fractal and fractional

Article

Conserved Quantities for Constrained Hamiltonian System within Combined Fractional Derivatives

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Abstract: Singular systems, which can be applied to gauge field theory, condensed matter theory, quantum field theory of anyons, and so on, are important dynamic systems to study. The fractional order model can describe the mechanical and physical behavior of a complex system more accurately than the integer order model. Fractional singular systems within mixed integer and combined fractional derivatives are established in this paper. The fractional Lagrange equations, fractional primary constraints, fractional constrained Hamilton equations, and consistency conditions are analyzed. Then Noether and Lie symmetry methods are studied for finding the integrals of the fractional constrained Hamiltonian systems. Finally, an example is given to illustrate the methods and results.

Keywords: fractional calculus; variational problem; constrained Hamiltonian system; Noether symmetry; Lie symmetry; conserved quantity

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1. Introduction

Fractional calculus is a hot topic lately, the advantage of which lies in its accuracy. Generally, the results obtained through the fractional order model are more precisely than those obtained by the integer order model. Fractional calculus has various applications in almost every field of science and engineering [1–10].

Since Riewe [11,12] found that fractional derivatives can be used to express dissipative forces, fractional calculus of variations with different fractional derivatives, such as the Riemann–Liouville fractional derivative [13–15], the Caputo fractional derivative [16,17], the symmetric fractional derivative [18], the Riesz fractional derivative [19–21], Agrawal's new operators [22–24], the combined fractional derivative [25–27], the mixed integer and fractional derivatives [28,29], and so on [30–36], have been investigated. It is noted that the combined fractional derivative is more general than most other fractional derivatives. For example, both the Riemann–Liouville and Caputo fractional derivatives are its special cases, as well as the Riesz fractional derivative.

In this paper, we plan to investigate the variational problems within mixed integer and combined fractional derivatives. After the fractional system is established, we consider the singular case. Singular systems, especially constrained Hamiltonian systems, play an important part in many aspects, such as the quantum field theory, the condensed matter theory, and the gauge field theory [37–39].

After the fractional constrained Hamilton equations are established, the symmetry method is considered. The symmetry method mainly contains the Noether symmetry method, the Lie symmetry method, and the Mei symmetry method [40–42]. This article pays attention to the first two symmetry methods. Under the infinitesimal transformations of time and coordinates, Noether symmetry means the invariance of the Hamilton action, while Lie symmetry means the invariance of the differential equations of motion. Noether symmetry can lead to a conserved quantity according to the Noether theory. A Lie symmetry can also lead to a conserved quantity under certain conditions. There are two kinds of conserved quantities obtained from the Lie symmetry. One is called the Hojman type conserved quantity, which is deduced directly from the Lie symmetry, and the other is called the Noether type conserved quantity, which is achieved with the help of the Noether symmetry. In this article, we discuss the latter one.

For the fractional conserved quantity, there are two definitions. One was given by Frederico and Torres [43], and the other was introduced by Atanacković et al. [44]. Fractional Noether theorems have been investigated on the basis of both definitions. For instance, the works [45,46] were achieved based on the former definition, and the results [30,31,47–52] were obtained on the basis of the latter one. However, Ferreira and Malinowska [53] proved that the fractional Noether theorem given in Ref. [43] was wrong through a counterexample. Later, Cresson and Szafrańska [54] made a detailed analysis to explain why and where the result given in Ref. [43] does not work. Furthermore, they also presented a fractional Noether theorem following their strategy, corrected the initial statement of Ref. [43], and achieved an alternative proof of the main result of Atanacković et al. [44]. There are also several results obtained for the fractional Lie symmetry. For example, Fu et.al [55,56] studied the Lie symmetry theorem of the fractional nonholonomic system on the basis of the combined Riemann–Liouville fractional derivative as well as the Lie symmetry and their inverse problem of the nonholonomic Hamiltonian system in terms of the Riemann–Liouville fractional derivative. Prakash and Sahadevan [57] gave a systematic investigation of finding Lie point symmetry of certain fractional linear and nonlinear ordinary differential equations. Nass [58] made use of Lie symmetry to solve fractional neutral ordinary differential equations. Jia and Zhang [21] studied Lie symmetry for the Birkhoffian system, etc.

In this paper, we investigate the fractional Noether theorem on the basis of Atanacković's definition for the fractional constrained Hamiltonian system within mixed integer and combined fractional derivatives, including mixed integer and combined Riemann–Liouville fractional derivatives (ICRL) and mixed integer and combined Caputo fractional derivatives (ICC). Lie symmetry with the corresponding Noether type conserved quantity is another topic in this paper.

This paper is organized as follows. Section 2 provides the preliminaries on the fractional derivatives. Based on the mixed integer and combined fractional derivatives, the fractional Lagrange equations, the fractional primary constraints, and the fractional constrained Hamilton equations are established in Sections 3–5, respectively. Then the fractional Noether symmetry and conserved quantity are studied in Section 6. Lie symmetry and the Noether type conserved quantity are investigated in Section 7. Section 8 gives an example to illustrate the methods and results.

2. Preliminaries on Fractional Derivatives

Combined fractional derivatives, which contain the combined Riemann–Liouville fractional derivative and the combined Caputo fractional derivative, are listed below [5,7,19,25,59].

Let $f(t)$ be a function, $t \in [t_1, t_2]$; then, the combined Riemann–Liouville fractional derivative and the combined Caputo fractional derivative are [25]

$$ {}^{RL}D_{\gamma}^{\alpha,\beta} f(t) = \gamma\, {}^{RL}_{t_1}D_t^{\alpha} f(t) + (-1)^n (1-\gamma)\, {}^{RL}_{t}D_{t_2}^{\beta} f(t) \tag{1} $$

$$ {}^{C}D_{\gamma}^{\alpha,\beta} f(t) = \gamma\, {}^{C}_{t_1}D_t^{\alpha} f(t) + (-1)^n (1-\gamma)\, {}^{C}_{t}D_{t_2}^{\beta} f(t) \tag{2} $$

where $n-1 \le \alpha, \beta < n$, and α and β denote the orders of the fractional derivatives; $\gamma \in [0,1]$, and γ determines the different amount of information from the past and the future; and ${}^{RL}_{t_1}D_t^{\alpha} f(t)$, ${}^{RL}_{t}D_{t_2}^{\beta} f(t)$, ${}^{C}_{t_1}D_t^{\alpha} f(t)$, and ${}^{C}_{t}D_{t_2}^{\beta} f(t)$ are the left and right Riemann–Liouville and Caputo fractional derivatives of $f(t)$, respectively. Their mathematical definitions are

$$ {}^{RL}_{t_1}D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_{t_1}^{t} (t-\xi)^{n-\alpha-1} f(\xi)\mathrm{d}\xi \tag{3} $$

$$ {}_{t}^{RL}D_{t_2}^{\beta}f(t) = \frac{1}{\Gamma(n-\beta)}\left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)^{n}\int_{t}^{t_2}(\xi-t)^{n-\beta-1}f(\xi)\mathrm{d}\xi \tag{4} $$

$$ {}_{t_1}^{C}D_{t}^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}\int_{t_1}^{t}(t-\xi)^{n-\alpha-1}\left(\frac{\mathrm{d}}{\mathrm{d}\xi}\right)^{n}f(\xi)\mathrm{d}\xi \tag{5} $$

$$ {}_{t}^{C}D_{t_2}^{\beta}f(t) = \frac{1}{\Gamma(n-\beta)}\int_{t}^{t_2}(\xi-t)^{n-\beta-1}\left(-\frac{\mathrm{d}}{\mathrm{d}\xi}\right)^{n}f(\xi)\mathrm{d}\xi \tag{6} $$

