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Sliding Mode Control for a Class of Nonlinear Fractional Order Systems with a Fractional Fixed-Time Reaching Law

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Abstract: In this paper, the sliding-mode control method was used to control a class of general nonlinear fractional-order systems which covers a wide class of chaotic systems. A novel sliding manifold with an additional nonlinear part which achieved better control performance was designed. Furthermore, a novel fixed-time reaching law with a fractional adaptive gain is proposed, where the reaching time to the sliding manifold is determined by the first positive zero of a Mittag–Leffler function and is independent of initial conditions. We have provided some instructions on tuning the parameters of the proposed reaching law to avoid exacerbating the chattering phenomenon. Finally, simulation examples are presented to validate all results.

Keywords: sliding mode control; fractional order systems; fixed-time reaching law; Mittag–Leffler function

1. Introduction

Recently, fractional calculus (an extension of integer order calculus) has played an important role in solving all kinds of science and engineering problems. Fractional calculus aids the precision and conciseness of modeling, and many practical plants have been validated to have fractional order properties, such as memristor [1], viscoelasticity [2], psoriasis [3], and abnormal diffusion process [4,5]. In addition, fractional order controllers have been shown to achieve better control performance, such as strong robustness and rapid convergence speed, compared with classical integer order controllers [6–8]. Moreover, some fractional order controllers, such as $PI^\lambda D^\mu$ controller [9], have been successfully applied in practice.

Sliding mode control (SMC) has been widely utilized to control perturbed nonlinear fractional-order systems due to its strong robustness to matched disturbance and system uncertainty [10–15]. For instance, in [16], finite-time inter-layer projective synchronization of fractional-order two-layer networks based on SMC technique was investigated. In [17], the authors discuss the problem of tracking and stabilization of a class of chained fractional-order nonlinear systems via SMC with a single input. SMC is used to realize the stabilization and synchronization of fractional chaotic systems [18–21]. For instance, in [22], SMC with an adaptive reaching law is presented to control a class of fractional chaotic systems, including fractional Lure systems, fractional Lorenz systems, and so on. In [19], the SMC method is used to control chaos synchronization between the sending end and the receiving end and improve robustness to the parameter uncertainties and disturbances in the system. In [23], a novel fixed-time SMC is designed for the secure communication of chaotic systems with disturbance and uncertainty.

Generally, SMC has two phases: the reaching phase and the sliding mode phase. Once the sliding manifold is reached, the system will enter the sliding mode phase, during which the system is robust to disturbance and uncertainty. Therefore, rapidly reaching the sliding manifold phase is desirable, and different reaching laws have been proposed based on

the traditional reaching law $\dot{s} = -ksgn(s)$, $k > 0$. A novel fractional-order reaching law was presented and analyzed in [24,25], where the sliding manifold can be reached in a shorter time than the traditional reaching law by appropriately tuning the parameters. However, the reaching time in almost all published works about SMC [24–27] is dependent upon the initial condition of the sliding manifold, i.e., $s(0)$. Consequently, there has been increasing attention paid to the fixed-time reaching law, where there is an upper bound for the reaching time with arbitrary initial values, making the reaching time independent of initial conditions [6,28,29]. In [6], we developed a novel second-order reaching law, which guarantees a fixed-time convergence with the sliding manifold. On this basis, two types of fixed-time reaching laws with an adaptive gain were then proposed in [30]. However, these mentioned reaching laws exacerbate the chattering phenomenon, compared with existing reaching laws. Recently, a reaching-phase-free approach was proposed to control a class of uncertain fractional-order systems, where the reaching phase was eliminated [15]. However, the initial condition must be exactly known in advance, which is an obstacle for practical usage.

The chattering phenomenon is always present when SMC is utilized. It is exacerbated by the pursuit of quick access to the sliding manifold phase, as the basic objective is to increase the reaching gain k, which mainly determines the chattering amplitude. Many articles concentrate on accelerating access to the sliding manifold without exacerbating chattering. A novel exponential reaching law was proposed to accelerate the access procedure without exacerbating the chattering phenomenon in [31]. In [32], a novel reaching law was proposed to attenuate the chattering phenomenon. Though it is proven that the fixed-time reaching law can guarantee quick access in [24], it will still exacerbate the chattering phenomenon more than the traditional reaching law in the presence of disturbance and uncertainty.

Motivated by aforementioned issues, SMC was utilized to control a class of general nonlinear fractional-order systems in this paper, which covers a wide class of fractional chaotic systems. A general design procedure for the sliding manifold is proposed. We found better system performance was achieved by introducing an additional nonlinear item. A novel fractional fixed-time reaching law is then proposed, where the reaching time is independent of initial conditions and determined by the first positive zero of a Mittag–Leffler function. We also provide some remarks on tuning the parameters of the fixed-time reaching law in order to attenuate the chattering phenomenon. The main contributions can be summarized as follows:

(1) The SMC method was used to control a class of fractional order systems covering a wide class of fractional chaotic systems, where a better control performance is achieved by introducing a nonlinear item to the sliding manifold.
(2) A novel fixed-time reaching law with a fractional adaptive gain is proposed, which reduces the fixed-time reaching law in [6] with some specific parameters.
(3) The proposed reaching law will not exacerbate the chattering phenomenon compared with the reaching law in [6] and shows more robustness to initial conditions compared with the reaching law in [29].

2. Preliminaries

In this section, some basic definitions about fractional calculus will be introduced [33]. The fractional order integral is defined as

$$_c\mathscr{I}_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_c^t (t-\tau)^{\alpha-1} f(\tau)d\tau, \tag{1}$$

where $\alpha > 0$, $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}dx$ is the Gamma function.

