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Boundedness of Fractional Integrals on Grand Weighted Herz–Morrey Spaces with Variable Exponent

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Abstract: In this paper, we introduce grand weighted Herz–Morrey spaces with a variable exponent and prove the boundedness of fractional integrals on these spaces.

Keywords: fractional integrals; grand Herz spaces; weighted Herz spaces; grand weighted Herz–Morrey spaces

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1. Introduction

In the last two decades, under the influence of some applications revealed in [1], there has been a vast amount of research into the so-called variable exponent spaces and the operators in them. The theory of such variable exponent Lebesgue, Orlicz, Lorentz and Sobolev function spaces has been developed—we refer to the books [2–4] and the surveying papers [5–8]. Herz spaces with a variable exponent have been recently introduced in [9–11]. In [12], variable parameters were used to define continual Herz spaces, and the boundedness of sublinear operators in these spaces was proved. The boundedness of other operators such as the Riesz potential operator and the Marcinkiewicz integrals was proved in [13,14].

The concept of Morrey spaces $L^{p,\lambda}$ was introduced by C. Morrey in 1938 (see [15]) in order to study regularity questions that appear in the calculus of variations. They describe local regularity more precisely than Lebesgue spaces and are widely used not just in harmonic analysis but also in PDEs. Meskhi introduced the idea of grand Morrey spaces $L^{r),\theta,\lambda}$ and derived the boundedness of a class of integral operators (Hardy–Littlewood maximal functions, Calderón–Zygmund singular integrals and potentials) in these spaces—see ([16]). Moreover, Izuki [11] defined the Herz–Morrey spaces with a variable exponent and proved the boundedness of vector-valued sublinear operators on these spaces.

In [17], the idea of grand variable Herz spaces $\dot{K}^{\alpha,p),\theta}_{q(\cdot)}(\mathbb{R}^n)$ was introduced, and the boundedness of sublinear operators $\dot{K}^{\alpha,p),\theta}_{q(\cdot)}(\mathbb{R}^n)$ was proved. Muckenhoupt in [18] established the theory of weights, called the Muckenhoupt A_p theory, in the study of weighted function spaces and greatly developed real analysis. Weighted norm inequalities for the maximal operator on variable Lebesgue spaces were proved in [19]. The boundedness of the fractional integrals on variable weighted Lebesgue spaces by using the extrapolation theorem can be checked in [20]. The idea of grand weighted Herz spaces with a variable exponent was introduced in [21], and the boundedness of fractional integrals on these spaces was proved. In this article, we introduce the concept of grand weighted Herz–Morrey spaces with a variable exponent and prove the boundedness of the fractional integral operator in these spaces. There are four sections in this article. The first section is dedicated to the introduction, and the second section contains some basic definitions and lemmas. We introduce the concept of grand weighted Herz–Morrey spaces in Section 3, and the boundedness of the fractional integral operator on grand weighted Herz–Morrey spaces is proved in the last section.

2. Preliminaries

For this section we refer to [2,3,10,11,22,23].

2.1. Lebesgue Space with Variable Exponent

Assume that $G \subseteq \mathbb{R}^n$ is an open set and $p(\cdot) : G \to [1, \infty)$ is a real-valued measurable function. Let the following condition hold:

$$1 \leq p_-(G) \leq p_+(G) < \infty, \tag{1}$$

where

(i) $p_- := \operatorname{ess\,inf}_{g \in G} \; p(g)$

(ii) $p_+ := \operatorname{ess\,sup}_{g \in G} \; p(g)$.

The Lebesgue space $L^{p(\cdot)}(G)$ is the space of measurable functions f_1 on G such that

$$I_{L^{p(\cdot)}}(f_1) = \int_G |f_1(g)|^{p(g)} dg < \infty,$$

where the norm is defined as

$$\|f_1\|_{L^{p(\cdot)}(G)} = \operatorname{ess\,inf}\left\{\gamma > 0 : I_{L^{p(\cdot)}}\left(\frac{f_1}{\gamma}\right) \leq 1\right\},$$

which is the Banach function space, and $p'(g) = \frac{p(g)}{p(g)-1}$ denotes the conjugate exponent of $p(g)$.

Next, we define the space $L^{p(\cdot)}_{\text{loc}}(G)$ as

$$L^{p(\cdot)}_{\text{loc}}(G) := \left\{\kappa : \kappa \in L^{p(\cdot)}(K) \; \textit{for all compact subsets } K \subset G\right\}.$$

Now, we define the log-condition,

$$|\eta(z_1) - \eta(z_2)| \leq \frac{C}{-\ln|z_1 - z_2|}, \quad |z_1 - z_2| \leq \frac{1}{2}, \quad z_1, z_2 \in G, \tag{2}$$

where $C = C(\eta) > 0$ is not dependent on z_1, z_2.

For the decay condition, let $\eta_\infty \in (1, \infty)$, such that

$$|\eta(z_1) - \eta_\infty| \leq \frac{C}{\ln(e + |z_1|)}, \tag{3}$$

$$|\eta(z_1) - \eta_0| \leq \frac{C}{\ln|z_1|}, |z_1| \leq \frac{1}{2}, \tag{4}$$

Equation (4) holds for $\eta_0 \in (1, \infty)$ in the case of homogenous Herz spaces. We adopt the following notations in this paper:

(i) The Hardy–Littlewood maximal operator M for $f \in L^1_{\text{loc}}(G)$ is defined as

$$Mf(g) := \sup_{t>0} t^{-n} \int_{D(g,r)} |f(g)|dg \quad (g \in G),$$

where $D(g,t) := \{y \in G : |g - y| < t\}$.