Under the condition $0<\alpha,\beta<1$, there are two relationships between Equations (3) and (5), as well as Equations (4) and (6),

$$ {}_{t_1}^{RL}D_{t}^{\alpha}f(t) = {}_{t_1}^{C}D_{t}^{\alpha}f(t) - \frac{1}{\Gamma(1-\alpha)}\frac{f(t_1)}{(t-t_1)^{\alpha}} \tag{7} $$

$$ {}_{t}^{RL}D_{t_2}^{\beta}f(t) = {}_{t}^{C}D_{t_2}^{\beta}f(t) + \frac{1}{\Gamma(1-\beta)}\frac{f(t_2)}{(t_2-t)^{\beta}} \tag{8} $$

When $\gamma=0$ or $\gamma=1$, we can find that the left and right Riemann–Liouville fractional derivatives and the left and right Caputo fractional derivatives are all special cases of the combined fractional derivatives. When $\alpha=\beta$, $\gamma=\frac{1}{2}$, we obtain

$$ {}^{RL}D_{1/2}^{\alpha,\beta}f(t) = \frac{1}{2\Gamma(n-\alpha)}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{n}\int_{t_1}^{t_2}|t-\xi|^{n-\alpha-1}f(\xi)\mathrm{d}\xi = {}_{t_1}^{R}D_{t_2}^{\alpha}f(t) \tag{9} $$

$$ {}^{C}D_{1/2}^{\alpha,\beta}f(t) = \frac{1}{2\Gamma(n-\alpha)}\int_{t_1}^{t_2}|t-\xi|^{n-\alpha-1}\left(\frac{\mathrm{d}}{\mathrm{d}\xi}\right)^{n}f(\xi)\mathrm{d}\xi = {}_{t_1}^{RC}D_{t_2}^{\alpha}f(t) \tag{10} $$

Equations (9) and (10) are the Riesz–Riemann–Liouville fractional derivative and the Riesz–Caputo fractional derivative of $f(t)$, which are also special cases of the combined fractional derivatives. Of course, different fractional derivatives can be obtained by selecting different values of γ. When $\alpha,\beta\to 1$, we have [7]

$$ \begin{gathered} {}_{t_1}^{RL}D_{t}^{1}f(t) = {}_{t_1}^{C}D_{t}^{1}f(t) = \tfrac{\mathrm{d}}{\mathrm{d}t}f(t),\ {}_{t}^{RL}D_{t_2}^{1}f(t) = {}_{t}^{C}D_{t_2}^{1}f(t) = -\tfrac{\mathrm{d}}{\mathrm{d}t}f(t) \\ {}^{RL}D_{\gamma}^{\alpha,\beta}f(t) = {}^{C}D_{\gamma}^{\alpha,\beta}f(t) = \tfrac{\mathrm{d}}{\mathrm{d}t}f(t) \end{gathered} \tag{11} $$

where $\frac{\mathrm{d}}{\mathrm{d}t}f(t)$ means the integer order derivative of $f(t)$.

In addition, the formulae of fractional integration by parts are [19]

$$ \int_{t_1}^{t_2}[*]{}_{t_1}^{RL}D_{t}^{\alpha}\eta\mathrm{d}t = \int_{t_1}^{t_2}\eta{}_{t}^{C}D_{t_2}^{\alpha}[*]\mathrm{d}t - \sum_{j=0}^{n-1}(-1)^{n+j}{}_{t_1}^{RL}D_{t}^{\alpha+j-n}\eta(t)D^{n-1-j}[*]\Big|_{t_1}^{t_2}, \tag{12} $$

$$ \int_{t_1}^{t_2}[*]{}_{t}^{RL}D_{t_2}^{\beta}\eta\mathrm{d}t = \int_{t_1}^{t_2}\eta{}_{t_1}^{C}D_{t}^{\beta}[*]\mathrm{d}t - \sum_{j=0}^{n-1}{}_{t}^{RL}D_{t_2}^{\beta+j-n}\eta(t)D^{n-1-j}[*]\Big|_{t_1}^{t_2}, \tag{13} $$

$$ \int_{t_1}^{t_2}[*]{}_{t_1}^{C}D_{t}^{\alpha}\eta\mathrm{d}t = \int_{t_1}^{t_2}\eta{}_{t}^{RL}D_{t_2}^{\alpha}[*]\mathrm{d}t + \sum_{j=0}^{n-1}{}_{t}^{RL}D_{t_2}^{\alpha+j-n}[*]D^{n-1-j}\eta(t)\Big|_{t_1}^{t_2}, \tag{14} $$

$$ \int_{t_1}^{t_2}[*]{}_{t}^{C}D_{t_2}^{\beta}\eta\mathrm{d}t = \int_{t_1}^{t_2}\eta{}_{t_1}^{RL}D_{t}^{\beta}[*]\mathrm{d}t + \sum_{j=0}^{n-1}(-1)^{n+j}{}_{t_1}^{RL}D_{t}^{\beta+j-n}[*]D^{n-1-j}\eta(t)\Big|_{t_1}^{t_2}, \tag{15} $$

$$ \int_{t_1}^{t_2}[*]{}_{t_1}^{R}D_{t_2}^{\alpha}\eta\mathrm{d}t = (-1)^{n}\int_{t_1}^{t_2}\eta{}_{t_1}^{RC}D_{t_2}^{\alpha}[*]\mathrm{d}t - \sum_{j=0}^{n-1}(-1)^{n+j}{}_{t_1}^{R}D_{t_2}^{\alpha+j-n}\eta(t)D^{n-1-j}[*]\Big|_{t_1}^{t_2}, \tag{16} $$

$$ \int_{t_1}^{t_2}[*]{}_{t_1}^{RC}D_{t_2}^{\alpha}\eta\mathrm{d}t = (-1)^{n}\int_{t_1}^{t_2}\eta{}_{t_1}^{R}D_{t_2}^{\alpha}[*]\mathrm{d}t + \sum_{j=0}^{n-1}(-1)^{j}{}_{t_1}^{R}D_{t_2}^{\alpha+j-n}[*]D^{n-1-j}\eta(t)\Big|_{t_1}^{t_2} \tag{17} $$

where $D = \frac{\mathrm{d}}{\mathrm{d}t}$ means the integer order derivative.

In this paper, we assume that $0 < \alpha, \beta < 1$.

3. Fractional Lagrange Equation

Fractional variational problems within mixed integer and combined fractional derivatives are studied in this section.

Based on ICRL, the fractional problem of the calculus of variations becomes finding the stationary function of the functional:

$$I_{RL}[\boldsymbol{q_{RL}}(\cdot)] = \int_{t_1}^{t_2} L_{RL}\left(t, \boldsymbol{q_{RL}}, \dot{\boldsymbol{q}}_{\boldsymbol{RL}}, {}^{\boldsymbol{RL}}\boldsymbol{D}_{\gamma}^{\boldsymbol{\alpha,\beta}}\boldsymbol{q_{RL}}\right)\mathrm{d}t \tag{18}$$

subject to given $2n$ boundary conditions $\boldsymbol{q_{RL}}(t_1) = \boldsymbol{q_{RL1}}$, $\boldsymbol{q_{RL}}(t_2) = \boldsymbol{q_{RL2}}$, where $[t_1, t_2] \subset \mathbb{R}$, $0 < \alpha, \beta < 1$, $\boldsymbol{q_{RL}} = (q_{RL1}, q_{RL2}, \cdots, q_{RLn})$, $\dot{\boldsymbol{q}}_{\boldsymbol{RL}} = (\dot{q}_{RL1}, \dot{q}_{RL2}, \cdots, \dot{q}_{RLn})$, $\dot{q}_{RLi} = \mathrm{d}q_{RLi}/\mathrm{d}t$, $i = 1, 2, \cdots, n$, ${}^{\boldsymbol{RL}}\boldsymbol{D}_{\gamma}^{\boldsymbol{\alpha,\beta}}\boldsymbol{q_{RL}} = \left({}^{RL}D_{\gamma}^{\alpha,\beta}q_{RL1}, {}^{RL}D_{\gamma}^{\alpha,\beta}q_{RL2}, \cdots, {}^{RL}D_{\gamma}^{\alpha,\beta}q_{RLn}\right)$, $\boldsymbol{q_{RL1}} = (q_{RL11}, q_{RL12}, \cdots, q_{RL1n})$, $\boldsymbol{q_{RL2}} = (q_{RL21}, q_{RL22}, \cdots, q_{RL2n})$, the Lagrangian $L_{RL} : [t_1, t_2] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ and $q_{RLi} : [t_1, t_2] \to \mathbb{R}$, $i = 1, 2, \cdots, n$, are assumed to be C^2 functions.