On this basis, Caputo's derivative definition is relevant to this study, which has the following form

$$_{c}\mathscr{D}_{t}^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)} \int_{c}^{t} \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau. \tag{2}$$

Its Laplace transform can be formulated as

$$\mathscr{L}\{_{0}\mathscr{D}_{t}^{\alpha}f(x)\} = p^{\alpha}F(p) - \sum_{k=0}^{n-1} p^{\alpha-k-1}f^{(k)}(0),$$

where p denotes the Laplace operator.

To simplify the notation, we denote the fractional order derivative of order α as $\mathscr{D}^{\alpha}$ instead of $_{0}\mathscr{D}_{t}^{\alpha}$ in this study. Similarly, fractional order integration $_{0}\mathscr{I}_{t}^{\alpha}$ is represented by $\mathscr{I}^{\alpha}$. In the entire following discussion, Caputo's derivative definition will be used.

The frequency-distributed model of the fractional-order integrator will be introduced in the following, which will contribute significantly to the analysis of the closed-loop stability.

Lemma 1 (Ref. [34]). *The fractional-order system $\mathscr{D}^{\alpha}y(t) = v(t)$ with $0 < \alpha < 1$, $v(t) \in \mathbb{R}$, $y(t) \in \mathbb{R}$ is a continuous linear frequency distributed system. Its frequency distributed state $z(\omega, t) \in \mathbb{R}$, which is also called the true state of fractional-order systems, satisfies*

$$\frac{\partial z(\omega, t)}{\partial t} = -\omega z(\omega, t) + v(t), \tag{3}$$

and the output $y(t)$ is the weighted integral

$$y(t) = \int_{0}^{\infty} \mu_{\alpha}(\omega)z(\omega, t)d\omega, \tag{4}$$

with the frequency weighting function

$$\mu_{\alpha}(\omega) = \frac{\sin(\alpha\pi)}{\omega^{\alpha}\pi} > 0. \tag{5}$$

Remark 1. *A fractional order system can be converted into its equivalent frequency distributed model based on Lemma 1. The stability of a frequency distributed model can be analyzed via the indirect Lyapunov method [35], and has been widely applied to analyze the stability of fractional order systems. Moreover, fractional order systems under Caputo's derivative definition and Riemann–Liouville's derivative definition are special cases of frequency-distributed systems (3) with initial conditions $z(\omega, 0)$ properly being settled [36,37] .*

The Mittag–Leffler function with two parameters, which is of great importance in the following, is defined as

$$\mathscr{E}_{,}(z) = \sum_{k=0}^{\infty} \frac{z^{k}}{\Gamma(\alpha k + \beta)}, \tag{6}$$

where $\alpha > 0$, $\beta > 0$.

Lemma 2 (Ref. [38]). *The following Laplace transform pair always holds*

$$\mathscr{L}^{-1}\left\{\frac{p^{\alpha-\beta}}{p^{\alpha}+\lambda}\right\} = t^{\beta-1}\mathscr{E}_{\alpha,\beta}(-\lambda t^{\alpha}), \tag{7}$$

where $\alpha > 0$, $\beta > 0$, $\lambda > 0$.

Lemma 3. *The following function*

$$g(t) = \mathscr{L}^{-1}\left\{ \frac{p^{\gamma}}{p^{1+\gamma} + \rho} \right\} = \mathscr{E}_{1+\gamma,1}\left(-\rho t^{1+\gamma}\right), \tag{8}$$

must have at least one positive zero for $t > 0$ with $0 < \gamma < 1$.

Proof. We will prove this lemma by contradiction and firstly assume that $g(t) > 0$ all the time. Consider the input function $\frac{t^{0.5\gamma}}{\Gamma(1+0.5\gamma)}$, whose Laplace transformation is $\frac{1}{p^{1+0.5\gamma}}$. Then define

$$y(t) = \mathscr{L}^{-1}\left\{ \frac{p^{\gamma}}{p^{1+\gamma} + \rho} \frac{1}{p^{1+0.5\gamma}} \right\} = \mathscr{L}^{-1}\left\{ \frac{p^{0.5\gamma}}{p(p^{1+\gamma} + \rho)} \right\}. \tag{9}$$

One can conclude that $y(\infty) = 0$ by Laplace's final value theorem.
On the other hand,

$$y(\infty) = \frac{1}{\Gamma(1+0.5\gamma)} \int_0^{\infty} g(t-\tau)\tau^{0.5\gamma}d\tau \tag{10}$$

is greater than zero since $g(t) > 0$ always holds due to the assumption.

By contradiction, it is known that $g(t)$ cannot always be greater than zero, which implies that it must have at least one zero. This completes the proof. □

3. Main Results

3.1. System Description

Consider the following controlled nonlinear fractional order systems

$$\begin{cases} \mathscr{D}^{\alpha_i} x_i = f_i(x) + \Delta_i + u_i, & i = 1, \cdots, n - m, \\ \mathscr{D}^{\alpha_i} x_i = f_i(x), & i = n - m + 1, \cdots, n, \end{cases} \tag{11}$$

where $0 < \alpha_i < 1$, $i = 1, 2, \cdots, n$, $x = [x_1, x_2, \cdots, x_n]^T$ represents the system states, $f_i(\cdot)$, $i = 1, 2, \cdots, n$ are the known nonlinear functions, which satisfy the Lipschitz condition to guarantee the existence and uniqueness of the solution of system (11); u_i, $i = 1, 2, \cdots, n - m$ are control inputs and $\Delta_i, i = 1, 2, \cdots, n - m$ are matched disturbances and uncertainties.

