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Backward and Non-Local Problems for the Rayleigh-Stokes Equation

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Abstract: This paper presents the method of separation of variables to find conditions on the right-hand side and on the initial data in the Rayleigh-Stokes problem, which ensure the existence and uniqueness of the solution. Further, in the Rayleigh-Stokes problem, instead of the initial condition, the non-local condition is considered: $u(x,T) = \beta u(x,0) + \varphi(x)$, where β is equal to zero or one. It is well known that if $\beta = 0$, then the corresponding problem, called the backward problem, is ill-posed in the sense of Hadamard , i.e., a small change in $u(x,T)$ leads to large changes in the initial data. Nevertheless, we will show that if we consider sufficiently smooth current information, then the solution exists, it is unique and stable. It will also be shown that if $\beta = 1$, then the corresponding non-local problem is well-posed and inequalities of coercive type are satisfied.

Keywords: the Rayleigh-Stokes problem; the backward problem; non-local problem; the method of separation of variables

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1. Introduction

The main object of study in this paper is the following Rayleigh-Stokes problem for a generalized second-grade fluid with a time-fractional derivative model:

$$\begin{cases} \partial_t u(x,t) - (1+\gamma\,\partial_t^\alpha)\Delta u(x,t) = f(x,t), \quad x\in\Omega, \quad 0<t\le T; \\ u(x,t) = 0, \quad x\in\partial\Omega, \quad 0<t\le T; \\ u(x,0) = \varphi(x), \quad x\in\Omega, \end{cases} \tag{1}$$

where $\gamma > 0$ is a fixed constant, the initial data φ and the source term $f(x,t)$ are given functions, $\partial_t = \partial/\partial t$, and ∂_t^α is the Riemann-Liouville fractional derivative of order $\alpha \in (0,1)$ defined by (see, e.g., [1]):

$$\partial_t^\alpha h(t) = \frac{d}{dt}\int_0^t \omega_{1-\alpha}(t-s)h(s)ds, \quad \omega_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}. \tag{2}$$

Here $\Gamma(\sigma)$ is the gamma function. Usually problem (1) is considered in the domain $\Omega \subset R^N$, $N = 1,2,3$, with a sufficiently smooth boundary $\partial\Omega$. Problem (1) is also called *the forward problem*.

Equation (1) when $\alpha = 1$ is also called the Haller equation. Such equations arise when describing the movement of soil moisture (see, for example, in [2], formulas (1.4) and (1.84) on p. 137 and 158, [3], formula (9.6.4) on p. 255, and [4], formula (2.6.1) on p. 59).

In recent years, much attention has been paid to the Rayleigh-Stokes problem (1) because of its importance for applications (see, e.g., [5,6]). This problem plays an important role in the study of the behavior of some non-Newtonian fluids. The fractional derivative ∂_t^α is used in equation (1) to describe the viscoelastic behavior of the flow (see e.g., [7,8]).

In order to get an idea of the behavior of the solution of this model, a number of authors have shown considerable interest in obtaining an exact solution in some special cases; see e.g., [7–9]. To find the exact solution to the problem, Shen et al. [8] used the sine Fourier transform and then applied the fractional Laplace transform. For the case of homogeneous initial and boundary conditions in a rectangular domain, Zhao and Yang [9] obtained exact solutions using the Fourier method. Note, here the eigenfunctions are written out explicitly. The authors of the above papers wrote out a formal solution, and the questions of the regularity of the solution were not specially studied.

Questions of the regularity of the solution were studied in the fundamental work Bazhlekova et al. [10]. The authors proved the Sobolev regularity of the homogeneous problem both for smooth and non-smooth initial data $\varphi(x)$, including $\varphi(x) \in L_2(\Omega)$.

The exact solutions obtained in the above papers include infinite series and special functions such as generalized Mittag-Leffler functions and are therefore inconvenient for numerical solution. In addition, explicit solutions are available only for a limited class of problem settings. Therefore, it is very important to develop efficient and maximally accurate numerical algorithms for the problem (1). Numerous works of specialists are devoted to this issue. An overview of the work in this direction is contained in the above work [10]. See also recent papers [11,12] and references therein.

Quite a lot of works are devoted to the study of the inverse problem of determining the right-hand side of the Raleigh-Stokes equation (see, e.g., [13–15] and the bibliography therein). However, the case when the right-hand side has the general form $f(x,t)$ has not yet been considered by anyone. In all known works, the right-hand side can be represented as $f(x,t) = h(t)g(x)$, where $h(t)$ is a given function, and $g(x)$ is a function to be determined. Since this inverse problem is an ill-posed problem in the sense of Hadamard, various regularization methods are proposed in the above papers. Also in these papers, the proposed regularized methods were tested by simple numerical experiments to check the error estimate between the desired solution and the regularized solution.

Note that such an inverse problem is ill-posed in the Hadamard sense also for the subdiffusion equation (see, e.g., [16–18]).

In the case when the initial condition $u(x,0) = \varphi(x)$ in problem (1) is replaced by $u(x,T) = \varphi(x)$, then the resulting problem is called *the backward problem*. The backward problem for the Rayleigh-Stokes equation is of great importance and it consists in determining the previous state of the physical field (for example, at t = 0) based on its current information (see, e.g., [19,20] and the bibliography therein). However, this problem (as well as the inverse problem of finding the right-hand side of the equation) is not stable. In other words, a small change in $u(x,T)$ leads to a large change in the original data. The authors of the above works proposed various regularization methods and tested these methods using numerical experiments.

Let us list the main findings of this paper.

(1) We will give a mathematically justified solution to the forward problem. A formal formula for the solution in the form of eigenfunction expansions was given in the paper [10], cited above (see also [15,19,20]), but the convergence of the series itself and the differentiated series has not been investigated.

(2) We will pay special attention to the backward problem, since in previous papers (see, e.g., [19,20]) the authors considered only the case $N \leq 3$. And this is connected with the method used in these works: if the dimension of the space is less than four, then for the eigenvalues λ_k of the Laplace operator with the Direchlet condition, the series $\sum_k \lambda_k^{-2}$ converges. In our reasoning, no conditions are imposed on the spectrum of the operator.

(3) It is well known that in order to choose the only solution of differential equations that simulates some phenomenon, the initial condition is used. However, there are also processes in which, instead of initial conditions, it is necessary to use non-local conditions, for example, an integral over time intervals, or a connection between the solution values at different points in time, for example, at the initial moment and at

the final moment of time. Of course, non-local conditions take into account some additional information and therefore they more accurately model some of the details of natural phenomena.

In the present paper we consider problem (1) with a non-local time condition:

$$u(x,T) = u(x,0) + \varphi(x). \tag{3}$$

In the case of classical diffusion equations (i.e., $\gamma = 0$), such a non-local problem has been studied by many researchers (see, for example, [21–23]). It should also be noted that various non-local boundary value problems for parabolic equations are reduced to a problem with condition (3) (see [24], Chapter 1). In the case when it is required that at the final time the temperature of the physical field differs from the initial temperature by the value φ, we come to the study of just such a non-local problem.