If $\boldsymbol{q_{RL}}(\cdot)$ is an extremal of Equation (18), then we have

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon_{RL}} I_{RL}[\boldsymbol{q_{RL}} + \varepsilon_{RL}\boldsymbol{h_{RL}}]|_{\varepsilon_{RL}=0} = 0 \tag{19}$$

where $\boldsymbol{h_{RL}} = (h_{RL1}, h_{RL2}, \cdots, h_{RLn})$, $\boldsymbol{h_{RL}}(\cdot) \in C^2([t_1, t_2]; \mathbb{R}^n)$, $\boldsymbol{h_{RL}}(t_1) = \boldsymbol{h_{RL}}(t_2) = 0$, and ε_{RL} is a small parameter.

From Equation (19), for $k = 1, 2, \cdots, n$, we obtain

$$\int_{t_1}^{t_2}\left(\frac{\partial L_{RL}}{\partial q_{RLk}} \cdot h_{RLk} + \frac{\partial L_{RL}}{\partial \dot{q}_{RLk}} \cdot \dot{h}_{RLk} + \frac{\partial L_{RL}}{\partial^{RL}D_{\gamma}^{\alpha,\beta}q_{RLk}} \cdot {}^{RL}D_{\gamma}^{\alpha,\beta}h_{RLk}\right)\mathrm{d}t = 0. \tag{20}$$

On the one hand, using the integer integration by parts formula and the fractional integration by parts formulae (Equations (12) and (13)) in the second and third terms of Equation (20), we have

$$\begin{aligned} &\int_{t_1}^{t_2}\left(\frac{\partial L_{RL}}{\partial \dot{q}_{RLk}} \cdot \dot{h}_{RLk}\right)\mathrm{d}t = \int_{t_1}^{t_2}\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L_{RL}}{\partial \dot{q}_{RLk}} \cdot h_{RLk}\right) - h_{RLk} \cdot \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L_{RL}}{\partial \dot{q}_{RLk}}\right]\mathrm{d}t \\ &= \frac{\partial L_{RL}}{\partial \dot{q}_{RLk}} \cdot h_{RLk}\Big|_{t=t_1}^{t=t_2} - \int_{t_1}^{t_2}\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L_{RL}}{\partial \dot{q}_{RLk}} \cdot h_{RLk}\right)\mathrm{d}t = -\int_{t_1}^{t_2}\left(h_{RLk} \cdot \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L_{RL}}{\partial \dot{q}_{RLk}}\right)\mathrm{d}t \end{aligned} \tag{21}$$

and

$$\begin{aligned} &\int_{t_1}^{t_2}\left(\frac{\partial L_{RL}}{\partial^{RL}D_{\gamma}^{\alpha,\beta}q_{RLk}} \cdot {}^{RL}D_{\gamma}^{\alpha,\beta}h_{RLk}\right)\mathrm{d}t \\ &= \int_{t_1}^{t_2}\frac{\partial L_{RL}}{\partial^{RL}D_{\gamma}^{\alpha,\beta}q_{RLk}} \cdot \left[\gamma\,{}_{t_1}^{RL}D_t^{\alpha}h_{RLk} - (1-\gamma)\,{}_t^{RL}D_{t_2}^{\beta}h_{RLk}\right]\mathrm{d}t \\ &= \gamma\int_{t_1}^{t_2}\left(\frac{\partial L_{RL}}{\partial^{RL}D_{\gamma}^{\alpha,\beta}q_{RLk}} \cdot {}_{t_1}^{RL}D_t^{\alpha}h_{RLk}\right)\mathrm{d}t - (1-\gamma)\int_{t_1}^{t_2}\left(\frac{\partial L_{RL}}{\partial^{RL}D_{\gamma}^{\alpha,\beta}q_{RLk}} \cdot {}_t^{RL}D_{t_2}^{\beta}h_{RLk}\right)\mathrm{d}t \\ &= \gamma\left\{\int_{t_1}^{t_2}\left(h_{RLk} \cdot {}_t^{C}D_{t_2}^{\alpha}\frac{\partial L_{RL}}{\partial^{RL}D_{\gamma}^{\alpha,\beta}q_{RLk}}\right)\mathrm{d}t + \left[{}_{t_1}^{RL}D_t^{-(1-\alpha)}h_{RLk}(t) \cdot \frac{\partial L_{RL}}{\partial^{RL}D_{\gamma}^{\alpha,\beta}q_{RLk}}\right]\Bigg|_{t=t_1}^{t=t_2}\right\} \\ &-(1-\gamma)\left\{\int_{t_1}^{t_2}\left(h_{RLk} \cdot {}_{t_1}^{C}D_t^{\beta}\frac{\partial L_{RL}}{\partial^{RL}D_{\gamma}^{\alpha,\beta}q_{RLk}}\right)\mathrm{d}t - \left[{}_t^{RL}D_{t_2}^{-(1-\beta)}h_{RLk}(t) \cdot \frac{\partial L_{RL}}{\partial^{RL}D_{\gamma}^{\alpha,\beta}q_{RLk}}\right]\Bigg|_{t=t_1}^{t=t_2}\right\} \\ &= -\int_{t_1}^{t_2}\left(h_{RLk} \cdot {}^{C}D_{1-\gamma}^{\beta,\alpha}\frac{\partial L_{RL}}{\partial^{RL}D_{\gamma}^{\alpha,\beta}q_{RLk}}\right)\mathrm{d}t + \frac{\gamma}{\Gamma(1-\alpha)}\int_{t_1}^{t_2}(t_2-t)^{-\alpha}h_{RLk}\mathrm{d}t \cdot \frac{\partial L_{RL}(t_2)}{\partial^{RL}D_{\gamma}^{\alpha,\beta}q_{RLk}} \\ &-\frac{1-\gamma}{\Gamma(1-\beta)}\int_{t_1}^{t_2}(t-t_1)^{-\beta}h_{RLk}\mathrm{d}t \cdot \frac{\partial L_{RL}(t_1)}{\partial^{RL}D_{\gamma}^{\alpha,\beta}q_{RLk}}, \end{aligned} \tag{22}$$

where $\frac{\partial L_{RL}(t_1)}{\partial^{RL}D_\gamma^{\alpha,\beta}q_{RLk}} = \frac{\partial L_{RL}}{\partial^{RL}D_\gamma^{\alpha,\beta}q_{RLk}}\left(t_1, \boldsymbol{q_{RL}}(t_1), \dot{\boldsymbol{q}}_{\boldsymbol{RL}}(t_1), {}^{\boldsymbol{RL}}\boldsymbol{D}_\gamma^{\boldsymbol{\alpha,\beta}}\boldsymbol{q_{RL}}(t_1)\right)$, $\frac{\partial L_{RL}(t_2)}{\partial^{RL}D_\gamma^{\alpha,\beta}q_{RLk}} = \frac{\partial L_{RL}}{\partial^{RL}D_\gamma^{\alpha,\beta}q_{RLk}}\left(t_2, \boldsymbol{q_{RL}}(t_2), \dot{\boldsymbol{q}}_{\boldsymbol{RL}}(t_2), {}^{\boldsymbol{RL}}\boldsymbol{D}_\gamma^{\boldsymbol{\alpha,\beta}}\boldsymbol{q_{RL}}(t_2)\right)$.

