fractal and fractional

Article

On the Analysis of a Neutral Fractional Differential System with Impulses and Delays

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Abstract: In this paper, we investigate the exact and approximate controllability, finite time stability, and β–Hyers–Ulam–Rassias stability of a fractional order neutral impulsive differential system. The controllability criteria is incorporated with the help of a fixed point approach. The famous generalized Grönwall inequality is used to study the finite time stability and β–Hyers–Ulam–Rassias stability. Finally, the main results are verified with the help of an example.

Keywords: fractional differential system; impulses and delays; mild solution; β–Hyers–Ulam–Rassias stability

MSC: 34D10; 45N05

Citation: Guo, L.; Ali Shah, K.; Bai, S.; Zada, A. On the Analysis of a Neutral Fractional Differential System with Impulses and Delays. *Fractal Fract.* **2022**, *6*, 673. https://doi.org/10.3390/fractalfract6110673

Academic Editor: Tassos C. Bountis

Received: 22 October 2022
Accepted: 11 November 2022
Published: 14 November 2022

1. Introduction

Fractional dynamical systems are systems that contain fractional differential equations of non-integer derivatives. Such systems are used to analyze fractional dynamics. Integrals and derivatives of fractional orders are used to illustrate objects that can be described by power-law non-locality, power-law long-range dependence, or fractal properties. Fractional order systems are useful in investigating the rules of dynamical systems in electrochemistry, physics, viscoelasticity, biology, and chaotic systems. In the past few decades, the growth of science and engineering systems has considerably stimulated the employment of fractional calculus in many subjects of the control theory, for example in stability, stabilization, controllability, observability, observer design, and fault estimation. In fact, the use of fractional calculus can improve and generalize well-established control methods. A variety of results have been established for the controllability of nonlinear fractional systems [1–9].

On the other hand, the stability theory of differential equations plays a vital role in the qualitative analysis of differential systems. There are different types of stability. Among these, one of the most important types is Hyers–Ulam stability (HUR) which was introduced by Ulam in 1940 and then generalized by Rassias in 1978 as Hyers–Ulam–Rassias stability (HURS). As this type of stability guarantees a bound between the exact and approximate solutions, it is often required in a variety of applications, including optimization, approximation, and numerical analysis; for more detals, we refer interested readers to [10–18]. Another important type of stability is finite time stability (FTS), which was first presented in 1953 [19]. It is concerned with the behavior of a system in a specified time interval. In order to extract sufficient conditions for FTS, researchers can employ the Lyapunov technique, characteristic equation method, or Grönwall approach [20–30].

Nawaz et al. [31] derived conditions for the controllability of a fractional differential system with control and state delay. Li and Wang [32] considered an explicit solution formula and derived the controllability criteria for a differential system with delay in the state. Sakthivel et al. [33] investigated fractional differential systems for approximate controllability. Their results were established by assuming the associated linear system to be approximately controllable. Denghao and Wei [34] studied the finite time stability of a neutral fractional system with time delay of the following form:

$$\begin{cases} {}^{C}D_0^{\delta}\varsigma(\mu) = \mathcal{H}_0\varsigma(\mu) + \mathcal{H}_1\varsigma(\mu-h) + \mathcal{H}_2{}^{C}D_0^{\delta}\varsigma(\mu-h), \;\; \mu \in [0,\tau], \\ \varsigma(\mu) = \phi(\mu), \;\; \mu \in [-h,0], \end{cases}$$

where ${}^{C}D_0^{\delta}$ provides the Caputo fractional derivative of order δ, $\varsigma(\mu) \in \mathfrak{R}^n$, $\mathcal{H}_0, \mathcal{H}_1, \mathcal{H}_2$ are system matrices of suitable dimensions, $h > 0$ represents the delay term which depends on past history, and $\phi(\mu)$ denotes a continuously differentiable function on $[-h, 0]$.

Motivated by the aforementioned works, in this paper we investigate a neutral impulsive and delay system for controllability and stability analysis, which takes the following form:

$$\begin{cases} {}^{C}D_0^{\delta}\varsigma(\mu) = \mathcal{H}_0\varsigma(\mu) + \mathcal{H}_1\varsigma(\mu-h) + \mathcal{B}\hat{\upsilon}(\mu) + \mathcal{H}_2{}^{C}D_0^{\delta}\varsigma(\mu-h) + z(\mu,\varsigma), \;\; \mu \in [0,\tau], \;\; \mu \neq \mu_k, \\ \varsigma(\mu) = \phi(\mu), \; \mu \in [-h,0], \\ \varsigma(\mu_k^+) = \varsigma(\mu_k^-) + \mathfrak{I}_k(\mu, \varsigma(\mu_k^-)), \;\; k = 1,2,\ldots,m, \end{cases} \tag{1}$$

where τ is a fixed number. Here, for $k \in M = \{1,2,\ldots,m\}$, μ_k satisfies $0 < \mu_1 < \mu_2 < \ldots < \mu_m$. In addition, $\mathcal{H}_0$ is the infinitesimal generator of a $\mathcal{C}_0$ semi-group $\mathcal{T}(\mu)$ on a Banach space Z, $\mathcal{H}_1$ and $\mathcal{H}_2$ are bounded linear operators, $\mathcal{B}$ is a bounded linear operator from U into Z, the control parameter $\hat{\upsilon}$ is provided in $\mathfrak{L}^2(\mathbb{I}, U)$, U is a Banach space, $z(\mu,\varsigma) : [0,\tau] \times Z \to Z$, and $\mathfrak{I}_k(\mu, \varsigma(\mu_k^-)) : [0,\tau] \times Z \to Z$ are given functions, which satisfy certain assumptions in the following sections. FTS has been thoroughly researched by scholars using various methodologies; however, the present study uses the generalized Grönwall approach. To the best of our knowledge, this is the first time that a the neutral impulsive fractional system has been investigated in the sense of β–HURS and FTS.

The rest of this paper is organized as follows. Section 2 presents the preliminaries and hypothesis. Section 3 provides results for the controllability of system (1). Section 4 deals with the stability analysis, while Section 5 provides a valid example. Finally, Section 6 briefly summarizes the outcomes.

2. Preliminaries

Consider the space of all continuous functions $\mathcal{C}(\mathbb{I}, Z)$, where $\mathbb{I} = [0,\tau] \subseteq \mathfrak{R}$, endowed with the norm:

$$||\varsigma||_{\mathcal{C}} = \sup_{\mu\in\mathbb{I}}\{||\varsigma(\mu)||, \text{ for all } \varsigma \in \mathcal{C}(\mathbb{I}, Z)\}.$$

In addition, consider the Banach space

$\mathcal{PC}(\mathbb{I}, Z) := \Big\{\varsigma : \mathbb{I} \to Z, \; \varsigma \in \mathcal{C}((\mu_k, \mu_{k+1}), Z), \; k = 0,1,\ldots,m\Big\}$ with norm defined by

$$||\varsigma||_{\mathcal{PC}} = \sup\Big\{||\varsigma(\mu)||, \text{for all } \mu \in \mathbb{I}\Big\}.$$

Definition 1. *For any linear space Z over a field F, $\|\cdot\|_\beta : Z \to [0,\infty)$ is said to be β-norm if:*

(i) $\|\varsigma\|_\beta = 0$ *if* $\varsigma = 0$;
(ii) $\|\varrho\varsigma\|_\beta = |\varrho|^\beta\|\varsigma\|_\beta$, *for any* $\varsigma \in Z$ *with* $\varrho \in F$;
(iii) $\|\varsigma + \xi\|_\beta \leq \|\varsigma\|_\beta + \|\xi\|_\beta$.

The space under consideration is then a $\mathcal{P}_\beta$-Banach space associated with norm $\|\varsigma\|_{\mathcal{P}_\beta} = \sup\{\|\varsigma(\mu)\|^\beta\}$.

Definition 2 ([35])**.** The fractional integral, in the Riemann-Liouville sense, of order $\delta \in \mathbb{R}^+$ with a lower limit zero of a function $z \in L^1(\mathbb{I}, \mathbb{R}^+)$, is provided by

$$I_0^{\delta}z(\mu) = \frac{1}{\Gamma(\delta)}\int_0^{\mu}(\mu - s)^{\delta-1}z(s)ds,$$

where

$$\Gamma(\delta) = \int_0^\infty \mu^{\delta-1} e^{-\mu} d\mu,\ \delta > 0.$$

Definition 3 ([35]). For a function $z \in C^n((0,\infty),\mathbb{R})$, the Caputo derivative of a fractional order $\delta \in \mathbb{R}^+$ is defined as

$${}^C D_0^\delta z(\mu) = \frac{1}{\Gamma(n-\delta)} \int_0^\mu (\mu - s)^{n-\delta-1} z^{(n)}(s) ds,$$

where $n = [\delta] + 1$, in which $[\delta]$ represents the integer part of δ and $C^n((0,\infty),\mathbb{R})$ is the space of all n-times continuously differentiable functions from $(0,\infty)$ to $\mathbb{R}$.

Lemma 1 ([35]). *The general solution of the fractional differential equation of the order $\delta > 0$ with the form*

$${}^C D_0^\delta z(\mu) = \beta(\mu),$$

is provided by

$$I_0^\delta[{}^C D_0^\delta z(\mu)] = I_0^\delta \beta(\mu) + K_0 + K_1\mu + K_2\mu^2 + \cdots + K_{n-1}\mu^{n-1},$$

where $n = [\delta] + 1$ and $K_i \in \mathbb{R}$, $i = 0, 1, 2, \ldots, n-1$.