It turns out that the non-local problem is well-posed. In other words, a solution to a non-local problem exists and is unique. Moreover, the solution depends continuously on the function $\varphi(x)$ in the non-local condition.

Thus, the problem of determining the previous state of the physical field on the basis of current information is an ill-posed problem. However, if, without knowing the initial and present states, we want them to differ by the value $\varphi(x)$, then such a problem turns out to be correct.

The remainder of this paper is composed of seven sections and Conclusion. In Section 3, we give exact formulations of the problems under study. In the next section, we introduce the Hilbert space of "smooth" functions with the help of the degree of the elliptic operator under consideration, and recall some properties of function $B_\alpha(\lambda, t)$, introduced in [10]. Section 4 is devoted to the study of the main problem (1) with operator A instead of the Laplace operator. Here, conditions are given on the right-hand side of the equation and on the initial function, under which the solution of problem (1) is found in the form of eigenfunction expansions. In Section 5, we study the backward problem. As noted above, this problem is not well-posed according to Hadamard. Nevertheless, this section shows that if we consider $u(T)$ in the class of smooth functions, then stability is preserved. Section 6 studies the conditional stability of the backward problem. The next two sections are devoted to the study of problem (1) with the non-local time condition (3).

2. Problem Statements

The method of our research is the method of separation of variables. Therefore, only eigenfunctions and eigenvalues are needed from the Laplace operator in the Rayleigh-Stokes problem (1). With this in mind, instead of the Laplace operator $(-\Delta)$ in problem (1), we consider an abstract positive operator.

Thus, consider an arbitrary unbounded positive self-adjoint operator A in the Hilbert space H. Let A have a complete in H system of orthonormal eigenfunctions $\{v_k\}$ and a countable set of positive eigenvalues $\lambda_k : 0 < \lambda_1 \le \lambda_2 \cdots \to +\infty$. Let $(\cdot,\cdot)$ be the scalar product and $||\cdot||$ be the norm in H. For a vector-valued functions (or simply functions) $h : \mathbb{R}_+ \to H$, we define the Riemann-Liouville fractional derivative of order $0 < \alpha < 1$ in the same way as (2) (see, e.g., [25]). Finally, let $C((a,b);H)$ stand for a set of continuous in $t \in (a,b)$ functions $u(t)$ with values in H.

Consider the following Rayleigh-Stokes problem

$$\begin{cases} \partial_t u(t) + (1 + \gamma \partial_t^\alpha) A u(t) = f(t), \quad 0 < t \le T; \\ u(T) = \beta u(0) + \varphi, \end{cases} \tag{4}$$

where $\gamma > 0$ is a fixed constant, $\varphi \in H$ and $f(t) \in C((0,T];H)$ and β is equal to 0 or 1. If $\beta = 0$ then this problem is called *the backward problem* and if $\beta = 1$, then it is called *the non-local problem*. We will consider both of these cases.

In a recent paper [26], the authors considered the subdiffusion equation with the Caputo-Fabrizio derivative on a N-dimensional sphere with a non-local condition $\beta u(T) = u(0) + \varphi$.

The motives for studying non-local boundary values are given. Cases $\beta = 0$ and $\beta > 0$ are studied separately. The solution limit at $\beta \to 0$ is also studied.

In the present paper, various initial problems are considered. The solution of these problems is defined in a standard way: all vector functions and their derivatives included in the equation and in the initial conditions must be continuous in the variable t, and all the corresponding equalities should be understood as the equality of the vectors from H at each point t. As an example, we give the definition of a solution to problem (4).

Definition 1. *We call a function $u(t) \in C([0,T];H)$ the solution of the non-local Rayleigh-Stokes problem (4), if $\partial_t u(t), \partial_t^\alpha Au(t) \in C((0,T);H)$ and it satisfies conditions (4).*

To solve problem (4) in the case when $\beta = 1$, we divide it into two auxiliary problems:

$$\begin{cases} \partial_t v(t) + (1+\gamma\,\partial_t^\alpha)Av(t) = f(t), \quad 0 < t \le T; \\ v(0) = 0, \end{cases} \tag{5}$$

and

$$\begin{cases} \partial_t w(t) + (1+\gamma\,\partial_t^\alpha)Aw(t) = 0, \quad 0 < t \le T; \\ w(T) = w(0) + \psi, \end{cases} \tag{6}$$

where $\psi \in H$ is a given function.

Problems (5) and (6) are a special case of problem (4), so solutions to problems (5) and (6) are defined similarly to Definition 1.

If $\psi = \varphi - v(\xi)$ and $v(t)$ and $w(t)$ are solutions of the formulated problems, then it is easy to check that the solution to problem (4) has the form $u(t) = v(t) + w(t)$. Therefore, it suffices to consider auxiliary problems.

Remark 1. *We note that, as operator A one can take, for example, the Laplace operator with the Dirichlet condition in an arbitrary N (not only ≤ 3)—dimensional domain with a sufficiently smooth boundary. For our reasoning, it is sufficient that operator A has the properties listed above.*

In [27,28], a similar non-local problem with an arbitrary parameter β was studied in detail for subdiffusion equations with Riemann-Liouville and Caputo derivatives. For the classical diffusion equation with the parameter $\beta = 1$, it was first considered in [21–23].

3. Preliminaries

In this section, using the degree of operator A, we introduce the Hilbert space and recall the main properties of function $B_\alpha(\lambda, t)$ introduced in [10] (see also [19]), which we will use further.

Let τ be an arbitrary real number. We introduce the power of operator A, acting in H according to the rule (note, operator A is positive and therefore $\lambda_k > 0$ for all k)

$$A^\tau h = \sum_{k=1}^{\infty} \lambda_k^\tau h_k v_k.$$

Here and everywhere below, for the vector $h \in H$, the symbol h_k will denote the Fourier coefficients of this vector: $h_k = (h, v_k)$. The domain of definition of this operator is determined from condition $A^\tau h \in H$ and has the form:

$$D(A^\tau) = \{h \in H : \sum_{k=1}^{\infty} \lambda_k^{2\tau} |h_k|^2 < \infty\}.$$

For elements of $D(A^\tau)$ we introduce the norm

$$||h||_\tau^2 = \sum_{k=1}^{\infty} \lambda_k^{2\tau} |h_k|^2 = ||A^\tau h||^2.$$

With this norm, the linear-vector space $D(A^\tau)$ becomes a Hilbert space.

Let $B_\alpha(\lambda,t)$ be a solution of the following Cauchy problem

$$Ly(t) \equiv y'(t) + \lambda(1+\gamma\partial_t^\alpha)y(t) = 0,\ t>0,\ \lambda>0,\ y(0)=1. \tag{7}$$

The solution of such an equation is expressed in terms of the generalized Wright function (see e.g., A.A. Kilbas et. al [1], Example 5.3, p. 289). But function $B_\alpha(\lambda,t)$, the solution of this Cauchy problem, is studied in detail in Bazhlekova, Jin, Lazarov, and Zhou [10]. See also Luc, Tuan, Kirane, Thanh [19], where very important lower bounds are obtained.

