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Maximum Principles for Fractional Differential Inequalities with Prabhakar Derivative and Their Applications

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Abstract: This paper is devoted to studying a class of fractional differential equations (FDEs) with the Prabhakar fractional derivative of Caputo type in an analytical manner. At first, an estimate of the Prabhakar fractional derivative of a function at its extreme points is obtained. This estimate is used to formulate and prove comparison principles for related fractional differential inequalities. We then apply these comparison principles to derive pre-norm estimates of solutions and to obtain a uniqueness result for linear FDEs. The solution of linear FDEs with constant coefficients is obtained in closed form via the Laplace transform. For linear FDEs with variable coefficients, we apply the obtained comparison principles to establish an existence result using the method of lower and upper solutions. Two well-defined monotone sequences that converge uniformly to the actual solution of the problem are generated.

Keywords: fractional differential equations; Prabhakar kernel; maximum principles; existence and uniqueness results

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1. Introduction

The simplest form of the Mittag-Leffler (M-L) function $E_\alpha(z)$ was introduced by the Swedish mathematician Magnus Gustaf Mittag-Leffler in 1903 in relation to methods for summation of divergent series [1], and was further investigated in terms of certain of its properties in [1,2]. The two-parameter M-L function first appeared in a paper by Wiman [3]. Subsequently, several generalizations of the M-L function have been proposed [4–7]. The importance of M-L functions has increased as a result of their significance and applications in fractional calculus (FC), FDEs, and integral equations (IEs) of Abel [8] type. The M-L function has a role comparable to solutions to FDEs, and can be seen as a generalization of the exponential function, as the exponential function solves ODEs with constant coefficients. A three-parameter M-L function, commonly called the Prabhakar function [9], has received a great deal of interest in recent years. It is defined as follows:

$$E^{\gamma}_{\alpha,\beta}(z)=\sum_{j=0}^{\infty}\frac{\Gamma(\gamma+j)}{\Gamma(\gamma)\Gamma(\alpha j+\beta)}\frac{z^j}{j!},\ \alpha,\beta,\gamma\in C, Re(\alpha)>0, \tag{1}$$

where $\Gamma(.)$ denotes Euler's gamma function.

Investigating the literature reveals the presence of several definitions of fractional integrals and derivatives. Among these definitions are the Riemann-Liouville (R-L) integral and the Caputo and R-L derivatives, which have been studied and implemented for applied models. A generalized definition of fractional derivatives was introduced by Hilfer, and is nowadays known as the Hilfer fractional derivative [10]. This definition is a very appropriate and reasonable way to generalize the R-L and Caputo derivatives by introducing one additional real parameter $\nu \in [0,1]$. Modifying the R-L integral operator by extending its kernel to the Prabhakar function results in the Prabhakar integral [9].

Recently, FDEs with Prabhakar and Hilfer–Prabhakar derivatives have attracted many researchers. The properties of the Hilfer–Prabhakar derivative and the applications of related FDEs were discussed in [11]. The solution of the Cattaneo–Hristove diffusion model with Hilfer–Prabhakar derivative was obtained using Elzaki and Fourier Sine transform in [12]. Garra and Garrappa extended the use of the Prabhakar derivative and applied it in the theory of nonlinear heat conduction equation with memory [13]. Samaraiz et. al. used the generalized Laplace transform (LT) to obtain analytical solutions to certain classes of FDEs that involve the Hilfer-Prabhakar derivative and appear in several fields of science, such as engineering, economics, and physics [14]. Agarwal et. al. established new formulas of FDEs involving a family of M-L functions which encompass several fractional operators as special cases [15].

Minimum–maximum principles have been used to analyze the solutions of various types of functional equations in an analytical manner. Recently, several authors have developed minimum–maximum principles for linear fractional differential inequalities and implemented them to analyze the solutions of related FDEs, including both linear and nonlinear equations. We refer readers to [16–24], to mention only a few out of many in the literature. These minimum–maximum principles have been used to extend the method of upper and lower solutions to FDEs and to establish several existence results [25–29].

This work is devoted to extending the idea of the maximum principle and the method of upper and lower solutions for FDEs involving the Prabhakar derivative. We analyze the solutions of a class of linear FDEs in an analytical manner. Because the Prabhakar kernel is a generalization of several kernels of fractional derivatives, several FDEs in the literature are considered as particular cases of the current study. Even though there are few studies devoted to FDE's with Prabhakar derivative, the theory of these equations are far from being completed. In Section 2, we define the Prabhakar fractional operators, estimate the fractional derivative of a function at the extreme points, and derive certain comparison principles for fractional differential inequalities. Section 3 is devoted to the analysis of solutions for related linear FDEs. The method of lower–upper solutions is extended in Section 4 to establish existence results for linear FDEs with variable coefficients. Finally, we close in Section 5 with an illustrative example, concluding remarks, and future research directions.

2. Comparison Principles

We start with the definition of Prabhakar operators. For more about Prabhakar operators, we refer readers to [9,11,30].