Remark 1 ([34]). *Using Lemma 1 and applying the integral on both sides of (1), the solution can be expressed in the form of the equivalent volterra integral equation*

$$\begin{aligned} \varsigma(\mu) =& \phi(0) + \mathcal{H}_2(\varsigma(\mu - h) - \varsigma(-h)) + \frac{1}{\Gamma(\delta)} \int_0^\mu (\mu - s)^{\delta-1} (\mathcal{H}_0\varsigma(s) + \mathcal{H}_1\varsigma(s-h) \\ &+ \mathcal{B}\hat{v}(s) + z(s,\varsigma))ds + \sum_{r=1}^m \Im_r(\mu_k, \varsigma(\mu_k)). \end{aligned} \tag{2}$$

Proceeding with the method followed by [36], the mild solution of System (1) (referring to [37], Definition 7) can be presented as follows:

$$\varsigma(\mu) = \begin{cases} \mathcal{P}_\delta(\mu)(\phi(0) - \mathcal{H}_2\varsigma(-h)) + \mathcal{H}_2\varsigma(\mu - h) \\ + \int_0^\mu (\mu - s)^{\delta-1} \mathcal{Q}_\delta(\mu - s)[\mathcal{H}_1\varsigma(s-h) + \mathcal{B}\hat{v}(s) + z(s, \varsigma(s-h))]ds \\ + \sum_{0<\mu_k<\mu} \mathcal{P}_\delta(\mu - \mu_k)\Im_k(\varsigma(\mu_k)), & \mu \in [0,\tau], \\ \phi(\mu), & \mu \in [-h, 0]. \end{cases} \tag{3}$$

where

$$\mathcal{P}_\delta(\mu) = \int_0^\infty \xi_\delta(\theta)\mathcal{T}(\mu^\delta\theta)d\theta, \quad \mathcal{Q}_\delta(\mu) = \delta\int_0^\infty \theta\xi_\delta(\theta)\mathcal{T}(\mu^\delta\theta)d\theta,$$

$$\xi_\delta(\theta) = \frac{1}{\pi}\sum_{n=1}^\infty (-1)^{n-1}\theta^{-n\delta-1}\frac{\Gamma(n\delta+1)}{n!}\sin(n\pi\delta), \quad \theta \in (0,\infty).$$

Lemma 2 ([38]). *The operators $\mathcal{P}_\delta(\mu)$, $\mathcal{Q}_\delta(\mu)$ appearing in Definition 1 have the following properties:*

(i) *For any $\mu \geq 0$, the operators $\mathcal{P}_\delta(\mu)$ and $\mathcal{Q}_\delta(\mu)$ are linear. Moreover, if $sup_{\mu\geq 0}\|\mathcal{T}(\mu)\| \leq \mathcal{K}$, then the operators $\mathcal{P}_\delta(\mu)$ and $\mathcal{Q}_\delta(\mu)$ are bounded, i.e., for any $\varsigma \in Z$, there holds*

$$\|\mathcal{P}_\delta(\mu)\varsigma\| \leq \mathcal{K}\|\varsigma\|, \quad \|\mathcal{Q}_\delta(\mu)\varsigma\| \leq \frac{\mathcal{K}}{\Gamma(\delta)}\|\varsigma\|.$$

(ii) *Operators $\mathcal{P}_\delta(\mu)(\mu \geq 0)$ and $\mathcal{Q}_\delta(\mu)(\mu \geq 0)$ are strongly continuous, i.e., for all $\varsigma \in Z$ and $0 \leq \mu_1 \leq \mu_2 \leq \tau$, we have*

$$\|\mathcal{P}_\delta(\mu_1)\varsigma - \mathcal{P}_\delta(\mu_2)\varsigma\| \to 0,\ \|\mathcal{Q}_\delta(\mu_1)\varsigma - \mathcal{Q}_\delta(\mu_2)\varsigma\| \to 0, \quad \text{as } \mu_1 \to \mu_2.$$

(iii) *For $\mu > 0$, $\mathcal{P}_\delta(\mu)$ and $\mathcal{Q}_\delta(\mu)$ are compact operators if $\mathcal{T}(\mu)$ is compact.*

Definition 4 (Exact Controllability [27])**.** *System (1) is known to be exactly controllable on $[0,\tau]$ if, for every $\phi \in \mathcal{C}([-h,0];Z)$, and $\varsigma_1 \in \mathcal{PC}(\mathbb{I},Z)$, there exists $\tilde{v} \in \mathcal{PC}(\mathbb{I},U)$ such that the solution $\varsigma(\mu)$ of (1) corresponding to $\tilde{v}$ satisfies $\varsigma(0) = \phi(0)$ and $\varsigma(\tau) = \varsigma_1$.*

Definition 5 (Approximate Controllability [39])**.** *System (1) is called approximately controllable on $[0,\tau]$ if, for every $\phi \in \mathcal{C}([-h,0];Z)$, $\varsigma_1 \in \mathcal{PC}(\mathbb{I},Z)$, and $\epsilon \geq 0$, there exists $\tilde{v} \in \mathcal{PC}(\mathbb{I},U)$ such that the corresponding solution $\varsigma(\mu)$ of (1) satisfies $\varsigma(0) = \phi(0)$ and $\|\varsigma(\tau) - \varsigma_1\|_Z \leq \epsilon$.*

Remark 2 ([40])**.** *A semilinear impulsive system is exactly controllable if, for any initial condition ς_0 and final condition ς_1, we are able to find a control $\tilde{v}$ such that the operator $\mathcal{F}$ defined by the right side of the system solution has a fixed point.*

Definition 6 (Finite time stability)**.** *For a system to be finite time stable with respect to $\{0,[0,\tau],q,\eta,\epsilon\}$, $\eta < \epsilon$, the following criteria must be fulfilled:*

$$\|\phi(\mu)\| \leq \eta, \text{ and } \|\hat{v}(\mu)\| \leq q_{\hat{v}}, \forall \mu \in [0,\tau],$$

which implies that

$$\|\varsigma(\mu)\| \leq \epsilon, \forall \mu \in [0,\tau],$$

where $\eta,\epsilon \in [0,\infty]$.

Definition 7 (β–Hyers Ulam–Rassias stability)**.** *System (1) is said to be β–HUR stable with respect to $(\psi^\beta,\varphi^\beta)$ if we can find a positive constant $\mathcal{Z}_{z,\varphi,\beta}$ such that for any solution $\varsigma \in \mathcal{PC}(\mathbb{I},Z) \cap \mathcal{C}(\mathbb{I},Z)$ of (11) and any $\epsilon > 0$ there exists a solution y of system (1) in $\mathcal{PC}(\mathbb{I},Z)$ satisfying*

$$||y(\mu) - \varsigma(\mu)||^\beta \leq \mathcal{Z}_{z,\varphi,\beta}\epsilon^\beta\left(\varphi^\beta(\mu) + \psi^\beta\right), \ \ \mu \in \mathbb{I}.$$

Lemma 3 (Grönwall lemma [41])**.** *For $\mu \geq 0$ with*

$$\hat{v}(\mu) \quad \leq \quad q(\mu) + \int_0^\mu p(x)\hat{v}(x)dx + \sum_{0<\mu_k<\mu} \gamma_r \hat{v}(\mu_k^-), \tag{4}$$

where $\gamma > 0$ and q is nondecreasing, it is the case that for $\mu \in \mathfrak{R}^+$, we have

$$\hat{v}(\mu) \quad \leq \quad q(\mu)\Big(1+\gamma_k\Big)^k \exp\Big(\int_0^\mu p(x)dx\Big), \text{ where } k \in M. \tag{5}$$

Remark 3. *If we replace γ_k with $\gamma_k(\mu)$, then*

$$\hat{v}(\mu) \quad \leq \quad q(\mu)\prod_{0<\mu_k<\mu}\Big(1+\gamma_k(\mu)\Big) \exp\Big(\int_0^\mu p(x)dx\Big), \text{ where } k \in M. \tag{6}$$

Lemma 4 (Generalized Grönwall Inequality [34])**.** *Suppose $\varsigma(\mu)$, $a(\mu)$ are non-negative and locally integrable on $0 \leq \mu < \tau$, $g(\mu) \leq \mathcal{K} = constant$, and $\delta > 0$ with*

$$\varsigma(\mu) \leq a(\mu) + g(\mu)\int_0^\mu (\mu - s)^{\delta-1}\varsigma(s)ds \tag{7}$$

on this interval; then,

$$\varsigma(\mu) \leq a(\mu) + g(\mu)\int_0^\mu \Big[\sum_{n=1}^\infty \frac{(g(\mu)\Gamma(\delta))^n}{\Gamma(n\delta)}(\mu-s)^{n\delta-1}a(s)\Big]ds, \ \ 0 \leq \mu < \tau. \tag{8}$$

Corollary 1. *Let $a(\mu)$ be a non decreasing function on $[0,\tau)$; then,*

$$\varsigma(\mu) \leq a(\mu) E_\delta(g(\mu)\Gamma(\delta)\mu^\delta)$$

where $E_\delta = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(k\beta+1)}, x \in \boldsymbol{C}, Re(\beta) > 0$.

Lemma 5 ([42])**.** *Let $\varsigma \in \mathcal{PC}(\mathbb{I}, Z)$ satisfy the following inequality:*

$$\|\varsigma(\mu)\| \leq c_1(\mu) + c_2 \int_0^\mu (\mu - s)^{q-1} \|\varsigma(s)\| ds + \sum_{0<\mu_k<\mu} \Im_k \|\varsigma(\mu_k)\|,$$

where $c_1(\mu)$ is non-negative continuous on $\mathbb{I}$, and c_2, θ_k are constants. Then,

$$\|\varsigma(\mu)\| \leq c_1(\mu)(1 + \Im E_\beta(c_2\Gamma(\beta)\mu^\beta))^k E_\beta(c_2\Gamma(\beta)\mu^\beta), \;\; for \;\; \mu \in (\mu_k, \mu_{k+1}].$$

Definition 8. *The function $f : U \to Z$ is called a contraction if, for every $\varsigma, \Theta \in U$, there exists a constant $0 \leq k < 1$ such that*

$$d(f(\varsigma), f(\Theta)) \leq k d(\varsigma, \Theta),$$

where (U, d) is a metric space.

