& \Bigg\{\frac{1}{\Gamma(\mathcal{P}_1)}\int_1^\kappa \left(In\frac{\kappa}{r}\right)^{\mathcal{P}_1-1}\frac{||\mathcal{G}_1(r,\psi(r),\varpi(r))||_\infty}{r}dr \\ &+\frac{(In\ \kappa)^{\mathcal{P}_1-1}}{\Delta_1}\Bigg[|\varepsilon_3| + \frac{|\varepsilon_1|}{\Gamma(\mathcal{Q}_1+\mathcal{P}_1)}\int_1^\zeta \left(In\frac{\zeta}{r}\right)^{\mathcal{Q}_1+\mathcal{P}_1-1}\frac{||\mathcal{G}_1(r,\psi(r),\varpi(r))||_\infty}{r}dr \\ &+\frac{|\varepsilon_2|}{\Gamma(\mathcal{P}_1)}\int_1^e \left(In\frac{e}{r}\right)^{\mathcal{P}_1-1}\frac{||\mathcal{G}_1(r,\psi(r),\varpi(r))||_\infty}{r}dr\Bigg]\Bigg\},\end{aligned} \quad (15)$$

using $(\mathcal{A}_2)$, $\forall\kappa \in [1,e]$ we have

$$||\psi(\kappa,\psi(\kappa),\varpi(\kappa))||_\infty \leq \mathcal{U}^*_\psi(\kappa)\mathfrak{H}_\psi(||\psi(\kappa)||_\infty + ||\varpi(\kappa)||_\infty) \\ \leq \mathcal{U}^*_\psi\mathfrak{H}_\psi(\Theta),$$

$$\begin{aligned} ||\mathcal{T}_1(\psi,\varpi)||_\infty =& \Bigg\{ \frac{1}{\Gamma(\mathcal{P}_1)} \int_1^\kappa \Big(In\frac{\kappa}{r}\Big)^{\mathcal{P}_1-1} \frac{\mathcal{U}^*_\psi(\kappa)\mathfrak{H}_\psi(||\psi(\kappa)||_\infty + ||\varpi(\kappa)||_\infty)}{r}dr \\ &+ \frac{(In\,\kappa)^{\mathcal{P}_1-1}}{\Delta_1}\Bigg[|\varepsilon_3| + \frac{|\varepsilon_1|}{\Gamma(\mathcal{Q}_1+\mathcal{P}_1)}\int_1^\zeta \Big(In\frac{\zeta}{r}\Big)^{\mathcal{Q}_1+\mathcal{P}_1-1}\frac{\mathcal{U}^*_\psi(\kappa)\mathfrak{H}_\psi(||\psi(\kappa)||_\infty + ||\varpi(\kappa)||_\infty)}{r}dr \\ &+ \frac{|\varepsilon_2|}{\Gamma(\mathcal{P}_1)}\int_1^e \Big(In\frac{e}{r}\Big)^{\mathcal{P}_1-1}\frac{\mathcal{U}^*_\psi(\kappa)\mathfrak{H}_\psi(||\psi(\kappa)||_\infty + ||\varpi(\kappa)||_\infty)}{r}dr\Bigg]\Bigg\}, \\ &\leq \mathcal{U}^*_\psi\mathfrak{H}_\psi(\Theta). \end{aligned} \tag{16}$$

Similarly,

$$\begin{aligned} ||\mathcal{T}_2(\psi,\varpi)||_\infty =& \Bigg\{ \frac{1}{\Gamma(\mathcal{P}_2)} \int_1^\kappa \Big(In\frac{\kappa}{r}\Big)^{\mathcal{P}_2-1} \frac{\mathcal{U}^*_\varpi(\kappa)\mathfrak{H}_\varpi(||\psi(\kappa)||_\infty + ||\varpi(\kappa)||_\infty)}{r}dr \\ &+ \frac{(In\,\kappa)^{\mathcal{P}_2-1}}{\Delta_2}\Bigg[|\delta_3| + \frac{|\delta_1|}{\Gamma(\mathcal{Q}_2+\mathcal{P}_2)}\int_1^\zeta \Big(In\frac{\zeta}{r}\Big)^{\mathcal{Q}_2+\mathcal{P}_2-1}\frac{\mathcal{U}^*_\varpi(\kappa)\mathfrak{H}_\varpi(||\psi(\kappa)||_\infty + ||\varpi(\kappa)||_\infty)}{r}dr \\ &+ \frac{|\delta_2|}{\Gamma(\mathcal{P}_2)}\int_1^e \Big(In\frac{e}{r}\Big)^{\mathcal{P}_2-1}\frac{\mathcal{U}^*_\varpi(\kappa)\mathfrak{H}_\varpi(||\psi(\kappa)||_\infty + ||\varpi(\kappa)||_\infty)}{r}dr\Bigg]\Bigg\}, \\ &\leq \mathcal{U}^*_\varpi\mathfrak{H}_\varpi(\Theta). \end{aligned} \tag{17}$$

(16) and (17) yields,

$$\begin{aligned} ||\mathcal{T}(\psi,\varpi)||_{\widehat{\mathcal{E}}} =& ||\mathcal{T}_1(\psi,\varpi)||_\infty + ||\mathcal{T}_2(\psi,\varpi)||_\infty \\ \leq& \mathcal{U}^*_\psi\mathcal{O}_1\mathfrak{H}_\psi(\Theta) + \mathcal{U}^*_\varpi\mathcal{O}_2\mathfrak{H}_\varpi(\Theta) \\ \leq& \Theta, \end{aligned} \tag{18}$$

that is $\mathcal{T}\mathcal{S}_\Theta \subset \mathcal{S}_\Theta$.