Substituting Equations (21) and (22) into Equation (20), we obtain

$$\begin{aligned}\int_{t_1}^{t_2}\left[\frac{\partial L_{RL}}{\partial q_{RLk}} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L_{RL}}{\partial \dot{q}_{RLk}} - {}^{C}D_{1-\gamma}^{\beta,\alpha}\frac{\partial L_{RL}}{\partial^{RL}D_\gamma^{\alpha,\beta}q_{RLk}} + \frac{\gamma(t_2-t)^{-\alpha}}{\Gamma(1-\alpha)}\frac{\partial L_{RL}(t_2)}{\partial^{RL}D_\gamma^{\alpha,\beta}q_{RLk}}\right.\\ \left. - \frac{(1-\gamma)(t-t_1)^{-\beta}}{\Gamma(1-\beta)}\frac{\partial L_{RL}(t_1)}{\partial^{RL}D_\gamma^{\alpha,\beta}q_{RLk}}\right]\cdot h_{RLk}\mathrm{d}t = 0.\end{aligned} \tag{23}$$

It follows from the fundamental lemma of the calculus of variations [60] that

$$\begin{aligned}\frac{\partial L_{RL}}{\partial q_{RLk}} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L_{RL}}{\partial \dot{q}_{RLk}} - {}^{C}D_{1-\gamma}^{\beta,\alpha}\frac{\partial L_{RL}}{\partial^{RL}D_\gamma^{\alpha,\beta}q_{RLk}} + \frac{\gamma(t_2-t)^{-\alpha}}{\Gamma(1-\alpha)}\frac{\partial L_{RL}(t_2)}{\partial^{RL}D_\gamma^{\alpha,\beta}q_{RLk}}\\ - \frac{(1-\gamma)(t-t_1)^{-\beta}}{\Gamma(1-\beta)}\frac{\partial L_{RL}(t_1)}{\partial^{RL}D_\gamma^{\alpha,\beta}q_{RLk}} = 0,\ k = 1, 2, \cdots, n.\end{aligned} \tag{24}$$

Equation (24) is called the fractional Lagrange equation within ICRL.

On the other hand, using the integer integration by parts formula, the fractional integration by parts formulae (Equations (14) and (15)), and the relationships (Equations (7) and (8)) in the second and third terms of Equation (20), as well as the fundamental lemma of the calculus of variations [60], we obtain

$$\frac{\partial L_{RL}}{\partial q_{RLk}} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L_{RL}}{\partial \dot{q}_{RLk}} - {}^{RL}D_{1-\gamma}^{\beta,\alpha}\frac{\partial L_{RL}}{\partial^{RL}D_\gamma^{\alpha,\beta}q_{RLk}} = 0,\ k = 1, 2, \cdots, n. \tag{25}$$

Equation (25) is also called the fractional Lagrange equation within ICRL. Equations (24) and (25) are two different forms.

Remark 1. *When* $\gamma = 1$, *Equation (25) reduces to the fractional Lagrange equation within the mixed integer and the left Riemann–Liouville fractional derivative, which is consistent with the result in Ref.* [29].

Similarly, based on ICC, the fractional problem of the calculus of variations becomes finding the stationary function of the functional:

$$I_C[\boldsymbol{q_C}(\cdot)] = \int_{t_1}^{t_2} L_C\left(t, \boldsymbol{q_C}, \dot{\boldsymbol{q}}_{\boldsymbol{C}}, {}^{\boldsymbol{C}}\boldsymbol{D}_\gamma^{\boldsymbol{\alpha,\beta}}\boldsymbol{q_C}\right)\mathrm{d}t \tag{26}$$

subject to given $2n$ boundary conditions $\boldsymbol{q_C}(t_1) = \boldsymbol{q_{C1}}$, $\boldsymbol{q_C}(t_2) = \boldsymbol{q_{C2}}$, where $[t_1, t_2] \subset \mathbb{R}$, $0 < \alpha, \beta < 1$, $\boldsymbol{q_C} = (q_{C1}, q_{C2}, \cdots, q_{Cn})$, $\dot{\boldsymbol{q}}_{\boldsymbol{C}} = (\dot{q}_{C1}, \dot{q}_{C2}, \cdots, \dot{q}_{Cn})$, $\dot{q}_{Ci} = \mathrm{d}q_{Ci}/\mathrm{d}t$, $i = 1, 2, \cdots, n$, ${}^{\boldsymbol{C}}\boldsymbol{D}_\gamma^{\boldsymbol{\alpha,\beta}}\boldsymbol{q_C} = \left({}^{C}D_\gamma^{\alpha,\beta}q_{C1}, {}^{C}D_\gamma^{\alpha,\beta}q_{C2}, \cdots, {}^{C}D_\gamma^{\alpha,\beta}q_{Cn}\right)$, $\boldsymbol{q_{C1}} = (q_{C11}, q_{C12}, \cdots, q_{C1n})$, $\boldsymbol{q_{C2}} = (q_{C21}, q_{C22}, \cdots, q_{C2n})$, the Lagrangian $L_C : [t_1, t_2] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ and $q_{Ci} : [t_1, t_2] \to \mathbb{R}$, $i = 1, 2, \cdots, n$, are assumed to be C^2 functions.

If $\boldsymbol{q_C}(\cdot)$ is an extremal for Equation (26), then we obtain

$$\frac{\partial L_C}{\partial q_{Ck}} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L_C}{\partial \dot{q}_{Ck}} - {}^{RL}D_{1-\gamma}^{\beta,\alpha}\frac{\partial L_C}{\partial^{C}D_\gamma^{\alpha,\beta}q_{Ck}} = 0,\ k = 1, 2, \cdots, n. \tag{27}$$

Equation (27) is called the fractional Lagrange equation within ICC.

Of course, we can also give another form of the fractional Lagrange equation within ICC, which we only refer to briefly here.

Remark 2. *When* $\gamma = 1$, *Equation (27) reduces to the fractional Lagrange equation within the mixed integer and left Caputo fractional derivative, which coincides with the result in Ref.* [28].

Remark 3. *Equations (24) and (27) are the two main fractional Lagrange equations obtained in this article. The combined Riemann–Liouville fractional derivative and the combined Caputo fractional derivative are general and universal because of γ, so we can obtain different results by selecting different values of γ.*

4. Fractional Primary Constraint

If a Lagrangian system is singular, then some inherent constraints exist when the Lagrangian system is represented by a Hamiltonian system. Fractional primary constraints within ICRL and ICC are presented in this section.