3. Controllability

The exact and approximate controllability of the fractional neutral system are proved in this section. Before stating our main results, the following conditions are imposed:
$[C_1]$: The semigroup $\mathcal{T}(\mu)$ generated by $\mathcal{H}_0$ is uniformly bounded on Z, i.e., there is a constant $\mathcal{K} > 0$ such that $\sup_{\mu \in [0,\infty)} \|\mathcal{T}(\mu)\| \leq \mathcal{K}$.
$[C_2]$: The nonlinear function $z(\mu, \varsigma)$ is continuous in μ for all $\varsigma \in Z$, while $\exists$ is a positive constant $\mathcal{L}_z$ such that

$$\|z(\mu, \varsigma) - z(\mu, y)\| \leq \mathcal{L}_z \|\varsigma - y\|,$$

for all $\varsigma, y \in Z$.
$[C_3]$: There exist constants $\mathcal{L}_{\Im_k}, k = 0, 1, \cdots$, such that

$$\|\Im_k(\mu, \varsigma) - \Im_k(\mu, y)\| \leq \mathcal{L}_{\Im_k} \|\varsigma - y\|, \forall \varsigma, y \in Z$$

and $\sum_{k=1}^m \mathcal{L}_{\Im_k} = \mathcal{L}_\Im$.
$[C_4]$: The function $z : \mathbb{I} \times Z \to Z$ is uniformly bounded, and $\exists\, N > 0$ such that $\|z(\mu, \varsigma)\| \leq N$ for all $(\mu, \varsigma) \in \mathbb{I} \times Z$.
$[C_5]$: $\mathcal{Q}_\delta(\mu)$ is compact.
$[C_6]$: The following inequalities hold:

$$\Big(\sigma_2 + \mathcal{K}\Big\{\frac{\sigma_{01}}{\Gamma(\delta)} \sqrt{\frac{\tau^{2\delta-1}}{2\delta - 1}} + \mathcal{L}_\Im\Big\}\Big) < \frac{1}{2}$$

and

$$\Big(\sigma_2 + \mathcal{K}\Big\{\frac{\sigma_{01} + \mathcal{L}_z}{\Gamma(\delta)} \sqrt{\frac{\tau^{2\delta-1}}{2\delta - 1}} + \mathcal{L}_\Im\Big\}\Big) < \frac{1}{2}.$$

3.1. Exact Controllability

We define the operator $\mathcal{F}$ by

$$(\mathcal{F}\varsigma)(\mu) = \begin{cases} \mathcal{P}_\delta(\mu)(\phi(0) - \mathcal{H}_2\varsigma(-h)) + \mathcal{H}_2\varsigma(\mu - h) + \int_0^\mu (\mu - s)^{\delta-1} \mathcal{Q}_\delta(\mu - s)[\mathcal{H}_0\varsigma(s) + \mathcal{H}_1\varsigma(s - h) \\ +\mathcal{B}\hat{v}(s) + z(s, \varsigma(s - h))]ds + \sum_{0<\mu_k<\mu} \mathcal{P}_\delta(\mu - \mu_k)\Im_k(\varsigma(\mu_k)), \;\; \mu \geq 0, \\ \phi(\mu), \;\; -h \leq \mu \leq 0. \end{cases} \tag{9}$$

In view of Remark 2, the problem of finding the exact controllability is reduced to finding a fixed point for $\mathcal{F}$. This is achieved with the help of Banach contraction mapping.

Theorem 1. *Let conditions $[C_1]-[C_3]$ hold true. Then, for a given control function $\hat{v}(\cdot)\in U$, the problem (1) is exactly controllable on $\mathcal{C}([-h,\tau];Z)$.*

Proof. Step 1: Consider the sphere $B_k=\{\varsigma(\cdot)\in\mathcal{C}([-h,\tau],Z):\|\varsigma\|\leq R\}$ such that

$$\max\{\|\phi\|,\mathcal{K}(1+\sigma_2)\|\phi\|+\frac{\mathcal{K}}{\Gamma(\delta)}\Big[\sqrt{\frac{\tau^{2\delta-1}}{2\delta-1}}\mathcal{K}_{\mathcal{B}}\|\hat{v}\|+\frac{\mathcal{L}_z\tau^{\delta}}{\delta}\Big]\}\leq\frac{R}{2}.$$

We can show that $\mathcal{F}(B_k)\subset B_k$. If $\mu\in[-h,0)$, it is readily obtainable that $\|\mathcal{F}\varsigma\|=\|\phi\|\leq R$. For any $\varsigma\in B_k$, if $\mu\in[0,\tau]$, then under assumption $[C_2]$ and per Lemma 2(i) we have

$$\begin{aligned}\|(\mathcal{F}\varsigma)(\mu)\|\leq&\|\mathcal{P}_\delta(\mu)\|\|(\phi(0)-\mathcal{H}_2\varsigma(-h))\|+\|\mathcal{H}_2\varsigma(\mu-h)\|+\int_0^\mu(\mu-s)^{\delta-1}\|\mathcal{Q}_\delta(\mu-s)\|\Big\|\Big\{\mathcal{H}_0\varsigma(s)+\mathcal{H}_1\varsigma(s-h)\\&+\mathcal{B}\hat{v}(s)+z(s,\varsigma(s))\Big\}\Big\|ds+\sum_{0<\mu_k<\mu}\|\mathcal{P}_\delta(\mu-\mu_k)\|\|\mathfrak{I}_k(\varsigma(\mu_k))\|\end{aligned}$$

Let $\sigma_0=\|\mathcal{H}_0\|,\ \sigma_1=\|\mathcal{H}_1\|,\ \sigma_2=\|\mathcal{H}_2\|,\ \sigma_{01}=max\{\|\mathcal{H}_0\|,\|\mathcal{H}_1\|\}$, then

$$\begin{aligned}\|(\mathcal{F}\varsigma)(\mu)\|\leq&\mathcal{K}\|\phi(0)\|+\mathcal{K}\sigma_2\|\varsigma(-h)\|+\sigma_2\|\varsigma(\mu-h)\|+\frac{\mathcal{K}}{\Gamma(\delta)}\Big[\mathcal{K}_{\mathcal{B}}\int_0^\mu(\mu-s)^{\delta-1}\|\hat{v}(s)\|ds\\&+(\sigma_0+\sigma_1)\int_0^\mu(\mu-s)^{\delta-1}\|\varsigma(s-h)\|ds+\mathcal{L}_z\int_0^\mu(\mu-s)^{\delta-1}ds\Big]+\mathcal{K}\mathcal{L}_{\mathfrak{I}}\|\varsigma(\mu)\|\\\leq&\mathcal{K}(\|\phi(0)\|+\sigma_2\|\phi(-h)\|)+\sigma_2R+\frac{\mathcal{K}}{\Gamma(\delta)}\Big[\sqrt{\frac{\tau^{2\delta-1}}{2\delta-1}}\Big(\mathcal{K}_{\mathcal{B}}\|\hat{v}\|+\sigma_{01}R\Big)+\mathcal{K}\mathcal{L}_{\mathfrak{I}}R+\frac{\mathcal{L}_z\tau^\delta}{\delta}\Big]\\\leq&\mathcal{K}(1+\sigma_2)\|\phi\|+\frac{\mathcal{K}}{\Gamma(\delta)}\Big[\sqrt{\frac{\tau^{2\delta-1}}{2\delta-1}}\mathcal{K}_{\mathcal{B}}\|\hat{v}\|+\frac{\mathcal{L}_z\tau^\delta}{\delta}\Big]+\Big[\sigma_2+\frac{\mathcal{K}\sigma_{01}}{\Gamma(\delta)}\sqrt{\frac{\tau^{2\delta-1}}{2\delta-1}}+\mathcal{K}\mathcal{L}_{\mathfrak{I}}\Big]R\\\leq&\kappa+\Big(\sigma_2+\mathcal{K}\Big\{\frac{\sigma_{01}}{\Gamma(\delta)}\sqrt{\frac{\tau^{2\delta-1}}{2\delta-1}}+\mathcal{L}_{\mathfrak{I}}\Big\}\Big)R,\end{aligned}$$

where

$$\kappa=\mathcal{K}(1+\sigma_2)\|\phi\|+\frac{\mathcal{K}}{\Gamma(\delta)}\Big[\sqrt{\frac{\tau^{2\delta-1}}{2\delta-1}}\mathcal{K}_{\mathcal{B}}\|\hat{v}\|+\frac{\mathcal{L}_z\tau^\delta}{\delta}\Big].$$

In view of the definition of R and condition $[C_6]$, we obtain $\|(\mathcal{F}\varsigma)(\mu)\|\leq R$. Therefore, $\mathcal{F}$ maps the ball B_k of radius R into itself.