Assumption 1. *The matched disturbances and uncertainties $\Delta_i, i = 1, \cdots, n - m$ are bounded and their upper bounds are known as d_i, i.e., $|\Delta_i| < d_i$, respectively.*

Assumption 2. *For the following subsystem*

$$\dot{x}_i = f_i(x), \quad i = n - m + 1, \cdots, n, \tag{12}$$

there exists a positive definite matrix $P \in \mathbb{R}^{m \times m}$, such that

$$\begin{cases} L = y^T P y, \\ \dot{L} = \sum\limits_{i=1}^{m} \vartheta_i(x)x_i + \psi(y), \\ \psi(y) \leq 0, \end{cases} \tag{13}$$

where $y = [x_{n-m+1}, x_{n-m+2}, \cdots, x_n]^T$, $\vartheta_i(x)$ and $\psi(y)$ are functions of x and y, respectively.

Remark 2. *Assumption 2 states that the uncontrolled part of system (11) is self-stabilized since $\psi(y)$ is irrelevant to the controlled states x_i, $i = 1, \cdots, n - m$, which is reasonable and necessary*

for guaranteeing the closed-loop stability. More interestingly, Assumption 2 is assumed to hold for the corresponding integer order system.

Remark 3. *System (11) under Assumption 2 covers a wide class of fractional chaotic systems, such as Lorenz systems, Chen systems, Financial systems, Liu systems, and Lure systems [22,39], which have been widely used in encrypted communication. Therefore, Assumption 2 is not strong and the controller design for the proposed nonlinear fractional order system is of great importance for practical usage.*

3.2. Sliding Manifold Design

For the controlled system (11), the following sliding manifolds can be designated

$$s_i = \mathscr{I} \mathscr{D}^{\alpha_i} x_i + \mathscr{I} [\vartheta_i(x) + l_i(x_i)], \tag{14}$$

where $i = 1, \cdots, n - m$, $\mathscr{I}$ denotes the conventional first-order integral and $x_i l_i(x_i) \geq 0$, $i = 1, 2, \cdots, n - m$ hold.

Theorem 1. *Once the sliding manifolds (14) are reached, the closed-loop system (11) is asymptotically stable.*

Proof. Once the sliding manifolds are reached i.e., $s_i = 0$, $i = 1, 2, \cdots, n - m$, we obtain $\dot{s}_i = \mathscr{D}^{\alpha_i} x_i + \vartheta_i(x) + l(x_i) = 0$, $i = 1, 2, \cdots, n - m$. The closed-loop system can then be transformed into

$$\begin{cases} \mathscr{D}^{\alpha_i} x_i = -\vartheta_i(x) - l_i(x_i), \ i = 1, \cdots, n - m, \\ \mathscr{D}^{\alpha_i} x_i = f_i(x), i = n - m + 1, \cdots, n. \end{cases} \tag{15}$$

One can obtain the following frequency distributed models of (15) by Lemma 1

$$\begin{cases} \frac{\partial z_i(\omega, t)}{\partial t} = -\omega z_i(\omega, t) - \vartheta_i(x) - l_i(x_i), \ i = 1, \cdots, n - m, \\ \frac{\partial z_i(\omega, t)}{\partial t} = -\omega z_i(\omega, t) + f_i(x), i = n - m + 1, \cdots, n, \\ x_i = \int_0^\infty \mu_{\alpha_i}(\omega) z_i(\omega, t) d\omega, i = 1, 2, \cdots, n. \end{cases} \tag{16}$$

Select the following Lyapunov function

$$V = \sum_{i=1}^{n-m} \int_0^\infty \mu_{\alpha_i}(\omega) z_i^2(\omega, t) d\omega + \int_0^\infty z^T(\omega, t) \mu(\omega) P z(\omega, t) d\omega, \tag{17}$$

where

$$\mu(\omega) = diag[\mu_{\alpha_{n-m+1}}(\omega), \mu_{\alpha_{n-m+2}}(\omega), \cdots, \mu_{\alpha_n}(\omega)],$$

and

$$z(\omega, t) = [z_{n-m+1}(\omega, t), z_{n-m+2}(\omega, t), \cdots, z_n(\omega, t)]^T.$$

Taking the first time derivative of V, one can obtain

$$
\begin{aligned}
\dot{V} =\ & \sum_{i=1}^{n-m} \int_0^\infty \mu_{\alpha_i}(\omega)\frac{\partial z_i(\omega,t)}{\partial t}d\omega \\
& + \int_0^\infty z^T(\omega,t)P\mu(\omega)\frac{\partial z(\omega,t)}{\partial t}d\omega \\
& + \int_0^\infty \frac{\partial z^T(\omega,t)}{\partial t}\mu(\omega)Pz(\omega,t)d\omega \\
=\ & -2\sum_{i=1}^{n-m}\int_0^\infty \omega\mu_{\alpha_i}(\omega)z_i^2(\omega,t)d\omega \\
& -2\int_0^\infty \omega z^T(\omega,t)\mu(\omega)Pz(\omega,t)d\omega \\
& -2\sum_{i=1}^{n-m}[\vartheta_i(x)+l_i(x_i)]x_i + 2\sum_{i=1}^{m}\vartheta_i(x)x_i + 2\psi(y) \\
=\ & -2\sum_{i=1}^{n-m}\int_0^\infty \omega\mu_{\alpha_i}(\omega)z_i^2(\omega,t)d\omega \\
& -2\int_0^\infty \omega z^T(\omega,t)\mu(\omega)Pz(\omega,t)d\omega \\
& -2\sum_{i=1}^{n-m}l_i(x_i)x_i + 2\psi(y).
\end{aligned}
\tag{18}
$$