We also note the work of Pskhu [29], where a more general Cauchy problem than (7) was studied. From the results of this paper one can obtain a representation for the solution of the Cauchy problem (7), which is very convenient for further research.

The authors of [10], in particular, proved the following lemma.

Lemma 1. *Let $B_\alpha(\lambda,t)$ be a solution of the Cauchy problem (7). Then*

1. $B_\alpha(\lambda,0)=1,\ 0<B_\alpha(\lambda,t)<1,\ t>0,$
2. $\partial_t B_\alpha(\lambda,t)<0,\ t\ge 0,$
3. $\lambda B_\alpha(\lambda,t) < C\min\{t^{-1}, t^{\alpha-1}\},\ t>0,$
4. $\int\limits_0^T B_\alpha(\lambda,t)dt \le \frac{1}{\lambda},\ T>0.$

It should only be noted that in paper [10], instead of assertion 2, a more general proposition was proved. But the assertion 2 in our lemma easily follows from the representation of function $B_\alpha(\lambda,t)$, also obtained in [10]:

$$B_\alpha(\lambda,t) = \int\limits_0^\infty e^{-rt} b_\alpha(\lambda,r)dr, \tag{8}$$

where

$$b_\alpha(\lambda,r) = \frac{\gamma}{\pi}\frac{\lambda r^\alpha \sin\alpha\pi}{(-r+\lambda\gamma r^\alpha\cos\alpha\pi+\lambda)^2 + (\lambda\gamma r^\alpha \sin\alpha\pi)^2}.$$

The following assertion is also implicitly contained in [10]. But in view of its importance in our reasoning, we present a proof.

Lemma 2. *The Cauchy problem*

$$y'(t) + \lambda(1+\gamma\partial_t^\alpha)y(t) = f(t),\ t>0,\ \lambda>0,\ y(0)=0, \tag{9}$$

has the only solution

$$y(t) = \int\limits_0^t B_\alpha(\lambda, t-\tau) f(\tau)d\tau. \tag{10}$$

Proof. Since $B_\alpha(\lambda,0)=1$, then

$$\partial_t y(t) = f(t) + \int\limits_0^t \partial_t B_\alpha(\lambda,t-\tau)f(\tau)d\tau.$$

Next we have

$$\partial_t^\alpha y(t) = \frac{1}{\Gamma(1-\alpha)}\partial_t \int\limits_0^t (t-\tau)^{-\alpha} \int\limits_0^\tau B_\alpha(\lambda,\tau-\xi)f(\xi)d\xi d\tau$$

(change the order of integration)

$$= \frac{1}{\Gamma(1-\alpha)}\partial_t \int_0^t f(\xi) \int_\xi^t (t-\tau)^{-\alpha} B_\alpha(\lambda, \tau-\xi) d\tau d\xi$$

(change of variables: $\tau - \xi = \eta$, $\tau = \xi + \eta$, $d\tau = d\eta$)

$$= \frac{1}{\Gamma(1-\alpha)}\partial_t \int_0^t f(\xi) \int_0^{t-\xi} (t-(\xi+\eta))^{-\alpha} B_\alpha(\lambda, \eta) d\eta d\xi$$

$$= \int_0^t f(\xi) \frac{1}{\Gamma(1-\alpha)}\partial_t \int_0^{t-\xi} (t-(\xi+\eta))^{-\alpha} B_\alpha(\lambda, \eta) d\eta d\xi = \int_0^t f(\xi) \partial_t^\alpha B_\alpha(\lambda, t-\xi) d\xi.$$

Thus, given that $B_\alpha(\lambda, t)$ is a solution to the Cauchy problem (7), we obtain

$$Ly(t) = f(t) + \int_0^t f(\xi) L B_\alpha(\lambda, \cdot - \xi) d\xi = f(t).$$

□

Corollary 1. *The Cauchy problem*

$$y'(t) + \lambda(1 + \gamma\partial_t^\alpha) y(t) = f(t),\ t > 0,\ \lambda > 0,\ y(0) = y_0, \tag{11}$$

has the only solution

$$y(t) = y_0 B_\alpha(\lambda, t) + \int_0^t B_\alpha(\lambda, t-\tau) f(\tau) d\tau. \tag{12}$$

We also need the following properties of function $B_\alpha(\lambda, t)$.

Lemma 3. *There is a constant $C > 0$, such that*

$$|\partial_t B_\alpha(\lambda, t)| \le \frac{C}{\lambda\, t^{2-\alpha}},\ t > 0.$$

Proof. From (8) we have

$$\partial_t B_\alpha(\lambda, t) = -\int_0^\infty r e^{-rt} b_\alpha(\lambda, r) dr.$$

Therefore, by the definition of b_α,

$$|\partial_t B_\alpha(\lambda, t)| \le \frac{\gamma}{\pi} \int_0^\infty \frac{\lambda r^\alpha \sin\alpha\pi}{(\lambda\gamma r^\alpha \sin\alpha\pi)^2} r e^{-rt} dr = \frac{1}{\gamma\pi\lambda \sin\alpha\pi} \int_0^\infty r^{1-\alpha} e^{-rt} dr$$

(change of variables: $\tau = rt$, $d\tau = t dr$)

$$= \frac{t^{\alpha-2}}{\gamma\pi\lambda\sin\alpha\pi} \int_0^\infty \tau^{1-\alpha} e^{-\tau} d\tau = \frac{t^{\alpha-2}}{\gamma\pi\lambda\sin\alpha\pi}\Gamma(2-\alpha) = \frac{C}{\lambda\, t^{2-\alpha}}.$$

□

One can also obtain an estimate for $\partial_t B_\alpha$ with a smaller singularity at $t = 0$.

Lemma 4. *There is a constant $C > 0$, such that*

$$|\partial_t B_\alpha(\lambda, t)| \le C\frac{\lambda}{t^\alpha}, \ t > 0.$$

Proof. Again by the definition of b_α one has

$$|\partial_t B_\alpha(\lambda, t)| \le \frac{\gamma}{\pi}\int_0^\infty \frac{\lambda r^\alpha \sin\alpha\pi}{r^2} re^{-rt}dr = \frac{\lambda\,\gamma\,\Gamma(\alpha)\,\sin\alpha\pi}{\pi t^\alpha}.$$

□

Now, by combining these two estimates, we can obtain the following statement, which we also apply further.