Step 2: We now show that $\mathcal{F}$ is a contraction mapping on $\mathcal{C}([-h,\tau];Z)$. If $\mu\in[-h,0)$, the claim is obviously valid. If $\mu\in[0,\tau]$, then for any $\varsigma,y\in\mathcal{C}([-h,\tau];Z)$, it follows from assumption (C_2) that we have

$$\begin{aligned}\|\mathcal{F}\varsigma(\mu)-\mathcal{F}y(\mu)\| \leq&\|\mathcal{H}_2\varsigma(\mu-h)-\mathcal{H}_2y(\mu-h)\|+\int_0^\mu(\mu-s)^{\delta-1}\|\mathcal{Q}_\delta(\mu-s)\|\Big[\|\mathcal{H}_0\varsigma(s)-\mathcal{H}_0y(s)\| \\ &+\|\mathcal{H}_1\varsigma(s-h)-\mathcal{H}_1y(s-h)\|\Big]ds+\int_0^\mu(\mu-s)^{\delta-1}\|\mathcal{Q}_\delta(\mu-s)\|\|z(s,\varsigma)-z(s,y)\|ds \\ &+\sum_{0<\mu_k<\mu}\|\mathcal{P}_\delta(\mu-\mu_k)\|\|\mathfrak{I}_k(\varsigma(\mu_k))-\mathfrak{I}_k(y(\mu_k))\| \\ \leq&\sigma_2\|\varsigma(s-h)-y(s-h)\|+\frac{\mathcal{K}}{\Gamma(\delta)}\Big[\sigma_{01}\int_0^\mu(\mu-s)^{\delta-1}\|\varsigma(s-h)-y(s-h)\|ds \\ &+\mathcal{L}_z\int_0^\mu(\mu-s)^{\delta-1}\|\varsigma(s)-y(s)\|ds)\Big]+\sum_{k=1}^m\mathcal{L}_{\mathfrak{I}_k}\|\varsigma(\mu_k)-y(\mu_k)\| \\ \leq&\Big(\sigma_2+\mathcal{K}\Big\{\frac{\sigma_{01}+\mathcal{L}_z}{\Gamma(\delta)}\sqrt{\frac{\tau^{2\delta-1}}{2\delta-1}}+\mathcal{L}_{\mathfrak{I}}\Big\}\Big)\|\varsigma-y\|.\end{aligned}$$

Hence, following $[C_6]$, $\mathcal{F}$ is a contraction on B_k. Therefore, $\mathcal{F}$ has a unique fixed point in B_k, which is the solution of the system. □

3.2. *Approximate Controllability*

Consider the linear fractional control system

$$\begin{aligned}{}^{C}D_0^\delta\varsigma(\mu)=&\mathcal{H}(\mu)\varsigma(\mu)+\mathcal{B}(\mu)\tilde{\vartheta}(\mu),\ \ \mu\in[0,\tau], \\ \varsigma(0)=&\phi(0).\end{aligned} \tag{10}$$

Let $\varsigma_\tau(\varsigma_0,\hat{\vartheta})$ be the state value of (1) at terminal time τ corresponding to $\hat{\vartheta}$ and the initial value ς_0. The set $\mathbf{R}(\tau,\varsigma_0)=\{\varsigma_\tau(\varsigma_0,\hat{\vartheta})(0):\hat{\vartheta}(\cdot)\in\mathfrak{L}^2(\mathbb{I},U)\}$ is known as the reachable set of system (1) at terminal time τ. The closure set is denoted by $\overline{\mathbf{R}(\tau,\varsigma_0)}$. A system is said to be approximately controllable if $\overline{\mathbf{R}(\tau,\varsigma_0)}=Z$, i.e., for any $\epsilon>0$, the system can steer from ς_0 to a neighborhood of ς_1 within a distance ϵ from all points in the state space Z at time τ.

We define the controllability Grammian operator by

$$\Gamma_0^\tau\varsigma=\int_0^\tau\mathcal{Q}_\delta(\tau,s)\mathcal{B}\mathcal{B}^*\mathcal{Q}_\delta^*(\tau,s)\varsigma ds,$$

and

$$\Upsilon(\varepsilon,\Gamma_0^\tau)=(\varepsilon I+\Gamma_0^\tau)^{-1}.$$

Here, $\mathcal{B}^*$ is the adjoint of $\mathcal{B}$ and $\mathcal{Q}_\delta^*$ is the adjoint of $\mathcal{Q}_\delta$.

Theorem 2. *Assume that* $[C_1]-[C_4]$ *hold; then, system (1) is approximately controllable on* $[0,\tau]$ *if the linear system (10) is approximately controllable on* $[0,\tau]$.

Proof. Let $\hat{\varsigma}_\varepsilon(\cdot)$ be a fixed point of $\mathcal{F}$ in B_k. Any fixed point of $\mathcal{F}$ is a mild solution of the system under control:

$$\hat{\vartheta}_\varepsilon(\mu)=\mathcal{B}^*\mathcal{Q}^*(\tau-\mu)\mathbf{R}(\varepsilon,\Gamma_0^\tau)p(\hat{\varsigma}_\varepsilon)$$

and satisfies

$$\hat{\varsigma}_\varepsilon(\tau)=\varsigma_\tau-\varepsilon\mathbf{R}(\varepsilon,\Gamma_0^\tau)p(\hat{\varsigma}_\varepsilon),$$

where

$$p(\hat{\varsigma}_\varepsilon)=\varsigma_\tau-\mathcal{P}(\tau)\phi(0)-\int_0^\tau(\tau-s)^{\delta-1}\mathcal{Q}_\delta(\tau-s)z(s,\varsigma(s))ds.$$

Per $[C_4]$, we have

$$\int_0^\tau \|z(s,\hat{\varsigma}_\varepsilon(s))\|^2 ds \leq \tau N^2.$$

Consequently, the sequence $\{z(s,\hat{\varsigma}_\varepsilon(s))\}$ is bounded in $\mathfrak{L}^2(\mathbb{I},Z)$. Thus, there is a subsequence $\{z(s,\hat{\varsigma}_\varepsilon(s))\}$ that converges weakly to $z(\mu)$ in $\mathfrak{L}^2(\mathbb{I},Z)$. The compactness of $\mathcal{Q}_\delta(\mu)$ now implies that $\mathcal{Q}_\delta(\tau-s)z(s,\hat{\varsigma}_\varepsilon(s)) \to \mathcal{Q}_\delta(\tau-s)z(s)$ in $\mathfrak{L}^2(\mathbb{I},Z)$, and accordingly we obtain

$$\begin{aligned}\|p(\hat{\varsigma}_\varepsilon)-w\| &= \|\int_0^\tau (\tau-s)^{\delta-1}\mathcal{Q}_\delta(\mu-s)[z(s,\hat{\varsigma}_\varepsilon(s))-z(s)]ds\| \\ &\leq \sup_{0\leq\mu\leq\tau}\|\int_0^\mu (\mu-s)^{\delta-1}\mathcal{Q}_\delta(\mu-s)[z(s,\hat{\varsigma}_\varepsilon(s))-z(s)]ds\|\end{aligned}$$

$\to 0$ as $\varepsilon \to 0$, where

$$w = \varsigma_\tau - \mathcal{P}(\tau)\phi(0) - \int_0^\tau (\mu-s)^{\delta-1}\mathcal{Q}_\delta(\tau-s)z(s)ds.$$

Then, we obtain

$$\begin{aligned}\|\varsigma^\varepsilon(\tau)-\varsigma_\tau\| &= \|\varepsilon\Upsilon(\varepsilon,\Gamma_0^\tau)(w)\| + \|\varepsilon\Upsilon(\varepsilon,\Gamma_0^\tau)\|\|p(\varsigma^\varepsilon)-w\| \\ &\leq \|\varepsilon\Upsilon(\varepsilon,\Gamma_0^\tau)(w)\| + \|p(\varsigma^\varepsilon)-w\| \to 0\end{aligned}$$

as $\varepsilon \to 0^+$.

Consequently, the approximate controllability is proved. □

4. Stability Results

This section deals with finite time stability and Ulam-type stability for system (1).

4.1. FTS Results

The finite time stability of system (1) is presented in the following theorem, given a necessary and sufficient condition.

Theorem 3. *The neutral fractional system (1) is finite time stable subject to the following condition:*

$$\Big(1+2\nu_2+\frac{\nu_{01}\mu^\delta}{\Gamma(\delta+1)}\Big)\eta + \frac{\mathcal{K}_\mathcal{B}q_{\hat{v}}+m}{\Gamma(\delta+1)}\mu^\delta(1+\mathcal{L}E_\delta\Big(\frac{\mathcal{L}_z+\nu_{01}}{\Gamma(\delta)}\Gamma(\delta)\mu^\delta)\Big)^k E_\delta(\frac{\mathcal{L}_z+\nu_{01}}{\Gamma(\delta)}\Gamma(\delta)\mu^\delta) < \epsilon.$$

Proof. We designate norm of an element $\phi \in \mathcal{C}$ by

$$\|\phi\| = \sup_{-h\leq\mu\leq 0}\|\phi(\mu)\|.$$

Let $Z = \mathcal{C}([-h,\tau],\mathfrak{R}^n)$ be equipped with norm

$$\|\varsigma(\mu)\| = \sup_{0\leq\mu\leq\tau}\varsigma(\mu), \text{ and } \|\varsigma_\mu\| = \sup_{-h\leq\mu\leq 0}\|\varsigma(\mu+\theta)\|,$$

where $\|\varsigma(\mu)\| \leq \|\varsigma_\mu\|$.

From Definition 1, solution (2) is provided by

$$\varsigma(\mu) = \begin{cases} \phi(\mu), \quad \mu\in[-h,0], \\ \phi(0)-\mathcal{H}_2\phi(-h)+\mathcal{H}_2\varsigma(\mu-h)+\frac{1}{\Gamma(\delta)}\int_0^\mu(\mu-s)^{\delta-1}(\mathcal{H}_0\varsigma(s)+\mathcal{H}_1\varsigma(s-h)+\mathcal{B}\hat{v}(s) \\ +z(\varsigma,s))ds+\sum_{0\leq\mu_k\leq\mu}\mathfrak{I}_k\varsigma(\mu_k), \quad \mu\in[0,\tau]. \end{cases}$$

Applying the norm on both sides, we have

$$\|\varsigma(\mu)\| \leq \|\phi(0)\| + \|\mathcal{H}_2\phi(-h)\| + \|\mathcal{H}_2\varsigma(\mu-h)\| + \frac{1}{\Gamma(\delta)}\int_0^\mu |\mu-s|^{\delta-1}\|\mathcal{H}_0\varsigma(s) + \mathcal{H}_1\varsigma(s-h)$$
$$+ \mathcal{B}\hat{v}(s) + z(\varsigma,s)\|ds + \sum_{0\leq\mu_k\leq\mu}\Im_k\varsigma(\mu_k),$$
$$\|\varsigma(\mu)\| \leq (1+\|\mathcal{H}_2\|)\|\phi\| + \|\mathcal{H}_2\|\|\varsigma(\mu-h)\| + \frac{1}{\Gamma(\delta)}\int_0^\mu |\mu-s|^{\delta-1}\|\mathcal{H}_0\|\|\varsigma(s)\|ds$$
$$+ \frac{1}{\Gamma(\delta)}\int_0^\mu |\mu-s|^{\delta-1}\|\mathcal{H}_1\|\|\varsigma(s-h)\|ds + \frac{1}{\Gamma(\delta)}\int_0^\mu |\mu-s|^{\delta-1}\|\mathcal{B}\|\|\hat{v}(s)\|ds$$
$$+ \frac{1}{\Gamma(\delta)}\int_0^\mu |\mu-s|^{\delta-1}\|z(\varsigma,s)\|ds + \sum_{0\leq\mu_k\leq\mu}\|\Im_k\|\|\varsigma(\mu_k)\|.$$

