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Construction of Optimal Split-Plot Designs for Various Design Scenarios

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Abstract: When performing fractional factorial experiments in a completely random order is impractical, fractional factorial split-plot designs are suitable options as an alternative. It is well recognized that the more there are lower order effects of interest at lower order confounding, the better the designs. From this viewpoint, this paper considers the construction of optimal regular two-level fractional factorial split-plot designs. The optimality criteria for two different design scenarios are proposed. Under the newly proposed optimality criteria, the theoretical construction methods of optimal regular two-level fractional factorial split-plot designs are then proposed. In addition, we also explore the theoretical construction methods of some optimal regular two-level fractional factorial split-plot designs under the widely adopted general minimum lower order confounding criterion.

Keywords: general minimum lower order confounding; regular two-level fractional factorial design; split-plot design

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1. Introduction

Regular two-level fractional factorial (FF) designs are commonly used for factorial experiments. When performing an FF design, it is required to perform the experimental runs in a completely random order. However, in some experiments, due to the reasons of being time-consuming or of economic cost, it is impractical or even impossible to perform the FF experimental runs in a completely random manner. For example, consider a modified experiment from [1] in which the purpose is to study the corrosion resistance of steel bars treated with two coatings, say $C1$ and $C2$, each at two furnace temperatures, 360 °C and 380 °C. It takes a long time to reset the furnace and reach a new equilibrium temperature. The factor furnace temperature is called a hard-to-change factor and the factor coating is called an easy-to-change factor. To save experimental time, it is desirable to reduce the times of resetting equilibrium temperature (the hard-to-change factor). To do so, regular two-level fractional factorial split-plot (FFSP) designs are practical design options. For more examples of the experiments which involve hard-to-change factors, one may refer to [1].

For choosing FFSP designs, Ref. [2] proposed the minimum aberration-FFSP (MA-FFSP) criterion by extending the MA criterion proposed in [3] for FF designs. Since then, a large amount of study on MA-FFSP designs has been carried out, including Ref. [4], which discussed the difference between the FF designs and FFSP designs and developed some theories on MA-FFSP designs; Ref. [5], which developed an algorithm for searching optimal MA-FFSP designs; Ref. [6], which studied MA-FFSP designs by developing a finite projective geometric formulation; Ref. [7], which considered the construction of FFSP designs in terms of consulting designs; Ref. [8], which extended the MA criterion to multi-level FFSP designs; Ref. [9], which proposed theoretical construction methods for MA orthogonal split-plot designs; Ref. [10], which considered the design scenario where the whole plot (WP) factors are more important than the sub-plot (SP) factors under the MA criterion; and Ref. [11], which constructed the MA FFSP designs for the design scenario considered in [10] via complementary designs.

According to the effect hierarchy principle and effect sparsity principle (see [12]), main effects and two-factor interactions (2FIs) are always of interest, assuming that the third- and higher-order interactions are negligible. A main effect or 2FI is said to be clear if it is not aliased with any other main effects or 2FIs. Based on the effect hierarchy principle, effect sparsity principle, and the concept of clear effects, some work on choosing optimal FFSP designs were carried out, including Ref. [13], which gave the conditions of an FFSP design to contain clear main effects and 2FIs; Ref. [14], which gave the bounds on the maximum number of clear effects of FFSP designs; Ref. [15,16], which studied the mixed-level FFSP designs with a four-level factor in WP or SP section respectively; Ref. [17] which investigated the conditions for the FFSP designs which involving some two-level factors and an eight-level factor to contain clear effects; Ref. [18], which studied the conditions of FFSP designs with some two-level factors and a 2^t-level factor containing various clear effects; and Ref. [19], which provided the conditions of FFSP designs with some s-level factors and an s^t-level factor containing various clear effects.

Apart from the MA and clear effect criterion for the FFSP designs, Ref. [20] extended the general minimum lower order confounding (GMC) criterion for the regular two-level FF designs in [21] to the regular two-level FFSP designs and proposed the GMC-FFSP criterion for assessing the regular two-level FFSP designs. However, the theoretical construction methods of the optimal regular two-level FFSP designs under the GMC-FFSP criterion have not been studied yet.

For a regular two-level FFSP design, the effect involving only WP factors is called a WP effect, and the effect involving at least one SP factor is called an SP effect. The studies on MA orthogonal FFSP designs in [9] were motivated by five different design scenarios; among them, two are presented as follows:

Scenario 1: the WP effects and SP effects are equally important.

Scenario 2: the SP effects are more important than the WP effects.

In this paper, we investigate the regular two-level FFSP designs for Scenario 1 and Scenario 2 based on a commonly adopted principle that the more there are lower order effects of interest at the lower order confounding, the better the regular two-level FFSP designs. This viewpoint is different from that considered in [9]. In addition, this paper also considers constructing optimal regular two-level FFSP designs under the GMC-FFSP criterion. The contributions of this paper are threefold:

(1) We develop suitable optimality criteria for choosing regular two-level FFSP designs for Scenarios 1 and Scenario 2 based on the assumption that the effects involving more than two factors are negligible.
(2) The construction methods of the optimal regular two-level FFSP designs under the newly proposed optimality criteria are provided.
(3) The construction methods of some optimal regular two-level FFSP designs under the GMC-FFSP criterion are derived.

The rest of the paper is organized as follows. Section 2 includes some useful notation, definitions, and the development of the optimality criteria for designs for Scenario 1 and Scenario 2, respectively. The construction of some optimal regular two-level FFSP designs are provided in Section 3. Conclusions are given in Section 4.

2. Optimality Criteria, Notation and Definitions

Let $k_1 = n_1 - m_1, k_2 = n_2 - m_2$, $k = k_1 + k_2$, and $N = 2^k$. Throughout the paper, we use the notation $2^{(n_1+n_2)-(m_1+m_2)}$ to denote a regular two-level FFSP design with n_1 WP factors/columns, n_2 SP factors/columns, and N runs. Since the factors are assigned to columns of designs, we do not differentiate between factors and columns. Denote $a_1, a_2, \ldots, a_{k_1}, b_1, b_2, \ldots, b_{k_2}$ as k independent $2^k \times 1$ columns at $+1$ and -1 levels. The saturated design $H = H(a_1, a_2, \ldots, a_{k_1}, b_1, b_2, \ldots, b_{k_2})$ with 2^k runs and $2^k - 1$ columns can be obtained by taking all possible component-wise products among the k independent columns. Let $H_a = H(a_1, a_2, \ldots, a_{k_1})$, without special statement; the columns in H and H_a are placed one after another in Yates order, i.e.,

$$\begin{aligned} H &= \{a_1, a_2, a_1a_2, a_3, a_1a_3, \ldots, a_1a_2\cdots a_{k_1}, b_1, a_1b_1, a_1a_2b_1, \ldots, a_1\cdots a_{k_1}b_1\cdots b_{k_2}\}, \\ H_a &= \{a_1, a_2, a_1a_2, a_3, a_1a_3, \ldots, a_1a_2\cdots a_{k_1}\}. \end{aligned}$$

Let $S \subset H$ and $\gamma \in H$; then we denote $B_i(S,\gamma) = \#\{(d_1,\ldots,d_i){:}d_1,\ldots,d_i \in S, d_1\cdots d_i = \gamma\}$ and $\bar{g}(S) = \#\{\gamma{:}\gamma \in H\backslash S, B_2(S,\gamma) > 0\}$, where # denotes the cardinality of a set, $d_1,\ldots,d_i$ are mutually different columns in S and $d_1\cdots d_i$ is the column genarated by taking component-wise products of columns $d_1,\ldots,d_i$. Let $T = (T_W, T_S)$ denote a $2^{(n_1+n_2)-(m_1+m_2)}$ design with $T_W = \{a_1, a_2, \ldots, a_{k_1}, a_{k_1+1}, \ldots, a_{n_1}\}$ and $T_S = \{b_1, b_2, \ldots, b_{k_2}, b_{k_2+1}, \ldots, b_{n_2}\}$, where T_W and T_S denote the WP section and SP section in the $2^{(n_1+n_2)-(m_1+m_2)}$ design, respectively. It is worth noting that we have set T_W to contain k_1 independent columns and T_S to contain k_2 independent columns here. Given any k independent columns $a_1, a_2, \ldots, a_{k_1}, b_1, b_2, \ldots, b_{k_2}$, choosing a $2^{(n_1+n_2)-(m_1+m_2)}$ design is equal to choosing $m(= m_1 + m_2)$ more columns $a_{k_1+1}, \ldots, a_{n_1}, b_{k_2+1}, \ldots, b_{n_2}$ from H. Certainly, the m columns $a_{k_1+1}, \ldots, a_{n_1}, b_{k_2+1}, \ldots, b_{n_2}$ can be generated by some of the previously stated k independent columns.