Step 2: We show the continuity of the operator $\mathcal{T}$. To do this, we let the sequence

$$\{\mathcal{V}_n = (\psi_n,\varpi_n)\} \in \mathcal{S}_\Theta, \quad \text{and show that } \mathcal{V}_n \to \mathcal{V} = (\psi,\varpi) \text{ as } n \to \infty.$$

Because of Carathéodory continuity of ψ, it is clear that

$$\psi(\cdot,\psi_n(\cdot),\varpi_n(\cdot)) \to \psi(\cdot,\psi(\cdot),\varpi(\cdot)) \text{ as } n \to \infty.$$

Recalling $(\mathcal{A}_2)$, we deduce that

$$\Big(In\frac{\kappa}{r}\Big)^{\mathcal{P}_1-1}||\psi(r,\psi_n(r),\varpi_n(r)) - \psi(r,\psi(r),\varpi(r))||_\infty \leq \mathcal{U}^*_\psi\mathfrak{H}_\psi(\Theta)\Big(In\frac{\kappa}{r}\Big)^{\mathcal{P}_1-1}. \tag{19}$$

Additionally, by the function's Lebesgue dominated convergence theorem and the fact that

$$\mathcal{M} \to \mathcal{U}^*_\psi\mathfrak{H}_\psi(\Theta)\Big(In\frac{\kappa}{r}\Big)^{\mathcal{P}_1-1}, \tag{20}$$

is Lebesgue integrable on $[1, e]$, we get

$$\begin{aligned}||\mathcal{T}_1(\psi,\varpi)||_\infty =&\Bigg\{\frac{1}{\Gamma(\mathcal{P}_1)}\int_1^\kappa\Big(In\frac{\kappa}{r}\Big)^{\mathcal{P}_1-1}\frac{||\psi(r,\psi_n(r),\varpi_n(r))-\psi(r,\psi(r),\varpi(r))||_\infty}{r}dr\\&+\frac{(In\,\kappa)^{\mathcal{P}_1-1}}{\Delta_1}\Bigg[|\varepsilon_3|+\frac{|\varepsilon_1|}{\Gamma(\mathcal{Q}_1+\mathcal{P}_1)}\int_1^\zeta\Big(In\frac{\zeta}{r}\Big)^{\mathcal{Q}_1+\mathcal{P}_1-1}\\&\times\frac{||\psi(r,\psi_n(r),\varpi_n(r))-\psi(r,\psi(r),\varpi(r))||_\infty}{r}dr\\&+\frac{|\varepsilon_2|}{\Gamma(\mathcal{P}_1)}\int_1^e\Big(In\frac{e}{r}\Big)^{\mathcal{P}_1-1}\frac{||\psi(r,\psi_n(r),\varpi_n(r))-\psi(r,\psi(r),\varpi(r))||_\infty}{r}dr\Bigg]\Bigg\},\end{aligned}\tag{21}$$

that is

$$||\mathcal{T}_1(\psi_n,\varpi_n)(\kappa)-\mathcal{T}_1(\psi,\varpi)(\kappa)||_\infty\to 0 \text{ as } n\to\infty\ \forall\ \kappa\in[1,e],$$

then

$$||\mathcal{T}_1(\psi_n,\varpi_n)-\mathcal{T}_1(\psi,\varpi)||_\infty\to 0 \text{ as } n\to\infty,\tag{22}$$

which means that the operator $\mathcal{T}_1$ is continuous.

Similarly

$$||\mathcal{T}_2(\psi_n,\varpi_n)-\mathcal{T}_2(\psi,\varpi)||_\infty\to 0 \text{ as } n\to\infty,\tag{23}$$

(22) and (23) yields,

$$||\mathcal{T}(\psi_n,\varpi_n)-\mathcal{T}(\psi,\varpi)||_{\widehat{\mathcal{E}}}\to 0 \text{ as } n\to\infty.\tag{24}$$

By getting (24) we conclude that the operator $\mathcal{T}$ is continuous.