Let the biggest singular value of matrix $(\cdot)$ be denoted by $\nu_{max}(\cdot)$. For simplicity, we denote $\nu_{max}(\mathcal{H}_0)$ by ν_0, $\nu_{max}(\mathcal{H}_1)$ by ν_1, $\nu_{max}(\mathcal{B})$ by $\mathcal{K}_\mathcal{B}$, $\nu_{max}(\mathcal{H}_2)$ by ν_2, and $\nu_{max}(\mathcal{H}_0) + \nu_{max}(\mathcal{H}_1)$ by ν_{01}. Therefore,

$$\|\varsigma(\mu)\| \leq (1+\nu_2)\|\phi\| + \nu_2\|\varsigma(\mu-h)\| + \frac{\nu_0}{\Gamma(\delta)}\int_0^\mu |\mu-s|^{\delta-1}\|\varsigma(s)\|ds$$
$$+ \frac{\nu_1}{\Gamma(\delta)}\int_0^\mu |\mu-s|^{\delta-1}\|\varsigma(s-h)\|ds + \frac{\mathcal{K}_\mathcal{B}}{\Gamma(\delta)}\int_0^\mu |\mu-s|^{\delta-1}\|\hat{v}(s)\|ds$$
$$+ \frac{1}{\Gamma(\delta)}\int_0^\mu |\mu-s|^{\delta-1}(m + \mathcal{L}_z\|\varsigma(\mu)\|)ds + \sum_{0\leq\mu_k\leq\mu}\mathcal{L}_{\Im_k}\|\varsigma(\mu_k)\|.$$

For $\mu \in [0,\tau]$, we have $\|\varsigma(\mu-h)\| \leq \|\phi\|$ and

$$\|\varsigma(\mu)\| \leq (1+2\nu_2)\|\phi\| + \frac{1}{\Gamma(\delta)}\int_0^\mu |\mu-s|^{\delta-1}\Big\{\nu_0\|\varsigma(s)\| + \nu_1\|\varsigma(s-h)\| + \mathcal{K}_\mathcal{B}\|\hat{v}(s)\|$$
$$+ (m + \mathcal{L}_z\|\varsigma(\mu)\|)\Big\}ds + \sum_{0\leq\mu_k\leq\mu}\mathcal{L}_{\Im_k}\|\varsigma(\mu_k)\|, \ \ 0 \leq \mu \leq \tau.$$

Using relation

$$\|\varsigma(\mu-h)\| \leq \sup_{\{\mu-h\leq\theta\leq\mu\}}\|\varsigma(\theta)\|,$$

we obtain

$$\|\varsigma(\mu)\| \leq (1+2\nu_2)\|\phi\| + \frac{1}{\Gamma(\delta)}\int_0^\mu |\mu-s|^{\delta-1}\Big\{\nu_{01}(\sup_{\{\mu-h\leq\theta\leq\mu\}}\|\varsigma(\theta)\| + \|\phi\|) + \mathcal{K}_\mathcal{B}\|\hat{v}(s)\|ds$$
$$+ (m + \mathcal{L}_z\|\varsigma(\mu)\|)\Big\}ds + \sum_{0\leq\mu_k\leq\mu}\mathcal{L}_{\Im_k}\|\varsigma(\mu_k)\|, \ \ 0 \leq \mu \leq \tau,$$

and

$$\|\varsigma(\mu)\| \leq \Big(1 + 2\nu_2 + \frac{\nu_{01}\mu^\delta}{\Gamma(\delta+1)}\Big)\|\phi\| + \frac{\mathcal{K}_\mathcal{B}q_{\hat{v}} + m}{\Gamma(\delta+1)}\mu^\delta + \frac{\mathcal{L}_z + \nu_{01}}{\Gamma(\delta)}\int_0^\mu |\mu-s|^{\delta-1}\Big\{\sup_{\{\mu-h\leq\theta\leq\mu\}}\|\varsigma(\theta)\|\Big\}ds$$
$$+ \sum_{0\leq\mu_k\leq\mu}\mathcal{L}_{\Im_k}\sup_{\{\mu_k-h\leq\theta_k\leq\mu_k\}}\|\varsigma(\theta_k)\|, \ \ 0 \leq \mu \leq \tau.$$

Let

$$a(\mu) = \Big(1 + 2\nu_2 + \frac{\nu_{01}\mu^\delta}{\Gamma(\delta+1)}\Big)\|\phi\| + \frac{\mathcal{K}_\mathcal{B}q_{\hat{v}} + m}{\Gamma(\delta+1)}\mu^\delta$$

and

$$g(\mu) = \frac{\mathcal{L}_z + \nu_{01}}{\Gamma(\delta)}.$$

Because the right-hand side of the above equation is a nondecreasing function, we have

$$\|\varsigma(\mu)\| \leq \sup_{\{\mu-h\leq\theta\leq\mu\}} \|\varsigma(\theta)\| \leq a(\mu) + g(\mu)\int_0^\mu |\mu-s|^{\delta-1} \sup_{\{\mu-h\leq\theta\leq\mu\}} \|\varsigma(\theta)\|ds + \sum_{0\leq\mu_k\leq\mu} \mathcal{L}_{\Im_k} \sup_{\{\mu_k-h\leq\theta_k\leq\mu_k\}} \|\varsigma(\theta)\|, \; 0\leq\mu\leq\tau,$$

or

$$\|\varsigma_\mu\| \leq a(\mu) + g(\mu)\int_0^\mu |\mu-s|^{\delta-1}\|\varsigma_\mu\|ds + \sum_{0\leq\mu_k\leq\mu} \mathcal{L}_{\Im_k}\|\varsigma_{\mu_k}(\theta)\|, \; 0\leq\mu\leq\tau.$$

Using the generalized Grönwall inequality, we obtain

$$\|\varsigma_\mu\| \leq a(\mu)(1+\mathcal{L}E_\delta\Big(g(\mu)\Gamma(\delta)\mu^\delta)\Big)^k E_\delta(g(\mu)\Gamma(\delta)\mu^\delta)$$

with $\mathcal{L} = max\{\mathcal{L}_k : k = 1,2,\ldots,m\}$. Taking $\|\phi\| < \eta$, we then have

$$\|\varsigma(\mu)\| \leq \Big(1+2\nu_2+\frac{\nu_{01}\mu^\delta}{\Gamma(\delta+1)}\Big)\eta + \frac{\mathcal{K}_{\mathcal{B}}q_{\hat{v}}+m}{\Gamma(\delta+1)}\mu^\delta(1+\mathcal{L}E_\delta\Big(\frac{\mathcal{L}_z+\nu_{01}}{\Gamma(\delta)}\Gamma(\delta)\mu^\delta)\Big)^k E_\delta(\frac{\mathcal{L}_z+\nu_{01}}{\Gamma(\delta)}\Gamma(\delta)\mu^\delta).$$

Hence, using the basic condition of Lemma 3, we have

$$\|\varsigma(\mu)\| \leq \epsilon.$$

□

4.2. HURS Results

The β–HURS of the given system is discussed by considering a few assumptions:
$[A_1]$: $z : \mathbb{I}\times Z \to Z$, which satisfies the Caratheodory conditions, and the $\exists$ constant $\mathcal{L}_z > 0$ such that

$$||z(\mu,\varsigma) - z(\mu,\varsigma')|| \leq \mathcal{L}_z||\varsigma-\varsigma'||$$

for every $\varsigma, \varsigma' \in Z$.
$[A_2]$: $\Im_k \in \mathcal{C}(\mathbb{I}, Z) : Z \to Z$, for $k = 1,2,\ldots,m$, where there exist constants $\mathcal{L}_{\Im_k} > 0$ such that

$$||\Im_k(\mu,\varsigma_k,\hat{v}_k) - \Im_k(\mu,\varsigma'_k,\hat{v}_k)|| \leq \mathcal{L}_{\Im_k}||\varsigma_k-\varsigma'_k||,$$

for each $\varsigma_k, \varsigma'_k \in Z$.
$[A_3]$: The inequality $\Big\{\sum_{r=1}^{m}\mathcal{L}_{\Im_k} + \sigma_2 + \frac{\tau^\delta}{\Gamma(\delta+1)}(\sigma_{01}+\mathcal{L}_z\tau)\Big\} < 1$ holds.