Definition 1 ([30])**.** *Let $z \in L(0,T)$; then, the Prabhakar fractional integral (PFI) is defined by*

$$(\mathbb{E}^{\gamma}_{\alpha,\beta,\omega;0^+}\, z)(t) = \int_0^t (t-\zeta)^{\beta-1} E^{\gamma}_{\alpha,\beta}(\omega(t-\zeta)^{\alpha}) z(\zeta) d\zeta, \;\; t>0,$$

where $\alpha,\beta,\gamma,\omega \in C, Re(\alpha), Re(\beta) > 0$ and $E^{\gamma}_{\alpha,\beta}(t)$ is the Prabhakar function defined in (1).

The Prabhakar fractional derivative (PFD) is then defined as the left inverse operator of the PFI operator.

Definition 2 ([30])**.** *Let $z \in W^{(n,1)}([0,T])$; then, the PFD of Caputo type is defined by*

$$\begin{aligned} ({}^{C}\mathbb{D}^{\gamma}_{\alpha,\beta,\omega,0^+} z)(t) &= \mathbb{E}^{-\gamma}_{\alpha,n-\beta,w;a^+} D^n z(t), \;\; t>0 \\ &= \int_0^t (t-\zeta)^{n-\beta-1} E^{-\gamma}_{\alpha,n-\beta}(w(t-\zeta)^{\alpha}) z^{(n)}(\zeta) d\zeta, \end{aligned}$$

where $n = [\beta]+1, \alpha,\beta,\gamma,\omega \in C, Re(\alpha),$ and $Re(\beta) > 0$.

Here, $W^{(n,1)}(\Omega)$ denotes the Sobolev space, which is defined as

$$W^{(n,1)}[0,T] = \{f \in L^1(0,T) : \frac{d^k}{dt^k} f \in L^1(0,T), k = 1, \cdots, n\}.$$

In the current paper, we consider $\alpha, \beta, \gamma, w \in \mathbb{R}$ with $\alpha > 0$. It was proven in [31] that the Prabhakar kernel $e^{\gamma}_{\alpha,\beta}(t) = t^{\beta-1} E^{\gamma}_{\alpha,\beta}(-t^{\alpha}),\ t > 0$ is a locally integrable and completely monotone function provided that

$$0 < \alpha \leq 1,\ 0 < \alpha\gamma \leq \beta \leq 1. \tag{2}$$

The Prabhakar function $E^{\gamma}_{\alpha,1-\beta}(-t),\ t > 0$ is completely monotone [32] provided that

$$(G1) \qquad 0 < \alpha \leq 1,\ \gamma > 0,\ 1 - \beta \geq \alpha\gamma. \tag{3}$$

For $0 < \beta < 1$ and $w = -v < 0$, we have

$$({}^{C}\mathbb{D}^{\gamma}_{\alpha,\beta,-v,0^+} z)(t) = \int_0^t (t-\zeta)^{-\beta} E^{-\gamma}_{\alpha,1-\beta}(-v(t-\zeta)^{\alpha}) z'(s) d\zeta.$$

Lemma 1 ([33])**.** *The following relations of the Laplace transform hold true*

$$\mathcal{L}[t^{\beta-1} E^{\gamma}_{\alpha,\beta}(\lambda t^{\alpha}); p] = \frac{p^{\alpha\gamma-\beta}}{(p^{\alpha} - \lambda)^{\gamma}},\ Re(p) > 0,\ |p| > |\lambda|^{1\ \alpha} \tag{4}$$

$$\mathcal{L}[({}^{C}\mathbb{D}^{\gamma}_{\alpha,\beta,w,0} f)(t); p] = p^{\beta-\alpha\gamma}(p^{\alpha} - w)^{\gamma} \Bigg[F(p) - \sum_{k=0}^{m-1} p^{-k-1} f^{(k)}(0) \Bigg]. \tag{5}$$

In this paper, we consider the Prabhakar derivative $\mathbb{D}^{\gamma}_{\alpha,\beta,-v,0^+}$, where $v > 0, 0 < \beta < 1$, and we assume that condition (G1) holds true.

Lemma 2. *Let (G1) hold true, and let $z \in W^{(1,1)}([0,T]) \cap C[0,T]$ attain global maximum at $t_* \in (0,T]$. Then, the following fractional inequality holds true:*

$$({}^{C}\mathbb{D}^{\gamma}_{\alpha,\beta,-v,0^+} z)(t_*) \geq t_*^{-\beta} E^{-\gamma}_{\alpha,1-\beta}(-v t_*^{\alpha})(z(t_*) - z(0)) \geq 0. \tag{6}$$

Proof. Consider the auxiliary function $\nu(t) = z(t) - z(t_*),\ t \in [0,T]$. Then, it holds that

$$({}^{C}\mathbb{D}^{\gamma}_{\alpha,\beta,-v,0^+} z)(t) = ({}^{C}\mathbb{D}^{\gamma}_{\alpha,\beta,-v,0^+} \nu)(t),\ \nu(t_*) = 0, \text{ and } \nu(t) \leq 0,\ t \in [0,t_*]. \tag{7}$$