(ii) The set $\mathcal{P}(G)$ is the collection of all $p(\cdot)$ satisfying $p_- > 1$ and $p_+ < \infty$.

(iii) A weight is a locally integrable and positive function that is defined on $\mathbb{R}^n$ and can be written as $\omega(G) := \int_G \omega(g)dg$ for a weight w and measurable set G.

(iv) The set of $p(\cdot)$ satisfying (3) and (4) is represented by $LH(\mathbb{R}^n)$.

C is a constant that is independent of the main parameters involved, and its value varies from line to line.

Lemma 1 (Generalized Hölder's inequality [17]). *Assume that G is a measurable subset of $\mathbb{R}^n$, and $1 \le p_-(G) \le p_+(G) \le \infty$. Then,*

$$\|fg\|_{L^{r(\cdot)}(G)} \le C\|f\|_{L^{p(\cdot)}(G)}\|g\|_{L^{q(\cdot)}(G)}$$

holds, where $f \in L^{p(\cdot)}(G)$, $g \in L^{q(\cdot)}(G)$ and $\frac{1}{r(z)} = \frac{1}{p(z)} + \frac{1}{q(z)}$ for every $z \in G$.

2.2. Herz Spaces with Variable Exponent

We adopt the following notations in this subsection:

(a) $\chi_k = \chi_{R_k}$.

(b) $R_k = D_k \setminus D_{k-1}$.

(c) $D_k = D(0, 2^k) = \{x \in \mathbb{R}^n : |x| < 2^k\}$ for all $k \in \mathbb{Z}$.

(d) $R_{t,\tau} := D(0,\tau) \setminus D(0,t)$.

Definition 1. *Let $r \in [1,\infty)$, $\alpha \in \mathbb{R}$ and $s(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. The homogenous Herz space $\dot{K}^{\alpha,r}_{s(\cdot)}(\mathbb{R}^n)$ is defined by*

$$\dot{K}^{\alpha,r}_{s(\cdot)}(\mathbb{R}^n) = \left\{ f \in L^{q(\cdot)}_{\text{loc}}(\mathbb{R}^n \setminus \{0\}) : \|f\|_{\dot{K}^{\alpha,r}_{s(\cdot)}(\mathbb{R}^n)} < \infty \right\}, \tag{5}$$

where

$$\|f\|_{\dot{K}^{\alpha,r}_{s(\cdot)}(\mathbb{R}^n)} = \left(\sum_{k=-\infty}^{k=\infty} \|2^{k\alpha} f\chi_k\|^r_{L^{s(\cdot)}} \right)^{\frac{1}{r}}.$$

Definition 2. *Let $r \in [1,\infty)$, $\alpha \in \mathbb{R}$ and $s(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. The non-homogenous Herz space $K^{\alpha,r}_{s(\cdot)}(\mathbb{R}^n)$ is defined by*

$$K^{\alpha,r}_{s(\cdot)}(\mathbb{R}^n) = \left\{ f \in L^{s(\cdot)}_{\text{loc}}(\mathbb{R}^n \setminus \{0\}) : \|f\|_{K^{\alpha,r}_{s(\cdot)}(\mathbb{R}^n)} < \infty \right\}, \tag{6}$$

where

$$\|f\|_{K^{\alpha,r}_{s(\cdot)}(\mathbb{R}^n)} = \left(\sum_{k=-\infty}^{k=\infty} \|2^{k\alpha} f\chi_k\|^r_{L^{s(\cdot)}} \right)^{\frac{1}{r}} + \|f\|_{L^{s(\cdot)}(D(0,1))}.$$

2.3. The Variable Exponent Muckenhoupt Weights

Let $r(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, and w is a weight. The weighted Lebesgue space $L^{r(\cdot)}$ is the collection of all complex-valued measurable functions f such that $fw^{\frac{1}{p(\cdot)}} \in L^{r(\cdot)}(\mathbb{R}^n)$. $L^{r(\cdot)}(w)$ is a Banach space, and its norm is given by

$$\|f\|_{L^{r(\cdot)}(w)} := \|fw^{\frac{1}{r(\cdot)}}\|_{L^{r(\cdot)}},$$

where $r'(\cdot)$ is the conjugate exponent of $r(\cdot)$ given by $\frac{1}{r(\cdot)} + \frac{1}{r'(\cdot)} = 1$. Next, we define Muckenhoupt classes by starting with classical Muckenhoupt weights.

Definition 3. *Suppose $r(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, a weight w is called an $A_{r(\cdot)}$ weight if*

$$\sup_{D:ball} \frac{1}{|D|} \|w^{\frac{1}{p(\cdot)}} \chi_D\|_{L^{r(\cdot)}} \|w^{\frac{-1}{r(\cdot)}} \chi_D\|_{L^{r'(\cdot)}} < \infty. \tag{7}$$

The set $A_{r(\cdot)}$ consists of all $A_{r(\cdot)}$ weights.

Now, we give the definitions of the Muckenhoupt classes A_r with $r = 1, \infty$.