From the assumptions that $x_i l_i(x_i) \geq 0$, $i = 1,2,\cdots,n-m$, and $\psi(y) \leq 0$, it is concluded that $\dot{V} \leq 0$ and the closed-loop system is asymptotically stable. This completes the proof. □

Remark 4. *The condition $x_i l_i(x_i) \geq 0$, $i = 1,2,\cdots,n-m$ is easy to satisfy and one may select $l_i(x_i)$ as rx_i, $r|x_i|^\beta sgn(x_i)$ with $r > 0$, $\beta > 0$. $l_i(x_i)$, $i = 1,2,\cdots,n-m$ are added from (18) to faster attenuate the energy of Lyapunov function V and achieve better control performance.*

3.3. Fractional Fixed-Time Reaching Law

In order to shorten the reaching time to the sliding manifold, we proposed a novel fixed-time reaching law with an adaptive gain in [6]. However, the proposed reaching law will exacerbate the chattering phenomenon. In the study, a novel fixed-time reaching law with a fractional update gain will be designed to attenuate the chattering amplitude, which can be formulated as

$$
\begin{cases}
\dot{s} = -(\theta|s|^{1-\beta} + d)sgn(s), \\
\mathscr{D}^\gamma\theta = \frac{\rho}{\beta}|s|^\beta,
\end{cases}
\tag{19}
$$

where $0 < \gamma < 1$, $\rho > 0$, $0 < \beta \leq 1$, d is the upper bound of the matched disturbance, $\theta(0) = 0$, and $\mathscr{D}^\gamma$ denotes the Caputo's derivative definition. Then the sliding mode controller is designated as

$$
\begin{cases}
u_i = -f_i(x) - \vartheta_i(x) - l_i(x_i) - (\theta_i|s_i|^\beta + d_i)sgn(s_i), \\
\mathscr{D}^\gamma\theta_i = \frac{\rho}{\beta}|s_i|^\beta,
\end{cases}
\tag{20}
$$

where $0 < \gamma < 1$, $\rho > 0$, $0 < \beta \leq 1$, d is the upper bound of the matched disturbance, $\theta(0) = 0$, $i = 1,2,\cdots,n-m$.

With sliding mode controller (20), one can obtain that

$$
\begin{aligned}
\dot{s}_i =\ & \mathscr{D}^\alpha x_i + \vartheta_i(x) + l_i(x_i) \\
=\ & f_i(x) + \Delta_i + u_i + \vartheta_i(x) + l_i(x_i) \\
=\ & f_i(x) + \Delta_i + \vartheta_i(x) + l_i(x_i) - f_i(x) \\
& -\vartheta_i(x) - l_i(x_i) - (\theta_i + d_i)sgn(s_i) \\
=\ & \Delta_i - (\theta_i + d_i)sgn(s_i).
\end{aligned}
\tag{21}
$$

The following analysis is established for each control input u_i in (20); thus, we only analyze the properties for a single input. The subscript i in (20) and (21) is omitted for convenience in the following discussion.

Theorem 2. *Controller (20) guarantees a fixed-time convergence to the sliding manifold, whose reaching time is shorter than the first positive zero of $\mathscr{E}_{1+\gamma,1}\left(-\rho t^{1+\gamma}\right)$.*

Proof. Similar to (21), we have the following equation

$$
\begin{cases}
\dot{s} = -(\theta|s|^{1-\beta} + d)sgn(s) + \Delta, \\
\mathscr{D}^\gamma \theta = \frac{\rho}{\beta}|s|^\beta,
\end{cases}
\tag{22}
$$

where the subscript i is omitted for convenience. Define a Lyapunov function as $V = \frac{1}{\beta}|s|^\beta$, and take first-order derivative versus time, yielding,

$$
\dot{V} = |s|^{\beta-1}\dot{s}\,sgn(s) = -\theta + |s|^{\beta-1}(\Delta - d).
\tag{23}
$$

Finally, we arrive at the following equalities

$$
\begin{cases}
\dot{V} = -\theta + m(t), \\
\mathscr{D}^\gamma \theta = \rho V,
\end{cases}
\tag{24}
$$

where $m(t) = |s|^{\beta-1}(\Delta - d) \leq 0$ due to the assumption $d \geq |\Delta|$. Laplace transform is performed on both sides of (24), yielding,

$$
\begin{cases}
pV(p) - V(0) = -\theta(p) + m(p), \\
p^\gamma \theta(p) = \rho V(p),
\end{cases}
\tag{25}
$$

where initial condition $\theta(0) = 0$ is used. On this basis, we obtain

$$
V(p) = \frac{p^\gamma}{p^{1+\gamma} + \rho}V(0) + \frac{p^\gamma}{p^{1+\gamma} + \rho}m(p).
\tag{26}
$$