Lemma 5. *There is a constant $C > 0$, such that for any ε, $0 \le \varepsilon \le 1$, we have*

$$|\partial_t B_\alpha(\lambda, t)| \le \frac{C\lambda^\varepsilon}{t^{1-\varepsilon(1-\alpha)}}, \ t > 0.$$

Proof. Let $0 \le \varepsilon \le 1$. Apply Lemmas 3 and 4 to get

$$|\partial_t B_\alpha(\lambda, t)| \le C\left(\frac{1}{\lambda t^{2-\alpha}}\right)^{\frac{1-\varepsilon}{2}} \cdot \left(\frac{\lambda}{t^\alpha}\right)^{\frac{1+\varepsilon}{2}} = \frac{C\lambda^\varepsilon}{t^{1-\varepsilon(1-\alpha)}}, \ t > 0.$$

□

In the future, instead of λ, we will have eigenvalues λ_k of operator A. The next lower bound for $B_\alpha(\lambda_k, t)$ was obtained in Luc, N.H., Tuan, N.H., Kirane, M., Thanh, D.D.X [19]. For the convenience of the reader, we present this proof.

Lemma 6. *The following estimate holds for all $t \in [0, T]$ and $k \ge 1$:*

$$B_\alpha(\lambda_k, t) \ge \frac{C(\alpha, \gamma, \lambda_1)}{\lambda_k},$$

where

$$C(\alpha, \gamma, \lambda_1) = \frac{\gamma\sin\alpha\pi}{4}\int_0^\infty \frac{r^\alpha e^{-rT}}{\frac{r^2}{\lambda_1^2} + \gamma^2 r^{2\alpha} + 1}dr.$$

Proof. Since the modulo of trigonometric functions does not exceed one and $(a + b + c)^2 \le 3(a^2 + b^2 + c^2)$, then

$$(-r + \lambda_k\gamma r^\alpha \cos\alpha\pi + \lambda_k)^2 + (\lambda_k\gamma r^\alpha \sin\alpha\pi)^2 \le 3(r^2 + \lambda_k^2\gamma^2 r^{2\alpha} + \lambda_k^2) + \lambda_k^2\gamma^2 r^{2\alpha}$$

$$\le 4\lambda_k^2(\frac{r^2}{\lambda_1^2} + \gamma^2 r^{2\alpha} + 1).$$

Therefore, (8) implies

$$B_\alpha(\lambda_k, t) \ge \frac{\gamma\sin\alpha\pi}{4\lambda_k}\int_0^\infty \frac{r^\alpha e^{-rT}}{\frac{r^2}{\lambda_1^2} + \gamma^2 r^{2\alpha} + 1}dr.$$

It should be noted that the improper integral converges. Indeed,

$$\int_0^\infty \frac{r^\alpha e^{-rT}}{\frac{r^2}{\lambda_1^2}+\gamma^2 r^{2\alpha}+1}dr \le \int_0^\infty r^\alpha e^{-rT}dr = T^{-(\alpha+1)}\int_0^\infty \tau^\alpha e^{-\tau}d\tau = T^{-(\alpha+1)}\Gamma(\alpha+1).$$

□

4. Forward Problem

Consider the following Cauchy problem:

$$\begin{cases} \partial_t v(t) + (1+\gamma\,\partial_t^\alpha)Av(t) = f(t), \quad 0 < t \le T; \\ v(0) = \varphi, \end{cases} \tag{13}$$

where φ is a given vector of H. We also call this problem *the forward problem.*

Theorem 1. *Let $\varepsilon \in (0,1)$ and $f(t) \in C([0,T]; D(A^\varepsilon))$. Then for any $\varphi \in H$ problem (13) has a unique solution*

$$v(t) = \sum_{k=1}^\infty B_\alpha(\lambda_k, t)\varphi_k v_k + \sum_{k=1}^\infty \left[\int_0^t B_\alpha(\lambda_k, t-\tau) f_k(\tau)d\tau\right] v_k. \tag{14}$$

The estimate of the coercive type is valid with some constants C and $C_\varepsilon > 0$:

$$||\partial_t v(t)||^2 + ||\partial_t^\alpha Av(t)||^2 \le Ct^{-2}||\varphi||^2 + C_\varepsilon \max_{t\in[0,T]} ||f||_\varepsilon^2, \quad 0 < t \le T. \tag{15}$$

Moreover, the estimate

$$||v(T)||_1^2 \le C_T\big(||\varphi||^2 + \max_{t\in[0,T]} ||f||^2\big), \tag{16}$$

is valid, where $C_T > 0$ is a constant depending on T.

As noted in the introduction, the Cauchy problem (13) in the case when A is the Laplace operator in the N-dimensional domain ($N = 1,2,3$) has been considered by many authors. So the authors of [10] establish the Sobolev regularity of the homogeneous problem for both smooth and non smooth initial data φ. In particular, the authors proved an estimate similar to (15) (see estimate (2.16), note that in this paper $f(t) \equiv 0$).

Formula (14) for solving problem (13) is formally given in the same article of Bazhlekova, Jin, Lazarov, and Zhou [10], but since this paper is devoted to the study of a homogeneous problem, the question for which φ and $f(t)$ it gives a solution to problem (13) is not considered. Note that functions φ and $f(t)$ must be such that, for example, the series (14) allows term-by-term differentiation with respect to the variable t.

The formula is also contained in papers [15], Equation (2.2), [19], Equation (2.6) and [20], Equation (8). Again, since these works are devoted to the study of other problems, the above issues were not considered.

Proof. According to the method of separation of variables, we expand the functions φ and $f(t)$ in terms of the system of eigenfunctions $\{v_k\}$ with coefficients φ_k and $f_k(t) = (f(t), v_k)$ correspondingly. The solution of problem (5) will be sought in the form of a series $\sum_k T_k(t)v_k$ with unknown coefficients $T_k(t)$.

It is easy to verify that functions $T_k(t)$ are solutions of the Cauchy problem (11) with the right-hand side $f(t) = f_k(t)$, the initial condition $y_0 = \varphi_k$ and with the parameter $\lambda = \lambda_k$. Therefore (see (10))

$$T_k(t) = B_\alpha(\lambda_k, t)\,\varphi_k + \int_0^t B_\alpha(\lambda_k, t-\tau) f_k(\tau)d\tau.$$

Thus, series (14) is a formal solution to problem (13). It remains to show that the series itself and the series obtained after differentiation also converge. Let us get started with this task.

By Parseval's equality, the first assertion of the Lemma 1, and Hölder's inequality, we have

$$\left|\left|\sum_{k=1}^{j}\left[\int_{0}^{t} B_\alpha(\lambda_k, t-\tau) f_k(\tau)d\tau\right] v_k\right|\right|^2 = \sum_{k=1}^{j}\left[\int_{0}^{t} B_\alpha(\lambda_k, t-\tau) f_k(\tau)d\tau\right]^2$$

$$\leq \sum_{k=1}^{j}\left[\int_{0}^{t} |f_k(\tau)|d\tau\right]^2 \leq C \max_{0\leq t\leq T} ||f(t)||^2.$$

By the same way,

$$\left|\left|\sum_{k=1}^{j} B_\alpha(\lambda_k, t)\varphi_k v_k\right|\right|^2 \leq C||\varphi||^2.$$