Let $\kappa(t) = t^{-\beta} E^{-\gamma}_{\alpha,1-\beta}(-v t^{\alpha})$; then, we have

$$({}^{C}\mathbb{D}^{\gamma}_{\alpha,\beta,-v,0^+} \nu)(t_*) = \int_0^{t_*} \kappa(t-\zeta) \nu'(\zeta) d\zeta.$$

Through integration by parts and using Equation (7) along with the fact that $\kappa'(t) < 0, t > 0$, we have

$$\begin{aligned}
({}^{C}\mathbb{D}^{\gamma}_{\alpha,\beta,-v,0^+}\nu)(t_*) &= \kappa(t_*-\zeta)\nu(\zeta)|_0^{t_*} + \int_0^{t_*} \kappa'(t_*-\zeta)\nu(\zeta)d\zeta \\
&= \lim_{t\to t_*} \kappa(t_*-\zeta)\nu(\zeta) - \kappa(t_*)\nu(0) + \int_0^{t_*} \kappa'(t_*-\zeta)\nu(\zeta)d\zeta \\
&= -\kappa(t_*)\nu(0) + \int_0^{t_*} \kappa'(t_*-\zeta)\nu(\zeta)d\zeta \geq -\kappa(t_*)\nu(0) \\
&= -t_*^{-\beta}E^{-\gamma}_{\alpha,1-\beta}(-vt_*^{\alpha})(z(0)-z(t_*)) \\
&= t_*^{-\beta}E^{-\gamma}_{\alpha,1-\beta}(-vt_*^{\alpha})(z(t_*)-z(0)),
\end{aligned}$$

which completes the proof. □

Remark 1. *Because $\nu(t) \in W^{(1,1)}([0,T]) \cap C[0,T]$ and $\nu(t_*) = \nu'(t_*) = 0$, it holds that $\nu(t) = (t-t_*)q(t)$ for some $q \in C[0,T]$. Thus, $\lim_{t\to t_*} \kappa(t_*-\zeta)\nu(\zeta) = 0$, and the improper integral $\int_0^{t_*} \kappa'(t_*-\zeta)\nu(\zeta)d\zeta$ is well-defined.*

The following comparison principle is a direct consequence of the result in Lemma 2.

Proposition 1. *Let (G1) hold true and let $z \in W^{(1,1)}[0,T] \cap C[0,T]$ satisfy*

$$\mathbb{P}(z) = ({}^{C}\mathbb{D}^{\gamma}_{\alpha,\beta,-v,0^+}z)(t) + k(t)z \leq 0, t > 0, \tag{8}$$

$$z(0) \leq 0, \tag{9}$$

where $k(t) > 0$, $t \in [0,T]$. Then, $z(t) \leq 0$, $t \in [0,T]$.

Proof. Assume by contradiction that the result is untrue. Then, there exists $t_m > 0$ with $z(t_m) = z_m = \max_{t\in[0,T]} z(t) > 0$, and thus $({}^{C}\mathbb{D}^{\gamma}_{\alpha,\beta,-v,0^+}z)(t_m) \geq 0$ by virtue of the result in Lemma 2. We then have

$$\mathbb{P}(z)(t_m) = ({}^{C}\mathbb{D}^{\gamma}_{\alpha,\beta,-v,0^+}z)(t_m) + k(t_m)z(t_m) \geq k(t_m)z(t_m) > 0,$$

which contradicts the assumption. □

In the following proposition, we prove that the comparison principle in Proposition 1 is valid in the case where $k(t)$ is not necessarily positive on $[0,T]$.

Proposition 2. *Let (G1) hold true and let $z \in W^{(1,1)}[0,T] \cap C[0,T]$ satisfy (8)–(9), where $-\frac{\Gamma(\beta+1)}{T^{\beta}} \leq k(t) \leq 0$, $t \in [0,T]$. Then, $z(t) \leq 0$, $t \in [0,T]$.*

Proof. Assume the result is untrue and we shall reach a contradiction. Because $z(0) \leq 0$, there exists $0 < t_0 < t_1$ and the following holds true:

$$z(t) \leq 0,\ t \in [0,t_0],\ \ z(t) \geq 0,\ t \in [t_0,t_1].$$

Let $z(t_M) = \max_{t\in[t_0,t_1]} z(t) > 0$.
Applying the Prabhakar fractional integral operator to Inequality (8) yields

$$(\mathbb{E}^{\gamma}_{\alpha,\beta,-v,0^+}\, {}^{C}\mathbb{D}^{\gamma}_{\alpha,\beta,-v,0^+}z)(t) + (\mathbb{E}^{\gamma}_{\alpha,\beta,-v,0^+}\, k\, z)(t) \leq 0,$$

or

$$z(t) - z(0) + (\mathbb{E}^{\gamma}_{\alpha,\beta,-v,0^+}\, k\, z)(t) \leq 0,$$

or

$$z(t) + (\mathbb{E}^{\gamma}_{\alpha,\beta,-v,0^+}\, k\, z)(t) \le z(0) \le 0.$$

Then, it holds that

$$z(t_M) + (\mathbb{E}^{\gamma}_{\alpha,\beta,-v,0^+}\, k\, z)(t_M) \le 0. \tag{10}$$