Step 3: We show that $\mathcal{T}$ is equicontinuous. Let $\kappa_1,\kappa_2\in[1,e]$ and $\forall(\psi,\varpi)\in\mathcal{S}_\Theta$, then

$$\begin{aligned}&||\mathcal{T}_1(\psi,\varpi)(\kappa_2)-\mathcal{T}_1(\psi,\varpi)(\kappa_1)||_\infty\\=&\Bigg\{\frac{1}{\Gamma(\mathcal{P}_1)}\int_1^{\kappa_1}\Big(In\frac{\kappa_2}{r}\Big)^{\mathcal{P}_1-1}-\Big(In\frac{\kappa_1}{r}\Big)^{\mathcal{P}_1-1}\frac{||\psi(r,\psi(r),\varpi(r))||_\infty}{r}dr\\&+\frac{1}{\Gamma(\mathcal{P}_1)}\int_{\kappa_1}^{\kappa_2}\Big(In\frac{\kappa_2}{r}\Big)^{\mathcal{P}_1-1}\frac{||\psi(r,\psi(r),\varpi(r))||_\infty}{r}dr\\&+\frac{(In\,\kappa_2)^{\mathcal{P}_1-1}-(In\,\kappa_1)^{\mathcal{P}_1-1}}{\Delta_1}|\varepsilon_3|\\&+\frac{(In\,\kappa_2)^{\mathcal{P}_1-1}-(In\,\kappa_1)^{\mathcal{P}_1-1}}{\Delta_1}\frac{|\varepsilon_1|}{\Gamma(\mathcal{Q}_1+\mathcal{P}_1)}\int_1^\zeta\Big(In\frac{\zeta}{r}\Big)^{\mathcal{Q}_1+\mathcal{P}_1-1}\frac{||\psi(r,\psi(r),\varpi(r))||_\infty}{r}dr\\&+\frac{(In\,\kappa_2)^{\mathcal{P}_1-1}-(In\,\kappa_1)^{\mathcal{P}_1-1}}{\Delta_1}\frac{|\varepsilon_2|}{\Gamma(\mathcal{P}_1)}\int_1^e\Big(In\frac{e}{r}\Big)^{\mathcal{P}_1-1}\frac{||\psi(r,\psi(r),\varpi(r))||_\infty}{r}dr\Bigg\},\end{aligned}\tag{25}$$

$$\begin{aligned}\le&\Bigg\{\frac{\mathcal{U}_\psi^*\mathfrak{H}_\psi(\Theta)}{\Gamma(\mathcal{P}_1)}\int_1^{\kappa_1}\Bigg[\Big(In\frac{\kappa_2}{r}\Big)^{\mathcal{P}_1-1}-\Big(In\frac{\kappa_1}{r}\Big)^{\mathcal{P}_1-1}\Bigg]\frac{dr}{r}\\&+\frac{1}{\Gamma(\mathcal{P}_1)}\int_{\kappa_1}^{\kappa_2}\Big(In\frac{\kappa_2}{r}\Big)^{\mathcal{P}_1-1}\frac{dr}{r}\\&+\frac{(In\,\kappa_2)^{\mathcal{P}_1-1}-(In\,\kappa_1)^{\mathcal{P}_1-1}}{\Delta_1}|\varepsilon_3|\\&+\frac{(In\,\kappa_2)^{\mathcal{P}_1-1}-(In\,\kappa_1)^{\mathcal{P}_1-1}}{\Delta_1}\mathcal{U}_\psi^*\mathfrak{H}_\psi(\Theta)\times\Bigg[\frac{|\varepsilon_1|}{\Gamma(\mathcal{Q}_1+\mathcal{P}_1)}\int_1^\zeta\Big(In\frac{\zeta}{r}\Big)^{\mathcal{Q}_1+\mathcal{P}_1-1}\frac{dr}{r}\\&+\frac{(In\,\kappa_2)^{\mathcal{P}_1-1}-(In\,\kappa_1)^{\mathcal{P}_1-1}}{\Delta_1}\frac{|\varepsilon_2|}{\Gamma(\mathcal{P}_1)}\int_1^e\Big(In\frac{e}{r}\Big)^{\mathcal{P}_1-1}\frac{dr}{r}\Bigg]\Bigg\}\to 0\ \ as\ \ \kappa_1\to\kappa_2.\end{aligned}\tag{26}$$