Let ${}^{\#}_{1}C_2^{(k)}(T)$ denote the number of main effects which are aliased with k 2FIs, where $k = 0, 1, \ldots, K$ with $K = \binom{n}{2}$. Let ${}^{\#}_{2}C_2^{(k)}(T)$ denote the number of 2FIs which are aliased with k 2FIs, where $k = 0, 1, \ldots, K-1$. Let ${}^{\#}_{1(s)}C_{(w)}^{(0)}$ and ${}^{\#}_{1(s)}C_{(w)}^{(1)}$ denote the number of SP main effects which are not aliased with any WP effects, and the number of SP main effects which are aliased with at least one WP effect, respectively. Let ${}^{\#}_{2(s)}C_{(w)}^{(0)}$ and ${}^{\#}_{2(s)}C_{(w)}^{(1)}$ denote the number of SP 2FIs which are not aliased with any WP effects, and the number of SP 2FIs which are aliased with at least one WP effect, respectively. With these notation, we provide the optimality criteria for choosing $2^{(n_1+n_2)-(m_1+m_2)}$ designs for Scenario 1 and Scenario 2, respectively, as follows. The $2^{(n_1+n_2)-(m_1+m_2)}$ designs which can sequentially maximize

$${}^{\#}_{1}C(T) = ({}^{\#}_{1(s)}C_{(w)}^{(0)}(T) = n_2, {}^{\#}_{1}C_2(T), {}^{\#}_{2}C_2(T)), \tag{1}$$

are optimal for Scenario 1, where ${}^{\#}_{1}C_2(T) = ({}^{\#}_{1}C_2^{(0)}(T), {}^{\#}_{1}C_2^{(1)}(T), \ldots, {}^{\#}_{1}C_2^{(K)}(T))$ and ${}^{\#}_{2}C_2(T) = ({}^{\#}_{2}C_2^{(0)}(T), {}^{\#}_{2}C_2^{(1)}(T), \ldots, {}^{\#}_{2}C_2^{(K-1)}(T))$. The $2^{(n_1+n_2)-(m_1+m_2)}$ designs which can sequentially maximize

$${}^{\#}_{2}C(T) = ({}^{\#}_{1(s)}C_{(w)}^{(0)}(T) = n_2, {}^{\#}_{1}C_2(T), {}^{\#}_{2(s)}C_{(w)}^{(0)}(T)) \tag{2}$$

are optimal for Scenario 2. By combining (1) and (2), the $2^{(n_1+n_2)-(m_1+m_2)}$ designs which can sequentially maximize

$${}^{\#}_{3}C(T) = ({}^{\#}_{1(s)}C_{(w)}^{(0)}(T) = n_2, {}^{\#}_{1}C_2(T), {}^{\#}_{2}C_2(T), {}^{\#}_{2(s)}C_{(w)}^{(0)}(T)) \tag{3}$$

are optimal under the GMC-FFSP criterion. Let 2^{n-m} denote a regular two-level FF design with n columns and $N = 2^{n-m}$ runs. For a 2^{n-m} design D, the notation ${}^{\#}_{1}C_2^{(k)}(D)$ and ${}^{\#}_{2}C_2^{(k)}(D)$ have the same meanings as ${}^{\#}_{1}C_2^{(k)}(T)$ and ${}^{\#}_{2}C_2^{(k)}(T)$, respectively. A 2^{n-m} design D which can sequentially maximize

$$({}^{\#}_{1}C_2(D), {}^{\#}_{2}C_2(D)) \tag{4}$$

is optimal under the GMC criterion. To avoid confusion, hereafter, we use the expression GMC-FF instead of GMC to present the contents relative to the 2^{n-m} designs.

Before introducing the theoretical results of this work, we introduce some more notation. Let $F_a = F(a_1, a_2, \ldots, a_{k_1})$ be the set of columns which are the component-wise products of all possible odd number of columns among the k_1 independent columns $a_1, a_2, \ldots, a_{k_1}$, i.e., $F_a = \{a_1, a_2, a_3, a_1a_2a_3, a_4, a_1a_2a_4, a_1a_3a_4, a_2a_3a_4, \ldots\}$. The set $F_b = F(b_1, b_2, \ldots, b_{k_2})$ and $F_{ab} = F(a_1, a_2, \ldots, a_{k_1}, b_1, b_2, \ldots, b_{k_2})$ are similarly defined. Denote $G_{ab} = F_{ab} \backslash F_a$. The columns in F_a, F_{ab} and G_{ab} are placed in Yates order, respectively. For any two sets A and B of columns from H, the notation $A \otimes B$ denotes the set which consists of all the mutually different columns generated by taking component-wise products between two columns in which one is from A and the other is from B. In [13], it is stated that $T = (T_W, T_S)$ is a $2^{(n_1+n_2)-(m_1+m_2)}$ design if and only if

$$\begin{cases} T_W \subseteq H_a, & T_S \subseteq H \backslash H_a, \\ \#(T_W) = n_1, & \#(T_S) = n_2, \end{cases} \tag{5}$$

where $\#(\cdot)$ denotes the number of columns in a design.

3. Construction of Optimal $2^{(n_1+n_2)-(m_1+m_2)}$ Designs

A $2^{(n_1+n_2)-(m_1+m_2)}$ design is said to have resolution R if no c-factor interaction is aliased with any other interaction involving fewer than $R - c$ factors. The resolution III $2^{(n_1+n_2)-(m_1+m_2)}$ designs have at least one main effect which is aliased with at least one 2FI. In the resolution R=IV $2^{(n_1+n_2)-(m_1+m_2)}$ designs, all the main effects are clear but there is at least one 2FI which is aliased with at least one 2FI. In Sections 3.1–3.3, we provide the construction methods of some optimal $2^{(n_1+n_2)-(m_1+m_2)}$ designs for Scenario 1, Scenario 2, and under the GMC-FFSP criterion.

3.1. Construction Methods of Optimal $2^{(n_1+n_2)-(m_1+m_2)}$ Designs for Scenario 1

We first provide a lemma which generalizes the construction of GMC-FF 2^{n-m} designs for given n and m with $\frac{5N}{16} + 1 \leq n \leq \frac{N}{2}$. Theorems 1 and 2 provide the construction methods of some optimal $2^{(n_1+n_2)-(m_1+m_2)}$ designs for Scenario 1.

Lemma 1. *For $k_1 \geq 2$, suppose D is a 2^{n-m} design with respect to*

$$\begin{cases} 2^{k_1-2} + 1 \leq n_1 \leq 2^{k_1-1}, \\ n_2 = \sum_{t=s}^{k-2} 2^t \text{ for } k_1 - 1 \leq s \leq k-2 \text{ and} \\ \frac{5N}{16} + 1 \leq n \leq \frac{N}{2}. \end{cases} \tag{6}$$

If D consists of the first n_1 columns of F_a and the last n_2 columns of G_{ab}, then D is optimal under the GMC-FF criterion.

Proof. According to [22,23], a 2^{n-m} design D with $D \subset F_{ab}$ must has resolution at least IV. Therefore, ${}_1^{\#}C_2^{(0)}(D) = n$, and ${}_1^{\#}C_2(D)$ is sequentially maximized. Next, we prove that ${}_2^{\#}C_2(D)$ is sequentially maximized among all the 2^{n-m} designs with respect to (6).