We have

$$\begin{aligned}
(\mathbb{E}^{\gamma}_{\alpha,\beta,-v,0^+}\, k\, z)(t_M) &= \int_0^{t_M} (t_M-\zeta)^{\beta-1} E^{\gamma}_{\alpha,\beta}(-v(t_M-\zeta)^{\alpha}) k(\zeta) z(\zeta) d\zeta \\
&= \int_0^{t_0} (t_M-\zeta)^{\beta-1} E^{\gamma}_{\alpha,\beta}(-v(t_M-\zeta)^{\alpha}) k(\zeta) z(\zeta) d\zeta \\
&\quad + \int_{t_0}^{t_M} (t_M-\zeta)^{\beta-1} E^{\gamma}_{\alpha,\beta}(-v(t_M-\zeta)^{\alpha}) k(\zeta) z(\zeta) d\zeta = I_1 + I_2.
\end{aligned}$$

Because $z(t), k(t) \le 0$ on $[0, t_0]$, we have $I_1 \ge 0$. On $[t_0, t_M]$, we have $0 \le z(t) \le z(t_M)$, and thus

$$\begin{aligned}
(\mathbb{E}^{\gamma}_{\alpha,\beta,-v,0^+}\, k\, z)(t_M) &\ge \int_{t_0}^{t_M} (t_M-\zeta)^{\beta-1} E^{\gamma}_{\alpha,\beta}(-v(t_M-\zeta)^{\alpha}) k(\zeta) z(\zeta) d\zeta \\
&\ge -\frac{\Gamma(\beta+1)}{T^{\beta}} z(t_M) \int_{t_0}^{t_M} (t_M-\zeta)^{\beta-1} E^{\gamma}_{\alpha,\beta}(-v(t_M-\zeta)^{\alpha}) d\zeta \\
&= -\frac{\Gamma(\beta+1)}{T^{\beta}} z(t_M) (t_M-t_0)^{\beta} E^{\gamma}_{\alpha,\beta+1}(-v(t_M-t_0)^{\alpha}) \\
&> -\frac{\Gamma(\beta+1)}{T^{\beta}} z(t_M) \frac{T^{\beta}}{\Gamma(\beta+1)} = -z(t_M).
\end{aligned} \tag{11}$$

The last equation yields

$$(\mathbb{E}^{\gamma}_{\alpha,\beta,-v,0^+}\, k\, z)(t_M) + z(t_M) > 0,$$

which contradicts the result in (10), and completes the proof. □

Combining the results in Propositions 1 and 2, we arrive at the following result.

Lemma 3. *Let (G1) hold true and let $z \in W^{(1,1)}[0,T] \cap C([0,T])$ satisfy*

$$\mathbb{P}(z) = ({}^{C}\mathbb{D}^{\gamma}_{\alpha,\beta,-v,0^+} z)(t) + k(t)z \le 0, t > 0,\ z(0) \le 0,$$

where $-\frac{\Gamma(\beta+1)}{T^{\beta}} \le k(t),\ t \in [0,T]$. Then, $z(t) \le 0,\ t \in [0,T]$.

Applying the above result to $-z$ yields

Lemma 4. *Let $z \in W^{(1,1)}[0,T] \cap C([0,T])$ satisfy*

$$\mathbb{P}(z) = ({}^{C}\mathbb{D}^{\gamma}_{\alpha,\beta,-v,0^+} z)(t) + k(t)z \ge 0, t > 0,\ z(0) \ge 0,$$

where $-\frac{\Gamma(\beta+1)}{T^{\beta}} \le k(t),\ t \in [0,T]$. Then, $z(t) \ge 0,\ t \in [0,T]$.

3. Linear Fractional Equations

We consider the linear fractional initial value problem (FIVP) with variable coefficients

$$\mathbb{P}(z) = ({}^{C}\mathbb{D}^{\gamma}_{\alpha,\beta,-v,0^+} z)(t) \quad + \quad k(t)z = g(t),\ t > 0,\ z(0) = z_0, \tag{12}$$

where $z \in W^{(1,1)}[0,T] \cap C[0,T], k, g \in C[0,T]$.

Lemma 5. *Let (G1) hold true and let $z \in W^{(1,1)}([0,T]) \cap C[0,T]$ be possible solutions to (12). If $k(t) > 0$, then it holds that*

$$||z||_{[0,T]} = \max_{t\in[0,T]} |z(t)| \leq M_A = \max_{t\in[0,T]} \{|\frac{g(t)}{k(t)}|, z_0\}, \tag{13}$$

provided that M_A exists.