Now consider the series after differentiation. Apply Parseval's equality to obtain

$$\left|\left|\sum_{k=1}^{j}\left[\partial_t\int_{0}^{t} B_\alpha(\lambda_k, t-\tau) f_k(\tau)d\tau\right] v_k\right|\right|^2 = \left|\left|\sum_{k=1}^{j}\left[f_k(t) - \int_{0}^{t} \partial_t B_\alpha(\lambda_k, t-\tau) f_k(\tau)d\tau\right] v_k\right|\right|^2$$

$$\leq 2 \max_{0\leq t\leq T} ||f(t)||^2 + 2\sum_{k=1}^{j}\left[\int_{0}^{t} \partial_t B_\alpha(\lambda_k, t-\tau) f_k(\tau)d\tau\right]^2.$$

Generalized Minkowski inequality and Lemma 5 imply

$$\sum_{k=1}^{j}\left[\int_{0}^{t} \partial_t B_\alpha(\lambda_k, t-\tau) f_k(\tau)d\tau\right]^2 \leq C\left(\int_{0}^{t} \tau^{\varepsilon(1-\alpha)-1}\left(\sum_{k=1}^{j}|\lambda_k^{\varepsilon} f_k(\tau)|^2\right)^{\frac{1}{2}} d\tau\right)^2$$

$$\leq C_\varepsilon \max_{t\in[0,T]} ||f(t)||_\varepsilon.$$

Similarly, apply Parseval's equality and Lemma 5 with $\varepsilon = 0$ to get

$$\left|\left|\sum_{k=1}^{j} \partial_t B_\alpha(\lambda_k, t)\varphi_k v_k\right|\right|^2 \leq \frac{C}{t^2}||\varphi||^2.$$

Equation (5) implies

$$||(1+\gamma\partial_t^\alpha)Av||^2 \leq ||\partial_t v||^2 + ||f(t)||^2.$$

Therefore, it follows from the above

$$\left|\left|(1+\gamma\partial_t^\alpha)A\sum_{k=1}^{j}\left[\int_{0}^{t} B_\alpha(\lambda_k, t-\tau) f_k(\tau)d\tau\right] v_k\right|\right|^2 \leq C_\varepsilon \max_{t\in[0,T]} ||f(t)||_\varepsilon,$$

and

$$\left|\left|(1+\gamma\partial_t^\alpha)A\sum_{k=1}^{j} B_\alpha(\lambda_k, t)\varphi_k v_k\right|\right|^2 \leq C(\frac{1}{t^2}||\varphi|| + ||f(t)||).$$

Note that $||f(t)|| \leq ||f(t)||_\varepsilon$, $\varepsilon > 0$.

Thus, it has been shown that the function defined by the series (14) is indeed a solution to problem (5).

Obviously, (15) follows from the established estimates.

As for the uniqueness of the solution of problem (5), it follows from completeness of the set of eigenfunctions $\{v_k\}$ in H. Indeed, suppose the problem has two solutions v^1 and v^2. Then the difference $v = v^1 - v^2$ is a solution of the homogeneous problem:

$$\begin{cases} \partial_t v(t) + (1+\gamma\,\partial_t^\alpha)Av(t) = 0, \quad 0 < t \le T; \\ v(0) = 0. \end{cases}$$

Let $v(t)$ be any solution of this problem. Consider the Fourier coefficients $T_k(t) = (v(t), v_k)$. It is not hard to see, that T_k is a solution of the Cauchy problem

$$\partial_t T_k(t) + \lambda_k(1+\gamma\,\partial_t^\alpha)T_k(t) = 0, \quad 0 < t \le T,\ T_k(0) = 0.$$

Corollary 1 implies that $T_k(t) \equiv 0$ for all k. Since the set of eigenfunctions $\{v_k\}$ complete in H, then $v(t) \equiv 0$.

It remains to prove estimate (16). Let $S_j(t)$ be a partial sum of the series (14). Then

$$||AS_j(T)|| \le \Bigg|\Bigg|\sum_{k=1}^{j} \lambda_k B_\alpha(\lambda_k, T)\varphi_k v_k\Bigg|\Bigg| + \Bigg|\Bigg|\sum_{k=1}^{j}\left[\int_0^T \lambda_k B_\alpha(\lambda_k, t-\tau) f_k(\tau)d\tau\right] v_k\Bigg|\Bigg|.$$

Apply Parseval's equality and estimate 3 of Lemma 1 to obtain

$$\Bigg|\Bigg|\sum_{k=1}^{j} \lambda_k B_\alpha(\lambda_k, T)\varphi_k v_k\Bigg|\Bigg|^2 \le C_T \sum_{k=1}^{j} |\varphi|^2 \le C_T||\varphi||^2.$$

Similarly,

$$\Bigg|\Bigg|\sum_{k=1}^{j}\left[\int_0^T \lambda_k B_\alpha(\lambda_k, t-\tau) f_k(\tau)d\tau\right] v_k\Bigg|\Bigg|^2 \le C_T \sum_{k=1}^{j}\left[\int_0^T \tau^{-\alpha} f_k(\tau)d\tau\right]^2$$

(apply generalized Minkowski inequality)

$$\le C_T\left(\int_0^T \tau^{-\alpha}\left(\sum_{k=1}^{j}|f_k(\tau)|^2\right)^{1/2} d\tau\right)^2 \le C_T \max_{t\in[0,T]} ||f(t)||^2. \tag{17}$$

The last two estimates imply (16).

The theorem is completely proven. □

5. Backward Problem

In the present paragraph we consider the following *backward* problem:

$$\begin{cases} \partial_t \omega(t) + (1+\gamma\,\partial_t^\alpha)A\omega(t) = f(t), \quad 0 < t < T; \\ \omega(T) = \psi, \end{cases} \tag{18}$$

where ψ is a given vector of H. It should be specially emphasized that for the backward problem the parameter γ in the Rayleigh-Stokes equation plays an important role. If this parameter is equal to zero, then we get the classical diffusion equation, for which the backward problem is strongly ill-posed: not for any $\psi \in H$ (even not for any $\psi \in D(A^k)$, $k \ge 1$) there is a solution, and if it exists, then a small change in ψ leads to a very strong change in the solution (see e.g., Chapter 8.2 of [30]). In the next Theorem 2, we show that the parameter $\gamma \ne 0$ "ennobles" the backward problem for the Rayleigh-Stokes equation and its solution already exists for all $\psi \in D(A)$.

Nevertheless the solution to problem (18) is also unstable, as is the solution to the classical diffusion equation (i.e., $\gamma = 0$). Indeed, let $f(t) \equiv 0$ and take

$$\omega(0) = \lambda_k^{-1+\varepsilon} \cdot \frac{1}{B_\alpha(\lambda_k, T)} \cdot v_k, \ \varepsilon > 0,$$

as a initial condition in problem (13). Then the unique solution to problem (13) is (see Theorem 13)

$$\omega(t) = \lambda_k^{-1+\varepsilon} \cdot \frac{B_\alpha(\lambda_k, t)}{B_\alpha(\lambda_k, T)} \cdot v_k.$$

This function is a unique solution of the backward problem (18) with $u(T) = \lambda_k^{-1+\varepsilon} v_k$. Therefore, on the one hand, $||u(T)|| = \lambda_k^{-1+\varepsilon}$ and it tends to zero as $k \to \infty$ (even $||u(T)||_a \to 0$ for any $a < 1 - \varepsilon$), and on the other hand, according to Lemma 6,

$$||\omega(0)|| \geq \lambda_k^{-1+\varepsilon} \cdot \frac{\lambda_k}{C(\alpha, \gamma, \lambda_1)} \to \infty \quad \text{when} \quad k \to \infty.$$

However, the situation changes if we consider the norm of $u(T)$ in the space $D(A)$; note that norm $||u(T)||_1 = \lambda_k^\varepsilon$ is unbounded as $k \to \infty$.