Definition 4. *(i) A weight w is called a Muckenhoupt A_1 weight if $Mw(z) \leq w(z)$ holds for almost every $z \in \mathbb{R}^n$. The set A_1 collection of all Muckenhoupt A_1 weights. For each $w \in A_1$*

$$[w]_{A_1} := \sup_{D:ball} \left(\frac{1}{|D|} \int_D w(z) dz \|w^{-1}\|_{L^\infty(D)} \right).$$

Then, the finite value of $[w]_{A_1}$ is called A_1 constant.

(ii) A weight is called Muckenhoupt weight A_∞ if the weight belongs to the following set:

$$A_\infty := \bigcup_{1<r<\infty} A_r.$$

Definition 5. *Suppose $r(\cdot) \in \mathcal{P}(\mathbb{R}^n)$; a weight is called a $A'_{r(\cdot)}$ weight if*

$$\sup_{D:ball} |D|^{-P_D} \|w\chi_D\|_{L^1} \|w^{-1}\chi_D\|_{L^{r'(\cdot)/r(\cdot)}} < \infty, \tag{8}$$

where $P_D := (\frac{1}{|D|} \int_D \frac{1}{r(z)} dz)^{-1}$ is the harmonic average of $r(\cdot)$ over D. The set $A'_{r(\cdot)}$ consists of all $A'_{r(\cdot)}$ weights.

Definition 6. *Let $0 < \alpha < n$ and $r_1(\cdot), r_2(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ such that $\frac{1}{r_2(z)} \equiv \frac{1}{r_1(z)} - \frac{\alpha}{n}$. A weight w is called $A(r_1(\cdot), r_2(\cdot))$ weight is*

$$\|w\chi_D\|_{L^{r_2(\cdot)}} \|w^{-1}\chi_D\|_{L^{r'_1(\cdot)}} \leq |D|^{1-\frac{\alpha}{n}},$$

holds for all balls $D \subset \mathbb{R}^n$.

Lemma 2 ([22]). *Assume that G is a Banach function space, and the Hardy–Littlewood maximal operator M is weakly bounded on G, such that following inequality holds for all $g \in G$ and $\lambda > 0$:*

$$\|\chi_{(Mg>\lambda)}\|_G \leq \lambda^{-1} \|g\|_G, \tag{9}$$

then, we have

$$\sup_{D:ball} \frac{1}{|D|} \|\chi_D\|_G \|\chi_D\|_{G'} < \infty. \tag{10}$$

Lemma 3 ([22]). *Let M be bounded on the associate space X', and X is a Banach function space. Then, there exists a constant $\delta \in (0,1)$ such that for all measurable sets $E \subset D$ and for all balls $D \subset \mathbb{R}^n$,*

$$\frac{\|\chi_E\|_X}{\|\chi_D\|_X} \leq \left(\frac{|E|}{|D|} \right)^\delta.$$

Let $r_2(\cdot) \in \mathcal{P}(\mathbb{R}^n) \cap LH(\mathbb{R}^n)$, $w^{r_2(\cdot)} \in A_{r_2(\cdot)}$, $w^{-r_2'(\cdot)} \in A_{r_2'(\cdot)}$, and for $\delta_1, \delta_2 \in (0,1)$

$$\frac{\|\chi_E\|_{L^{r_2(\cdot)}(w^{r_2(\cdot)})}}{\|\chi_D\|_{L^{r_2(\cdot)}(w^{r_2(\cdot)})}} = \frac{\|\chi_E\|_{L^{r_2'(\cdot)}(w^{-r_2'(\cdot)})'}}{\|\chi_D\|_{L^{r_2'(\cdot)}(w^{-r_2'(\cdot)})'}} \leq \left(\frac{|E|}{|D|}\right)^{\delta_1}. \tag{11}$$

$$\frac{\|\chi_E\|_{L^{r_2'(\cdot)}(w^{r_2'(\cdot)})'}}{\|\chi_D\|_{L^{r_2'(\cdot)}(w^{r_2'(\cdot)})'}} \leq \left(\frac{|E|}{|D|}\right)^{\delta_2}. \tag{12}$$

For more details, see [22].

3. Grand Weighted Herz–Morrey Spaces with Variable Exponent

In this section, we first define grand Herz spaces and then introduce the concept of grand weighted Herz–Morrey spaces with a variable exponent.

Definition 7 ([17])**.** *Let $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$, $r \in [1,\infty)$, $s : \mathbb{R}^n \to [1,\infty)$, $\theta > 0$. A grand Herz space with variable exponent $\dot{K}^{\alpha(\cdot),r),\theta}_{s(\cdot)}(\mathbb{R}^n)$ is defined by*

$$\dot{K}^{\alpha(\cdot),r),\theta}_{s(\cdot)}(\mathbb{R}^n) = \left\{ g \in L^{s(\cdot)}_{\text{loc}}(\mathbb{R}^n \setminus \{0\}) : \|g\|_{\dot{K}^{\alpha(\cdot),r),\theta}_{s(\cdot)}(\mathbb{R}^n)} < \infty \right\},$$

where

$$\begin{aligned} \|g\|_{\dot{K}^{\alpha(\cdot),r),\theta}_{s(\cdot)}(\mathbb{R}^n)} &= \sup_{\delta>0} \left(\delta^\theta \sum_{k\in\mathbb{Z}} 2^{k\alpha(\cdot)r(1+\epsilon)} \|g\chi_k\|^{r(1+\delta)}_{L^{s(\cdot)}} \right)^{\frac{1}{r(1+\delta)}} \\ &= \sup_{\delta>0} \delta^{\frac{\theta}{r(1+\delta)}} \|g\|_{\dot{K}^{\alpha(\cdot),r(1+\delta)}_{s(\cdot)}(\mathbb{R}^n)}. \end{aligned}$$

Now, we define variable exponent-weighted Lebesgue space.