Proof. Because $k(t) > 0$, we have $M_A k(t) > |g(t)|$. Let $w_1(t) = z(t) - M_A$; then, it holds that

$$\begin{aligned} \mathbb{P}(w_1) = \mathbb{P}(z - M_A) &= ({}^{C}\mathbb{D}^{\gamma}_{\alpha,\beta,-v,0^+} z)(t) + k(t)z - k(t)M_A \\ &= g(t) - k(t)M_A \leq 0. \end{aligned}$$

The last equation with $w_1(0) = z(0) - M_A \leq 0$ proves that $w_1 \leq 0$ on $[0,T]$, or

$$z(t) \leq M_A, \;\; t \in [0,T]. \tag{14}$$

Let $w_2(t) = -z - M_A$. Then, it holds that

$$\mathbb{P}(w_2) \;\; = \;\; -g(t) - k(t)M_A \leq 0.$$

Because $M_A \geq |z(0)|$, then $w_2(0) = -z(0) - M_A \leq 0$. Thus, $w_2 \leq 0$ on $[0,T]$, or

$$-z(t) \leq M_A, \;\; t \in [0,T]. \tag{15}$$

The proof is completed by combining the results in Equations (14) and (15). □

Lemma 6. *(Uniqueness Result) Let (G1) hold true and let $z_1, z_2 \in W^{(1,1)}([0,T]) \cap C[0,T]$ be possible solutions to (12). If $k(t) > 0$, then $z_1 = z_2$, $t \in [0,T]$.*

Proof. Let $w(t) = z_1(t) - z_2(t)$, $t \in [0,T]$. One can easily show that $w(t)$ satisfies the following:

$$\mathbb{P}(w) \;\; = \;\; ({}^{C}\mathbb{D}^{\gamma}_{\alpha,\beta,-v,0^+} w)(t) + k(t)w = 0, \; t > 0, \; w(0) = 0.$$

Using Equation (13), we have $||w||_{[0,T]} = 0$, which completes the proof. □

In the following result, we obtain the solution of linear FDEs with constant coefficients using the Laplace transform. This is a particular case of the Cauchy problem discussed in [11], and we present it here for the seek of completeness.

Lemma 7. *Let $z \in W^{(1,1)}(\Omega)$ and let f be piece-wise continuous for which the Laplace transform exists. The solution of*

$$({}^{C}\mathbb{D}^{\gamma}_{\alpha,\beta,-v,0^+} z)(t) \;\; + \;\; \mu z = f(t), \; \mu \in \mathbb{R}, \;\; t > 0 \tag{16}$$

$$z(0) \;\; = \;\; z_0, \tag{17}$$

is provided by

$$z(t) = z_0 h_1(t) + f * h_2(t), \tag{18}$$

where

$$h_1(t) = \sum_{j=0}^{\infty}(-\mu)^j t^{\beta j} E^{\gamma j}_{\alpha,\beta j+1}(-vt^\alpha) \tag{19}$$

$$h_2(t) = \sum_{j=0}^{\infty}(-\mu)^j t^{\beta(j+1)-1} E^{\gamma(j+1)}_{\alpha,\beta(j+1)}(-vt^\alpha). \tag{20}$$

Proof. Because $z \in W^{(1,1)}(\Omega)$, the Laplace transform of $({}^C\mathbb{D}^{\gamma}_{\alpha,\beta,-v,0^+}z)(t)$ exists. Applying the Laplace transform to Equation (16) and using Equation (5), we have

$$\begin{aligned} \mathcal{L}\left({}^C\mathbb{D}^{\gamma}_{\alpha,\beta,-v,0^+}z\right)(t) + \mu\mathcal{L}(z;p) &= \mathcal{L}(f;p) \\ p^{\beta-\alpha\gamma}(p^\alpha+v)^\gamma(\mathcal{L}(z;p) - p^{-1}z_0) + \mu\mathcal{L}(z;p) &= \mathcal{L}(f;p), \end{aligned}$$

or

$$\left(p^{\beta-\alpha\gamma}(p^\alpha+v)^\gamma + \mu\right)\mathcal{L}(z;p) = z_0 p^{\beta-\alpha\gamma-1}(p^\alpha+v)^\gamma + \mathcal{L}(f;p)$$

which yields

$$\mathcal{L}(z;p) = z_0\frac{p^{\beta-\alpha\gamma-1}(p^\alpha+v)^\gamma}{p^{\beta-\alpha\gamma}(p^\alpha+v)^\gamma+\mu} + \frac{\mathcal{L}(f;p)}{p^{\beta-\alpha\gamma}(p^\alpha+v)^\gamma+\mu}, \tag{21}$$

$$= z_0 H_1(p) + \mathcal{L}(f;p)H_2(p). \tag{22}$$

We have

$$\begin{aligned} H_1(p) &= \frac{p^{\beta-\alpha\gamma-1}(p^\alpha+v)^\gamma}{p^{\beta-\alpha\gamma}(p^\alpha+v)^\gamma+\mu} = \sum_{j=0}^{\infty}(-\mu)^j\frac{p^{\alpha\gamma j-(\beta j+1)}}{(p^\alpha+v)^{\gamma j}}, \quad \left|\frac{\mu p^{\alpha\gamma-\beta}}{(p^\alpha+v)^\gamma}\right| < 1, \\ H_2(p) &= \frac{1}{p^{\beta-\alpha\gamma}(p^\alpha+v)^\gamma+\mu} = \sum_{j=0}^{\infty}(-\mu)^j\frac{p^{\alpha\gamma(p+1)-\beta(j+1)}}{(p^\alpha+v)^{\gamma(j+1)}}, \quad \left|\frac{\mu p^{\alpha\gamma-\beta}}{(p^\alpha+v)^\gamma}\right| < 1. \end{aligned}$$