For Equation (24), the integer generalized momentum, the fractional generalized momentum, and the Hamiltonian can be defined as

$$p_{RLk} = \frac{\partial L_{RL}\left(t, \boldsymbol{q_{RL}}, \dot{\boldsymbol{q}}_{\boldsymbol{RL}}, {}^{\boldsymbol{RL}}\boldsymbol{D}_{\gamma}^{\boldsymbol{\alpha},\boldsymbol{\beta}}\boldsymbol{q_{RL}}\right)}{\partial \dot{q}_{RLk}},\ p_{RLk}^{(\alpha,\beta)} = \frac{\partial L_{RL}\left(t, \boldsymbol{q_{RL}}, \dot{\boldsymbol{q}}_{\boldsymbol{RL}}, {}^{\boldsymbol{RL}}\boldsymbol{D}_{\gamma}^{\boldsymbol{\alpha},\boldsymbol{\beta}}\boldsymbol{q_{RL}}\right)}{\partial {}^{RL}D_{\gamma}^{\alpha,\beta} q_{RLk}}, \tag{28}$$

$$H_{RL} = p_{RLk}\dot{q}_{RLk} + p_{RLk}^{(\alpha,\beta)} \cdot {}^{RL}D_{\gamma}^{\alpha,\beta} q_{RLk} - L_{RL}\left(t, \boldsymbol{q_{RL}}, \dot{\boldsymbol{q}}_{\boldsymbol{RL}}, {}^{\boldsymbol{RL}}\boldsymbol{D}_{\gamma}^{\boldsymbol{\alpha},\boldsymbol{\beta}}\boldsymbol{q_{RL}}\right) \tag{29}$$

In this paper, we assume that ${}^{RL}D_{\gamma}^{\alpha,\beta}\boldsymbol{q_{RL}} = \boldsymbol{u_{RL}}\left(t, \boldsymbol{q_{RL}}, \dot{\boldsymbol{q}}_{\boldsymbol{RL}}, \boldsymbol{p}_{\boldsymbol{RL}}^{(\boldsymbol{\alpha},\boldsymbol{\beta})}\right)$, which means ${}^{\boldsymbol{RL}}\boldsymbol{D}_{\gamma}^{\boldsymbol{\alpha},\boldsymbol{\beta}}\boldsymbol{q_{RL}}$ can always be described by the function $\boldsymbol{u_{RL}}$ depending on the elements of t, $\boldsymbol{q_{RL}}$, $\dot{\boldsymbol{q}}_{\boldsymbol{RL}}$ and $\boldsymbol{p}_{\boldsymbol{RL}}^{(\boldsymbol{\alpha},\boldsymbol{\beta})}$, where $\boldsymbol{p}_{\boldsymbol{RL}}^{(\boldsymbol{\alpha},\boldsymbol{\beta})} = \left(p_{RL1}^{(\alpha,\beta)}, p_{RL2}^{(\alpha,\beta)}, \cdots, p_{RLn}^{(\alpha,\beta)}\right)$, $\boldsymbol{u_{RL}} = (u_{RL1}, u_{RL2}, \cdots, u_{RLn})$.

The element H_{RLij} of the Hessian matrix $\left[H_{RLij}\right]$ is defined as

$$H_{RLij} = \frac{\partial^2 L_{RL}}{\partial \dot{q}_{RLi}\partial \dot{q}_{RLj}},\ i,j = 1,2,\cdots,n. \tag{30}$$

If $\det\left[H_{RLij}\right] \neq 0$, then the Hessian matrix $\left[H_{RLij}\right]$ is called a nondegenerate matrix, and the corresponding Lagrangian L_{RL} is called a regular Lagrangian. In this case, $\dot{q}_{RLk}$, $k = 1,2,\cdots,n$, can be expressed by a function that depends on the elements of t, $\boldsymbol{q_{RL}}$, $\dot{\boldsymbol{q}}_{\boldsymbol{RL}}$, and $\boldsymbol{p}_{\boldsymbol{RL}}^{(\boldsymbol{\alpha},\boldsymbol{\beta})}$ from Equation (28). If $\det\left[H_{RLij}\right] = 0$, then the Hessian matrix $\left[H_{RLij}\right]$ is called a degenerate matrix, and the corresponding Lagrangian L_{RL} is called a singular Lagrangian. In this case, we assume that $\operatorname{rank}\left[H_{RLij}\right] = R$, and we know that $0 \leq R < n$. Then we divide R into two cases to discuss, one case is $1 \leq R < n$, and the other is $R = 0$.

When $1 \leq R < n$, i.e., only $\dot{q}_{RL\sigma}$, $\sigma = 1,2,\cdots,R$ can be determined, while $\dot{q}_{RL\rho}$, $\rho = R+1, R+2,\cdots,n$ are random. From Equation (28), $\dot{q}_{RL\sigma}$, $\sigma = 1,2,\cdots,R$ can be expressed as

$$\dot{q}_{RL\sigma} = f_{RL}^{\sigma}\left(t, \boldsymbol{q_{RL}}, \boldsymbol{p}_{\boldsymbol{RL}}^{(\boldsymbol{\alpha},\boldsymbol{\beta})}, \boldsymbol{p_{RLA}}, \dot{\boldsymbol{q}}_{\boldsymbol{RLB}}\right),\ \sigma = 1,2,\cdots,R, \tag{31}$$

or

$$\dot{\boldsymbol{q}}_{\boldsymbol{RLA}} = \boldsymbol{f}_{\boldsymbol{RL}}^{\boldsymbol{A}}\left(t, \boldsymbol{q_{RL}}, \boldsymbol{p}_{\boldsymbol{RL}}^{(\boldsymbol{\alpha},\boldsymbol{\beta})}, \boldsymbol{p_{RLA}}, \dot{\boldsymbol{q}}_{\boldsymbol{RLB}}\right), \tag{32}$$

where $\boldsymbol{p_{RLA}} = (p_{RL1}, p_{RL2}, \cdots, p_{RLR})$, $\boldsymbol{p_{RLB}} = (p_{RLR+1}, p_{RLR+2}, \cdots, p_{RLn})$, $\dot{\boldsymbol{q}}_{\boldsymbol{RLA}} = (\dot{q}_{RL1}, \dot{q}_{RL2}, \cdots, \dot{q}_{RLR})$, $\dot{\boldsymbol{q}}_{\boldsymbol{RLB}} = (\dot{q}_{RLR+1}, \dot{q}_{RLR+2}, \cdots, \dot{q}_{RLn})$, and $\boldsymbol{f}_{\boldsymbol{RL}}^{\boldsymbol{A}} = (f_{RL}^1, f_{RL}^2, \cdots, f_{RL}^R)$. Substituting Equation (32) into Equation (28), we have

$$\begin{aligned} p_{RLk} &= g_{RLk}\left(t, \boldsymbol{q_{RL}}, \dot{\boldsymbol{q}}_{\boldsymbol{RL}}, {}^{\boldsymbol{RL}}\boldsymbol{D}_{\gamma}^{\boldsymbol{\alpha},\boldsymbol{\beta}}\boldsymbol{q_{RL}}\right) = g_{RLk}\left(t, \boldsymbol{q_{RL}}, \dot{\boldsymbol{q}}_{\boldsymbol{RLA}}, \dot{\boldsymbol{q}}_{\boldsymbol{RLB}}, \boldsymbol{p}_{\boldsymbol{RL}}^{(\boldsymbol{\alpha},\boldsymbol{\beta})}\right) \\ &= g_{RLk}\left(t, \boldsymbol{q_{RL}}, \boldsymbol{f}_{\boldsymbol{RL}}^{\boldsymbol{A}}\left(t, \boldsymbol{q_{RL}}, \boldsymbol{p}_{\boldsymbol{RL}}^{(\boldsymbol{\alpha},\boldsymbol{\beta})}, \boldsymbol{p_{RLA}}, \dot{\boldsymbol{q}}_{\boldsymbol{RLB}}\right), \dot{\boldsymbol{q}}_{\boldsymbol{RLB}}, \boldsymbol{p}_{\boldsymbol{RL}}^{(\boldsymbol{\alpha},\boldsymbol{\beta})}\right) \\ &= g_{RLk}\left(t, \boldsymbol{q_{RL}}, \boldsymbol{p_{RLA}}, \dot{\boldsymbol{q}}_{\boldsymbol{RLB}}, \boldsymbol{p}_{\boldsymbol{RL}}^{(\boldsymbol{\alpha},\boldsymbol{\beta})}\right) \end{aligned} \tag{33}$$