Definition 8 ([22])**.** *Let $\Omega \subset \mathbb{R}^n$ be a measurable set and w a positive and locally integrable function on Ω. The $L^{r(\cdot)}_{\text{loc}}(\Omega, w)$ is the collection of all functions g that satisfy the following condition: for all compact sets $E \subset \Omega$, there is a constant $\lambda > 0$ such that*

$$\int_E \left| \frac{g(z)}{\lambda} \right|^{r(z)} w(z)dz < \infty.$$

Definition 9. *Let $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, $0 < r < \infty$, $0 \leq \lambda < \infty$, $\alpha \in \mathbb{R}$, $\theta > 0$. The homogeneous grand weighted Herz–Morrey spaces with variable exponents $M\dot{K}^{\alpha,r),\theta}_{\lambda,q(\cdot)}(w)$ are the collection of $L^{q(\cdot)}_{\text{loc}}(\mathbb{R}^n/\{0\}, w)$ such that*

$$M\dot{K}^{\alpha,r),\theta}_{\lambda,q(\cdot)}(w) := \left\{ L^{q(\cdot)}_{\text{loc}} \mathbb{R}^n/(\{0\}) : \|f\|_{M\dot{K}^{\alpha,r),\theta}_{\lambda,q(\cdot)}(w)} < \infty \right\}, \tag{13}$$

where

$$\|f\|_{M\dot{K}^{\alpha,r),\theta}_{\lambda,q(\cdot)}(w)} = \sup_{\delta>0} \sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda} \left(\delta^\theta \sum_{k\in\mathbb{Z}} 2^{k\alpha r(1+\delta)} \|f\chi_k\|^{r(1+\delta)}_{L^{q(\cdot)}(w)} \right)^{\frac{1}{r(1+\delta)}}.$$

Non-homogeneous grand weighted Herz–Morrey spaces can be defined in a similar way. When $\lambda = 0$, the grand weighted Herz–Morrey spaces with a variable exponent become grand weighted Herz spaces with a variable exponent; see [21].

4. Boundedness of the Fractional Integrals

Definition 10. *Fractional integrals are given as follows:*

Let $0 < \zeta < n$. Then, the fractional integral operator I^{ζ} is defined by

$$I^{\zeta} f(z_1) := \int\limits_{\mathbb{R}^n} \frac{f(z_2)}{|z_1 - z_2|^{n-\zeta}} dz_2. \tag{14}$$

Theorem 1 ([22]). *Let $r_1(\cdot) \in \mathcal{P}(\mathbb{R}^n) \cap LH(\mathbb{R}^n)$, $0 < \zeta < n/r_1+$ and $\sigma := (n/\zeta)'$. Define $r_2(\cdot)$ by $1/r_2(\cdot) = 1/r_1(\cdot) - \zeta/n$. Then, for all weights w such that $(r_2(\cdot)/\sigma, w^{\sigma})$ is an M-pair, I^{ζ} is bounded from $L^{r_1(\cdot)}(w^{r_1(\cdot)})$ to $L^{r_2(\cdot)}(w^{r_2(\cdot)})$.*

Theorem 2. *Let $1 < r < \infty$, $q_2(\cdot) \in \mathcal{P}(\mathbb{R}^n) \cap LH(\mathbb{R}^n)$, $w^{q_2(\cdot)} \in A_1$, $\delta_1, \delta_2 \in (0,1)$ be the constants appearing in (11) and (12), respectively. α and ζ are such that*

(i) $-n\delta_1 < \alpha < n\delta_2 - \zeta$

(ii) $0 < \zeta < n(\delta_1 + \delta_2)$.

Define $q_1(\cdot)$ by $1/q_2(\cdot) = 1/q_1(\cdot) - \zeta/n$. Then, fractional integral operator I^{ζ} is a bounded operator from $M\dot{K}^{\alpha,r),\theta}_{\lambda,q_2(\cdot)}(w^{q_2(\cdot)})$ to $M\dot{K}^{\alpha,r),\theta}_{\lambda,q_1(\cdot)}(w^{q_1(\cdot)})$.