Using Equation (4), we have

$$\begin{aligned} \mathcal{L}^{-1}(H_1;t) &= \sum_{j=0}^{\infty}(-\mu)^j t^{\beta j} E^{\gamma j}_{\alpha,\beta j+1}(-vt^\alpha) = h_1(t), \\ \mathcal{L}^{-1}(H_2;t) &= \sum_{j=0}^{\infty}(-\mu)^j t^{\beta(j+1)-1} E^{\gamma(j+1)}_{\alpha,\beta(j+1)}(-vt^\alpha) = h_2(t). \end{aligned}$$

The result follows by applying the convolution result of the Laplace transform. □

4. Monotone Iterative Sequences of Lower and Upper Solutions

We consider the FIVP with variable coefficients

$$\mathbb{P}(z) = ({}^C\mathbb{D}^{\gamma}_{\alpha,\beta,-v,0^+}z)(t) \quad + \quad k(t)\,z = g(t),\ t>0,\ z(0) = z_0, \tag{23}$$

where g is piece-wise continuous and its Laplace transform exists.

Definition 3. *A function* $\underline{z} \in W^{(1,1)}[0,T] \cap C[0,T]$ *is called a lower solution to problem (23) if it satisfies*

$$\mathbb{P}(\underline{z}) = ({}^C\mathbb{D}^{\gamma}_{\alpha,\beta,-v,0^+}\underline{z})(t) + k(t)\underline{z} \le g(t),\ t>0,\ \underline{z}(0) \le z_0.$$

Analogously, a function $\widehat{z} \in W^{(1,1)}[0,T] \cap C[0,T]$ that satisfies the reversed inequalities is called an upper solution. If it holds that

$$\underline{z} \leq \widehat{z}, \; \forall \, t \in [0,T],$$

then $\underline{z}$ and $\widehat{z}$ are ordered lower and upper solutions. In the following, we assume that there exist ordered lower $\underline{z}^{(0)}$ and upper $\hat{z}^{(0)}$ solutions for the problem (23) and that C is an arbitrary constant that satisfies

$$(G2) \qquad C \geq \max_{t\in[0,T]} k(t) \geq -\frac{\Gamma(\beta+1)}{T^{\beta}}. \tag{24}$$

Under the above condition and using the result in Lemma 4, one can easily show that any lower and upper solutions are ordered.

Lemma 8. *Let (G1) and (G2) hold true. For $n \in \mathbb{N}$, let $\underline{z}^{(n)} \in W^{(1,1)}[0,T] \cap C[0,T]$ be the solution of*

$$({}^{C}\mathbb{D}^{\gamma}_{\alpha,\beta,-v,0^+}\underline{z}^{(n)})(t) \;+\; C\underline{z}^{(n)} = (C-k(t))\underline{z}^{(n-1)} + g(t), \tag{25}$$

$$\underline{z}^{(n)}(0) \;=\; z_0. \tag{26}$$

Then, it holds that

1. *$\underline{z}^{(n)}; n \in \mathbb{N}$, is an increasing sequence;*
2. *$\underline{z}^{(n)} \leq u$, on $[0,T]$ for all $n \in \mathbb{N}$.*

Proof. We prove the result by induction argument. We first show that the result holds true for $n = 1$. Because $\underline{z}^{(0)}$ is a lower solution to problem (23), it holds that

$$({}^{C}\mathbb{D}^{\gamma}_{\alpha,\beta,-v,0^+}\underline{z}^{(0)})(t) + k(t)\underline{z}^{(0)} \leq g(t), \; t > 0, \; \underline{z}^{(0)}(0) \leq z_0. \tag{27}$$

Furthemore, using Equations (25) and (26) for $n = 1$, we have

$$({}^{C}\mathbb{D}^{\gamma}_{\alpha,\beta,-v,0^+}\underline{z}^{(1)})(t) + C\underline{z}^{(1)} = (C-k(t))\underline{z}^{(0)} + g(t), \; t > 0, \; \underline{z}^{(1)}(0) = z_0. \tag{28}$$

The above two equations yield

$$({}^{C}\mathbb{D}^{\gamma}_{\alpha,\beta,-v,0^+}\big(\underline{z}^{(1)} - \underline{z}^{(0)}\big))(t) + C(\underline{z}^{(1)} - \underline{z}^{(0)}) \geq 0,$$

which together with $\underline{z}^{(1)}(0) \geq \underline{z}^{(0)}$ proves that $\underline{z}^{(1)} \geq \underline{z}^{(0)}$, $\forall t \in [0,T]$ by virtue of the result in Lemma 4. We now assume that the result is true for $k \leq n$ and prove it holds true for $k = n+1$. From Equations (25) and (26), we have

$$({}^{C}\mathbb{D}^{\gamma}_{\alpha,\beta,-v,0^+}\big(\underline{z}^{(n+1)} - \underline{z}^{(n)}\big))(t) + C\big(\underline{z}^{(n+1)} - \underline{z}^{(n)}\big)(t) = (C-k(t))\big(\underline{z}^{(n)} - \underline{z}^{(n-1)}\big)(t).$$