Suppose E is a 2^{n-m} design which consists of the first n columns of F_{ab}. According to [24], E is a GMC-FF design which sequentially maximizes ${}_2^{\#}C_2(E)$ among all the 2^{n-m} designs with respect to (6). Let $r = \lfloor n/2^{k_1-1} \rfloor$. Write $D = (\bar{D}_1, D_1)$, where D_1 contains the last $r \times 2^{k_1-1}$ columns of D, and $\bar{D}_1 = D \backslash D_1$. Write $E = (E_1, \bar{E}_1)$, where E_1 contains the first $r \times 2^{k_1-1}$ columns of E, and $\bar{E}_1 = E \backslash E_1$. We can always find $\gamma_1, \gamma_2 \in H \backslash F_{ab}$ such that $D_1 = \gamma_1 E_1$ and $H \backslash D_1 = \gamma_2 (H \backslash E_1)$, implying that $\bar{D}_1 = \gamma_2 \bar{E}_1$. Rewrite D_1 as $D_1 = \{d_1F_a, d_2F_a, \ldots, d_rF_a\}$, where d_1 is the grand mean and $d_2, \ldots, d_r$ are from $H \backslash F_{ab}$. Rewrite E_1 as $E_1 = \{e_1F_a, e_2F_a, \ldots, e_rF_a\}$, where $e_1, e_2, \ldots, e_r$ are from $H \backslash F_{ab}$. Actually, there exists the facts that

(1) $D_1 \otimes D_1 = E_1 \otimes E_1 = H \backslash F_{ab}$,
(2) $D_1 \otimes \bar{D}_1 = E_1 \otimes \bar{E}_1 \subset D_1 \otimes D_1 (= E_1 \otimes E_1)$,
(3) $\bar{D}_1 \otimes \bar{D}_1 = \bar{E}_1 \otimes \bar{E}_1 \subset D_1 \otimes D_1 (= E_1 \otimes E_1)$, and
(4) $(\bar{D}_1 \otimes \bar{D}_1) \cap (D_1 \otimes \bar{D}_1) = \emptyset$

due to the following reasons.

For (1). According to Lemma A.3 in [25], since $2^{k-2}+1 \leq \#(D_1) \leq 2^{k-1}$ and D_1 has k independent columns, then $D_1 \otimes D_1 = H\backslash F_{ab}$. Similarly, we can also obtain $E_1 \otimes E_1 = H\backslash F_{ab}$.

For (2). Let l_1 denote the first column of $\bar{D}_1$, then

$$\begin{aligned} l_1 \otimes (F_{ab}\backslash l_1) &= (l_1 \otimes ((F_{ab}\backslash D_1)\backslash l_1)) \cup (l_1 \otimes D_1) \\ &= (l_1 \otimes ((F_{ab}\backslash D_1)\backslash l_1)) \cup (\bar{D}_1 \otimes D_1) \\ &= H\backslash F_{ab}, \end{aligned}$$

where the second equality is because $l_1 \otimes D_1 = \bar{D}_1 \otimes D_1$ due to $\bar{D}_1 \subset F_a$ and the structure of D_1. Therefore, $\bar{D}_1 \otimes D_1 \subset D_1 \otimes D_1$. Similarly, we obtain that

$$\begin{aligned} q_1 \otimes (F_{ab}\backslash q_1) &= (q_1 \otimes ((F_{ab}\backslash E_1)\backslash q_1)) \cup (q_1 \otimes E_1) \\ &= (q_1 \otimes ((F_{ab}\backslash E_1)\backslash q_1)) \cup (\bar{E}_1 \otimes E_1) \\ &= H\backslash F_{ab} \end{aligned}$$

and $\bar{E}_1 \otimes E_1 \subset D_1 \otimes D_1 (= E_1 \otimes E_1)$, where q_1 is the first column in $\bar{E}_1$. Note that $(\bar{D}_1 \otimes D_1) \cup (\bar{D}_1 \otimes (F_{ab}\backslash D)) = H\backslash F_{ab}$, and $(\bar{D}_1 \otimes D_1) \cap (\bar{D}_1 \otimes (F_{ab}\backslash D)) = \emptyset$. Similarly, there exists $(\bar{E}_1 \otimes E_1) \cup (\bar{E}_1 \otimes (F_{ab}\backslash E)) = H\backslash F_{ab}$ and $(\bar{E}_1 \otimes E_1) \cap (\bar{E}_1 \otimes (F_{ab}\backslash E)) = \emptyset$. Since $\bar{E}_1 \otimes (F_{ab}\backslash E) = \bar{D}_1 \otimes (F_{ab}\backslash D)$ as $F_{ab}\backslash E = \gamma_2(F_{ab}\backslash D)$, we have $\bar{D}_1 \otimes D_1 = \bar{E}_1 \otimes E_1$. This obtains the fact (2).

For (3). Since $\bar{D}_1 \subset F_a$ and $\bar{D}_1 = \gamma_2 \bar{E}_1$, it is easy to obtain that $\bar{D}_1 \otimes \bar{D}_1 = \bar{E}_1 \otimes \bar{E}_1 \subset H_a\backslash F_a$. This completes the proof for (3).

For (4). Note that $\bar{D}_1 \otimes \bar{D}_1 \subset H_a\backslash F_a$ and any two-column interaction with one column from $\bar{D}_1$ and the other from D_1 is not in $H_a\backslash F_a$. Therefore, $(\bar{D}_1 \otimes \bar{D}_1) \cap (D_1 \otimes \bar{D}_1) = \emptyset$.

Based on the analysis above, the 2FIs of D and E can be classified into three disjoint groups, respectively, as

G_1: $D_1 \otimes \bar{D}_1 = E_1 \otimes \bar{E}_1$,
G_2: $\bar{D}_1 \otimes \bar{D}_1 = \bar{E}_1 \otimes \bar{E}_1$ and
G_3: $(D_1 \otimes D_1)\backslash((D_1 \otimes \bar{D}_1) \cup (\bar{D}_1 \otimes \bar{D}_1)) = (E_1 \otimes E_1)\backslash((E_1 \otimes \bar{E}_1) \cup (\bar{E}_1 \otimes \bar{E}_1))$.

From (1) and (2), for any $\gamma \in G_1$, there are $\#(\bar{D}_1)$ two-column pairs (α_1, β_1) with $\alpha_1 \in D_1$ and $\beta_1 \in \bar{D}_1$ such that $\gamma = \alpha_1\beta_1$, and there are $\#(\bar{E}_1)$ two-column pairs (α_2, β_2) with $\alpha_2 \in E_1$ and $\beta_2 \in \bar{E}_1$ such that $\gamma = \alpha_2\beta_2$, where $\#(\bar{D}_1) = \#(\bar{E}_1)$; if there are t_1 two-column pairs (α_1, β_1) with $\alpha_1 \in D_1$ and $\beta_1 \in D_1$ such that $\gamma = \alpha_1\beta_1$, there must be t_1 two-column pairs (α_2, β_2) with $\alpha_2 \in E_1$ and $\beta_2 \in E_1$ such that $\gamma = \alpha_2\beta_2$ due to $D_1 = \gamma_1 E_1$.

From (1) and (3), for any $\gamma \in G_2$, if there are t_3 two-column pairs (α_1, β_1) with $\alpha_1 \in \bar{D}_1$ and $\beta_1 \in \bar{D}_1$ such that $\gamma = \alpha_1\beta_1$, there must be t_3 two-column pairs (α_2, β_2) with $\alpha_2 \in \bar{E}_1$ and $\beta_2 \in \bar{E}_1$ such that $\gamma = \alpha_2\beta_2$, due to that $\bar{D}_1 = \gamma_2 \bar{E}_1$; if there are t_4 two-column pairs (α_1, β_1) with $\alpha_1 \in D_1$ and $\beta_1 \in D_1$ such that $\gamma = \alpha_1\beta_1$, there must be t_4 two-column pairs (α_2, β_2) with $\alpha_2 \in E_1$ and $\beta_2 \in E_1$ such that $\gamma = \alpha_2\beta_2$ due to that $D_1 = \gamma_1 E_1$.