Choose $\epsilon > 0$, φ, and $\psi \geq 0$ from $\mathcal{PC}(\mathbb{I}, Z)$. Assume the following inequality holds:

$$\begin{cases} \|{}^C D_0^\delta\varsigma(\mu) - \mathcal{H}_0\varsigma(\mu) + \mathcal{H}_1\varsigma(\mu-h) + \mathcal{B}\hat{v}(\mu) + \mathcal{H}_2{}^C D_0^\delta\varsigma(\mu-h) + z(\varsigma,\mu)\| \leq \epsilon\varphi(\mu), \; \mu\in[0,\tau], \; \mu\neq\mu_k, \\ \|\varsigma(\mu)-\phi(\mu)\| \leq \epsilon\psi, \; \mu\in[-h,0], \\ \|\varsigma(\mu_k^+) - \varsigma(\mu_k^-) - \Im_k(\mu,\varsigma(\mu_k^-))\| \leq \epsilon\psi_k, \; k=1,2,\ldots,m. \end{cases} \tag{11}$$

Remark 4. *Inequality (11) indicates that a function $y \in \mathcal{PC}([0,\tau],Z)\bigcap C([0,\tau],Z)$ is the solution to inequality (11) if and only if we can find $\mu \in C([0,\tau])$, $\psi \geq 0$ and a sequence μ_k, $k \in M$ satisfying*

$$\begin{cases} ||h(\mu)|| \leq \epsilon\varphi(\mu) \text{ and } ||h_r|| \leq \epsilon\psi, \ \mu \in [0,\tau], \mu \neq \mu_k \text{ and } k \in M, \\ {}^{C}D_0^{\delta}\varsigma(\mu) = \mathcal{H}_0\varsigma(\mu) + \mathcal{H}_1\varsigma(\mu - h) + \mathcal{B}\hat{v}(\mu) + CD^{\delta}\varsigma(\mu - h) + z(\varsigma, \mu) + h(\mu), \ \ \mu \in [0,\tau], \mu \neq \mu_k, \\ \varsigma(\mu) = \phi(\mu), \ \ \mu \in [-h, 0], \\ \varsigma(\mu_k^+) = \varsigma(\mu_k^-) + \Im_k(\mu, \varsigma(\mu_k^-)) + h(\mu_k), \ k = 1, 2, \ldots, m. \end{cases} \tag{12}$$

Remark 4 concludes that the solution of System (12) is

$$\begin{aligned} \varsigma(\mu) \ &= \ \phi(0) + \mathcal{H}_2\phi(h) + \mathcal{H}_2\varsigma(\mu - h) + \frac{1}{\Gamma(\delta)} \int_0^{\mu} (\mu - s)^{\delta - 1} (\mathcal{H}_0\varsigma(s) + \mathcal{H}_1\varsigma(s - h) \\ &\quad + \mathcal{B}\hat{v}(s) + z(s, \varsigma) + h(s))ds + \sum_{k=1}^{m} \Im_k(\mu_k, \varsigma(\mu_k)) + h(\mu_k)). \end{aligned}$$

Inequality (11) leads to

$$\begin{aligned} &||\varsigma(\mu) - \phi(0) - \mathcal{H}_2\phi(h) - \mathcal{H}_2\varsigma(\mu - h) - \frac{1}{\Gamma(\delta)} \int_0^{\mu} (\mu - s)^{\delta - 1} \{\mathcal{H}_0\varsigma(s) + \mathcal{H}_1\varsigma(s - h) \\ &+ \mathcal{B}\hat{v}(s) + z(s, \varsigma)\}ds - \sum_{k=1}^{m} \Im_k(\mu_k, \varsigma(\mu_k)))|| \\ = \ & ||\frac{1}{\Gamma(\delta)} \int_0^{\mu} (\mu - s)^{\delta - 1} h(s)ds + \sum_{k=1}^{m} h(\mu_k)|| \\ \leq \ & \frac{\mu^{\delta}}{\Gamma(\delta + 1)} ||h(s)||ds + \sum_{k=1}^{m} ||h(\mu_k)|| \\ \leq \ & \frac{\mu^{\delta}}{\Gamma(\delta + 1)} \epsilon\varphi(\mu) + \sum_{k=1}^{m} \epsilon\psi \\ \leq \ & \epsilon\Big(m\psi + \frac{\mu^{\delta}}{\Gamma(\delta + 1)} \varphi(\mu)\Big), \text{ where } \mu \in (\mu_k, \mu_{k+1}]. \end{aligned}$$

Theorem 4. *Let assumptions $[A_1] - [A_4]$ hold. Then, System (1) has a unique solution $\varsigma \in \mathcal{PC}(I)$.*

Proof. Define an operator $\mathcal{R} : \mathcal{PC}(\mathbb{I}, Z) \to \mathcal{PC}(\mathbb{I}, Z)$ by

$$(\mathcal{R}\varsigma)(\mu) = \begin{cases} \phi(\mu), \quad \mu \in [-h, 0], \\ \varsigma(0) + \mathcal{H}_2(\varsigma(\mu - h) - \varsigma(-h)) + \frac{1}{\Gamma(\delta)} \int_0^{\mu} (\mu - s)^{\delta - 1} (\mathcal{H}_0\varsigma(s) + \mathcal{H}_1\varsigma(s - h) \\ +\mathcal{B}\hat{v}(s) + z(\varsigma, s))ds + \sum_{0 \leq \mu_k \leq \mu} \Im_k\varsigma(\mu_k). \end{cases}$$

For any $\varsigma, \varsigma' \in \mathcal{PC}(\mathbb{I}, Z)$ and $\mu \in [-h, 0]$, we have

$$||(\mathcal{R}\varsigma)(\mu) - (\mathcal{R}\varsigma')(\mu)|| = 0.$$

For $\mu \in (\mu_m, \tau]$, we have

$$
\begin{aligned}
||(\mathcal{R}\varsigma)(\mu)-(\mathcal{R}\varsigma')(\mu)|| \leq & \|\mathcal{H}_2(\varsigma(\mu-h)-\varsigma'(\mu-h))+\frac{1}{\Gamma(\delta)}\int_0^{\mu}(\mu-s)^{\delta-1}\Big[\mathcal{H}_0(\varsigma(s)-\varsigma'(s)) \\
& +\mathcal{H}_1(\varsigma(s-h)-\varsigma'(s-h))+(z(\varsigma,s)-z(\varsigma',s))\Big]ds \\
& +\sum_{0\leq\mu_k\leq\mu}\Im_k(\varsigma(\mu_k)-\varsigma'(\mu_k))\| \\
\leq & \|\mathcal{H}_2\|\|\varsigma-\varsigma'\|+\frac{1}{\Gamma(\delta)}\int_0^{\mu}(\mu-s)^{\delta-1}(\|\mathcal{H}_0\|\|\varsigma-\varsigma'\|+\|\mathcal{H}_1\|\|\varsigma-\varsigma'\| \\
& +\mathcal{L}_z\|\varsigma-\varsigma'\|)ds+\sum_{0\leq\mu_k\leq\mu}\mathcal{L}_{\Im_k}\|\varsigma-\varsigma'\| \\
\leq & \Big\{\sum_{k=1}^{m}\mathcal{L}_{\Im_k}+\sigma_2+\tfrac{\tau^{\delta}}{\Gamma(\delta+1)}(\sigma_{01}+\mathcal{L}_z\tau)\Big\}||\varsigma-\varsigma'|| \\
< & ||\varsigma-\varsigma'||.
\end{aligned}
$$

Then, $\mathcal{R}$ is contractive with respect to $||\cdot||_{\mathcal{PC}}$. Therefore, $\mathcal{R}$ has a unique fixed point, which is the solution of System (1). □

Consider the following assumptions:
$[A_1^*]: z:\mathbb{I}\times Z\to Z$, which satisfies the Carathéodory condition, and there exists a function $\mathcal{L}_z\in\mathcal{C}(\mathbb{I},Z)$ such that

$$||z(\mu,\varsigma)-z(\mu,\varsigma')||\leq\mathcal{L}_z(\mu)||\varsigma-\varsigma'||,$$

for every $\mu\in\mathbb{I}$ and $\varsigma,\varsigma'\in Z$.

Considering the above assumptions and inequality (11), we present our result.

Theorem 5. *Let $[A_1^*],[A_2]$, and $[A_3]$ hold. Then, System (1) is β–HUR stable with respect to $(\psi^{\beta},\varphi^{\beta})$.*

Proof. Let inequality (11) result in y as its solution. Then,

$$||y(\mu)-\phi(\mu)||=0,\quad \mu\in[-h,0].$$

For each $\mu\in(\mu_k,\mu_{k+1}]$, we have

$$
\begin{aligned}
& ||y(\mu)-\varsigma(0)-\mathcal{H}_2(\varsigma(\mu-h)-\varsigma(-h))-\frac{1}{\Gamma(\delta)}\int_0^{\mu}(\mu-s)^{\delta-1}(\mathcal{H}_0\varsigma(s)+\mathcal{H}_1\varsigma(s-h) \\
& +\mathcal{B}\hat{v}(s)+z(\varsigma,s))ds-\sum_{0\leq\mu_k\leq\mu}\Im_k\varsigma(\mu_k)|| \\
\leq\ & \epsilon\Big(m\psi+\frac{\mu^{\delta}}{\Gamma(\delta+1)}\varphi(\mu)\Big) \\
\leq\ & \epsilon\Big(m+\frac{\mu^{\delta}}{\Gamma(\delta+1)}\Big)\Big(\varphi(\mu)+\psi\Big).
\end{aligned}
$$

Therefore, for every $\mu\in(\mu_k,\mu_{k+1}]$ we obtain

$$
\begin{aligned}
& ||y(\mu)-\varsigma(\mu)||^{\beta} \\
= \; & ||y(\mu)-\varsigma(0)-\mathcal{H}_2(\varsigma(\mu-h)-\varsigma(-h))-\frac{1}{\Gamma(\delta)}\int_0^{\mu}(\mu-s)^{\delta-1}(\mathcal{H}_0\varsigma(s) \\
& +\mathcal{H}_1\varsigma(s-h)+\mathcal{B}\hat{v}(s)+z(\varsigma,s))ds-\sum_{0\leq\mu_k\leq\mu}\Im_k\varsigma(\mu_k)||^{\beta} \\
\leq \; & \Big(||y(\mu)-\mathcal{H}_2(y(\mu-h))-\frac{1}{\Gamma(\delta)}\int_0^{\mu}(\mu-s)^{\delta-1}(\mathcal{H}_0y(s)+\mathcal{H}_1y(s-h) \\
& +\mathcal{B}\hat{v}(s)+z(y,s))ds-\sum_{0\leq\mu_k\leq\mu}\Im_k y(\mu_k)||\Big)^{\beta} \\
& +\Big(\|\mathcal{H}_2\|\|(y(\mu-h)-(\varsigma(\mu-h)\|+\frac{1}{\Gamma(\delta)}\int_0^{\mu}(\mu-s)^{\delta-1}\Big[\|\mathcal{H}_0\|\|y(s)-\varsigma(s)\| \\
& +\|\mathcal{H}_1\|\|y(s-h)-\varsigma(s-h)\|+\|z(s,y(s))-z(s,\varsigma(s))\|\Big]ds\Big)^{\beta} \\
& +\Big(\sum_{k=1}^{m}||\Im_k(\mu_k,y(\mu_k))-\Im_k(\mu_k,\varsigma(\mu_k))||\Big)^{\beta} \\
\leq \; & \Big(\epsilon(m+\frac{\mu^{\delta}}{\Gamma(\delta+1)})(\varphi(\mu)+\psi)\Big)^{\beta}+\Big(\sigma_2||y(\mu)-\varsigma(\mu)||+\frac{1}{\Gamma(\delta)}\int\limits_0^{\mu}[\sigma_{01}+\mathcal{L}_z]||y(x)-\varsigma(x)||dx\Big)^{\beta} \\
& +\Big(\sum_{k=1}^{m}\mathcal{L}_{\Im_k}||y(\mu_k)-\varsigma(\mu_k)||\Big)^{\beta}.
\end{aligned}
$$