Using the induction argument along with fact that $C \geq k(t)$, we have

$$({}^{C}\mathbb{D}^{\gamma}_{\alpha,\beta,-v,0^+}\big(\underline{z}^{(n+1)} - \underline{z}^{(n)}\big))(t) + C\big(\underline{z}^{(n+1)} - \underline{z}^{(n)}\big)(t) \geq 0,$$

which together with $\big(\underline{z}^{(n+1)} - \underline{z}^{(n)}\big))(0) = 0$ proves that $\underline{z}^{(n+1)}(t) \geq \underline{z}^{(n)}(t)$, $\forall t \in [0,T]$ and completes the proof.

By subtracting Equation (25) from Equation (23), we have

$$({}^{C}\mathbb{D}^{\gamma}_{\alpha,\beta,-v,0^+}\big(z - \underline{z}^{(n)}\big))(t) + k(t)z - Cz^{(n)} = -(C-k(t))z^{(n-1)}.$$

The above equation yields

$$({}^{C}\mathbb{D}^{\gamma}_{\alpha,\beta,-v,0^+}(z-\underline{z}^{(n)}))(t)+k(t)(z-z^{(n)})=(C-k(t))(z^{(n)}-z^{(n-1)})\geq 0,$$

as $\{z^{(n)};n\in\mathbb{N}\}$ is an increasing sequence and $C\geq k(t)$. The last inequality with $z(0)=z^{(n)}(0)$ proves that $z\geq z^{(n)}$ on $[0,T]$.
□

Applying analogous arguments, one can reach the following result.

Lemma 9. *Let (G1) and (G2) hold true. For $n\in\mathbb{N}$, let $\hat{z}^{(n)}\in W^{(1,1)}[0,T]\cap C[0,T]$ be the solution of*

$$({}^{C}\mathbb{D}^{\gamma}_{\alpha,\beta,-v,0^+}\hat{z}^{(n)})(t) + C\hat{z}^{(n)} = (C-k(t))\hat{z}^{(n-1)}+g(t), \tag{29}$$

$$\hat{z}^{(n)}(0) = z_0. \tag{30}$$

Then, it holds that

1. *$\hat{z}^{(n)};n\in\mathbb{N}$, is a decreasing sequence;*
2. *$\hat{z}^{(n)}\geq z$, on $[0,T]$, for all $n\in\mathbb{N}$.*

Lemma 10. *For the two sequences defined in (25)–(26) and (29)–(30), the following hold true:*

1. *$\underline{z}^{(n)};n\in\mathbb{N}$ is a sequence of lower solutions to problem (23);*
2. *$\hat{z}^{(n)};n\in\mathbb{N}$ is a sequence of upper solutions to problem (23);*
3. *$\underline{z}^{(n)}\leq\hat{z}^{(n)},\ n\in\mathbb{N}$.*

Proof. Using Equation (25) along with the facts that $\underline{z}^{(n)};n\in\mathbb{N}$ is an increasing sequence and $C\geq k(t)$, we have

$$\begin{aligned}({}^{C}\mathbb{D}^{\gamma}_{\alpha,\beta,-v,0^+}\underline{z}^{(n)})(t)+k(t)\underline{z}^{(n)}(t) &= (C-k(t))\underline{z}^{(n-1)}(t)+g(t)-C\underline{z}^{(n)}(t)+k(t)\underline{z}^{(n)}(t)\\ &= -(C-k(t))(\underline{z}^{(n)}(t)-\underline{z}^{(n-1)}(t))+g(t)\leq g(t).\end{aligned}$$

The above inequality with $\underline{z}^{(n)}(t)=z_0$ proves that $\underline{z}^{(n)}(t)$ is a lower solution for each $n\in\mathbb{N}$;

Analogous to the proof of (1);

The result is true for $n=0$; assuming that the result holds for $k=n-1$, we have

$$({}^{C}\mathbb{D}^{\gamma}_{\alpha,\beta,-v,0^+}(\hat{z}^{(n)}-\underline{z}^{(n)}))(t)+C(\hat{z}^{(n)}-\underline{z}^{(n)})(t)=(C-k(t))(\hat{z}^{(n-1)}-\underline{z}^{(n-1)})(t)\geq 0,$$

which together with $\hat{z}^{(n)}(0)=\underline{z}^{(n)}(0)$ proves that $\hat{z}^{(n)}(t)\geq\underline{z}^{(n)}(t),\ t\in[0,T]$.
□

Lemma 11. *Let (G1) and (G2) hold true. The sequences of the lower and upper solutions defined in (25), (26), (29) and (30) converge uniformly to the actual solution of problem (23).*

Proof. Because (G_1) and (G_2) hold true, then per Lemmas 8 and 9 the sequence $\{\underline{z}^{(n)}\}\in W^{(1,1)}[0,T]$ of lower solutions of problem (23) is an increasing and bounded sequence of continuous functions on the compact set $[0,T]$. It follows that it is convergent to a continuous function, say, z^*.