In a similar manner , we have

$$
\begin{aligned}
&||\mathcal{T}_2(\psi,\varpi)(\kappa_2)-\mathcal{T}_2(\psi,\varpi)(\kappa_1)||_\infty \\
&\le\bigg\{\frac{\mathcal{U}^*_\varpi\mathfrak{H}_\varpi(\Theta)}{\Gamma(\mathcal{P}_2)}\int_1^{\kappa_1}\bigg[\Big(In\frac{\kappa_2}{r}\Big)^{\mathcal{P}_2-1}-\Big(In\frac{\kappa_1}{r}\Big)^{\mathcal{P}_2-1}\bigg]\frac{dr}{r} \\
&+\frac{1}{\Gamma(\mathcal{P}_2)}\int_{\kappa_1}^{\kappa_2}\Big(In\frac{\kappa_2}{r}\Big)^{\mathcal{P}_2-1}\frac{dr}{r} \\
&+\frac{(In\,\kappa_2)^{\mathcal{P}_2-1}-(In\,\kappa_1)^{\mathcal{P}_2-1}}{\Delta_2}|\delta_3| \\
&+\frac{(In\,\kappa_2)^{\mathcal{P}_2-1}-(In\,\kappa_1)^{\mathcal{P}_2-1}}{\Delta_2}\mathcal{U}^*_\varpi\mathfrak{H}_\varpi(\Theta)\times\bigg[\frac{|\delta_1|}{\Gamma(\mathcal{Q}_2+\mathcal{P}_2)}\int_1^{\zeta}\Big(In\frac{\zeta}{r}\Big)^{\mathcal{Q}_2+\mathcal{P}_2-1}\frac{dr}{r} \\
&+\frac{(In\,\kappa_2)^{\mathcal{P}_2-1}-(In\,\kappa_1)^{\mathcal{P}_2-1}}{\Delta_2}\frac{|\delta_2|}{\Gamma(\mathcal{P}_2)}\int_1^{e}\Big(In\frac{e}{r}\Big)^{\mathcal{P}_2-1}\frac{dr}{r}\bigg]\bigg\}\to 0\ \ as\ \ \kappa_1\to\kappa_2.
\end{aligned}
\tag{27}
$$

From (26) and (27) we noted that both inequalities are independent of $(\psi,\varpi)\in\mathcal{S}_\Theta$, that led us to deduce that the operator $\mathcal{T}$ is bounded and equicontinuous.

Step 4: To satisfy all conditions of Mönch's fixed point, finally, we let $\Phi=\Phi_1\cap\Phi_2;\Phi_1,\Phi_2\subset\mathcal{S}_\Theta$. Furthermore , Φ_1 and Φ_2 are assumed to be bounded and equicontinuous.

We show that

$$
\Phi_1\subset\overline{conv}(\mathcal{T}_1(\Phi_1)\cup\{o\}),\ \text{ and }\ \Phi_2\subset\overline{conv}(\mathcal{T}_1(\Phi_1)\cup\{o\}).
$$

Thus, the functions

$$
\begin{aligned}
\Pi_1(\kappa)&=k(\Phi_1(\kappa)),\\
\Pi_2(\kappa)&=k(\Phi_2(\kappa)),
\end{aligned}
$$

are continuous on $[1,e]$. By the Kuratowski Lemma (1) and $(\mathcal{A}_3)$, we write

$$
\begin{aligned}
\Pi_1(\kappa)=&k(\Phi_1(\kappa))\\
\le&k(\overline{conv}(\mathcal{T}_1(\Phi_1)\cup\{o\}))\\
\le&k(\mathcal{T}_1\Phi_1(\kappa)) \qquad (28)\\
\le&k\bigg\{\frac{1}{\Gamma(\mathcal{P}_1)}\int_1^{\kappa}\Big(In\frac{\kappa}{r}\Big)^{\mathcal{P}_1-1}\frac{||\psi(r,\psi(r),\varpi(r))||_\infty}{r}dr\\
&+\frac{(In\,\kappa)^{\mathcal{P}_1-1}}{\Delta_1}\bigg[-\frac{|\varepsilon_1|}{\Gamma(\mathcal{Q}_1+\mathcal{P}_1)}\int_1^{\zeta}\Big(In\frac{\zeta}{r}\Big)^{\mathcal{Q}_1+\mathcal{P}_1-1}\frac{||\psi(r,\psi(r),\varpi(r))||_\infty}{r}dr\\
&-\frac{|\varepsilon_2|}{\Gamma(\mathcal{P}_1)}\int_1^{e}\Big(In\frac{e}{r}\Big)^{\mathcal{P}_1-1}\frac{||\psi(r,\psi(r),\varpi(r))||_\infty}{r}dr:(\psi,\varpi)\in\Phi\bigg]\bigg\}, \qquad (29)
\end{aligned}
$$

$$
\begin{aligned}
&\le\bigg\{\frac{1}{\Gamma(\mathcal{P}_1)}\int_1^{\kappa}\Big(In\frac{\kappa}{r}\Big)^{\mathcal{P}_1-1}\frac{\psi(r,\Phi_1(r))}{r}dr\\
&+\frac{(In\,\kappa)^{\mathcal{P}_1-1}}{\Delta_1}\bigg[-\frac{|\varepsilon_1|}{\Gamma(\mathcal{Q}_1+\mathcal{P}_1)}\int_1^{\zeta}\Big(In\frac{\zeta}{r}\Big)^{\mathcal{Q}_1+\mathcal{P}_1-1}\frac{\psi(r,\Phi_1(r))}{r}dr\\
&-\frac{|\varepsilon_2|}{\Gamma(\mathcal{P}_1)}\int_1^{e}\Big(In\frac{e}{r}\Big)^{\mathcal{P}_1-1}\frac{\psi(r,\Phi_1(r))}{r}dr:(\psi,\varpi)\in\Phi\bigg]\bigg\}.\\
&\le\mathcal{U}^*_\psi\mathcal{O}_1||\Pi_1||_\infty.
\end{aligned}
$$

That is

$$
||\Pi_1||_\infty\le\mathcal{U}^*_\psi\mathcal{O}_1||\Pi_1||_\infty,
$$

but it is supposed that $\max\{\mathcal{U}_\psi^*\mathcal{O}_1, \mathcal{U}_\varpi^*\mathcal{O}_2\} < 1$, yields $||\Pi_1||_\infty = 0$, so $\Pi_1(\kappa) = 0, \ \forall \ \kappa \in [1,e]$, in a similar manner, we get $\Pi_2(\kappa) = 0, \ \forall \ \kappa \in [1,e]$.