Theorem 2. *Let $\varepsilon \in (0, 1)$ and $f(t) \in C([0, T]; D(A^\varepsilon))$. Then for any $\psi \in D(A)$ problem (18) has a unique solution. Moreover there exists a constant $C = C(\alpha, \gamma, \lambda_1, T) > 0$, depending on $\alpha, \gamma, \lambda_1$ and T, such that*

$$||\omega(t)|| \leq C \left(||\omega(T)||_1 + \max_{t \in [0,T]} |f(t)|| \right). \tag{19}$$

Let $f(t) \equiv 0$. Then there exist constants $C_1, C_2 > 0$, such that

$$C_1 ||\omega(0)|| \leq ||\omega(T)||_1 \leq C_2 ||\omega(0)||. \tag{20}$$

Note that in work [31] the backward problem for the subdiffusion equation $\partial_t^\alpha u + Au = f$ was studied. It is shown that an estimate similar to (20) is valid only with the norm $||\omega(T)||_2$. However, if we take the derivative in the sense of Caputo in the subdiffusion equation, then the estimate (20) with the norm $||\omega(T)||_1$ is valid (see ([32])).

Proof. Let us take a solution (14) with an unknown initial function φ and use condition $\omega(T) = \psi$ to determine this unknown function. Then the Fourier coefficients of φ has the form

$$\varphi_k = \frac{1}{B_\alpha(\lambda_k, T)} \left(\psi_k - \int_0^T B_\alpha(\lambda_k, t - \tau) f_k(\tau) d\tau \right), \ k \geq 1, \tag{21}$$

and the unique formal solution of the backward problem can be written as

$$\omega(t) = \sum_{k=1}^\infty \frac{B_\alpha(\lambda_k, t)}{B_\alpha(\lambda_k, T)} \left(\psi_k - \int_0^T B_\alpha(\lambda_k, t - \tau) f_k(\tau) d\tau \right) v_k + \sum_{k=1}^\infty \left[\int_0^t B_\alpha(\lambda_k, t - \tau) f_k(\tau) d\tau \right] v_k.$$

According to Theorem 1, this formal solution will indeed be a unique solution to the backward problem if the condition $\varphi \in H$ is satisfied. Let us check this condition.

Apply Lemma 6 to obtain

$$\sum_{k=1}^j \left| \frac{\psi_k}{B_\alpha(\lambda_k, T)} \right|^2 \leq \frac{1}{C(\alpha, \gamma, \lambda_1)} \sum_{k=1}^j |\lambda_k \psi_k|^2 \leq \frac{1}{C(\alpha, \gamma, \lambda_1)} ||\psi||_1^2.$$

Again this lemma and estimate 3 of Lemma 1 imply (see (17))

$$\sum_{k=1}^{j}\left|\frac{1}{B_\alpha(\lambda_k,T)}\right|^2\left(\int_0^T B_\alpha(\lambda_k,t-\tau)f_k(\tau)d\tau\right)^2 \le \frac{C_T}{C(\alpha,\gamma,\lambda_1)}\max_{t\in[0,T]}||f(t)||^2.$$

Thus $\varphi = \sum_k \varphi_k v_k \in H$ and therefore, function $\omega(t)$ defined above is indeed a solution to the backward problem ((13).

Let us pass to the proof of estimate (19). We note right away that in order to estimate the norm in H, by virtue of the Parseval equality, it is sufficient to estimate a series of Fourier coefficients.

Application of Lemmas 1 and 6 gives

$$\sum_{k=1}^{j}\left|\frac{B_\alpha(\lambda_k,t)}{B_\alpha(\lambda_k,T)}\psi_k\right|^2 \le \frac{1}{C(\alpha,\gamma,\lambda_1)}\sum_{k=1}^{j}|\lambda_k\psi_k|^2 \le \frac{1}{C(\alpha,\gamma,\lambda_1)}||\psi||^2.$$

Apply Lemma 6 for B_α in the denominator, estimate (1) of Lemma 1 for B_α in the numerator, and finally estimate (3) of Lemma 1 for B_α under the integral sign (see (17)). Then

$$\sum_{k=1}^{j}\left|\frac{B_\alpha(\lambda_k,t)}{B_\alpha(\lambda_k,T)}\right|^2\left(\int_0^T B_\alpha(\lambda_k,t-\tau)f_k(\tau)d\tau\right)^2 \le \frac{C_T}{C(\alpha,\gamma,\lambda_1)}\max_{t\in[0,T]}||f(t)||^2.$$

Similarly, by virtue of estimate (3) of Lemma 1 we obtain

$$\sum_{k=1}^{j}\left[\int_0^t B_\alpha(\lambda_k,t-\tau)f_k(\tau)d\tau\right]^2 \le C_T\max_{t\in[0,T]}||f(t)||^2.$$

As for estimate (20), its left part follows from the just proved estimate (19), while the right part is contained in (16). □

As noted in the introduction, statements similar to Theorem 2 were known earlier (see, e.g., [19,20] and the bibliography therein). In these papers, operator A is the Laplace operator with the Dirichlet condition in the domain $\Omega \subset R^N$, $N \le 3$.

Let us make a few remarks:

1. The left side of the estimate (20) means that a small change of $\omega(T)$ in the norm $||\cdot||_1$ entails a small change in the norm $||\cdot||$ of the initial data.
2. The right side of this estimate asserts the unimprovability of the left side, i.e., it is impossible to replace the norm $||\cdot||_1$ with the norm $||\cdot||_{1-\varepsilon}$ with any $\varepsilon > 0$.
3. For the classical diffusion equation (i.e., $\gamma = 0$), the statement of Theorem 2 naturally does not hold, since $C(\alpha, 0, \lambda_1) = 0$ (see the definition of this constant in Lemma 6). It should also be noted that if we consider the backward problem for the classical diffusion equation, then for its solution estimates of the type (20) on the scales of spaces $D(A^a)$ are generally impossible (see, e.g., Chapter 8.2 of [30]).

6. Conditional Stability

Following work Luc, Huynh, O'Regan and Can [20], in this section we consider the problem of conditional stability of the backward problem. In other words, suppose that the initial data $\omega(0) = \varphi$ satisfies the following a priori estimate:

$$||\varphi||_\varepsilon^2 = \sum_{k=1}^{\infty}\lambda_k^{2\varepsilon}|\varphi_k|^2 \le \Phi_0^2, \tag{22}$$

where ε and Φ_0 are positive constants, and consider a class of functions that satisfy this condition.