Similarly, using Lemmas 8 and 9, the sequence $\{\hat{z}^{(n)}\}\in W^{(1,1)}[0,T]$ of upper solutions of problem (23) is a decreasing and bounded sequence of continuous functions on the compact set $[0,T]$. It follows that it is convergent to a continuous function, say, z_*.

Using Dini's Theorem (see [34], p. 150) and the fact that $\{\underline{z}^{(n)}\}$ and $\{\hat{z}^{(n)}\}$ are monotone convergent sequences of continuous functions defined on the compact set $[0, T]$, it follows that the convergence is uniform. Because $\{\underline{z}^{(n)}\}$ converges uniformly, it holds that

$$\lim_{n\to\infty}({}^{C}\mathbb{D}^{\gamma}_{\alpha,\beta,-v,0^+}\underline{z}^{(n)})(t) = ({}^{C}\mathbb{D}^{\gamma}_{\alpha,\beta,-v,0^+}z_*)(t).$$

Taking the limit in Equation (25), we have

$$\begin{aligned}\lim_{n\to\infty}({}^{C}\mathbb{D}^{\gamma}_{\alpha,\beta,-v,0^+}\underline{z}^{(n)})(t) &+ C\lim_{n\to\infty}\underline{z}^{(n)} = (C-k(t))\lim_{n\to\infty}\underline{z}^{(n-1)} + g(t),\\ ({}^{C}\mathbb{D}^{\gamma}_{\alpha,\beta,-v,0^+}z_*)(t) &+ Cz_* = (C-k(t))z_* + g(t),\end{aligned}$$

or

$$({}^{C}\mathbb{D}^{\gamma}_{\alpha,\beta,-v,0^+}z_*)(t) = k(t)z_* + g(t),$$

which together with $z_*(0) = z(0)$ proves that z_* is a solution to problem (23). By applying analogous steps, one can prove that z^* is a solution to problem (23), which completes the proof. $\square$

5. Discussion

As an illustrative example, we consider the FIVP

$$\mathbb{P}(z) = ({}^{C}\mathbb{D}^{\gamma}_{\alpha,\beta,-v,0^+}z)(t) + e^{-t}z = g(t),\ t > 0,\ z(0) = 0, \tag{31}$$

where $g(t) \geq 0 \in C[0, T]$. Let $\hat{z}$ be the solution of

$$({}^{C}\mathbb{D}^{\gamma}_{\alpha,\beta,-v,0^+}\hat{z})(t) + \hat{z} = g(t),\ t > 0,\ \hat{z}(0) = 0.$$

The solution $\hat{z}$ is provided by Equation (18) in Lemma 7. By virtue of the result in Lemma 4, we have $\hat{z} \geq 0$ on $[0, T]$. Because $0 \leq e^{-t} \leq 1,\ t \geq 0$, we have

$$\mathbb{P}(\hat{z}) = ({}^{C}\mathbb{D}^{\gamma}_{\alpha,\beta,-v,0^+}\hat{z})(t) + e^{-t}\hat{z} \leq ({}^{C}\mathbb{D}^{\gamma}_{\alpha,\beta,w,0^+}\hat{z})(t) + \hat{z} = g(t),$$

which together with $z(\hat{0}) = 0$ proves that $\hat{z}$ is an upper solution to (31). Let $\underline{z}$ be the solution of

$$({}^{C}\mathbb{D}^{\gamma}_{\alpha,\beta,-v,0^+}\underline{z})(t) = g(t),\ t > 0,\ \underline{z}(\hat{0}) = 0.$$

Then, $\underline{z} \geq 0$ on $[0, T]$, and it holds that

$$\mathbb{P}(\underline{z}) = ({}^{C}\mathbb{D}^{\gamma}_{\alpha,\beta,-v,0^+}\underline{z})(t) + e^{-t}\underline{z} \geq ({}^{C}\mathbb{D}^{\gamma}_{\alpha,\beta,-v,0^+}\underline{z})(t) = g(t),$$

which proves that $\underline{z}$ is a lower solution to (31). We have $\underline{z} \leq \hat{z}$ ordered lower and upper solutions to (31), and thus the existence of a solution is guaranteed by virtue of the result in Lemma 11.

As a conclusion, we have successfully extended the idea of maximum principles to deal with FDEs with the Prabhakar derivative. The Prabhakar kernel involves several parameters, and thus dealing with FDEs with the Prabhakar derivative is more difficult in comparison to those with standard derivatives, such as Riemann-Liouville and Caputo derivatives. Several analytical results, including existence and uniqueness results, have been formulated and proven in this paper. These analytical results provide better understanding of FDEs with the Prabhakar derivative, and include several existing results in the literature as particular cases. The analysis carried out in this paper can be extended to nonlinear FDEs and multi-term FDEs, which will be performed in future work. Another research direction is to study the current FDEs without imposing the necessary condition (G1) in Equation (3). The fixed-point theory can be extended to establish existence and uniqueness results. We refer readers to [35] for recent developments in fixed point theory and to [36] for the extensive literature on techniques applied to nonlinear systems.

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