Proof. Let $M\dot{K}^{\alpha,r),\theta}_{\lambda,q_2(\cdot)}(w^{q_2(\cdot)})$, $f_j := f\chi_j$ for any $j \in \mathbb{Z}$; then, $f = \sum\limits_{j=-\infty}^{\infty} f_j$, and we have

$$\begin{aligned}
\|I^{\zeta} f\|_{M\dot{K}^{\alpha,r),\theta}_{\lambda,q_2(\cdot)}(w^{q_2(\cdot)})} &= \sup_{\delta>0} \sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda} \left(\delta^{\theta} \sum_{k\in\mathbb{Z}} 2^{k\alpha r(1+\delta)} \|\chi_k I^{\zeta} f\|^{r(1+\delta)}_{L^{q_2(\cdot)}(w^{q_2(\cdot)})} \right)^{\frac{1}{r(1+\delta)}} \\
&= \sup_{\delta>0} \sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda} \left(\delta^{\theta} \sum_{k\in\mathbb{Z}} 2^{k\alpha r(1+\delta)} \sum_{j=-\infty}^{\infty} \|\chi_k (I^{\zeta} f_j)\|^{r(1+\delta)}_{L^{q_2(\cdot)}(w^{q_2(\cdot)})} \right)^{\frac{1}{r(1+\delta)}} \\
&\leq \sup_{\delta>0} \sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda} \left(\delta^{\theta} \sum_{k\in\mathbb{Z}} 2^{k\alpha r(1+\delta)} \sum_{j\leq k-2} \|\chi_k (I^{\zeta} f_j)\|^{r(1+\delta)}_{L^{q_2(\cdot)}(w^{q_2(\cdot)})} \right)^{\frac{1}{r(1+\delta)}} \\
&\quad + \sup_{\delta>0} \sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda} \left(\delta^{\theta} \sum_{k\in\mathbb{Z}} 2^{k\alpha r(1+\delta)} \sum_{j=k-1}^{k+1} \|\chi_k (I^{\zeta} f_j)\|^{r(1+\delta)}_{L^{q_2(\cdot)}(w^{q_2(\cdot)})} \right)^{\frac{1}{r(1+\delta)}} \\
&\quad + \sup_{\delta>0} \sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda} \left(\delta^{\theta} \sum_{k\in\mathbb{Z}} 2^{k\alpha r(1+\delta)} \sum_{j\geq k+2} \|\chi_k (I^{\zeta} f_j)\|^{r(1+\delta)}_{L^{q_2(\cdot)}(w^{q_2(\cdot)})} \right)^{\frac{1}{r(1+\delta)}} \\
&=: E_1 + E_2 + E_3.
\end{aligned}$$

As operator I^{ζ} is bounded on a weighted Lebesgue space, and so for E_2,

$$\begin{aligned}
E_2 &\leq \sup_{\delta>0} \sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda} \left(\delta^{\theta} \sum_{k\in\mathbb{Z}} 2^{k\alpha r(1+\delta)} \sum_{j=k-1}^{k+1} \|\chi_k (I^{\zeta} f_j)\|^{r(1+\delta)}_{L^{q_2(\cdot)}(w^{q_2(\cdot)})} \right)^{\frac{1}{r(1+\delta)}} \\
&\leq \sup_{\delta>0} \sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda} \left(\delta^{\theta} \sum_{k\in\mathbb{Z}} 2^{k\alpha r(1+\delta)} \sum_{j=k-1}^{k+1} \|(f\chi_j)\|^{r(1+\delta)}_{L^{q_1(\cdot)}(w^{q_1(\cdot)})} \right)^{\frac{1}{r(1+\delta)}} \\
&\leq \sup_{\delta>0} \sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda} \left(\delta^{\theta} \sum_{k\in\mathbb{Z}} 2^{k\alpha r(1+\delta)} \|(f\chi_k)\|^{r(1+\delta)}_{L^{q_1(\cdot)}(w^{q_1(\cdot)})} \right)^{\frac{1}{r(1+\delta)}} \\
&\leq \|f\|_{M\dot{K}^{\alpha,r),\theta}_{\lambda,q_1(\cdot)}(w^{q_1(\cdot)})}.
\end{aligned}$$

For E_1, by using the size condition and Hölder's inequality, we have

$$|I^{\zeta}(f_j)(z_1)|\chi_k(z_1) \le \chi_k(z_1)\int_{\mathbb{R}^n}|z_1-z_2|^{\zeta-n}|f_j(z_2)|dz_2$$
$$\le 2^{k(\zeta-n)}\|f_j\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}\|\chi_j\|_{(L^{q_1(\cdot)}(w^{q_1(\cdot)}))'}\cdot\chi_k(z_1).$$

By using Lemma 2, we get

$$\|\left(I^{\zeta}f_j\right)\chi_k\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})} \le 2^{k\zeta}\|f_j\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}\|\chi_j\|_{L^{(q_1(\cdot)}(w^{q_1(\cdot)}))'}2^{-kn}\|\chi_{D_k}\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}$$
$$\le 2^{k\zeta}\|f_j\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}\|\chi_j\|_{(L^{q_1(\cdot)}(w^{q_1(\cdot)}))'}\|\chi_{D_k}\|^{-1}_{(L^{q_2(\cdot)}(w^{q_2(\cdot)}))'}.$$

By using (12), we have

$$\|\left(I^{\zeta}f_j\right)\chi_k\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}$$
$$\le 2^{k\zeta}\|f_j\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}\|\chi_j\|_{(L^{q_1(\cdot)}(w^{q_1(\cdot)}))'}\|\chi_{D_k}\|^{-1}_{(L^{q_2(\cdot)}(w^{q_2(\cdot)}))'}$$
$$=2^{k\zeta}\|f_j\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}\|\chi_j\|_{(L^{q_1(\cdot)}(w^{q_1(\cdot)}))'}\|\chi_{D_k}\|^{-1}_{(L^{q_2(\cdot)}(w^{q_2(\cdot)}))'}\frac{\|\chi_{D_j}\|_{(L^{q_2(\cdot)}(w^{q_2(\cdot)}))'}}{\|\chi_{D_k}\|_{(L^{q_2(\cdot)}(w^{q_2(\cdot)}))'}}$$
$$\le 2^{k\zeta}2^{n\delta_2(j-k)}\|f_j\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}\|\chi_j\|_{(L^{q_1(\cdot)}(w^{q_1(\cdot)}))'}\|\chi_{D_j}\|^{-1}_{(L^{q_2(\cdot)}(w^{q_2(\cdot)}))'}.$$