Consequently $k(\Phi(\kappa)) \leq k(\Phi_1(\kappa)) = 0$, and $k(\Phi(\kappa)) \leq k(\Phi_1(\kappa)) = 0$, implying $\Phi(\kappa)$ is relatively compact in $\widehat{\mathcal{E}} \times \widehat{\mathcal{E}}$, based on the Arzila–Ascoli theorem we obtain that Φ is relatively compact in $\mathcal{S}_\Theta$.

Now all conditions of Mönch's fixed point Theorem applied, therefore $\mathcal{T}$ has fixed point (ψ, ϖ) on $\mathcal{S}_n$. □

4. Stability Results for the Problem

Let us define nonlinear operator $\mathcal{Z}_1, \mathcal{Z}_2 \in \mathcal{C}([1,e], \mathcal{R}_e) \times \mathcal{C}([1,e], \mathcal{R}_e) \to \mathcal{C}([1,e], \mathcal{R}_e)$, where $\mathcal{T}_1$ and $\mathcal{T}_2$ are define by (12) and (13)

$$\begin{cases} {}^{\mathcal{H}}\mathcal{D}^{\mathcal{P}_1}\psi(\kappa) - \mathcal{G}_1(\kappa, \psi(\kappa), \varpi(\kappa)) = \mathcal{Z}_1(\psi, \varpi)(\kappa), \quad \kappa \in (1,e), \ \mathcal{P}_1 \in (1,2], \\ {}^{\mathcal{H}}\mathcal{D}^{\mathcal{P}_2}\varpi(\kappa) - \mathcal{G}_2(\kappa, \psi(\kappa), \varpi(\kappa)) = \mathcal{Z}_2(\psi, \varpi)(\kappa). \end{cases}$$

For $\kappa \in [1,e]$. For some $\varsigma_1, \varsigma_2 > 0$, we consider the following inequities:

$$||\mathcal{Z}_1, (\psi, \varpi)|| \leq \varsigma_1, \quad ||\mathcal{Z}_2, (\psi, \varpi)|| \leq \varsigma_2. \tag{30}$$

Definition 5. *The coupled system (1) is said to be stable in the H-U sense, if $\mathcal{M}_1, \mathcal{M}_2 > 0$ exist that there is a unique solution $(\psi, \varpi) \in \mathcal{C}([1,e], \mathcal{R}_e) \times \mathcal{C}([1,e], \mathcal{R}_e)$ of problem (1) with*

$$||(\psi, \varpi) - (\hat{\psi}, \hat{\varpi})|| \leq \mathcal{M}_1\varsigma_1 + \mathcal{M}_2\varsigma_2,$$

for every solution $(\hat{\psi}, \hat{\varpi})$ belongs to $\mathcal{C}([1,e], \mathcal{R}_e) \times \mathcal{C}([1,e], \mathcal{R}_e)$ of inequality.

Theorem 3. *Suppose that $(\mathcal{A}_2)$ hold. Then the BVP (1) is H-U stable.*

Proof. Let $(\psi, \varpi) \in \mathcal{C}([1,e], \mathcal{R}_e) \times ([1,e], \mathcal{R}_e)$ be the (1) the solution of the problem that satisfies (4) and (5). Let $(\hat{\psi}, \hat{\varpi})$ be any solution satisfying (30). For $\kappa \in [1,e]$,

$$\begin{cases} {}^{\mathcal{H}}\mathcal{D}^{\mathcal{P}_1}\psi(\kappa) = \mathcal{G}_1(\kappa, \psi(\kappa), \varpi(\kappa)) + \mathcal{Z}_1(\psi, \varpi)(\kappa), \\ {}^{\mathcal{H}}\mathcal{D}^{\mathcal{P}_2}\varpi(\kappa) = \mathcal{G}_2(\kappa, \psi(\kappa), \varpi(\kappa)) + \mathcal{Z}_2(\psi, \varpi)(\kappa). \end{cases}$$