Inverse Laplace transform is performed on both sides of (26), resulting in

$$
V(t) = \mathscr{E}_{1+\gamma,1}\left(-\rho t^{1+\gamma}\right) * (V(0) + m(t)),
\tag{27}
$$

where $*$ denotes the convolution operator. According to Lemma 1, t_0 is supposed to be the first positive zero of $\mathscr{E}_{1+\gamma,1}\left(-\rho t^{1+\gamma}\right)$ and then $\mathscr{E}_{1+\gamma,1}\left(-\rho t^{1+\gamma}\right) > 0$ and $m(t) \leq 0$ hold for any $0 < t \leq t_0$. Therefore, for any $0 < t \leq t_0$, we obtain $\mathscr{E}_{1+\gamma,1}\left(-\rho t^{1+\gamma}\right) * m(t) < 0$ and then

$$
V(t) < \mathscr{E}_{1+\gamma,1}\left(-\rho t^{1+\gamma}\right)V(0).
$$

It is concluded that $V(t)$ must reach zero in a time shorter than t_0, the first positive zero of $\mathscr{E}_{1+\gamma,1}\left(-\rho t^{1+\gamma}\right)$, which is independent of initial conditions. Moreover, we obtain that $\theta(t) \geq 0$ always holds since $\theta(0) = 0$ and $\rho|s| \geq 0$, which indicates that $V(t)$ is non-increasing. Combining with the fact that $V(t) \geq 0$, once $V(t)$ reaches zero, it is concluded that $V(t)$ will be maintained thereafter. Thus, we determine that $V(t)$ will reach zero and will be maintained thereafter in a fixed time shorter than t_0, which is independent of initial conditions and indicates a fixed-time convergence. This completes the proof. □

Remark 5. *In Theorem 2, only the reaching time for a single sliding manifold is provided. It can be extended to multi sliding manifolds directly by using the following Lyapunov function*

$$
V = \frac{1}{\beta}\sum_{i=1}^{n-m}|s|^\beta.
$$

Defining $\Theta = \sum\limits_{i=1}^{n-m} \theta_i$, it follows that

$$\begin{cases} \dot{V} \le \Theta, \\ \mathscr{D}^{\gamma}\Theta = \rho V. \end{cases}$$

Consequently, the reaching time for multi sliding manifolds can similarly be derived, where the reaching time is shorter than the first positive zero of $\mathscr{E}_{1+\gamma,1}\left(-\rho t^{1+\gamma}\right)$. Additionally, the proposed reaching law will reduce to the fixed-time reaching law in [6] when $\beta = \gamma = 1$.

Theorem 2 indicates a fixed-time convergence to the sliding manifold, whose convergence time mainly depends on the designated parameters ρ, γ, and β. As is known, the amplitude of the sign function in (19), i.e., $\theta|s|^{1-\beta} + d$, determines the chattering phenomenon. For $0 < \beta < 1$, the fractional reaching law will not exacerbate the chattering phenomenon, since $\theta|s|^{1-\beta} = 0$ when $s = 0$. However, for $\beta = 1$, additional chattering will be introduced since θ is greater than zero. Consequently, a means to reduce the integral value of θ quickly after reaching the sliding manifold would be meaningful. The next theorem may reveal the relation between the decaying rate of θ and fractional order γ, and guide suitable selections for γ, ρ when designing a sliding mode controller.

Theorem 3. *For reaching law (19) with $\beta = 1$, if the convergence time is T, then for any given $\Delta T > 0$ and $\delta > 0$, one can always find a suitable γ such that $\theta(t) < \delta$ holds for any $t \ge \Delta T + T$.*

Proof. We assume that the sliding manifold converges to zero in a fixed time T according to Theorem 2 and following equation can be derived

$$\theta(t) = \frac{\rho}{\Gamma(\gamma)} \int_0^T (t-\tau)^{\gamma-1} |s(\tau)| d\tau. \tag{28}$$

Defining $\kappa = \sup\limits_{0 < t \le T} |s(t)|$, one obtains

$$\begin{aligned} \theta(t) &= \frac{\rho}{\Gamma(\gamma)} \int_0^T (t-\tau)^{\gamma-1} |s(\tau)| d\tau \\ &\le \frac{\rho}{\Gamma(\gamma)} \int_0^T (\Delta T)^{\gamma-1} \kappa d\tau \\ &\le \frac{\rho \kappa T}{\Gamma(\gamma)} (\Delta T)^{\gamma-1} \\ &\le \frac{\rho \kappa T}{\Gamma(\gamma)} \frac{1}{\min\{\Delta T, 1\}}. \end{aligned} \tag{29}$$

Finally, γ is selected such that $\Gamma(\gamma) \ge \frac{\rho \kappa T}{\delta \min\{\Delta T, 1\}}$, which could always be found since $\lim\limits_{\gamma \to 0} \Gamma(\gamma) = \infty$, and it is then concluded that $\theta \le \delta$. This completes the proof. $\square$

Remark 6. *For $0 < \beta < 1$, the fractional reaching law will not exacerbate the chattering phenomenon, since $\theta|s|^{1-\beta} = 0$ after the sliding manifold is reached in a fixed time. For $\beta = 1$, one can set γ sufficiently small at first to guarantee the quick decay of θ according to Theorem 3. After determining γ, gain ρ can be designated by numerical simulation such that the Mittag–Leffler function $\mathscr{E}_{1+\gamma,1}\left(-\rho t^{1+\gamma}\right)$ contains a positive zero smaller than the desired reaching time. Compared with the fixed-time reaching law in [6] which is a special case of (19) with $\beta = \gamma = 1$, the fractional update law will attenuate the gain θ after the sliding manifold is reached.*

Remark 7. *Though reaching-phase free SMC has been proposed to totally eliminate the reaching phase in [15], the initial conditions must be exactly known, which obstructs practical usage. The proposed fixed-time reaching law controls the upper bound of the convergence time with arbitrary initial conditions.*