When $k = 1, 2, \cdots, R$, Equation (33) obviously holds, while when $k = R+1, R+2, \cdots, n$, g_{RLk} will not depend on $\dot{\boldsymbol{q}}_{\boldsymbol{RLB}}$; otherwise, it contradicts the assumption rank$\left[H_{RLij}\right] = R$. In this case, we have

$$p_{RL\rho} = g_{RL\rho}\left(t, \boldsymbol{q_{RL}}, \boldsymbol{p_{RLA}}, \boldsymbol{p}_{\boldsymbol{RL}}^{(\boldsymbol{\alpha},\boldsymbol{\beta})}\right),\ \rho = R+1, R+2, \cdots, n. \tag{34}$$

For simplicity, let $a = 1, 2, \cdots, n-R$; Equation (34) can be written as

$$\phi_{RLa}\left(t, \boldsymbol{q_{RL}}, \boldsymbol{p_{RL}}, \boldsymbol{p}_{\boldsymbol{RL}}^{(\boldsymbol{\alpha},\boldsymbol{\beta})}\right) = p_{RLa} - g_{RLa}\left(t, \boldsymbol{q_{RL}}, \boldsymbol{p_{RLA}}, \boldsymbol{p}_{\boldsymbol{RL}}^{(\boldsymbol{\alpha},\boldsymbol{\beta})}\right) = 0. \tag{35}$$

When $R = 0$, we can obtain

$$\phi_{RLa}\left(t, \boldsymbol{q_{RL}}, \boldsymbol{p_{RL}}, \boldsymbol{p}_{\boldsymbol{RL}}^{(\boldsymbol{\alpha},\boldsymbol{\beta})}\right) = 0,\ a = 1, 2, \cdots, n. \tag{36}$$

Therefore, from Equations (35) and (36), we have

$$\phi_{RLa}\left(t, \boldsymbol{q_{RL}}, \boldsymbol{p_{RL}}, \boldsymbol{p}_{\boldsymbol{RL}}^{(\boldsymbol{\alpha},\boldsymbol{\beta})}\right) = 0,\ a = 1, 2, \cdots, n-R,\ 0 \leq R < n. \tag{37}$$

Equation (37) is called the fractional primary constraint within ICRL.

Similarly, for Equation (27), the integer generalized momentum, the fractional generalized momentum, and the Hamiltonian can be defined as

$$p_{Ck} = \frac{\partial L_C\left(t, \boldsymbol{q_C}, \dot{\boldsymbol{q}}_{\boldsymbol{C}}, {}^{\boldsymbol{C}}\boldsymbol{D}_{\gamma}^{\boldsymbol{\alpha},\boldsymbol{\beta}}\boldsymbol{q_C}\right)}{\partial \dot{q}_{Ck}},\ p_{Ck}^{(\alpha,\beta)} = \frac{\partial L_C\left(t, \boldsymbol{q_C}, \dot{\boldsymbol{q}}_{\boldsymbol{C}}, {}^{\boldsymbol{C}}\boldsymbol{D}_{\gamma}^{\boldsymbol{\alpha},\boldsymbol{\beta}}\boldsymbol{q_C}\right)}{\partial^C D_{\gamma}^{\alpha,\beta} q_{Ck}} \tag{38}$$

$$H_C = p_{Ck}\dot{q}_{Ck} + p_{Ck}^{(\alpha,\beta)} \cdot {}^C D_{\gamma}^{\alpha,\beta} q_{Ck} - L_C\left(t, \boldsymbol{q_C}, \dot{\boldsymbol{q}}_{\boldsymbol{C}}, {}^{\boldsymbol{C}}\boldsymbol{D}_{\gamma}^{\boldsymbol{\alpha},\boldsymbol{\beta}}\boldsymbol{q_C}\right),\ k = 1, 2, \cdots, n, \tag{39}$$

and we can also obtain the fractional primary constraint within ICC

$$\phi_{Ca}\left(t, \boldsymbol{q_C}, \boldsymbol{p_C}, \boldsymbol{p}_{\boldsymbol{C}}^{(\boldsymbol{\alpha},\boldsymbol{\beta})}\right) = 0,\ a = 1, 2, \cdots, n-R,\ 0 \leq R < n, \tag{40}$$

where $\boldsymbol{p}_{\boldsymbol{C}}^{(\boldsymbol{\alpha},\boldsymbol{\beta})} = \left(p_{C1}^{(\alpha,\beta)}, p_{C2}^{(\alpha,\beta)}, \cdots, p_{Cn}^{(\alpha,\beta)}\right)$, $\boldsymbol{p_C} = (p_{C1}, p_{C2}, \cdots, p_{Cn})$.

Remark 4. *The fractional primary constraints (Equations (37) and (40)) come from the definitions of the integer generalized momenta (Equations (28) and (38)) rather than the fractional Euler–Lagrange equations (Equations (24) and (27)).*

Remark 5. From *Equations (37) and (40), the fractional primary constraints within different fractional derivatives can be obtained due to the various values of γ.*

After the fractional primary constraints (Equations (37) and (40)) have been investigated, we begin to express the singular systems (Equations (24) and (27)) in the form of the Hamiltonian description.

5. Fractional Constrained Hamilton Equation

We begin with the fractional constrained Hamilton equation within ICRL.

On the one hand, taking isochronous variation of the Hamiltonian (Equation (29)) and using Equation (28), we have

$$\delta H_{RL} = \dot{q}_{RLk} \cdot \delta p_{RLk} + \delta p_{RLk}^{(\alpha,\beta)} \cdot {}^{RL} D_{\gamma}^{\alpha,\beta} q_{RLk} - \frac{\partial L_{RL}}{\partial q_{RLk}} \delta q_{RLk},\ k = 1, 2, \cdots, n. \tag{41}$$

On the other hand, it follows from Equations (28) and (29) that the Hamiltonian $H_{RL} = H_{RL}\left(t, \boldsymbol{q_{RL}}, \boldsymbol{p_{RL}}, \boldsymbol{p_{RL}^{(\alpha,\beta)}}\right)$; therefore,

$$\delta H_{RL} = \frac{\partial H_{RL}}{\partial q_{RLk}} \cdot \delta q_{RLk} + \frac{\partial H_{RL}}{\partial p_{RLk}} \cdot \delta p_{RLk} + \frac{\partial H_{RL}}{\partial p_{RLk}^{(\alpha,\beta)}} \cdot \delta p_{RLk}^{(\alpha,\beta)},\ k = 1, 2, \cdots, n. \tag{42}$$

It follows from Equations (41) and (42) that

$$\left(\dot{q}_{RLk} - \frac{\partial H_{RL}}{\partial p_{RLk}}\right)\delta p_{RLk} + \left({}^{RL}D_{\gamma}^{\alpha,\beta} q_{RLk} - \frac{\partial H_{RL}}{\partial p_{RLk}^{(\alpha,\beta)}}\right)\delta p_{RLk}^{(\alpha,\beta)} - \left(\frac{\partial L_{RL}}{\partial q_{RLk}} + \frac{\partial H_{RL}}{\partial q_{RLk}}\right)\delta q_{RLk} = 0. \tag{43}$$