For any $\gamma \in G_3$, if there are t_5 two-column pairs (α_1, β_1) with $\alpha_1 \in D_1$ and $\beta_1 \in D_1$ such that $\gamma = \alpha_1\beta_1$, there must be t_5 two-column pairs (α_2, β_2) with $\alpha_2 \in E_1$ and $\beta_2 \in E_1$ such that $\gamma = \alpha_2\beta_2$ due to that $D_1 = \gamma_1 E_1$.

Therefore, we have ${}_2^{\#}C_2(D) = {}_2^{\#}C_2(E)$ which is sequentially maximized among all the 2^{n-m} designs with respect to (6) as E is a GMC-FF design according to [24]. This completes the proof. □

Remark 1. *In [24], it is stated that a 2^{n-m} design with $\frac{5N}{16}+1 \leq n \leq \frac{N}{2}$ is a GMC-FF design if this design consists of the first (or last) n columns of F_{ab}. Lemma 1 generalizes their construction methods for GMC-FF 2^{n-m} designs with $\frac{5N}{16}+1 \leq n \leq \frac{N}{2}$.*

Based on Lemma 1, the following Theorems 1 and 2 provide construction methods of some optimal $2^{(n_1+n_2)-(m_1+m_2)}$ designs for Scenario 1.

Theorem 1. *Suppose $T = (T_W, T_S)$ is a $2^{(n_1+n_2)-(m_1+m_2)}$ design with respect to*

$$\begin{cases} 2^{k_1-2}+1 \leq n_1 \leq 2^{k_1-1}, \\ n_2 = \sum_{t=s}^{k-2} 2^t \text{ for } k_1 - 1 \leq s \leq k-2 \text{ and} \\ \frac{5N}{16}+1 \leq n \leq \frac{N}{2}. \end{cases}$$

If T_W consists of the first n_1 columns of F_a and T_S consists of the last n_2 columns of G_{ab}, then $T = (T_W, T_S)$ is optimal for Scenario 1.

Proof. Clearly, $T = (T_W, T_S)$ is a $2^{(n_1+n_2)-(m_1+m_2)}$ design as it satisfies (5); thus, ${}^{\#}_{1(s)}C^{(0)}_{(w)}(T) = n_2$. According to Lemma 1, we obtain that T can sequentially maximize $({}^{\#}_{1}C_2(T), {}^{\#}_{2}C_2(T))$. This completes the proof. □

Example 1 shows the application of Theorem 1.

Example 1. *Consider constructing an optimal $2^{(6+8)-(2+7)}$ design for Scenario 1. Without loss of generality, let $a_1 = 5, a_2 = 15, a_3 = 25, a_4 = 35$ and $b_1 = 45$, then $F_a = \{5, 15, 25, 125, 35, 135, 235, 1235\}$ and $G_{ab} = \{45, 145, 245, 1245, 345, 1345, 2345, 12345\}$. Let $T_W = \{5, 15, 25, 125, 35, 135\}$ and $T_S = \{45, 145, 245, 1245, 345, 1345, 2345, 12345\}$. According to Theorem 1, $T = (T_W, T_S)$ is optimal for Scenario 1.*

Theorem 2. *Suppose $T = (T_W, T_S)$ is a $2^{(n_1+n_2)-(m_1+m_2)}$ design with $n_1 = 2^{k_1-1}$, $n_2 \leq 2^{k-1} - 2^{k_1-1}$ and $\frac{5N}{16}+1 \leq n \leq \frac{N}{2}$. Let $T_W = F_a$ and T_S consists of the first n_2 columns of G_{ab}, then $T = (T_W, T_S)$ is optimal for Scenario 1.*

Proof. Clearly, the design T in this theorem is a $2^{(n_1+n_2)-(m_1+m_2)}$ design; thus, ${}^{\#}_{1(s)}C^{(0)}_{(w)}(T) = n_2$. Note that T consists of the first n columns of F_{ab}; thus, T sequentially maximizes $({}^{\#}_{1}C_2(T), {}^{\#}_{2}C_2(T))$ as it is also a GMC-FF design according to [24]. This completes the proof. □

Example 2. *Consider constructing an optimal $2^{(4+8)-(1+6)}$ design for Scenario 1. Without loss of generality, let $a_1 = 5$, $a_2 = 15$, $a_3 = 25$, $b_1 = 35$ and $b_2 = 45$, then $F_a = \{5, 15, 25, 125\}$ and $G_{ab} = \{35, 135, 235, 1235, 45, 145, 245, 1245, 345, 1345, 2345, 12345\}$. Let $T_W = \{5, 15, 25, 125\}$ and $T_S = \{35, 135, 235, 1235, 45, 145, 245, 1245\}$. According to Theorem 2, $T = (T_W, T_S)$ is optimal for Scenario 1.*

In Theorem 3, we build the connection between GMC-FF 2^{n-m} designs and the optimal $2^{(n_1+n_2)-(m_1+m_2)}$ designs for Scenario 1. Before introducing Theorem 3, we first give a useful lemma.

Lemma 2. *Suppose D and B are two 2^{n-m} designs from F_{ab}. If D can be divided into two disjoint parts D_1 and D_2 such that*

(i) $B_1 = \gamma_1 D_1$, $B_2 = \gamma_2 D_2$ *and* $B = B_1 \cup B_2$ *with* $B_1 \cap B_2 = \emptyset$,
(ii) $(D_1 \otimes D_2) \cap ((D_1 \otimes D_1) \cup (D_2 \otimes D_2)) = \emptyset$, *and*
(iii) $(B_1 \otimes B_2) \cap ((B_1 \otimes B_1) \cup (B_2 \otimes B_2)) = \emptyset$,

then $(-\bar{g}(D), {}^{\#}_{1}C_2(D), {}^{\#}_{2}C_2(D)) = (-\bar{g}(B), {}^{\#}_{1}C_2(B), {}^{\#}_{2}C_2(B))$, where each of γ_1 and γ_2 can be the grand mean or any column from $H \backslash F_{ab}$, and $\emptyset$ denotes the empty set.

Proof. Since $B_1 = \gamma_1 D_1$ and $B_2 = \gamma_2 D_2$, we have that $D_1 \otimes D_1 = B_1 \otimes B_1$, $D_2 \otimes D_2 = B_2 \otimes B_2$, and $\gamma_1\gamma_2(D_1 \otimes D_2) = B_1 \otimes B_2$. More specifically, if there are t_1 two-column pairs (α_1, α_2) with $\alpha_1 \in D_1$ and $\alpha_2 \in D_1$ such that $\nu = \alpha_1\alpha_2$, then there are must be t_1 two-column pairs (β_1, β_2) with $\beta_1 \in B_1$ and $\beta_2 \in B_1$ such that $\nu = \beta_1\beta_2$; for any $\nu \in D_2 \otimes D_2$, if there are t_2 two-column pairs (α_1, α_2) with $\alpha_1 \in D_2$ and $\alpha_2 \in D_2$ such that $\nu = \alpha_1\alpha_2$, then there must be t_2 two-column pairs (β_1, β_2) with $\beta_1 \in B_2$ and $\beta_2 \in B_2$

such that $\nu = \beta_1\beta_2$; for any $\nu \in D_1 \otimes D_2$, if there are t_3 two-column pairs (α_1, α_2) with $\alpha_1 \in D_1$ and $\alpha_2 \in D_2$ such that $\nu = \alpha_1\alpha_2$, then there must be t_3 two-column pairs (β_1, β_2) with $\beta_1 \in B_1$ and $\beta_2 \in B_2$ such that $\gamma_1\gamma_2\nu = \beta_1\beta_2$.