4. Illustrative Examples

In this section, some simulation examples will be presented to demonstrate the effectiveness of the proposed approaches. The fractional integrator for these numerical examples is implemented via the integer-order approximation algorithm in the frequency domain. Refer to [40] for more details about the approximate algorithm. The widely used fractional order Lorenz system is considered, which is described as:

$$\begin{cases} \mathscr{D}^{\alpha_1}x = 10y - 10x + \Delta + u, \\ \mathscr{D}^{\alpha_2}y = x(28 - z) - y, \\ \mathscr{D}^{\alpha_3}z = xy - 8/3z, \end{cases} \tag{30}$$

with $\Delta = 0.1 - 0.1\sin(\pi x) + 0.1\cos(t)$ when simulating. Taking P as a unitary matrix, it is obtained from (13) that

$$\begin{cases} L = y^2 + z^2, \\ \dot{L} = 56xy - 2y^2 - 16/3z^2, \end{cases} \Rightarrow \begin{cases} \vartheta = 56y, \\ \psi = -2y^2 - 16/3z^2. \end{cases}$$

The sliding manifold and controller can be designated as

$$\begin{cases} s = \mathscr{I}\mathscr{D}^{\alpha_1}x + \mathscr{I}[56y + l(x)], \\ u = 10x - 66y - l(x) - (\theta + d)sgn(s), \\ \mathscr{D}^\gamma\theta = \rho|s|, \end{cases} \tag{31}$$

where parameters γ, ρ, d, and $l(x)$ will be given later.

Example 1. *In this example, our purpose is to compare the proposed sliding mode controller with the controller in [39] and show the superiority of our proposed sliding manifold, where the same reaching law is utilized. The controller designated in [39] can be described as*

$$\begin{cases} s = \mathscr{I}\mathscr{D}^{\alpha_1}x + \mathscr{I}[28y + 10x], \\ u = -38y - (\theta + d)sgn(s), \\ \mathscr{D}^\gamma\theta = \rho|s|. \end{cases} \tag{32}$$

Take $\alpha_1 = 0.985$, $\alpha_2 = 0.993$, $\alpha_3 = 0.99$, which guarantees the chaotic behaviour of the uncontrolled system, and system initial conditions as $x(0) = y(0) = z(0) = 1$. When simulating, select $\gamma = 1$, $\rho = 1$, $d = 0.1$, $l(x) = 35x$.

Simulation results are shown in Figures 1 and 2. It is observed that our method performs better in system output from Figure 1 than the method by Yuan, with a smaller overshooting and a shorter settling time. Moreover, control input with our method is much steadier, as shown in Figure 2. All these results demonstrate that our method is superior to the method by Yuan in both control input and system output.

Since reaching law is usually independent of the sliding manifold, we will consider the same Lorenz system in the following examples. State responses are not shown as the results are similar to Example 1.
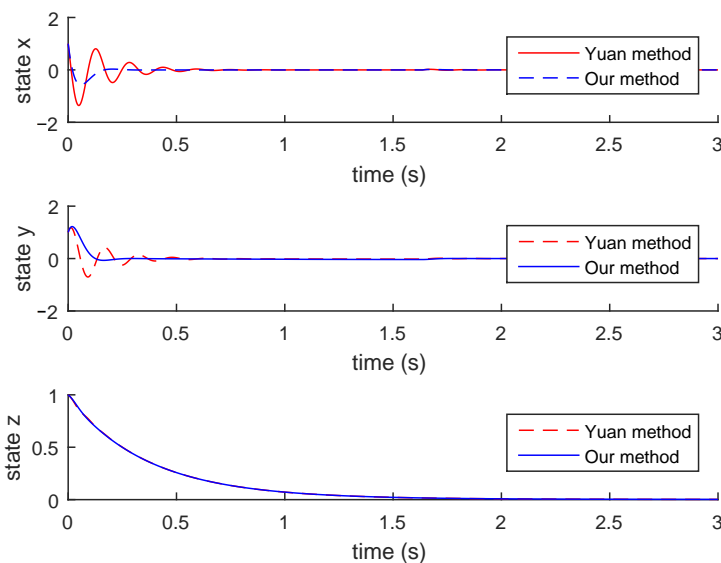
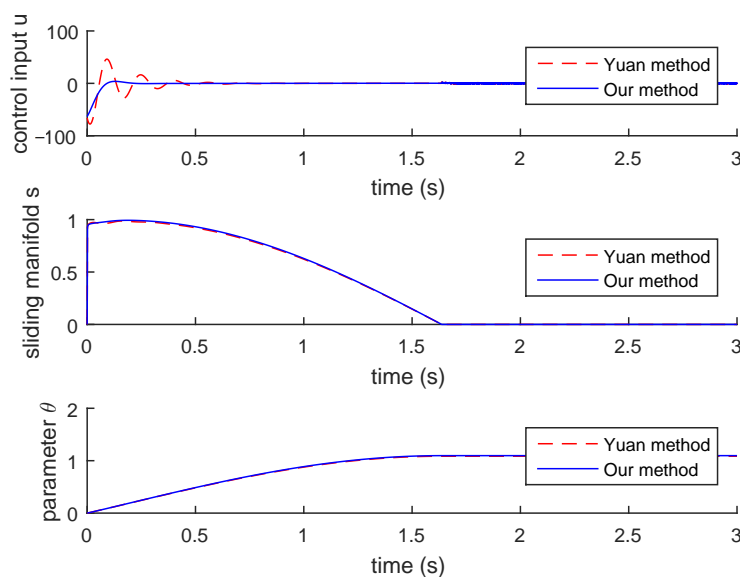**Figure 1.** State responses of x, y, z in Example 1.

Figure 2. Control inputs and sliding manifolds in Example 1.