Using

$$(p+q+r)^{\gamma}\leq 3^{\gamma-1}(p^{\gamma}+q^{\gamma}+r^{\gamma}), \text{where } p,q,r\geq 0, \text{and } \gamma>1,$$

and applying Grönwall's Lemma 3, we have

$$
\begin{aligned}
||y(\mu)-\varsigma(\mu)|| \leq \; & \frac{3^{\frac{1}{\beta}-1}}{1-3^{\frac{1}{\beta}-1}\sigma_2}\Big(\epsilon\Big((m+\frac{\mu^{\delta}}{\Gamma(\delta+1)})(\varphi(\mu)+\psi)\Big)\Big) \\
& \Big(1+\frac{3^{\frac{1}{\beta}-1}}{1-3^{\frac{1}{\beta}-1}\sigma_2}\mathcal{L}_{\Im}E_{\delta}((\sigma_{01}+\mathcal{L}_z)\Gamma(\delta)\mu^{\delta})\Big)^{k} \\
& \Big(\frac{3^{\frac{1}{\beta}-1}}{1-3^{\frac{1}{\beta}-1}\sigma_2}E_{\delta}((\sigma_{01}+\mathcal{L}_z)\Gamma(\delta)\mu^{\delta})\Big),
\end{aligned}
$$

where $\mathcal{L}_{\Im}=\sum\limits_{k=1}^{m}\mathcal{L}_{\Im_k}$. Hence, we obtain

$$
\begin{aligned}
||y(\mu)-\varsigma(\mu)||^{\beta} \leq \; & \frac{3^{1-\beta}}{(1-3^{\frac{1}{\beta}-1}\sigma_2)^{\beta}}\Big(\epsilon\Big(m+\frac{\mu^{\delta}}{\Gamma(\delta+1)}\Big)\Big)^{\beta}\Big(\varphi(\mu)+\psi\Big)^{\beta} \\
& \Big(1+\frac{3^{\frac{1}{\beta}-1}}{1-3^{\frac{1}{\beta}-1}\sigma_2}\mathcal{L}_{\Im}E_{\delta}(\sigma_{01}+\mathcal{L}_z)\Gamma(\delta)\mu^{\delta}\Big)^{k\beta} \\
& \Big(\frac{3^{\frac{1}{\beta}-1}}{1-3^{\frac{1}{\beta}-1}\sigma_2}E_{\delta}(\sigma_{01}+\mathcal{L}_z)\Gamma(\delta)\mu^{\delta}\Big)^{\beta} \\
\leq \; & \mathcal{Z}_{z,\varphi,\psi}\epsilon^{\beta}\Big(\varphi^{\beta}(\mu)+\psi^{\beta}\Big),
\end{aligned}
$$

where

$$\mathcal{Z}_{z,\varphi,\psi} = \frac{3^{1-\beta}}{(1-3^{\frac{1}{\beta}-1}\sigma_2)^\beta}\Big(\epsilon\Big(m+\frac{\mu^\delta}{\Gamma(\delta+1)}\Big)\Big)^\beta\Big(1+\frac{3^{\frac{1}{\beta}-1}}{1-3^{\frac{1}{\beta}-1}\sigma_2}\mathcal{L}_{\Im}E_\delta(\sigma_{01}+\mathcal{L}_z)\Gamma(\delta)\mu^\delta\Big)^{k\beta}$$
$$\Big(\frac{3^{\frac{1}{\beta}-1}}{1-3^{\frac{1}{\beta}-1}\sigma_2}E_\delta(\sigma_{01}+\mathcal{L}_z)\Gamma(\delta)\mu^\delta\Big)^\beta.$$

□

5. Example

Consider the following fractional problem:

$$\begin{cases} {}^C D_0^\delta \varsigma(\mu) = \mathcal{H}_0\varsigma(\mu) + \mathcal{H}_1\varsigma(\mu-h) + \mathcal{B}\hat{v}(\mu) + \mathcal{H}_2 D^\delta\varsigma(\mu-h) + z(\mu,\varsigma), \ \ \mu\in[0,1]/\{\frac{1}{3}\}, \\ \varsigma(\mu) = \phi(\mu), \ \mu\in[-h,0], \\ \varsigma((\frac{1}{3})^+) = \varsigma((\frac{1}{3})^-) + \frac{1}{20}\varsigma(\frac{1}{3}), \end{cases} \tag{13}$$

where

$$\mathcal{H}_0 = \begin{pmatrix} 0.2 & 0 \\ 0 & 0.2 \end{pmatrix}, \mathcal{H}_1 = \begin{pmatrix} 0.3 & 0 \\ 0.1 & 0.1 \end{pmatrix}, \mathcal{H}_2 = \begin{pmatrix} 0.2 & 0 \\ 0 & 0.1 \end{pmatrix}, \mathcal{B} = \begin{pmatrix} 1 & 0 \\ 0 & 0.5 \end{pmatrix}, \hat{v} = \begin{pmatrix} 0.4 \\ 0.3 \end{pmatrix}, z(\mu,\varsigma) = \frac{\cos\mu}{40}\begin{pmatrix} \varsigma_1 \\ \varsigma_2 \end{pmatrix}.$$

Clearly (A_1) and (A_2) hold for the reason that

$$\begin{aligned} \|z(\mu,\varsigma) - z(\mu,\varsigma')\| \leq & |\frac{\cos\mu}{40}|\Big\|\varsigma-\varsigma'\Big\| \\ \leq & \frac{1}{40}\|\varsigma-\varsigma'\|, \end{aligned}$$

and

$$\begin{aligned} \|\Im_1\Big(\varsigma\Big(\frac{1}{3}\Big)\Big) - \Im_1\Big(\varsigma'\Big(\frac{1}{3}\Big)\Big)\| \leq & \Big\|\frac{\varsigma(\frac{1}{3})}{20} - \frac{\varsigma'(\frac{1}{3})}{20}\Big\| \\ \leq & \frac{1}{20}\|\varsigma(\frac{1}{3}) - \varsigma'(\frac{1}{3})\|. \end{aligned}$$

Entering $\delta = \frac{4}{5}$, we have

$$\begin{aligned} & \sum_{k=1}^{m}\mathcal{L}_{\Im_k} + \sigma_2 + \frac{\tau^\delta}{\Gamma(\delta+1)}(\sigma_{01} + \mathcal{L}_z\tau) \\ = & \frac{1}{20} + 0.2 + \frac{\tau^{\frac{4}{5}}}{\Gamma(\frac{4}{5}+1)}\Big(0.6 + \frac{1}{40}\Big) \\ < & 1, \end{aligned}$$

and thus (A_3) holds. Therefore, the system has a unique solution.

Now, taking $m = 1, \beta = 1/2, \varphi = 2^{\mu}, \psi = 1, \epsilon = 0.01$, we have

$$
\begin{aligned}
||y(\mu) - \varsigma(\mu)||^{\beta} &\leq \frac{3^{1-\beta}}{(1-3^{\frac{1}{\beta}-1}\sigma_2)^{\beta}}\Big(\epsilon\Big(m + \frac{\mu^{\delta}}{\Gamma(\delta+1)}\Big)\Big)^{\beta}\Big(\varphi(\mu)+\psi\Big)^{\beta} \\
&\Big(1 + \frac{3^{\frac{1}{\beta}-1}}{1-3^{\frac{1}{\beta}-1}\sigma_2}\mathcal{L}_{\mathfrak{J}}E_{\delta}(\sigma_{01}+\mathcal{L}_z)\Gamma(\delta)\mu^{\delta}\Big)^{k\beta} \\
&\Big(\frac{3^{\frac{1}{\beta}-1}}{1-3^{\frac{1}{\beta}-1}\sigma_2}E_{\delta}(\sigma_{01}+\mathcal{L}_z)\Gamma(\delta)\mu^{\delta}\Big)^{\beta} \\
&\leq 4.28\epsilon^{\frac{1}{2}}(\varphi^{\frac{1}{2}}(\mu)+\psi)\Big(1 + 4.76\Big(\frac{1}{20}\Big)E_{\frac{4}{5}}\Big(0.6+\frac{1}{40}\Big)\Gamma\Big(\frac{4}{5}\Big)\mu^{\frac{4}{5}}\Big)^{\frac{1}{2}} \\
&\Big(4.76E_{\frac{4}{5}}\Big(0.6+\frac{1}{40}\Big)\Gamma\Big(\frac{4}{5}\Big)\mu^{\frac{4}{5}}\Big)^{1/2} \\
&\leq (20.4)(0.01)^{\frac{1}{2}}\Big(\varphi^{\frac{1}{2}}(\mu)+\psi^{\frac{1}{2}}\Big).
\end{aligned}
$$

Therefore, the system is $\frac{1}{2}$–HURS with respect to $(\varphi^{\frac{1}{2}}, \psi^{\frac{1}{2}})$, with $\mathcal{Z}_{z,\varphi,\psi} = 20.4$.