So what does conditional stability mean? We saw above that the solution to problem (18) is not stable, i.e., a small change in $||\omega(T)||$ leads to a large change in the original data $||\omega(0)||$. If for some class of functions $\omega(0)$ (see (22)) stability takes place, then the problem is called conditionally stable.

The following statement is true:

Theorem 3. *Let $\varphi \in D(A^\varepsilon)$ satisfy condition (22). Then there is a constant C depending on $\alpha, \gamma, \lambda_1$ and T such, that*

$$||\varphi|| \leq C\big[||\psi|| + \max_{0\leq t\leq T} ||f(t)||\big]^{\frac{\varepsilon}{1+\varepsilon}} \Phi_0^{\frac{1}{1+\varepsilon}}. \tag{23}$$

This theorem for the case of the Laplace operator with the Dirichlet condition in N, $N \leq 3$,—dimensional domain was proved in the above paper [20] (note that the condition of the theorem in [20] is slightly different). Note that in this case the estimates $\lambda_k \geq Ck^{\frac{2}{N}}$ hold for the eigenvalues of the Laplace operator, and the proof in [20] is based on the convergence of the series

$$\sum_{k=1}^{\infty} \frac{1}{\lambda_k^2} \leq C \sum_{k=1}^{\infty} \frac{1}{k^{\frac{4}{N}}} < \infty.$$

The proof of Theorem 3 repeats the proof of [20] with a slight change. For the convenience of the reader, we present this proof.

We emphasize that the assertion of Theorem 3 is valid without any conditions on the spectrum of operator A.

Proof. Set (see (21))

$$\Phi_k = \psi_k - \int_0^T B_\alpha(\lambda_k, t-\tau) f_k(\tau) d\tau, \; k \geq 1.$$

Then

$$||\varphi||^2 = \sum_{k=1}^{\infty} \left| \frac{\Phi_k}{B_\alpha(\lambda_k, t)} \right|^2 = \sum_{k=1}^{\infty} \frac{|\Phi_k|^{\frac{2\varepsilon}{1+\varepsilon}} |\Phi_k|^{\frac{2}{1+\varepsilon}}}{|B_\alpha(\lambda_k, t)|^2}.$$

Apply the Hölder inequality with parametrs $p = \frac{1+\varepsilon}{\varepsilon}$ and $q = 1+\varepsilon$ to obtain

$$||\varphi||^2 \leq \left(\sum_{k=1}^{\infty} |\Phi_k|^2 \right)^{\frac{\varepsilon}{1+\varepsilon}} \left(\sum_{k=1}^{\infty} \frac{1}{|B_\alpha(\lambda_k, t)|^{2\varepsilon}} \left| \frac{\Phi_k}{B_\alpha(\lambda_k, t)} \right|^2 \right)^{\frac{1}{1+\varepsilon}}.$$

We use Lemma 6 to get

$$\sum_{k=1}^{\infty} \frac{1}{|B_\alpha(\lambda_k, t)|^{2\varepsilon}} \left| \frac{\Phi_k}{B_\alpha(\lambda_k, t)} \right|^2 \leq \frac{1}{C^{2\varepsilon}(\alpha, \gamma, \lambda_1)} \sum_{k=1}^{\infty} |\lambda_k^\varepsilon \varphi_k|^2 \leq \frac{\Phi_0^2}{C^{2\varepsilon}(\alpha, \gamma, \lambda_1)}.$$

Finally estimate 1 of Lemma 1 and the Hölder inequality give

$$\sum_{k=1}^{\infty} |\Phi_k|^2 \leq 2||\psi||^2 + 2\sum_{k=1}^{\infty} \left[\int_0^T |f_k(\tau)| d\tau \right]^2 \leq 2||\psi||^2 + 2T \int_0^T \sum_{k=1}^{\infty} |f_k(\tau)|^2 d\tau$$

$$\leq 2||\psi||^2 + 2T \max_{0\leq t\leq T} ||f(t)||^2.$$

The last two estimates imply estimate (22). □

7. Auxiliary Problem (5)

If we set $\varphi = 0$, then from Theorem 1 we have the following result for our auxiliary problem (5).

Theorem 4. *Let $\varepsilon \in (0,1)$ and $f(t) \in C([0,T]; D(A^\varepsilon))$. Then problem (5) has a unique solution*

$$v(t) = \sum_{k=1}^{\infty} \left[\int_0^t B_\alpha(\lambda_k, t-\tau) f_k(\tau) d\tau \right] v_k. \tag{24}$$

Moreover, there is a constant $C_\varepsilon > 0$, such that

$$||\partial_t v(t)||^2 + ||\partial_t^\alpha A v(t)||^2 \le C_\varepsilon \max_{t\in[0,T]} ||f||_\varepsilon^2, \quad 0 \le t \le T. \tag{25}$$

Note that Theorem 5 remains valid for all $f \in H$ independent of t.

Corollary 2. *For any $f \in H$ problem (5) has a unique solution*

$$v(t) = \sum_{k=1}^{\infty} f_k v_k \int_0^t B_\alpha(\lambda_k, \tau) d\tau. \tag{26}$$

Moreover, the estimate of the coercive type is valid with some constant C:

$$||\partial_t v(t)||^2 + ||\partial_t^\alpha A v(t)||^2 \le C||f||^2, \quad 0 < t \le T. \tag{27}$$

Proof. Estimate 1 of Lemma 1 implies

$$\left|\left| \sum_{k=1}^{j} f_k v_k \partial_t \int_0^t B_\alpha(\lambda_k, \tau) d\tau \right|\right|^2 = \sum_{k=1}^{j} |f_k|^2 |B_\alpha(\lambda_k, t)|^2 \le ||f||^2.$$

Using this estimate and repeating the reasoning as in the proof of Theorem 5, it is easy to check that (26) is the only solution to problem (5) and estimate (27) is valid. □

Note that the integral in (26) can be rewritten in the form

$$\int_0^t B_\alpha(\lambda_k, \tau) d\tau = \frac{1}{\lambda_k}(1 - A_\alpha(\lambda_k, t)),$$

where function $A_\alpha(\lambda_k, t))$ is also studied in [10]. In particular, it is shown that the Laplace transform of this function is

$$\mathcal{L}\{A_\alpha(\lambda_k, \cdot)\}(z) = \frac{1 + \gamma\lambda_k z^{\alpha-1}}{z + \gamma\lambda_k z^\alpha + \lambda_k},$$

and the estimates

$$0 < A_\alpha(\lambda_k, t) \le 1$$

hold.

8. The Second Auxiliary and Non-Local Problems

In this section, we first consider the auxiliary problem (6) and then obtain the result for the non-local problem (4).