Therefore,

$$\begin{aligned} \hat{\psi}(\kappa) =& \mathcal{T}_1(\hat{\psi}, \hat{\varpi})(\kappa) + \Bigg\{ \frac{1}{\Gamma(\mathcal{P}_1)} \int_1^\kappa \left(In\frac{\kappa}{r} \right)^{\mathcal{P}_1 - 1} \mathcal{Z}_1(r, \psi(r), \varpi(r)) \frac{dr}{r} \\ &+ \frac{(In\ \kappa)^{\mathcal{P}_1 - 1}}{\Delta_1} \Bigg[|\varepsilon_3| + \frac{|\varepsilon_1|}{\Gamma(\mathcal{Q}_1 + \mathcal{P}_1)} \int_1^\zeta \left(In\frac{\zeta}{r} \right)^{\mathcal{Q}_1 + \mathcal{P}_1 - 1} \mathcal{Z}_1(r, \psi(r), \varpi(r)) \frac{dr}{r} \\ &+ \frac{|\varepsilon_2|}{\Gamma(\mathcal{P}_1)} \int_1^e \left(In\frac{e}{r} \right)^{\mathcal{P}_1 - 1} \mathcal{Z}_1(r, \psi(r), \varpi(r)) \frac{dr}{r} \Bigg] \Bigg\}, \end{aligned}$$

it follows that

$$\begin{aligned}|\mathcal{T}_1(\hat{\psi},\hat{\varpi})(\kappa)-\hat{\psi}(\kappa)| \leq & \Bigg\{\frac{1}{\Gamma(\mathcal{P}_1)}\int_1^{\kappa}\Big(In\frac{\kappa}{r}\Big)^{\mathcal{P}_1-1}\varsigma_1\frac{dr}{r} \\ & +\frac{(In\,\kappa)^{\mathcal{P}_1-1}}{\Delta_1}\Bigg[|\varepsilon_3|+\frac{|\varepsilon_1|}{\Gamma(\mathcal{Q}_1+\mathcal{P}_1)}\int_1^{\zeta}\Big(In\frac{\zeta}{r}\Big)^{\mathcal{Q}_1+\mathcal{P}_1-1}\varsigma_1\frac{dr}{r} \\ & +\frac{|\varepsilon_2|}{\Gamma(\mathcal{P}_1)}\int_1^{e}\Big(In\frac{e}{r}\Big)^{\mathcal{P}_1-1}\varsigma_1\frac{dr}{r}\Bigg]\Bigg\}, \\ \leq & \frac{1}{\Gamma(\mathcal{P}_1+1)}+\frac{1}{|\Delta_1|}\Bigg[\frac{|\varepsilon_1|(In\,\zeta)^{\mathcal{P}_1+\mathcal{Q}_1}}{\Gamma(\mathcal{P}_1+\mathcal{Q}_1+1)}+\frac{|\varepsilon_2|}{\Gamma(\mathcal{P}_1+1)}\Bigg]\varsigma_1 \\ \leq & \mathcal{O}_1\varsigma_1.\end{aligned}$$

In a similar manner

$$\begin{aligned}|\mathcal{T}_2(\hat{\psi},\hat{\varpi})(\kappa)-\hat{\varpi}(\kappa)| \leq & \Bigg\{\frac{1}{\Gamma(\mathcal{P}_2)}\int_1^{\kappa}\Big(In\frac{\kappa}{r}\Big)^{\mathcal{P}_2-1}\varsigma_2\frac{dr}{r} \\ & +\frac{(In\,\kappa)^{\mathcal{P}_2-1}}{\Delta_2}\Bigg[|\delta_3|+\frac{|\delta_1|}{\Gamma(\mathcal{Q}_2+\mathcal{P}_2)}\int_1^{\zeta}\Big(In\frac{\zeta}{r}\Big)^{\mathcal{Q}_2+\mathcal{P}_2-1}\varsigma_2\frac{dr}{r} \\ & +\frac{|\delta_2|}{\Gamma(\mathcal{P}_2)}\int_1^{e}\Big(In\frac{e}{r}\Big)^{\mathcal{P}_2-1}\varsigma_2\frac{dr}{r}\Bigg]\Bigg\}, \\ \leq & \frac{1}{\Gamma(\mathcal{P}_2+1)}+\frac{1}{|\Delta_2|}\Bigg[\frac{|\delta_1|(In\,\zeta)^{\mathcal{P}_2+\mathcal{Q}_2}}{\Gamma(\mathcal{P}_2+\mathcal{Q}_2+1)}+\frac{|\delta_2|}{\Gamma(\mathcal{P}_2+1)}\Bigg]\varsigma_2 \\ \leq & \mathcal{O}_2\varsigma_2.\end{aligned}$$