6. Conclusions

In the present article, we have explained the exact and approximate controllability of a neutral system of differential equations containing impulses and delays. Our results are dominated by fixed point theory. The finite time stability and β–Hyers–Ulam–Rassias stability of the aforementioned system are discussed by employing Grönwall-type inequality. Our obtained results are quite significant, as controllability is a qualitative property which plays a central role in control problems. It provides feedback to stabilize an unstable system. Finite time stability requires prescribed bounds on system variables. For systems that are known to operate only over a finite interval of time, this means that whenever, based on practical considerations, the system's variables must lie within the specific bounds, the Hyers-Ulam-Rassias stability of fractional differential systems guarantees a bound between the exact and approximate solutions. Therefore, such an approach may be required in a number of applications, including optimization, approximation, and numerical analysis. In the future, this study may be extended to include neutral integral fractional differential systems.

Author Contributions: Conceptualization, L.G., K.A.S., S.B. and A.Z.; formal analysis, L.G., K.A.S., S.B. and A.Z.; writing—original draft preparation, L.G., K.A.S., S.B. and A.Z.; writing—review and editing, L.G., K.A.S., S.B. and A.Z.; funding acquisition, L.G., K.A.S., S.B. and A.Z. All authors have read and agreed to the published version of the manuscript.

Funding: This research was supported by the National Natural Science Foundation of China (12101086), Changzhou Science and Technology Planning Project (CJ20210133), Natural Science Foundation of Chongqing (cstc2020jcyj-msxmX0123), and Technology Research Foundation of Chongqing Educational Committee (KJQN 202000528).

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Balachandran, K.; Park, J.Y.; Trujillo, J.J. Controllability of nonlinear fractional dynamical systems. *Nonlinear Anal. Theory Methods Appl.* **2012**, *75*, 1919–1926. [CrossRef]
2. Balachandran, K.; Kokila, J. On the controllability of fractional dynamical systems. *Int. J. Appl. Math. Comput. Sci.* **2012**, *22*, 523–531. [CrossRef]
3. Mahmudov, N.I. Approximate controllability of fractional neutral evolution equations in Banach spaces. *Abstr. Appl. Anal.* **2013**, *2013*, 531894. [CrossRef]

4. Pervaiz, B.; Zada, A.; Etemad, S.; Rezapour, S. An analysis on the controllability and stability to some fractional delay dynamical systems on time scales with impulsive effects. *Adv. Diff. Equ.* **2021**, *2021*, 1–36. [CrossRef]
5. Sakthivel, R.; Ganesh, R.; Ren, Y.; Anthoni, S.M. Approximate controllability of nonlinear fractional dynamical systems. *Commun. Nonlinear Sci. Numer. Simul.* **2013**, *18*, 3498–3508. [CrossRef]
6. Li, X.; Liu, Z.; Tisdell, C.C. Approximate controllability of fractional control systems with time delay using the sequence method. *Electron. J. Diff. Equ.* **2017**, *272*, 1–11.
7. Waheed, H.; Zada, A.; Rizwan, R.; Popa, I. Controllability of coupled fractional integrodifferential equations. *Int. J. Nonlinear Sci. Numer. Simul.* **2022**, *epub ahead of print*. [CrossRef]
8. Wang, J.; Zhou, Y. Complete controllability of fractional evolution systems. *Commun. Nonlinear Sci. Numer. Simul.* **2012**, *17*, 4346–4355. [CrossRef]
9. Zhenhai, L.; Li, X. Approximate Controllability of Fractional Evolution Systems with Riemann–Liouville Fractional Derivatives. *SIAM J. Control Optim.* **2015**, *53*, 1920–1933.
10. Aoki, T. On the stability of the linear transformation in Banach spaces. *J. Math. Soc. Jpn.* **1950**, *2*, 64–66. [CrossRef]
11. El-Hady, E.S.; Ogrekci, S. On Hyers-Ulam-Rassias stability of fractional differential equations with Caputo derivative. *J. Math. Comput. Sci.* **2021**, *22*, 325–332. [CrossRef]
12. Lee, Y.; Jun, K.W. A generalization of the Hyers-Ulam-Rassias stability of Jensen's equation. *J. Math. Anal. Appl.* **1999**, *238*, 305–315. [CrossRef]
13. Sousa, J.V.C.; De Oliveira, E.C. Ulam-Hyers stability of a nonlinear fractional Volterra integro-differential equation. *Appl. Math. Lett.* **2018**, *81*, 50–56. [CrossRef]
14. Wang, X.; Arif, M.; Zada, A. β–Hyers–Ulam–Rassias stability of semilinear nonautonomous impulsive system. *Symmetry* **2019**, *11*, 231. [CrossRef]
15. Wang, J.; Li, X. Ulam–Hyers stability of fractional Langevin equations. *Appl. Math. Comput.* **2015**, *258*, 72–83. [CrossRef]
16. Wang, J.; Li, X. E_α–Ulam type stability of fractional order ordinary differential equations. *J. Appl. Math. Comput.* **2014**, *45*, 449–459. [CrossRef]
17. Wang, J.; Li, X. A uniformed method to Ulam–Hyers stability for some linear fractional equations. *Mediterr. J. Math.* **2016**, *13*, 625–635. [CrossRef]
18. Zada, A.; Shah, S.O. Hyers-Ulam stability of first-order non-linear delay differential equations with fractional integrable impulses. *Hacet. J. Math. Stat.* **2018**, *47*, 1196–1205. [CrossRef]
19. Kamenkov, G. On stability of motion over a finite interval of time. *J. Appl. Math. Mech.* **1953**, *17*, 529–540.
20. Bhat, S.P.; Bernstein, D.S. Finite-time stability of homogeneous systems. In Proceedings of the 1997 American Control Conference (Cat. No.97CH36041), Albuquerque, NM, USA, 6 June 1997; IEEE: Piscataway, NJ, USA, 1997; Volume 4.
21. Bhat, S.P.; Bernstein, D.S. Finite-time stability of continuous autonomous systems. *SIAM J. Control Optim.* **2000**, *38*, 751–766. [CrossRef]
22. Hong, Y. Finite-time stabilization and stability of a class of controllable systems. *Syst. Control Lett.* **2002**, *46*, 231–236. [CrossRef]
23. Li, M.; Wang, J. Exploring delayed Mittag-Leffler type matrix functions to study finite time stability of fractional delay differential equations. *Appl. Math. Comput.* **2018**, *324*, 254–265. [CrossRef]
24. Li, M.; Wang, J. Finite time stability of fractional delay differential equations. *Appl. Math. Lett.* **2017**, *64*, 170–176. [CrossRef]
25. Moulay, E.; Dambrine, M.; Yeganefar, N.; Perruquetti, W. Finite-time stability and stabilization of time-delay systems. *Syst. Control Lett.* **2008**, *57*, 561–566. [CrossRef]
26. Phat, V.N.; Thanh, N.T. New criteria for finite-time stability of nonlinear fractional-order delay systems: A Gronwall inequality approach. *Appl. Math. Lett.* **2018**, *83*, 169–175. [CrossRef]
27. Shah, K.A.; Zada, A. Controllability and stability analysis of an oscillating system with two delays. *Math. Methods Appl. Sci.* **2021**, *44*, 14733–14765. [CrossRef]
28. Wang, Q.; Lu, D.; Fang, Y. Stability analysis of impulsive fractional differential systems with delay. *Appl. Math. Lett.* **2015**, *40*, 1–6. [CrossRef]
29. Weiss, L.; Infante, E. Finite time stability under perturbing forces and on product spaces. *IEEE Trans. Automat. Control* **1967**, *12*, 54–59. [CrossRef]
30. Wu, R.; Lu, Y.; Chen, L. Finite-time stability of fractional delayed neural networks. *Neurocomputing* **2015**, *149*, 700–707. [CrossRef]
31. Nawaz, M.; Wei, J.; Jiale, S. The controllability of fractional differential system with state and control delay. *Adv. Diff. Equ.* **2020**, *2020*, 30. [CrossRef]
32. Li, M.; Debbouche, A.; Wang, J. Relative controllability in fractional differential equations with pure delay. *Math. Methods Appl. Sci.* **2018**, *41*, 8906–8914. [CrossRef]
33. Sakthivel, R.; Ren, Y.; Mahmudov, N.I. On the approximate controllability of semilinear fractional differential systems. *Comput. Math. Appl.* **2011**, *62*, 1451–1459. [CrossRef]
34. Denghao, P.; Wei, J. Finite-time stability of neutral fractional time-delay systems via generalized Gronwalls inequality. *Abstr. Appl. Anal.* **2014**, *2014*, 610547. [CrossRef]
35. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. Theory and Applications of Fractional Differential Equations. In *North–Holland Mathematics Studies*; Elsevier: Amsterdam, The Netherlands, 2006; Volume 204.

36. Jneid, M. Approximate controllability of semilinear integro-differential control systems with nonlocal conditions. *Appl. Math. Sci.* **2017**, *11*, 1441–1453. [CrossRef]
37. Mahmudov, N.I.; Mckibben, M.A. On approximately controllable systems. *Appl. Comput. Math.* **2016**, *15*, 247–264.
38. Zhou, Y.; Jiao, F. Existence of mild solutions for fractional neutral evolution equations. *Comput. Math. Appl.* **2010**, *59*, 1063–1077. [CrossRef]
39. Wang, J.; Ibrahim, A.G.; O'Regan, D. Finite approximate controllability of hilfer fractional semilinear differential equations. *Miskolc Math. Notes* **2020**, *21*, 489–507. [CrossRef]
40. Nieto, J.J.; Tisdell, C.C. On Exact Controllability of First-Order Impulsive Differential Equations. *Adv. Diff. Equ.* **2010**, *2010*, 1–9. [CrossRef]
41. Samoilenko, A.M.; Perestyuk, N.A. Impulsive differential equations. In *World Scientific Series on Nonlinear Science: Monographs and Treatises*; World Scientific: Singapore, 1995; Volume 14.
42. Wang, J.; Zhou, Y.; Feckan, M. Nonlinear impulsive problems for fractional differential equations and Ulam stability. *Comput. Math. Appl.* **2012**, *64*, 3389–3405. [CrossRef]