With the analysis above, we first prove that $-\bar{g}(D) = -\bar{g}(B)$. Recalling the definition of $\bar{g}(D)$, we have

$$\begin{aligned}\bar{g}(D) &= \#\{\nu : \nu \in H\backslash D, B_2(D_1 \cup D_2, \nu) > 0\} \\ &= \#\{\nu : \nu \in H\backslash D, \nu \in (D_1 \otimes D_1) \cup (D_2 \otimes D_2) \cup (D_1 \otimes D_2)\} \\ &= \#\{\nu : \nu \in H\backslash D, \nu \in (D_1 \otimes D_1) \cup (D_2 \otimes D_2)\} + \#\{\nu : \nu \in H\backslash D, \nu \in D_1 \otimes D_2\} \\ &= \#\{\tau : \tau \in H\backslash B, \tau \in (B_1 \otimes B_1) \cup (B_2 \otimes B_2)\} + \#\{\tau : \tau \in H\backslash B, \tau \in B_1 \otimes B_2\}, \\ &= \bar{g}(B) \end{aligned} \tag{7}$$

where in the fourth equality $\#\{\tau : \tau \in H\backslash B, \tau \in B_1 \otimes B_2\} = \#\{\nu : \nu \in H\backslash D, \nu \in D_1 \otimes D_2\}$ is due to the fact that for any $\nu_0 \in H\backslash D$ with $\nu_0 \in D_1 \otimes D_2$ we have $\tau_0 = \gamma_1\gamma_2\nu_0 \in H\backslash B$. This obtains that $-\bar{g}(D) = -\bar{g}(B)$.

Since any 2^{n-m} design from F_{ab} has resolution IV, then ${}_1^{\#}C_2(D) = {}_1^{\#}C_2(B)$.

Next, we give the proof that ${}_2^{\#}C_2(D) = {}_2^{\#}C_2(B)$. According to the analysis in the first paragraph, for any $\nu_0 = \alpha_1\alpha_2 \in (D_1 \otimes D_1) \cup (D_2 \otimes D_2)$, we have $\tau_0 = \nu_0 = (\gamma_i\alpha_1)(\gamma_i\alpha_2) \in (B_1 \otimes B_1) \cup (B_2 \otimes B_2)$. Therefore, we have

$$\#\{\nu : \nu \in (D_1 \otimes D_1) \cup (D_2 \otimes D_2), B_2(D, \nu) = k\} = \#\{\tau : \tau \in (B_1 \otimes B_1) \cup (B_2 \otimes B_2), B_2(B, \tau) = k\},$$

where $k = 0, 1, \ldots, K$. Similarly, for any $\nu_0 = \alpha_1\alpha_2 \in D_1 \otimes D_2$, we have $\tau_0 = \gamma_1\gamma_2\nu_0 = (\gamma_1\alpha_1)(\gamma_2\alpha_2) \in B_1 \otimes B_2$. Therefore, we have

$$\#\{\nu : \nu \in D_1 \otimes D_2, B_2(D, \nu) = k\} = \#\{\tau : \tau \in B_1 \otimes B_2, B_2(B, \tau) = k\},$$

where $k = 0, 1, \ldots, K$. This obtains that ${}_2^{\#}C_2(D) = {}_2^{\#}C_2(B)$ and the proof is completed. □

With Lemma 2, we immediately obtain Theorem 3, which connects optimal FFSP designs for Scenario 1 with GMC-FF 2^{n-m} designs.

Theorem 3. *Suppose $T = (T_W, T_S) \subset F_{ab}$ and $B \subset F_{ab}$ are $2^{(n_1+n_2)-(m_1+m_2)}$ and GMC-FF 2^{n-m} designs with $\frac{5N}{16} + 1 \le n \le \frac{N}{2}$, respectively. For $\bar{T} = F_{ab}\backslash T$ and $\bar{B} \subset F_{ab}\backslash B$, if $\bar{T}$ can be divided into two disjoint parts T_1 and T_2 such that*

(i) $B_1 = \gamma_1 T_1$, $B_2 = \gamma_2 T_2$ *and* $\bar{B} = B_1 \cup B_2$ *with* $B_1 \cap B_2 = \emptyset$;
(ii) $(T_1 \otimes T_2) \cap ((T_1 \otimes T_1) \cup (T_2 \otimes T_2)) = \emptyset$, *and*
(iii) $(B_1 \otimes B_2) \cap ((B_1 \otimes B_1) \cup (B_2 \otimes B_2)) = \emptyset$,

then T is optimal for Scenario 1, where each of γ_1 and γ_2 can be the grand mean or any column from $H\backslash F_{ab}$.

Proof. On one hand, according to Lemma 1 of [24], sequentially maximizing ${}_2^{\#}C_2(T)$ is equal to sequentially maximizing $(-\bar{g}(\bar{T}), {}_2^{\#}C_2(\bar{T}))$. On the other hand, according to Lemma 2, we obtain that $(-\bar{g}(\bar{T}), {}_2^{\#}C_2(\bar{T})) = (-\bar{g}(\bar{B}), {}_2^{\#}C_2(\bar{B}))$ indicating that $(-\bar{g}(\bar{T}), {}_2^{\#}C_2(\bar{T}))$ is sequentially maximized. This is because $(-\bar{g}(\bar{B}), {}_2^{\#}C_2(\bar{B}))$ is sequentially maximized among all the 2^{n-m} designs with $\frac{5N}{16} + 1 \le n \le \frac{N}{2}$. Therefore, we obtain that T can sequentially maximize (1) among all the $2^{(n_1+n_2)-(m_1+m_2)}$ designs and thus it is optimal for Scenario 1. □

Theorem 3 provides an approach to conforming that a $2^{(n_1+n_2)-(m_1+m_2)}$ design is optimal for Scenario 1. The following example illustrates the application of Theorem 3.

Example 3. *For a given $2^{(6+6)-(2+5)}$ design $T = (T_W, T_S)$ with $T_W = \{5, 15, 25, 125, 35, 135\}$ and $T_S = \{45, 145, 245, 1245, 345, 1345\}$, we have $\bar{T} = F_{ab}\backslash T = \{235, 1235, 2345, 12345\}$. Divide $\bar{T}$ into two disjoint subsets as $\bar{T} = T_1 \cup T_2$ with $T_1 = \{235, 1235\}$ and $T_2 = \{2345, 12345\}$, then T_1 and T_2 satisfy $((T_1 \otimes T_1) \cup (T_2 \otimes T_2)) \cap (T_1 \cup T_2) = \emptyset$. Let $B_1 = 24T_1 = \{345, 1345\}$,*

$B_2 = T_2$ and $\bar{B} = B_1 \cup B_2$ then $B = F_{ab} \backslash \bar{B} = \{5, 15, 25, 125, 35, 135, 235, 1235, 45, 145, 245, 1245\}$ which is composed of the first 12 columns of F_{ab}. According to Theorem 3, we obtain that T sequentially maximizes (1) among all $2^{(6+6)-(2+5)}$ FFSP designs. Therefore, design T is optimal for Scenario 1.

3.2. Construction Methods of Optimal $2^{(n_1+n_2)-(m_1+m_2)}$ Designs for Scenario 2

Lemmas 3 and 4 below derive some properties for $2^{(n_1+n_2)-(m_1+m_2)}$ designs which is useful for deriving the construction methods of optimal $2^{(n_1+n_2)-(m_1+m_2)}$ designs for Scenario 2.