By the boundednesss of $I^{\zeta}: L^{q_1(\cdot)}(w^{q_1(\cdot)}) \to L^{q_2(\cdot)}(w^{q_2(\cdot)})$, using the inequality $2^{j\zeta}\chi_{D_j} \le (I^{\zeta}f_{D_j})(x)$, we have

$$\|\chi_{D_j}\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})} \le 2^{-j\zeta}\|I^{\zeta}\chi_{B_j}\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})} \le 2^{-j\zeta}\|\chi_{D_j}\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}.$$

By using Lemma 2 again, we obtain

$$\|\chi_{D_j}\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})} \le 2^{-j\zeta}\|\chi_{D_j}\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})} \le 2^{j(n-\zeta)}\|\chi_{D_j}\|^{-1}_{(L^{q_1(\cdot)}(w^{q_1(\cdot)}))'}$$
$$\le 2^{j(n-\zeta)}\|\chi_j\|^{-1}_{(L^{q_1(\cdot)}(w^{q_1(\cdot)}))'}.$$

By using the above inequalities, we get

$$\|\left(I^{\zeta}f_j\right)\chi_k\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}$$
$$\le 2^{k\zeta}2^{n\delta_2(j-k)}2^{j(n-\zeta)}\|f_j\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}\|\chi_{D_j}\|^{-1}_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}\|\chi_{D_j}\|^{-1}_{(L^{q_2(\cdot)}(w^{q_2(\cdot)}))'}$$
$$=2^{(\zeta-n\delta_2)(k-j)}\|f_j\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}\left(2^{-jn}\|\chi_{D_j}\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}\|\chi_{D_j}\|_{(L^{q_2(\cdot)}(w^{q_2(\cdot)}))'}\right)^{-1}$$
$$\le 2^{(\zeta-n\delta_2)(k-j)}\|f_j\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}.$$

It is known that $\zeta - n\delta_2 + \alpha < 0$; thus, we consider two cases $1 < r(1+\delta) < \infty$ and $0 < r(1+\delta) \leq 1$. Now, consider the first case $1 < r(1+\delta) < \infty$; by applying Hölder's inequality, we get

$$
\begin{aligned}
E_1 &\leq \sup_{\delta>0}\sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda}\left(\delta^\theta \sum_{k=-\infty}^{\infty} 2^{k\alpha r(1+\delta)}\left(\sum_{j=-\infty}^{k-2}\|\chi_k I^\zeta(f_j)\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}\right)^{r(1+\delta)}\right)^{\frac{1}{r(1+\delta)}} \\
&\leq \sup_{\delta>0}\sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda}\left(\delta^\theta \sum_{k=-\infty}^{\infty}\left(2^{\alpha j}\sum_{j=-\infty}^{k-2} 2^{(\zeta-n\delta_2+\alpha)(k-j)}\|f_j\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}\right)^{r(1+\delta)}\right)^{\frac{1}{r(1+\delta)}} \\
&\leq \sup_{\delta>0}\sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda}\left[\delta^\theta \sum_{k=-\infty}^{\infty}\left(\sum_{j=-\infty}^{k-2} 2^{\alpha jr(1+\delta)}\|f_j\|^{r(1+\delta)}_{L^{q_1(\cdot)}(w^{q_1(\cdot)})} 2^{(\zeta-n\delta_2+\alpha)(k-j)r(1+\delta)/2}\right)\right. \\
&\quad\times\left.\left(\sum_{j=-\infty}^{k-2} 2^{(\zeta-n\delta_2+\alpha)(k-j)(r(1+\delta))'/2}\right)^{\frac{r(1+\delta)}{(r(1+\delta))'}}\right]^{\frac{1}{r(1+\delta)}}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \sup_{\delta>0}\sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda}\left(\delta^\theta \sum_{k=-\infty}^{\infty}\sum_{j=-\infty}^{k-2} 2^{\alpha jr(1+\delta)}\|f_j\|^{r(1+\delta)}_{L^{q_1(\cdot)}(w^{q_1(\cdot)})} 2^{(\zeta-n\delta_2+\alpha)(k-j)r(1+\delta)/2}\right)^{\frac{1}{r(1+\delta)}} \\
&\leq \sup_{\delta>0}\sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda}\left(\delta^\theta \sum_{j=-\infty}^{\infty} 2^{\alpha jr(1+\delta)}\|f_j\|^{r(1+\delta)}_{L^{q_1(\cdot)}(w^{q_1(\cdot)})} \sum_{k\leq j-2} 2^{(\zeta-n\delta_2+\alpha)(k-j)r(1+\delta)/2}\right)^{\frac{1}{r(1+\delta)}} \\
&\leq \sup_{\delta>0}\sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda}\left(\delta^\theta \sum_{l=-\infty}^{\infty} 2^{\alpha jr(1+\delta)}\|f_j\|^{r(1+\delta)}_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}\right)^{\frac{1}{r(1+\delta)}} \\
&\leq \|f\|_{M\dot{K}^{\alpha,r),\theta}_{\lambda,q_1(\cdot)}(w^{q_1(\cdot)})}.
\end{aligned}
$$