Thus, the operator $\mathcal{T}$, which is given by (12) and (13), can be extracted from the fixed point property, as follows:

$$\begin{aligned}|\psi(\kappa)-\psi*(\kappa)| = & |\psi(\kappa)-\mathcal{T}_1(\psi*,\varpi*)(\kappa)+\mathcal{T}_1(\psi*,\varpi*)(\kappa)-\psi*(\kappa)| \\ \leq & |\mathcal{T}_1(\psi,\varpi)(\kappa)-\mathcal{T}_1(\psi*,\varpi*)(\kappa)|+|\mathcal{T}_1(\psi*,\varpi*)(\kappa)-\psi*(\kappa)| \\ \leq & ((\mathcal{O}_1\vartheta_1+\mathcal{O}_1\hat{\vartheta}_1)+(\mathcal{O}_1\vartheta_2+\mathcal{O}_1\hat{\vartheta}_2))||(\psi,\varpi)-(\psi*,\varpi*)|| \\ & +\mathcal{O}_1\hat{\varsigma_1}+\mathcal{O}_1\hat{\varsigma_2}.\end{aligned} \quad (31)$$

$$\begin{aligned}|\varpi(\kappa)-\varpi*(\kappa)| = & |\varpi\kappa)-\mathcal{T}_2(\psi*,\varpi*)(\kappa)+\mathcal{T}_2(\psi*,\varpi*)(\kappa)-\varpi*(\kappa)| \\ \leq & |\mathcal{T}_2(\psi,\varpi)(\kappa)-\mathcal{T}_2(\psi*,\varpi*)(\kappa)|+|\mathcal{T}_2(\psi*,\varpi*)(\kappa)-\varpi*(\kappa)| \\ \leq & ((\mathcal{O}_2\vartheta_1+\mathcal{O}_2\hat{\vartheta}_1)+(\mathcal{O}_2\vartheta_2+\mathcal{O}_2\hat{\vartheta}_2))||(\psi,\varpi)-(\psi*,\varpi*)|| \\ & +\mathcal{O}_2\hat{\varsigma_1}+\mathcal{O}_2\hat{\varsigma_2}.\end{aligned} \quad (32)$$

From the above Equations (31) and (32) it follows that

$$\begin{aligned}||(\psi,\varpi)-(\psi*,\varpi*)|| \leq & \frac{(\mathcal{O}_1+\mathcal{O}_2)\hat{\varsigma_1}+(\mathcal{O}_1+\mathcal{O}_2)\hat{\varsigma_2}}{1-((\mathcal{O}_1+\mathcal{O}_2)(\vartheta_1+\vartheta_2)+(\mathcal{O}_1+\mathcal{O}_2)(\hat{\vartheta}_1+\hat{\vartheta}_2))}, \\ \leq & \mathcal{V}_1\hat{\varsigma_1}+\mathcal{V}_2\hat{\varsigma_2},\end{aligned}$$

with

$$\mathcal{V}_1=\frac{(\mathcal{O}_1+\mathcal{O}_2)}{1-((\mathcal{O}_1+\mathcal{O}_2)(\vartheta_1+\vartheta_2)+(\mathcal{O}_1+\mathcal{O}_2)(\hat{\vartheta}_1+\hat{\vartheta}_2))},$$

$$\mathcal{V}_2=\frac{(\mathcal{O}_1+\mathcal{O}_2)}{1-((\mathcal{O}_1+\mathcal{O}_2)(\vartheta_1+\vartheta_2)+(\mathcal{O}_1+\mathcal{O}_2)(\hat{\vartheta}_1+\hat{\vartheta}_2))}.$$

Hence, the problem (1) is U-H stable. □

5. Example

Define $\psi_0 = \{\psi = (\psi_1, \psi_2, \psi_3 \cdots, \psi_n, \cdots) : \lim_{n\to\infty} \psi_n = 0\}$, it is obvious that ψ_0 is a Banach space with $||\psi||_\infty = \sup_{n\geq 1} |\psi_n|$.

Example 1. *Consider the following system:*

$$\begin{cases} {}^{\mathcal{H}}\mathcal{D}^{\mathcal{P}_1}\psi(\kappa) = \mathcal{G}_1(\kappa, \psi(\kappa), \varpi(\kappa)), \quad \kappa \in (1, e), \ \ \mathcal{P}_1 \in (1, 2], \\ {}^{\mathcal{H}}\mathcal{D}^{\mathcal{P}_2}\varpi(\kappa) = \mathcal{G}_2(\kappa, \psi(\kappa), \varpi(\kappa)), \\ \psi(1) = 0, \ \ \varepsilon_1 {}^{\mathcal{H}}\mathcal{I}^{\mathcal{Q}_1}\psi(\zeta) + \varepsilon_2\psi(e) = \varepsilon_3, \\ \varpi(1) = 0, \ \ \delta_1 {}^{\mathcal{H}}\mathcal{I}^{\mathcal{Q}_2}\varpi(\xi) + \delta_2\varpi(e) = \delta_3. \end{cases} \tag{33}$$