Lemma 3. *For any $2^{(n_1+n_2)-(m_1+m_2)}$ design $T = (T_W, T_S)$, there must be $n_1 n_2 \le {}^{\#}_{2(s)}C^{(0)}_{(w)}(T) \le \binom{n_2}{2} + n_1 n_2$.*

Proof. For any $2^{(n_1+n_2)-(m_1+m_2)}$ design, the number of 2FIs which have two SP factors and the number of 2FIs which have only one SP factor are $\binom{n_2}{2}$ and $n_1 n_2$, respectively. Therefore, ${}^{\#}_{2(s)}C^{(0)}_{(w)} \le \binom{n_2}{2} + n_1 n_2$. As aforementioned, the generator which contains only one SP factor is not allowed, implying that all the 2FIs which have only one SP factor are not aliased with any WP effects. Therefore, we have $n_1 n_2 \le {}^{\#}_{1(s)}C^{(0)}_{(w)}(T)$. This completes the proof. □

Lemma 4. *For any $2^{(n_1+n_2)-(m_1+m_2)}$ design $T = (T_W, T_S)$ with $k_2 = 1$, there must be ${}^{\#}_{2(s)}C^{(0)}_{(w)}(T) = n_1 n_2$.*

Proof. The formula $k_2 = 1$ indicates that there is only one independent SP factor denoted as b_1. Therefore, the SP dependent factors $b_2, b_3, \ldots, b_{n_2}$ can be expressed as $b_i = b_1 a_{i_1} a_{i_2} \cdots a_{i_j}$, where $i = 2, 3, \ldots, n_2$ and $i_1, i_2, \ldots, i_j = 1, 2, \ldots, k_1$. Therefore, all of the $\binom{n_2}{2}$ 2FIs which contain two SP factors are aliased with WP effects. As aforementioned, for any $2^{(n_1+n_2)-(m_1+m_2)}$ design, the 2FIs which contain only one SP factor are not aliased with any WP effects. Therefore, we have ${}^{\#}_{2(s)}C^{(0)}_{(w)}(T) = n_1 n_2$. This completes the proof. □

With Lemma 3, Theorems 4 blow provides construction methods of some FFSP designs which are optimal for Scenario 2.

Theorem 4. *Suppose $T = (T_W, T_S)$ is a $2^{(n_1+n_2)-(m_1+m_2)}$ design with $n_1 \le 2^{k_1-1}$ and $n_2 = k_2$, i.e., $m_2 = 0$, if $T_W \subset F_a$ and $T_S \subset G_{ab}$, then $T = (T_W, T_S)$ is optimal for Scenario 2.*

Proof. Note that $T \subset F_{ab}$, then T has resolution at least IV. Therefore, T sequentially maximizes $({}^{\#}_{1(s)}C^{(0)}_{(w)}(T) = n_2, {}^{\#}_{1}C_2(T))$. The formula $m_2 = 0$ implies that no SP 2FI is aliased with WP effects meaning that ${}^{\#}_{2(s)}C^{(0)}_{(w)}(T) = \binom{n_2}{2} + n_1 n_2$ which is the upper bound of ${}^{\#}_{2(s)}C^{(0)}_{(w)}(\cdot)$. Therefore, T sequentially maximizes $({}^{\#}_{1(s)}C^{(0)}_{(w)}(T) = n_2, {}^{\#}_{1}C_2(T), {}^{\#}_{2(s)}C^{(0)}_{(w)}(T))$ meaning that T is optimal for Scenario 2. □

Example 4. *Consider constructing a $2^{(4+2)-(1+0)}$ design which is optimal for Scenario 2. Without loss of generality, we set $a_1 = 5, a_2 = 15, a_3 = 25, b_1 = 35$ and $b_2 = 45$. Let $T_W = \{5, 15, 25, 125\}$ and $T_S = \{35, 45\}$. According to Theorem 4, the design $T = (T_W, T_S)$ is an optimal $2^{(4+2)-(1+0)}$ design for Scenario 2.*

With Lemma 3, we obtain Theorem 5 below.

Theorem 5. *Suppose $T = (T_W, T_S)$ is a $2^{(n_1+n_2)-(m_1+m_2)}$ design with $n_1 \le 2^{k_1-1}$, $n_2 \le 2^{k_2} - 1$ and $m_2 \ge 1$. Let $T_W \subset F_a$ and $T_S \subset F(a_1, b_1, \ldots, b_{k_2}) \backslash a_1$, then $T = (T_W, T_S)$ is optimal for Scenario 2.*

Proof. Clearly, T is a $2^{(n_1+n_2)-(m_1+m_2)}$ design as $T_W \subset H_a$ and $T_S \subset F_{ab} \backslash H_a$. Since any two-column interaction of $F(a_1, b_1, \ldots, b_{k_2}) \backslash a_1$ is not in H_a, and any two-column interaction with one column from F_a and the other from $F(a_1, b_1, \ldots, b_{k_2}) \backslash a_1$ is not in H_a, then T has no SP 2FI which is aliased with any WP effects. Therefore, we have ${}^{\#}_{2(s)}C^{(0)}_{(w)}(T) = n_1 n_2 + \binom{n_2}{2}$ which is the upper bound for every $2^{(n_1+n_2)-(m_1+m_2)}$ design according to Lemma 3. This completes the proof, noting that T sequentially maximizes ${}^{\#}_{1}C_2(T)$ due to its resolution IV. □

Example 5 below illustrates the application of Theorem 5.

Example 5. *Consider constructing a $2^{(2+7)-(0+4)}$ design which is optimal for Scenario 2. Without loss of generality, we set $a_1 = 5, a_2 = 15, b_1 = 25, b_2 = 35$, and $b_3 = 45$. Then $F_a = \{5, 15\}$ and $F(a_1, b_1, \ldots, b_{k_2}) \backslash a_1 = \{25, 35, 45, 235, 245, 345, 2345\}$. According to Theorem 5, any $2^{(n_1+n_2)-(m_1+m_2)}$ design $T = (T_W, T_S)$ with $T_W \subset F_a$ and $T_S \subset F(a_1, b_1, \ldots, b_{k_2}) \backslash a_1$ is an optimal $2^{(2+7)-(0+4)}$ design for Scenario 2.*

With Theorems 4 and 5, the following corollary is obtained.

Corollary 1. *The $2^{(n_1+n_2)-(m_1+m_2)}$ designs constructed by Theorems 4 and 5 have ${}^{\#}_{1(s)}C^{(0)}_{(w)}(T) = n_2$, ${}^{\#}_{1}C^{(t)}_{2}(T) = 0$ for $t = 1, 2, \ldots, K$, and ${}^{\#}_{2(s)}C^{(0)}_{(w)}(T) = \binom{n_2}{2} + n_1 n_2$.*

With Lemma 4, we can immediately obtain the results in Theorem 6.

Theorem 6. *Suppose $T = (T_W, T_S)$ is a $2^{(n_1+n_2)-(m_1+m_2)}$ design with $k - 1 \leq n_1 \leq 2^{k-2}$, $n_2 \leq 2^{k-2}$ and $m_2 = n_2 - 1$. Let $T_W \subset F_a$ and T_S contains any n_2 columns of G_{ab}, then T is optimal for Scenario 2.*

Example 6. *Consider constructing a $2^{(5+2)-(1+1)}$ design which is optimal for Scenario 2. Without loss of generality, we set $a_1 = 5, a_2 = 15, a_3 = 25, a_4 = 35$ and $b_1 = 45$. Then $F_a = \{5, 15, 25, 125, 35, 135, 235, 1235\}$ and $G_{ab} = \{45, 145, 245, 1245, 345, 1345, 2345, 12345\}$. According to Theorem 6, any $2^{(n_1+n_2)-(m_1+(n_2-1))}$ design $T = (T_W, T_S)$ with $T_W \subset F_a$ and $T_S \subset G_{ab}$ is an optimal $2^{(5+2)-(1+1)}$ design for Scenario 2. Without loss of generality, let $T_W = \{5, 15, 25, 125, 35\}$ and $T_S = \{145, 245\}$, then $T = (T_W, T_S)$ is optimal for Scenario 2.*

3.3. Construction Methods of GMC-FFSP $2^{(n_1+n_2)-(m_1+m_2)}$ Designs

With Theorem 1 and Lemma 4, we immediately obtain Theorem 7 below, which constructs some GMC-FFSP $2^{(n_1+n_2)-(m_1+m_2)}$ designs.