Theorem 5. *For any $\psi \in H$ problem (6) has a unique solution*

$$w(t) = \sum_{k=1}^{\infty} \frac{B_\alpha(\lambda_k, t)}{B_\alpha(\lambda_k, T) - 1} \psi_k v_k. \tag{28}$$

Moreover, the estimate of the coercive type is valid with some constant C:

$$||\partial_t w(t)||^2 + ||\partial_t^\alpha Aw(t)||^2 \le C\frac{1}{t^2}||\psi||^2, \quad 0 < t \le T. \tag{29}$$

Proof. The solution to problem (6) will be sought in the form of eigenfunction expansions

$$w(t) = \sum_{k=1}^{\infty} T_k(t) v_k,$$

where $T_k(t)$ is the solution to the non-local problem

$$\begin{cases} \partial_t T(t) + \lambda_k(1 + \gamma\, \partial_t^\alpha) T_k(t) = 0, \quad 0 < t < T; \\ T_k(T) = T_k(0) + \psi_k, \end{cases} \tag{30}$$

Assuming $T_k(0) = h_k$ to be known, we write a solution to equation (30) with this initial condition (see (1))

$$T_k(t) = h_k B_\alpha(\lambda_k, t).$$

Now, using the non-local condition in (30), we obtain an equation for finding the unknowns h_k:

$$h_k(B_\alpha(\lambda_k, T) - 1) = \psi_k.$$

Since $T > 0$ and $\lambda_k > 0$, then $0 < B_\alpha(\lambda_k, T) < 1$ (see Lemma 1). Therefore

$$h_k = \frac{\psi_k}{B_\alpha(\lambda_k, T) - 1}, \ |h_k| \le C|\psi_k|, \ k \ge 1,$$

and (28) is a formal solution of problem (6).

The fact that the series (28) converges is beyond doubt. It remains to show that this series can be term-by-term differentiated.

First apply Parseval's equality and Lemma 5 with $\varepsilon = 0$ to get

$$\left|\left| \sum_{k=1}^{j} \frac{\partial_t B_\alpha(\lambda_k, t)}{B_\alpha(\lambda_k, T) - 1} \psi_k v_k \right|\right|^2 \le \frac{C}{t^2}||\psi||^2, \ t > 0.$$

On the other hand equation (6) implies

$$||(1 + \gamma\partial_t^\alpha) Aw||^2 \le ||\partial_t w||^2.$$

Therefore, it follows from the above

$$\left|\left| (1 + \gamma\partial_t^\alpha) A \sum_{k=1}^{j} \frac{B_\alpha(\lambda_k, t)}{B_\alpha(\lambda_k, T) - 1} \psi_k v_k \right|\right|^2 \le C\frac{1}{t^2}||\psi||, \ t > 0.$$

The uniqueness of the problem solution (6) is based on the completeness of the set of eigenfunctions $\{v_k\}$ in H and the estimate $B_\alpha(\lambda_k, T) < 1$. Indeed, if we assume that there are two solutions w^1 and w^2, then for the difference $w = w^1 - w^2$ we have

$$\begin{cases} \partial_t w(t) + (1 + \gamma\, \partial_t^\alpha) Aw(t) = 0, \quad 0 < t \le T; \\ w(T) = w(0). \end{cases}$$

Let $w(t)$ be any solution of this problem. Consider the Fourier coefficients $T_k(t) = (w(t), v_k)$. It is not hard to see, that T_k is a solution of the problem

$$\partial_t T_k(t) + \lambda_k(1 + \gamma\, \partial_t^\alpha) T_k(t) = 0, \quad 0 < t \le T, \ T_k(T) = T_k(0).$$

The solution of the Cauchy problem with the initial data $T_k(0) = h_k$ has the form $T_k(t) = h_k B_\alpha(\lambda_k, t)$ (see Corollary 1). The non-local condition implies $h_k B_\alpha(\lambda_k, T) = h_k$. Therefore $T_k(t) \equiv 0$ for all k. Since the set of eigenfunctions $\{v_k\}$ complete in H, then $w(t) \equiv 0$. □

Combining the statements of the last two theorems, we obtain the following assertion for the non-local problem (4).

Theorem 6. *Let $\varepsilon \in (0,1)$ and $f(t) \in C([0,T]; D(A^\varepsilon))$. Then for any $\varphi \in H$ problem (4) has a unique solution*

$$u(t) = \sum_{k=1}^{\infty} \left[\frac{\varphi_k - y_k(T)}{B_\alpha(\lambda_k, T) - 1} B_\alpha(\lambda_k, t) + y_k(t) \right] v_k, \tag{31}$$

where

$$y_k(t) = \int_0^t B_\alpha(\lambda_k, t-\tau) f_k(\tau) d\tau.$$

Moreover, the estimate of the coercive type is valid with some constants C and C_ε:

$$||\partial_t u(t)||^2 + ||\partial_t^\alpha A u(t)||^2 \le C \frac{1}{t^2} ||\varphi|| + C_\varepsilon \max_{t\in[0,T]} ||f||_\varepsilon^2, \quad 0 < t < T. \tag{32}$$

Proof. Let $f(t) \in C([0,T]; D(A^\varepsilon))$ with some $\varepsilon \in (0,1)$ and $\varphi \in H$. As we noted above, if we set $\psi = \varphi - v(T) \in H$ and $v(t)$ and $w(t)$ are the solutions of problems (5) and (6) correspondingly, then function $u(t) = v(t) + w(t)$ is a solution of problem (4). Therefore, (31) is the unique solution of problem (4).

Estimate (32) follows from estimates (25) and (29). □

9. Conclusions

In many papers, explicit solutions of the simplest Rayleigh-Stokes problems are constructed. In [10], the regularity of the solution was studied for the first time and a formal formula for solving an in-homogeneous problem in a three-dimensional domain was given. Also, the backward problem was studied by the authors in domains of dimension less than four. Some proofs of these results essentially use the fact that the dimension of the domain under consideration is $N \le 3$.

In this paper, all these questions are investigated for the case of an abstract operator instead of the Laplace operator in the Rayleigh-Stokes problem. In addition, a new non-local problem for equation Rayleigh-Stokes is considered and it is shown that, unlike the backward problem, it is well-posed. As the operator A, we can take the Laplace operator with the Dirichlet condition in an arbitrary N-dimensional domain with a sufficiently smooth boundary. It should be specially emphasized that our reasoning does not depend on the behavior of the eigenvalues of the operator under consideration.

We considered problem (4) for two β values: 0 and 1. It is ill-posed for $\beta = 0$ and well-posed for $\beta = 1$. Of course, it would be interesting to indicate, as in works [27,28], the boundary value $\beta^\star$ separating the two types of behavior of the "semi-inverse" problem under consideration? But for this it is necessary to study the behavior of the derivative $\partial_\lambda B(\lambda, t)$, which has not been possible so far.

In this connection, we note that in [27,28] a similar question was studied for the subdiffusion equations. It turns out that in all values $\beta \notin (0,1)$ the non-local problem is well-posed, and if $\beta \in (0,1)$ then for the existence of a solution to the problem, the functions φ and $f(x,t)$ must be orthogonal to some eigenfunctions of operator A.

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