Making use of Equations (24) and (28), the term $\partial L_{RL}/\partial q_{RLk}$ in Equation (43) can be replaced by $\dot{p}_{RLk} + {}^{C}D_{1-\gamma}^{\beta,\alpha} p_{RLk}^{(\alpha,\beta)} - \frac{\gamma(t_2-t)^{-\alpha}}{\Gamma(1-\alpha)} p_{RLk}^{(\alpha,\beta)}(t_2) + \frac{(1-\gamma)(t-t_1)^{-\beta}}{\Gamma(1-\beta)} p_{RLk}^{(\alpha,\beta)}(t_1)$; therefore, for $k = 1, 2, \cdots, n$, we have

$$\begin{gathered}\left(\dot{q}_{RLk} - \frac{\partial H_{RL}}{\partial p_{RLk}}\right) \cdot \delta p_{RLk} + \left({}^{RL}D_{\gamma}^{\alpha,\beta} q_{RLk} - \frac{\partial H_{RL}}{\partial p_{RLk}^{(\alpha,\beta)}}\right) \cdot \delta p_{RLk}^{(\alpha,\beta)} - \Big[\dot{p}_{RLk} + {}^{C}D_{1-\gamma}^{\beta,\alpha} p_{RLk}^{(\alpha,\beta)} \\ - \frac{\gamma(t_2-t)^{-\alpha}}{\Gamma(1-\alpha)} p_{RLk}^{(\alpha,\beta)}(t_2) + \frac{(1-\gamma)(t-t_1)^{-\beta}}{\Gamma(1-\beta)} p_{RLk}^{(\alpha,\beta)}(t_1) + \frac{\partial H_{RL}}{\partial q_{RLk}}\Big]\delta q_{RLk} = 0.\end{gathered} \tag{44}$$

When the system (Equation (24)) is singular, because of the existence of the fractional primary constraint within ICRL (Equation (37)), we cannot let the coefficients of δp_{RLk}, $\delta p_{RLk}^{(\alpha,\beta)}$, and δq_{RLk} in Equation (44) be equal to 0. The fractional primary constraint within ICRL (Equation (37)) should be considered. Taking the isochronous variation of Equation (37) and introducing the Lagrangian multiplier $\lambda_{RLa}(t)$, $a = 1, 2, \cdots, n-R$, $0 \le R < n$, we have

$$\lambda_{RLa}\frac{\partial \phi_{RLa}}{\partial q_{RLk}} \cdot \delta q_{RLk} + \lambda_{RLa}\frac{\partial \phi_{RLa}}{\partial p_{RLk}} \cdot \delta p_{RLk} + \lambda_{RLa}\frac{\partial \phi_{RLa}}{\partial p_{RLk}^{(\alpha,\beta)}} \cdot \delta p_{RLk}^{(\alpha,\beta)} = 0. \tag{45}$$

It follows from Equations (44) and (45) that

$$\begin{gathered}\dot{p}_{RLk} = -\frac{\partial H_{RL}}{\partial q_{RLk}} - \lambda_{RLa}\frac{\partial \phi_{RLa}}{\partial q_{RLk}} - {}^{C}D_{1-\gamma}^{\beta,\alpha} p_{RLk}^{(\alpha,\beta)} + \frac{\gamma(t_2-t)^{-\alpha}}{\Gamma(1-\alpha)} p_{RLk}^{(\alpha,\beta)}(t_2) \\ - \frac{(1-\gamma)(t-t_1)^{-\beta}}{\Gamma(1-\beta)} p_{RLk}^{(\alpha,\beta)}(t_1),\ {}^{RL}D_{\gamma}^{\alpha,\beta} q_{RLk} = \frac{\partial H_{RL}}{\partial p_{RLk}^{(\alpha,\beta)}} + \lambda_{RLa}\frac{\partial \phi_{RLa}}{\partial p_{RLk}^{(\alpha,\beta)},} \\ \dot{q}_{RLk} = \frac{\partial H_{RL}}{\partial p_{RLk}} + \lambda_{RLa}\frac{\partial \phi_{RLa}}{\partial p_{RLk}},\ a = 1, 2, \cdots, n-R,\ 0 \le R < n,\ k = 1, 2, \cdots, n.\end{gathered} \tag{46}$$

Equation (46) is called the fractional constrained Hamilton equation within ICRL.

Similarly, we can also obtain the fractional constrained Hamilton equation within ICC:

$$\begin{gathered}\dot{p}_{Ck} = -\frac{\partial H_C}{\partial q_{Ck}} - \lambda_{Ca}\frac{\partial \phi_{Ca}}{\partial q_{Ck}} - {}^{RL}D_{1-\gamma}^{\beta,\alpha} p_{Ck}^{(\alpha,\beta)},\ \dot{q}_{Ck} = \frac{\partial H_C}{\partial p_{Ck}} + \lambda_{Ca}\frac{\partial \phi_{Ca}}{\partial p_{Ck}}, \\ {}^{C}D_{\gamma}^{\alpha,\beta} q_{Ck} = \frac{\partial H_C}{\partial p_{Ck}^{(\alpha,\beta)}} + \lambda_{Ca}\frac{\partial \phi_{Ca}}{\partial p_{Ck}^{(\alpha,\beta)}},\ a = 1, 2, \cdots, n-R,\ 0 \le R < n,\ k = 1, 2, \cdots, n.\end{gathered} \tag{47}$$

Remark 6. *From Equations (46) and (47), different fractional constrained Hamilton equations in terms of fractional derivatives can be obtained due to the various values of* γ.

It follows from the methods introduced above for establishing fractional constrained Hamilton equations that the Lagrangian multipliers are the key points. In other words, Lagrangian multipliers must be calculated before establishing the fractional constrained Hamilton equations. Lagrangian multipliers can be calculated through the fractional Poisson bracket, which is presented as follows:

Let $F = F\left(t, \boldsymbol{q}, \boldsymbol{p}, \boldsymbol{p}^{(\alpha,\beta)}\right)$, $G = G\left(t, \boldsymbol{q}, \boldsymbol{p}, \boldsymbol{p}^{(\alpha,\beta)}\right)$; we define the fractional Poisson bracket as

$$\{F, G\} = \frac{\partial F}{\partial q_k}\frac{\partial G}{\partial p_k} - \frac{\partial F}{\partial p_k}\frac{\partial G}{\partial q_k},\ k = 1, 2, \cdots, n, \tag{48}$$

where $\boldsymbol{q} = (q_1, q_2, \cdots, q_n)$, $\boldsymbol{p} = (p_1, p_2, \cdots, p_n)$, $\boldsymbol{p}^{(\alpha,\beta)} = \left(p_1^{(\alpha,\beta)}, p_2^{(\alpha,\beta)}, \cdots, p_n^{(\alpha,\beta)}\right)$. Then, from the fractional primary constraint within ICRL (Equation (37)) and the fractional Poisson bracket, we have

$$\begin{gathered}\lambda_{RLb}\{\phi_{RLa}, \phi_{RLb}\} + \{\phi_{RLa}, H_{RL}\} - \tfrac{\partial \phi_{RLa}}{\partial p_{RLk}} \cdot \Big[{}^{C}D^{\beta,\alpha}_{1-\gamma} p^{(\alpha,\beta)}_{RLk} - \tfrac{\gamma (t_2 - t)^{-\alpha}}{\Gamma(1-\alpha)} p^{(\alpha,\beta)}_{RLk}(t_2) \\ + \tfrac{(1-\gamma)(t-t_1)^{-\beta}}{\Gamma(1-\beta)} p^{(\alpha,\beta)}_{RLk}(t_1)\Big] + \tfrac{\partial \phi_{RLa}}{\partial t} + \tfrac{\partial \phi_{RLa}}{\partial p^{(\alpha,\beta)}_{RLk}} \dot{p}^{(\alpha,\beta)}_{RLk} = 0,\end{gathered} \tag{49}$$

where $a, b = 1, 2, \cdots, n - R$, $0 \le R < n$, $k = 1, 2, \cdots, n$. Equation (49) is called the consistency condition of the fractional primary constraint within ICRL.