Theorem 7. *Suppose $T = (T_W, T_S)$ is a $2^{(n_1+n_2)-(m_1+m_2)}$ design with $2^{k-3} + 1 \leq n_1 \leq 2^{k-2}$, $n_2 = 2^{k-2}$, $\frac{5N}{16} + 1 \leq n \leq \frac{N}{2}$ and $m_2 = n_2 - 1$. If T_W consists of the first n_1 columns of F_a and $T_S = G_{ab}$, then T is a GMC-FFSP design.*

Example 7. *Consider constructing a $2^{(5+8)-(1+7)}$ GMC-FFSP design by Thereom 7. Without loss of generality, we set $a_1 = 5, a_2 = 15, a_3 = 25, a_4 = 35$ and $b_1 = 45$. Then $F_a = \{5, 15, 25, 125, 35, 135, 235, 1235\}$ and $G_{ab} = \{45, 145, 245, 1245, 345, 1345, 2345, 12345\}$. Let $T_W = \{5, 15, 25, 125, 35\}$ and $T_S = \{45, 145, 245, 1245, 345, 1345, 2345, 12345\}$, then $T = (T_W, T_S)$ is a $2^{(5+8)-(1+7)}$ GMC-FFSP design.*

With Theorem 2 and Lemma 4, Theorem 8 below provides construction methods of some GMC-FFSP $2^{(n_1+n_2)-(m_1+m_2)}$ designs.

Theorem 8. *Suppose $T = (T_W, T_S)$ is a $2^{(n_1+n_2)-(m_1+m_2)}$ design with $n_1 = 2^{k-2}$, $n_2 \leq 2^{k-2}$, $\frac{5N}{16} + 1 \leq n \leq \frac{N}{2}$ and $m_2 = n_2 - 1$. If $T_W = F_a$ and T_S consists of the first n_2 columns of G_{ab}, then T is a GMC-FFSP design.*

Proof. The formula $n_1 = 2^{k-1}$ indicates that T_W consists of $k-1$ independent columns, i.e., $k_1 = k-1$. Therefore, we have $k_2 = 1$. In Theorem 2, it is proved that T can sequentially maximize $({}^{\#}_{1(s)}C^{(0)}_{(w)}(T) = n_2, {}^{\#}_{1}C_2(T), {}^{\#}_{2}C_2(T))$. According to Lemma 4, for any $2^{(n_1+n_2)-(m_1+m_2)}$ design with $k_2 = 1$, we have ${}^{\#}_{2(s)}C^{(0)}_{(w)}(\cdot) = n_1 n_2$. This completes the proof. □

Example 8. *Consider constructing a $2^{(8+3)-(4+2)}$ GMC-FFSP design by Theorem 8. Without loss of generality, we set $a_1 = 5, a_2 = 15, a_3 = 25, a_4 = 35$ and $b_1 = 45$. Then $F_a = \{5, 15, 25, 125, 35, 135, 235, 1235\}$ and $G_{ab} = \{45, 145, 245, 1245, 345, 1345, 2345, 12345\}$. Let $T_W = \{5, 15, 25, 125, 35, 135, 235, 1235\}$ and $T_S = \{45, 145, 245\}$, then $T = (T_W, T_S)$ is a $2^{(8+3)-(4+2)}$ GMC-FFSP design.*

Similar to Theorem 3, the theorem below provides an approach to conforming that some $2^{(n_1+n_2)-(m_1+m_2)}$ designs are GMC-FFSP designs.

Theorem 9. *For $\frac{5N}{16} + 1 \le n \le \frac{N}{2}$, suppose $T = (T_W, T_S)$ is a $2^{(n_1+n_2)-(m_1+m_2)}$ design with $T \subset F_{ab}$ and $m_2 = n_2 - 1$. If there exists a GMC-FF design $D \subset F_{ab}$ such that*

(i) $T_1 = \gamma_1 D_1$, $T_2 = \gamma_2 D_2$ and $\bar{T} = T_1 \cup T_2$ with $T_1 \cap T_2 = \emptyset$;
(ii) $(D_1 \otimes D_2) \cap \{(D_1 \otimes D_1) \cup (D_2 \otimes D_2)\} = \emptyset$, and
(iii) $(T_1 \otimes T_2) \cap \{(T_1 \otimes T_1) \cup (T_2 \otimes T_2)\} = \emptyset$,

then T is a GMC-FFSP design, where $\bar{T} = F_{ab} \backslash T$, $\bar{D} = F_{ab} \backslash D$, $\bar{D} = D_1 \cup D_2$ with $D_1 \cap D_2 = \emptyset$, each of γ_1 and γ_2 can be the grand mean or any column from $H \backslash F_{ab}$, and $\emptyset$ denotes the empty set.

Example 9. *For a given $2^{(12+12)-(7+11)}$ design $T = (T_W, T_S)$ with $T_W = \{6, 16, 26, 126, 36, 136, 236, 1236, 46, 146, 246, 1246\}$ and $T_S = \{56, 156, 256, 1256, 356, 1356, 2356, 12356, 456, 1456, 2456, 12456\}$, we have $\bar{T} = F_{ab} \backslash T = \{346, 1346, 2346, 12346, 3456, 13456, 23456, 123456\}$. Divide $\bar{T}$ into two disjoint subsets as $\bar{T} = T_1 \cup T_2$ with $T_1 = \{346, 1346, 2346, 12346\}$ and $T_2 = \{3456, 13456, 23456, 123456\}$, then T_1 and T_2 satisfy $((T_1 \otimes T_1) \cup (T_2 \otimes T_2)) \cap (T_1 \cup T_2) = \emptyset$. Let $D_1 = 35T_1 = \{456, 1456, 2456, 12456\}$, $D_2 = T_2$ and $\bar{D} = D_1 \cup D_2$ then $D = F_{ab} \backslash \bar{D} = \{6, 16, 26, 126, 36, 136, 236, 1236, 46, 146, 246, 1246, 346, 1346, 2346, 12346, 56, 156, 256, 1256, 356, 1356, 2356, 12356\}$ which is composed of the first 24 columns of F_{ab}. According to Theorem 9, we obtain that T is a $2^{(12+12)-(7+11)}$ GMC-FFSP design.*

3.4. Some More Illustrative Examples and Further Discussions

In this section, we provide some more examples to illustrate how to recognize the superiority of an FFSP design over another under criteria (1), (2), and (3), respectively.

Consider the following two $2^{(2+7)-(0+4)}$ designs represented by their independent defining words

$$D_1 : I = a_1 a_2 b_2 b_3 b_4 = a_1 a_2 b_2 b_5 = a_1 b_1 b_2 b_6 = a_2 b_1 b_2 b_7 \text{ and}$$
$$D_2 : I = a_1 b_1 b_2 b_4 = a_1 b_2 b_3 b_5 = a_1 b_2 b_3 b_6 = b_1 b_2 b_3 b_7,$$

respectively. With some calculations we obtain that

$$ {}^{\#}_{1(s)}C^{(0)}_{(w)}(D_1) = 7,\ {}^{\#}_{1}C_2(D_1) = 9,\ {}^{\#}_{2}C_2(D_1) = (15, 0, 21),\ {}^{\#}_{2(s)}C^{(0)}_{(w)}(D_1) = 33 \text{ and}$$
$$ {}^{\#}_{1(s)}C^{(0)}_{(w)}(D_2) = 7,\ {}^{\#}_{1}C_2(D_2) = 9,\ {}^{\#}_{2}C_2(D_2) = (8, 0^2, 28),\ {}^{\#}_{2(s)}C^{(0)}_{(w)}(D_2) = 35.$$

Under criterion (1), D_1 is better than D_2 due to the following reasons. Note that ${}^{\#}_{2}C^{(0)}_{2}(\cdot)$ is the first component, in (1), such that ${}^{\#}_{2}C^{(0)}_{2}(D_1) \neq {}^{\#}_{2}C^{(0)}_{2}(D_2)$ and ${}^{\#}_{2}C^{(0)}_{2}(D_1) = 15 > {}^{\#}_{2}C^{(0)}_{2}(D_2) = 8$. Therefore, D_1 is better than D_2 under criterion (1).