For $0 < r(1+\delta) \leq 1$, we get

$$
\begin{aligned}
E_1 &\leq \sup_{\delta>0}\sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda}\left(\delta^\theta \sum_{k=-\infty}^{\infty}\left(2^{\alpha j}\sum_{j=-\infty}^{k-2} 2^{(\zeta-n\delta_2+\alpha)(k-j)}\|f_j\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}\right)^{r(1+\delta)}\right)^{\frac{1}{r(1+\delta)}} \\
&\leq \sup_{\delta>0}\sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda}\left(\delta^\theta \sum_{k=-\infty}^{\infty}\sum_{j=-\infty}^{k-2} 2^{\alpha jr(1+\delta)}\|f_j\|^{r(1+\delta)}_{L^{q_1(\cdot)}(w^{q_1(\cdot)})} 2^{(\zeta-n\delta_2+\alpha)(k-j)r(1+\delta)}\right)^{\frac{1}{r(1+\delta)}} \\
&\leq \sup_{\delta>0}\sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda}\left(\delta^\theta \sum_{j=-\infty}^{\infty} 2^{\alpha jr(1+\delta)}\|f_j\|^{r(1+\delta)}_{L^{q_1(\cdot)}(w^{q_1(\cdot)})} \sum_{k\leq j-2} 2^{(\zeta-n\delta_2+\alpha)(k-j)r(1+\delta)}\right)^{\frac{1}{r(1+\delta)}} \\
&\leq \sup_{\delta>0}\sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda}\left(\delta^\theta \sum_{l=-\infty}^{\infty} 2^{\alpha jr(1+\delta)}\|f_j\|^{r(1+\delta)}_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}\right)^{\frac{1}{r(1+\delta)}} \\
&\leq \|f\|_{M\dot{K}^{\alpha,r),\theta}_{\lambda,q_1(\cdot)}(w^{q_1(\cdot)})}.
\end{aligned}
$$

Now, we estimate E_3; by using the size condition and Hölder's inequality, for $j,k \in \mathbb{Z}$ with $j \geq k+2$, we have

$$|I^{\zeta}(f_j)(z_1)|\chi_k(z_1) \leq \chi_k(z_1)\int_{D_j}|z_1-z_2|^{\zeta-n}|f_j(z_2)|dz_2$$
$$\leq 2^{j(\zeta-n)}\|f_j\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}\|\chi_j\|_{L^{q_1'(\cdot)}(w^{-q_1'(\cdot)})}\cdot\chi_k(z_1).$$

By taking the $L^{p_2(\cdot)}(w^{p_2(\cdot)})$-norm, we get

$$\begin{aligned}&\left\|\left(I^{\zeta}f_j\right)\chi_k\right\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}\\ \leq&2^{j(-n+\zeta)}\|f_j\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}\|\chi_j\|_{L^{q_1'(\cdot)}(w^{-q_1'(\cdot)})}\|\chi_k\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}\\ \leq&2^{j(-n+\zeta)}\|f_j\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}\|\chi_j\|_{L^{q_1'(\cdot)}(w^{-q_1'(\cdot)})}\|\chi_j\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}\frac{\|\chi_k\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}}{\|\chi_j\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}}\\ \leq&2^{j(-n+\zeta)}2^{n\delta_1(k-j)}\|f_j\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}\|\chi_j\|_{L^{q_1'(\cdot)}(w^{-q_1'(\cdot)})}\|\chi_j\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}.\end{aligned}$$

By using the definition of $A(p_1(\cdot),p_2(\cdot))$, we obtain

$$\begin{aligned}\|\chi_j\|_{L^{q_1'(\cdot)}(w^{-q_1'(\cdot)})}\|\chi_j\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})} &\leq \|\chi_{D_j}\|_{L^{q_1'(\cdot)}(w^{-q_1'(\cdot)})}\|\chi_{D_j}\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}\\ &\leq \|w^{-1}\chi_{D_j}\|_{L^{q_1'(\cdot)}}\|w\chi_{D_j}\|_{L^{q_2(\cdot)}}\\ &\leq 2^{jn(1-\zeta/n)}.\end{aligned}$$

Hence, we have

$$\begin{aligned}&\left\|\left(I^{\zeta}f_j\right)\chi_k\right\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}\\ &\leq 2^{j(-n+\zeta)}2^{n\delta_1(k-j)}\|f_j\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}\|\chi_j\|_{L^{q_1'(\cdot)}(w^{-q_1'(\cdot)})}\|\chi_j\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}\\ &\leq 2^{j(-n+\zeta)}2^{n\delta_1(k-j)}2^{jn(1-\zeta/n)}\|f_j\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}\\ &\leq 2^{n\delta_1(k-j)}\|f_j\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}.\end{aligned}$$