Here $\mathcal{P}_1 = \frac{97}{50}, \mathcal{P}_2 = \frac{41}{25}, \Delta_1 = 0.2387, \Delta_2 = 0.25357, \varepsilon_1 = \frac{1}{8}, \varepsilon_2 = \frac{3}{25}, \zeta = \frac{36}{25}, \xi = \frac{44}{25}, \mathcal{Q}_1 = \frac{9}{40}, \mathcal{Q}_2 = \frac{8}{40}$, *and*

$$\mathcal{G}_1(\kappa, \psi(\kappa), \varpi(\kappa)) = \left\{ \frac{1}{In\kappa + 10} \left(\frac{1}{4^n} + In(1 + |\psi_n| + |\varpi_n|) \right) \right\}, n \geq 1,$$
$$\mathcal{G}_2(\kappa, \psi(\kappa), \varpi(\kappa)) = \left\{ \frac{1}{10} \left(\frac{1}{n^4} + In(1 + |\psi_n| + |\varpi_n|) \right) \right\}, n \geq 1,$$

$\forall \kappa \in [1, 3]$ *with* $\{\psi_n\}_{n\geq 1}, \{\varpi_n\}_{n\geq 1} \in \psi_0$, *the hypothesis* $\mathcal{A}_2$ *of theorem 2 is verified. Also,*

$$\begin{aligned} ||\mathcal{G}_1(\kappa, \psi(\kappa), \varpi(\kappa))||_\infty &\leq \left\| \left\{ \frac{1}{In\kappa + 10} \left(\frac{1}{4^n} + In(1 + |\psi_n| + |\varpi_n|) \right) \right\} \right\|_\infty \\ &\leq \frac{1}{In\ \kappa + 10}(||\psi|| + 1) \\ &= \mathcal{U}_\psi(\kappa)\mathfrak{H}_\psi(||\psi||_\infty). \end{aligned}$$

Similarly,

$$\begin{aligned} ||\mathcal{G}_2(\kappa, \psi(\kappa), \varpi(\kappa))||_\infty &\leq \left\| \left\{ \frac{1}{10} \left(\frac{1}{n^4} + In(1 + |\psi_n| + |\varpi_n|) \right) \right\} \right\|_\infty \\ &\leq \frac{1}{10}(||\psi|| + 1) \\ &= \mathcal{U}_\varpi(\kappa)\mathfrak{H}_\varpi(||\varpi||_\infty). \end{aligned}$$

as a result, Theorem 2 condition $(\mathcal{A}_2)$ *is also verified.*

Next, by relying on the bounded subset $\mathcal{S} \subset \widehat{\mathcal{E}} \times \widehat{\mathcal{E}}$, *we get to*

$$\begin{aligned} \mathcal{K}(\psi(\kappa, \mathcal{S})) &\leq \mathcal{U}_\psi(\kappa)\mathcal{K}(\mathcal{S}), \\ \mathcal{K}(\varpi(\kappa, \mathcal{S})) &\leq \mathcal{U}_\varpi(\kappa)\mathcal{K}(\mathcal{S}), \end{aligned}$$

where in our case, we have $\mathcal{U}_\psi(\kappa) = \frac{1}{In\ \kappa + 9}$, $\mathcal{U}_\varpi(\kappa) = \frac{\kappa}{10}$; *the latter two inequalities show that the condition* $(\mathcal{A}_2)$ *of the Theorem 2 is satisfied.*

Finally, we calculate

$$\mathcal{U}_{\psi}^{*}(\kappa)=\frac{1}{10},\mathcal{O}_1\leq\frac{1}{\Gamma(\mathcal{P}_1+1)}+\frac{1}{|\Delta_1|}\left[\frac{|\varepsilon_1|(In\,\zeta)^{\mathcal{P}_1+\mathcal{Q}_1}}{\Gamma(\mathcal{P}_1+\mathcal{Q}_1+1)}+\frac{|\varepsilon_2|}{\Gamma(\mathcal{P}_1+1)}\right]=1.7961117138,$$

$$\mathcal{U}_{\varpi}^{*}(\kappa)=\frac{3}{10},\mathcal{O}_2\leq\frac{1}{\Gamma(\mathcal{P}_2+1)}+\frac{1}{|\Delta_2|}\left[\frac{|\delta_1|(In\,\xi)^{\mathcal{P}_2+\mathcal{Q}_2}}{\Gamma(\mathcal{P}_2+\mathcal{Q}_2+1)}+\frac{|\delta_2|}{\Gamma(\mathcal{P}_2+1)}\right]=1.363009035,$$

then, $\max\{\mathcal{O}_1\mathcal{U}_{\varpi}(\kappa),\mathcal{O}_2\mathcal{U}_{\varpi}(\kappa)\}=\max\{0.1796111,0.4089027\}=0.1796111<1$. *Thus, the Theorem 2 requirements are all satisfied, that is the Equation (33) has at least one solution* $(\psi,\varpi)\in\mathcal{C}([1,3],\psi_0)\times\mathcal{C}([1,3],\psi_0)$.

6. Conclusions

We have proved based on Mӧnch's fixed point theorem that there is a solution to the system of fractional differential equations. In addition to verifying the stability of the solutions for this system using the method of Ulam-Hyers. We concluded the work with an applied example that makes it easier for the reader to understand the theoretical results. For future work, Those interested in the field can also investigate the existence of these solutions for the studied system using new fractional derivatives such as Caputo-Hadmard, Katugambula, and ψ-Caputo.

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