In contrast, the FFSP design D_2 is better than D_1 under criterion (2). Note that criterion (2) prefers FFSP designs with resolution of at least IV, which have more SP 2FIs that are not aliased with any WP effect regardless of ${}^{\#}_{2}C_2(\cdot)$. With this point in mind, since ${}^{\#}_{1(s)}C^{(0)}_{(w)}(D_1) = {}^{\#}_{1(s)}C^{(0)}_{(w)}(D_2)$, ${}^{\#}_{1}C_2(D_1) = {}^{\#}_{1}C_2(D_2)$ and ${}^{\#}_{2(s)}C^{(0)}_{(w)}(D_2) = 35 > {}^{\#}_{2(s)}C^{(0)}_{(w)}(D_1) = 33$, then design D_2 is better than D_1 under criterion (2).

As for criterion (3), it is clear that, if an FFSP design is better than another under criterion (1), then it is always the case when they are compared under criterion (3), noting that criterion (3) concerns one more component ${}^{\#}_{2(s)}C^{(0)}_{(w)}(\cdot)$ apart from the three common components ${}^{\#}_{1(s)}C^{(0)}_{(w)}(\cdot) = n_2$, ${}^{\#}_{1}C_2(\cdot)$ and ${}^{\#}_{2}C_2(\cdot)$ shared by (1) and (3). Therefore, design D_1 is better than D_2 under criterion (3). To show how to identify a better design under criterion (3), we consider two more examples represented by their independent defining words:

$$\begin{aligned}
D_3 : I &= a_1b_1b_2b_3b_4 = a_2b_1b_2b_3b_5 = a_3b_1b_2b_3b_6 = a_1a_2a_3b_1b_2b_3b_7 = a_1a_2b_1b_8 \\
&= a_1a_3b_1b_9 = a_2a_3b_1b_{10} = a_1a_2b_2b_{11} = a_1a_3b_2b_{12} = a_2a_3b_2b_{12} \\
&= a_1a_2b_3b_{14} = a_1a_3b_3b_{15} = a_2a_3b_3b_{16} = a_1a_2a_3a_4, \\
D_4 : I &= a_1b_1b_2b_3b_4 = a_1a_2b_1b_2b_5 = a_1a_3b_1b_2b_6 = a_1a_2a_3b_1b_2b_3b_7 = a_1a_2b_3b_8 \\
&= a_1a_3b_3b_9 = a_1a_2a_3a_4 = a_2b_1b_3b_{10} = a_3b_1b_3b_{11} = a_2a_3b_1b_{12} \\
&= a_2b_2b_3b_{13} = a_3b_2b_3b_{14} = a_2a_3b_2b_{15} = a_2a_3b_3b_{16},
\end{aligned}$$

where D_3 and D_4 are two $2^{(4+16)-(1+13)}$ FFSP designs, respectively. With some calculations, we obtain that

$$\begin{aligned}
&{}^{\#}_{1(s)}C^{(0)}_{(w)}(D_3) = 16,\ {}^{\#}_{1}C_2(D_3) = 20,\ {}^{\#}_{2}C_2(D_3) = (0^3, 160, 0^5, 30),\ {}^{\#}_{2(s)}C^{(0)}_{(w)}(D_3) = 160 \text{ and} \\
&{}^{\#}_{1(s)}C^{(0)}_{(w)}(D_4) = 16,\ {}^{\#}_{1}C_2(D_4) = 20,\ {}^{\#}_{2}C_2(D_4) = (0^3, 160, 0^5, 30),\ {}^{\#}_{2(s)}C^{(0)}_{(w)}(D_4) = 171.
\end{aligned}$$

Although D_3 and D_4 have equal performance under criterion (1) due to that ${}^{\#}_{1(s)}C^{(0)}_{(w)}(D_3) = {}^{\#}_{1(s)}C^{(0)}_{(w)}(D_4)$, ${}^{\#}_{1}C_2(D_3) = {}^{\#}_{1}C_2(D_4)$ and ${}^{\#}_{2}C_2(D_3) = {}^{\#}_{2}C_2(D_4)$, design D_4 is better than D_3 under criterion (3) as ${}^{\#}_{2(s)}C^{(0)}_{(w)}(D_4) = 171 > {}^{\#}_{2(s)}C^{(0)}_{(w)}(D_3) = 160$.

The study of this paper is substantially different from the Refs. [7–11,18,19,26]. More specifically, Ref. [7] considered the regular symmetrical or mixed-level FFSP designs under the minimum secondary aberration criterion, which concerns only the number of SP-factor interactions in the WP alias sets; Ref. [8] studied the matrix presentation for FFSP designs at s levels as well as the maximum resolution and minimum aberration properties for such FFSP designs, where s is a prime number; Ref. [9] proposed generalized minimum aberration criteria for two-level orthogonal FFSP designs in five different design scenarios and tabulated a catalog of optimal 12-, 16-, 20-, and 24-run FFSP designs under their generalized minimum aberration criteria by computer algorithm; Refs. [10,11] both considered construction of FFSP designs under the WP-minimum aberration criterion, which assumes that the whole plot factor are more important. The criteria considered in our paper is different from those in [7–11] . These differences lead to that, for two-level regular FFSP designs, the optimal ones under the criteria considered in [7–11] may not be optimal under criteria (1), (2), and (3), and vice versa. Ref. [18,19] proposed some sufficient and necessary conditions for the asymmetrical split-plot designs to contain various types of clear effects, while our work considers developing theoretical construction methods of regular two-level FFSP designs under the optimality criteria (1), (2), and (3). Ref. [26] mainly focused on the regular two-level FFSP designs with replicated settings of the level combinations for WP factors, while the level combinations for the regular two-level FFSP design in our work are not replicated.

Due to the complex structure of FFSP designs, although we provide a series of theoretical construction methods for optimal FFSP designs under criteria (1), (2), and (3), there are still many optimal $2^{(n_1+n_2)-(m_1+m_2)}$ FFSP designs which cannot be constructed by our methods. For example, the theoretical construction methods for optimal $2^{(n_1+n_2)-(m_1+m_2)}$

FFSP designs, under criteria (1), (2), and (3), which satisfy $\frac{N}{4} + 1 \leq n \leq \frac{5N}{16}$ are not covered in this paper. This is a future research direction worthy of study.

4. Conclusions

The $2^{(n_1+n_2)-(m_1+m_2)}$ designs enjoy a wide application when performing a 2^{n-m} design in a completely random order is impractical. A large body of work on choosing $2^{(n_1+n_2)-(m_1+m_2)}$ designs under the MA criterion and clear effect criterion was proposed. The GMC-FFSP criterion is a widely used criterion for assessing $2^{(n_1+n_2)-(m_1+m_2)}$ designs. This criterion advocates the FFSP designs with more effects at lower order confounding. The FFSP designs chosen under the GMC-FFSP criterion are preferable when we have prior information on the importance ordering of some effects. However, the theoretical construction methods of optimal $2^{(n_1+n_2)-(m_1+m_2)}$ designs under the GMC-FFSP criterion have not been studied yet.

This paper investigates theoretical construction methods of GMC-FFSP $2^{(n_1+n_2)-(m_1+m_2)}$ designs. In addition, from the angle that the more there are lower order effects of interest at lower order confounding, the better the $2^{(n_1+n_2)-(m_1+m_2)}$ designs, we propose optimality criteria for two kinds of design scenarios stated in the Introduction section. Some optimal $2^{(n_1+n_2)-(m_1+m_2)}$ designs for these two kinds of design scenarios are also theoretically constructed under the newly proposed optimality criteria. In the supplementary material, the R code for the proposed designs is provided.

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Abbreviations

The following abbreviations are used in this manuscript:

FF	fractional factorial
MA	minimum aberration
GMC	general minimum lower order confounding
FFSP	fractional factorial split-plot
WP	whole plot
SP	subplot
2FI	two-factor interaction

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