Therefore, we get

$$\begin{aligned}E_3 &= \sup_{\delta>0}\sup_{k_o\in\mathbb{Z}}2^{-k_0\lambda}\left(\delta^{\theta}\sum_{k\in\mathbb{Z}}2^{k\alpha r(1+\delta)}\sum_{j\geq k+2}\|\chi_k(I^{\zeta}f_j)\|^{r(1+\delta)}_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}\right)^{\frac{1}{r(1+\delta)}}\\ &\leq \sup_{\delta>0}\sup_{k_o\in\mathbb{Z}}2^{-k_0\lambda}\left(\delta^{\theta}\sum_{k\in\mathbb{Z}}2^{k\alpha r(1+\delta)}\sum_{j\geq k+2}2^{n\delta_1(k-j)}\|f_j\|^{r(1+\delta)}_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}\right)^{\frac{1}{r(1+\delta)}}\\ &\leq \sup_{\delta>0}\sup_{k_o\in\mathbb{Z}}2^{-k_0\lambda}\left(\delta^{\theta}\sum_{k\in\mathbb{Z}}\left(\sum_{j\geq k+2}2^{(\alpha+n\delta_1)(k-j)}2^{\alpha j}\|f_j\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}\right)^{r(1+\delta)}\right)^{\frac{1}{r(1+\delta)}}.\end{aligned}$$

For $\alpha + n\delta_1 > 0$, we consider the two cases: $1 < r(1+\delta) < \infty$ and $0 < r(1+\delta) \leq 1$. Now, consider the first case $1 < r(1+\delta) < \infty$; by applying Hölder's inequality, we get

$$
\begin{aligned}
E_3 &\leq \sup_{\delta>0}\sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda}\left(\delta^\theta \sum_{k\in\mathbb{Z}}\left(\sum_{j\geq k+2} 2^{(\alpha+n\delta_1)(k-j)}2^{\alpha j}\|f_j\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}\right)^{r(1+\delta)}\right)^{\frac{1}{r(1+\delta)}} \\
&\leq \sup_{\delta>0}\sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda}\left[\delta^\theta \sum_{k=-\infty}^{\infty}\left(\sum_{j\geq k+2} 2^{\alpha jr(1+\delta)}\|f_j\|^{r(1+\delta)}_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}2^{(n\delta_1+\alpha)(k-j)r(1+\delta)/2}\right)\right. \\
&\quad\times\left.\left(\sum_{j\geq k+2} 2^{(n\delta_1+\alpha)(k-j)(r(1+\delta))'/2}\right)^{\frac{r(1+\delta)}{(r(1+\delta))'}}\right]^{\frac{1}{r(1+\delta)}} \\
&\leq \sup_{\delta>0}\sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda}\left(\delta^\theta \sum_{k=-\infty}^{\infty}\sum_{j\geq k+2} 2^{\alpha jr(1+\delta)}\|f_j\|^{r(1+\delta)}_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}2^{(n\delta_1+\alpha)(k-j)r(1+\delta)/2}\right)^{\frac{1}{r(1+\delta)}} \\
&\leq \sup_{\delta>0}\sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda}\left(\delta^\theta \sum_{j=-\infty}^{\infty} 2^{\alpha jr(1+\delta)}\|f_j\|^{r(1+\delta)}_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}\sum_{k\leq j-2}2^{(n\delta_1+\alpha)(k-j)r(1+\delta)/2}\right)^{\frac{1}{r(1+\delta)}} \\
&\leq \sup_{\delta>0}\sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda}\left(\delta^\theta \sum_{j=-\infty}^{\infty} 2^{\alpha jr(1+\delta)}\|f_j\|^{r(1+\delta)}_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}\right)^{\frac{1}{r(1+\delta)}} \\
&\leq \|f\|_{M\dot{K}^{\alpha,r),\theta}_{\lambda,q_1(\cdot)}(w^{q_1(\cdot)})}.
\end{aligned}
$$

For $0 < r(1+\delta) \leq 1$, we get

$$
\begin{aligned}
E_3 &\leq \sup_{\delta>0}\sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda}\left(\delta^\theta \sum_{k\in\mathbb{Z}}\left(\sum_{j\geq k+2} 2^{(\alpha+n\delta_1)(k-j)}2^{\alpha j}\|f_j\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}\right)^{r(1+\delta)}\right)^{\frac{1}{r(1+\delta)}} \\
&\leq \sup_{\delta>0}\sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda}\left(\delta^\theta \sum_{k=-\infty}^{\infty}\sum_{j\geq k+2} 2^{\alpha jr(1+\delta)}\|f_j\|^{r(1+\delta)}_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}2^{(n\delta_1+\alpha)(k-j)r(1+\delta)}\right)^{\frac{1}{r(1+\delta)}} \\
&\leq \sup_{\delta>0}\sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda}\left(\delta^\theta \sum_{j=-\infty}^{\infty} 2^{\alpha jr(1+\delta)}\|f_j\|^{r(1+\delta)}_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}\sum_{k\leq j-2}2^{(n\delta_1+\alpha)(k-j)r(1+\delta)}\right)^{\frac{1}{r(1+\delta)}} \\
&\leq \sup_{\delta>0}\sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda}\left(\delta^\theta \sum_{j=-\infty}^{\infty} 2^{\alpha jr(1+\delta)}\|f_j\|^{r(1+\delta)}_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}\right)^{\frac{1}{r(1+\delta)}} \\
&\leq \|f\|_{M\dot{K}^{\alpha,r),\theta}_{\lambda,q_1(\cdot)}(w^{q_1(\cdot)})},
\end{aligned}
$$

which completes the proof. □

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