Example 2. *In this example, we will compare the proposed fractional reaching law with [6] to show the effectiveness of the fractional update law for attenuating chattering. Take $\beta = 1$ and $d = 0.3$ when simulating. Consider following three parameter settings*

$$\begin{cases} \textit{Case 1} \;:\; \gamma = 1,\, \rho = 10,\, \beta = 1 \\ \textit{Case 2} \;:\; \gamma = 0.05,\, \rho = 10,\, \beta = 1 \\ \textit{Case 3} \;:\; \rho = 0. \end{cases} \tag{33}$$

Results are shown in Figures 3 and 4. All the reaching laws can guarantee a finite time convergence of the sliding manifold. It is found that the proposed fractional reaching law (Case 2) presents a faster convergence rate than the traditional reaching law (Case 3) without exacerbating chattering. Compared with the reaching law in [6] (Case 1), the proposed fractional reaching law has a similar convergence time but the chattering amplitude is quickly attenuated after the sliding manifold is reached as shown in Figure 3, which demonstrates the conclusion of Theorem 3.
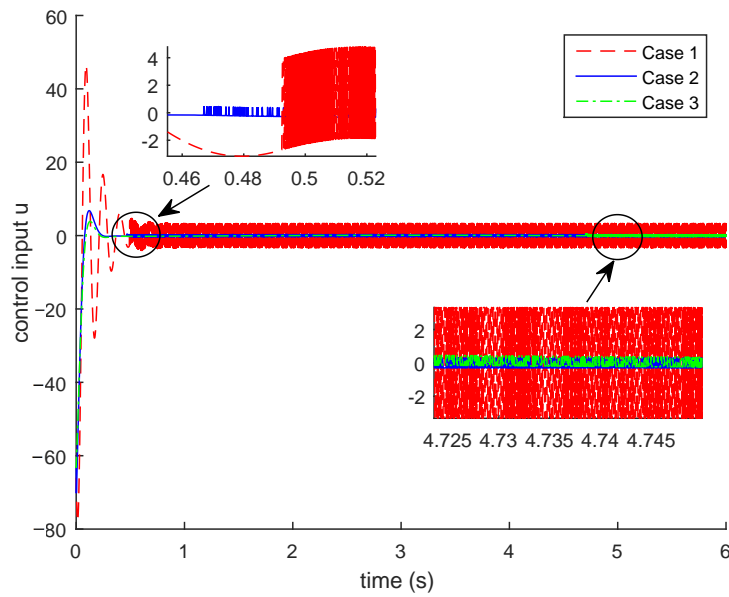
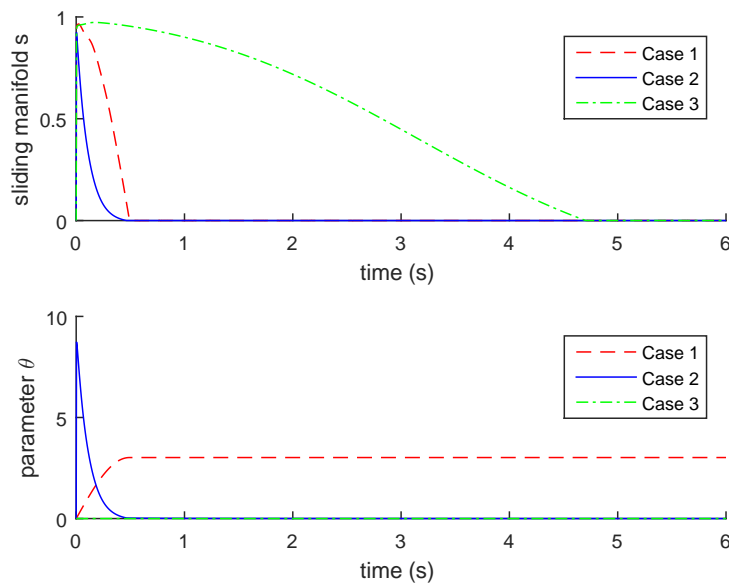**Figure 3.** Control inputs in Example 2.

Figure 4. Sliding manifolds in Example 2.

Example 3. *In this example, we will show the fixed-time convergence of the sliding manifold with different initial conditions and the same sliding manifold parameters as in Example 1. Set $\beta = 0.5$ and $\gamma = 0.3$ (since $\beta \neq 1$, the proposed reaching law will not introduce additional chattering). If the reaching time is required to be shorter than 1 s , one can choose ρ such that the first positive zero of the Mittag–Leffler function $\mathscr{E}_{1+\gamma,1}\left(-\rho t^{1+\gamma}\right)$ is smaller than the desired reaching time in the simulation. By simulation, it is found that the zero of $\mathscr{E}_{1+\gamma,1}\left(-\rho t^{1+\gamma}\right)$ is 1 s when $\rho = 2.28$. Take $\rho = 2.5$ when simulating and consider following three cases*

$$\begin{cases} Case\ 1\ :\ x = y = z = 1, \\ Case\ 2\ :\ x = y = z = 2, \\ Case\ 3\ :\ x = y = z = 3. \end{cases} \tag{34}$$

Results are presented in Figure 5, showing that the reaching time for the three cases is less than 1 (s). Moreover, the reaching times with different initial conditions are all close to 0.85 s. Unlike

the conventional case, a larger initial condition of the sliding manifold always results in a longer reaching time, which shows the robustness to the initial conditions.
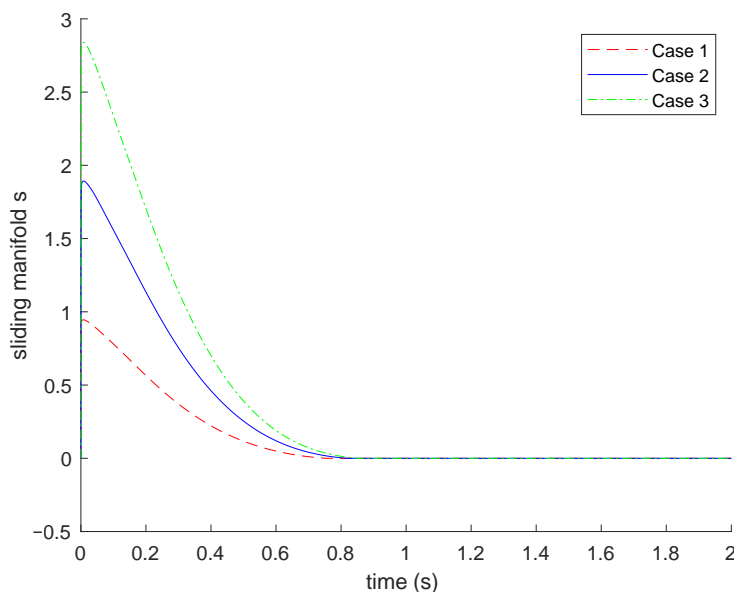

Figure 5. Sliding manifolds in Example 3.

Example 4. *In this example, we will compare the proposed reaching law with the reaching law in [29], which can be expressed as*

$$\dot{s} = -(k_1|s|^{\kappa_1} + k_2|s|^{\kappa_2} + d)sgn(s), \tag{35}$$

where $k_1 > 0$, $k_2 > 0$, $0 < \kappa_1 < 1$, $1 < \kappa_2 < 2$ are designated parameters. When simulating, set $k_1 = k_2 = 3$, $\kappa_1 = 0.5$, $\kappa_2 = 1.5$, with the parameters for reaching law (19) the same as in Example 3. Results are shown in Figures 6 and 7. It is found that both reaching laws could achieve a fixed-time convergence to the sliding manifold with different initial conditions. Notably, it was found that the reaching times of the fractional reaching law (19) under different initial conditions are almost the same. However, for the reaching law (35), reaching times with different initial conditions are different although there is an upper bound for the reaching time.
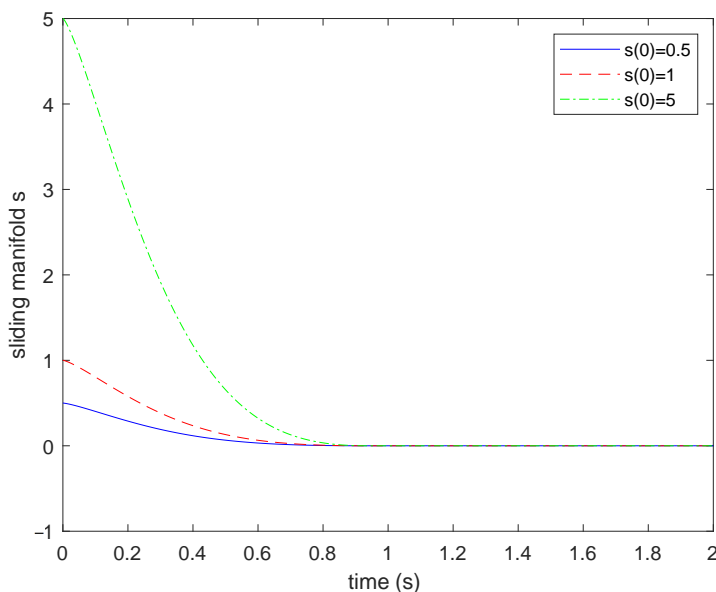

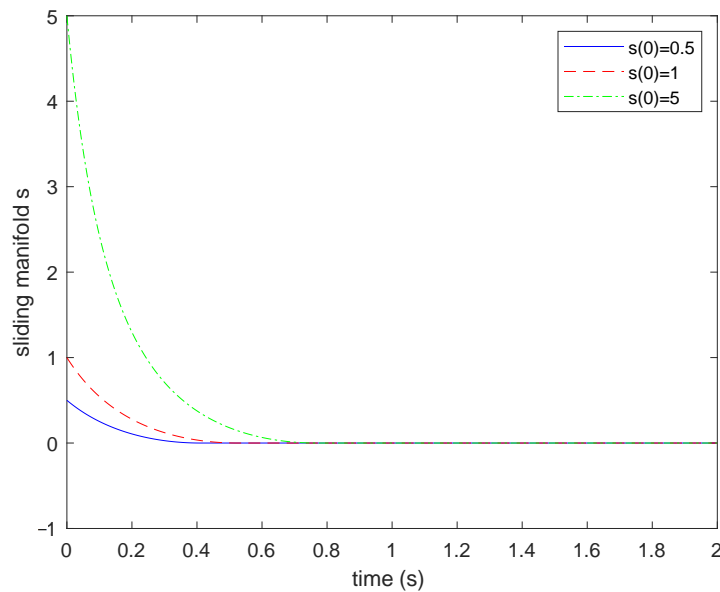**Figure 6.** Sliding manifolds with reaching law (19) in Example 4.

Figure 7. Sliding manifolds with reaching law (35) in Example 4.

5. Conclusions

A sliding mode controller has been designed to control a class of general nonlinear fractional-order systems, covering a wide class of typical chaotic systems. A general design procedure of the sliding manifold with an additional nonlinear part is proposed, where better system performance is achieved. A novel fractional fixed-time reaching law is then proposed, where the reaching time to the sliding manifold is independent of initial conditions. Moreover, the proposed reaching law will not exacerbate the chattering phenomenon compared with the traditional reaching law. All the conclusions are carefully validated by simulation results. A promising topic for future research is the use of the proposed fixed-time reaching law in practical applications.

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