Similarly, from the fractional primary constraint within ICC (Equation (40)) and the fractional Poisson bracket, we have

$$\begin{gathered}\lambda_{Cb}\{\phi_{Ca}, \phi_{Cb}\} + \{\phi_{Ca}, H_C\} - \tfrac{\partial \phi_{Ca}}{\partial p_{Ck}} \cdot {}^{RL}D^{\beta,\alpha}_{1-\gamma} p^{(\alpha,\beta)}_{Ck} + \tfrac{\partial \phi_{Ca}}{\partial t} + \tfrac{\partial \phi_{Ca}}{\partial p^{(\alpha,\beta)}_{Ck}} \dot{p}^{(\alpha,\beta)}_{Ck} = 0, \\ a, b = 1, 2, \cdots, n - R,\ 0 \le R < n,\ k = 1, 2, \cdots, n.\end{gathered} \tag{50}$$

Equation (50) is called the consistency condition of the fractional primary constraint within ICC.

If $\det[\{\phi_{RLa}, \phi_{RLb}\}] \neq 0$ (resp. $\det[\{\phi_{Ca}, \phi_{Cb}\}] \neq 0$), $a, b = 1, 2, \cdots, n - R$, and $0 \le R < n$; then, all the Lagrangian multipliers can be calculated from Equation (49) (resp. Equation (50)). If $\det[\{\phi_{RLa}, \phi_{RLb}\}] = 0$ (resp. $\det[\{\phi_{Ca}, \phi_{Cb}\}] = 0$), we assume $\mathrm{rank}[\{\phi_{RLa}, \phi_{RLb}\}] = m$ (resp. $\mathrm{rank}[\{\phi_{Ca}, \phi_{Cb}\}] = m$), $m < n - R$, and $0 \le R < n$; then, new constraints will be deduced because $n - R - m$ Lagrangian multipliers cannot be determined. The new constraints are called fractional secondary constraints, which arise from the consistency conditions of the fractional primary constraints. Then, the consistency condition of the fractional secondary constraints may also lead to some new fractional secondary constraints. However, for a system with finite degrees of freedom, no new fractional secondary constraints will be produced after a finite number of steps.

If we cannot solve all the Lagrangian multipliers, then the fractional constrained Hamilton equation within ICRL (Equation (46)) (resp. ICC (Equation (47))) is invalid. In this case, there is another way to construct a significant fractional constrained Hamilton equation within ICRL (resp. ICC). We only refer to it briefly here.

6. Noether Symmetry and Conserved Quantity

Noether symmetry means the invariance of the fractional Hamilton action under infinitesimal transformations. Noether symmetry always leads to a conserved quantity.

Definition 1. *A quantity C is called a conserved quantity if and only if* $\mathrm{d}C/\mathrm{d}t = 0$ *holds.*

6.1. Noether Symmetry and Conserved Quantity within ICRL

Hamilton action within ICRL is defined as

$$I_{RL} = \int_{t_1}^{t_2} \left[p_{RLk}\dot{q}_{RLk} + p^{(\alpha,\beta)}_{RLk} \cdot {}^{RL}D^{\alpha,\beta}_{\gamma} q_{RLk} - H_{RL}\left(t, \boldsymbol{q}_{\boldsymbol{RL}}, \boldsymbol{p}_{\boldsymbol{RL}}, \boldsymbol{p}^{(\alpha,\beta)}_{\boldsymbol{RL}}\right)\right] \mathrm{d}t. \tag{51}$$

The infinitesimal transformations are given as

$$\begin{gathered}\bar{t} = t + \Delta t,\ \bar{q}_{RLk}(\bar{t}) = q_{RLk}(t) + \Delta q_{RLk},\ \bar{p}_{RLk}(\bar{t}) = p_{RLk}(t) + \Delta p_{RLk}, \\ \bar{p}^{(\alpha,\beta)}_{RLk}(\bar{t}) = p^{(\alpha,\beta)}_{RLk}(t) + \Delta p^{(\alpha,\beta)}_{RLk},\ k = 1, 2, \cdots, n,\end{gathered} \tag{52}$$

and the expanded expression of Equation (52) is

$$\begin{aligned}
\bar{t} &= t + \theta_{RL}\xi_{RL0}\left(t, \boldsymbol{q_{RL}}, \boldsymbol{p_{RL}}, \boldsymbol{p_{RL}^{(\alpha,\beta)}}\right) + o(\theta_{RL}), \\
\bar{q}_{RLk}(\bar{t}) &= q_{RLk}(t) + \theta_{RL}\xi_{RLk}\left(t, \boldsymbol{q_{RL}}, \boldsymbol{p_{RL}}, \boldsymbol{p_{RL}^{(\alpha,\beta)}}\right) + o(\theta_{RL}), \\
\bar{p}_{RLk}(\bar{t}) &= p_{RLk}(t) + \theta_{RL}\eta_{RLk}\left(t, \boldsymbol{q_{RL}}, \boldsymbol{p_{RL}}, \boldsymbol{p_{RL}^{(\alpha,\beta)}}\right) + o(\theta_{RL}), \\
\bar{p}_{RLk}^{(\alpha,\beta)}(\bar{t}) &= p_{RLk}^{(\alpha,\beta)}(t) + \theta_{RL}\eta_{RLk}^{(\alpha,\beta)}\left(t, \boldsymbol{q_{RL}}, \boldsymbol{p_{RL}}, \boldsymbol{p_{RL}^{(\alpha,\beta)}}\right) + o(\theta_{RL}),
\end{aligned} \tag{53}$$

where θ_{RL} is a small parameter, ξ_{RL0}, ξ_{RLk}, η_{RLk}, and $\eta_{RLk}^{(\alpha,\beta)}$ are called infinitesimal generators within ICRL, and $o(\theta_{RL})$ means the higher order of θ_{RL}.

The Hamilton action within ICRL (Equation (51)) changes from I_{RL} to $\bar{I}_{RL}$ under the infinitesimal transformations; denoting as $\Delta I_{RL} = \bar{I}_{RL} - I_{RL}$, without considering the higher order of θ_{RL}, we have

$$\begin{aligned}
\Delta I_{RL} = \theta_{RL}\int_{t_1}^{t_2} \Big[& p_{RLk}^{(\alpha,\beta)} \cdot {}^{RL}D_{\gamma}^{\alpha,\beta}(\xi_{RLk} - \dot{q}_{RLk}\xi_{RL0}) + \left(p_{RLk}^{(\alpha,\beta)} \cdot {}^{RL}D_{\gamma}^{\alpha,\beta}q_{RLk} - H_{RL}\right)\dot{\xi}_{RL0} \\
& + p_{RLk}\dot{\xi}_{RLk} + \left(p_{RLk}^{(\alpha,\beta)}\frac{\mathrm{d}}{\mathrm{d}t}{}^{RL}D_{\gamma}^{\alpha,\beta}q_{RLk} - \frac{\partial H_{RL}}{\partial t}\right)\xi_{RL0} - \frac{\partial H_{RL}}{\partial q_{RLk}}\xi_{RLk} + \lambda_{RLa}\frac{\partial \phi_{RLa}}{\partial p_{RLk}}\eta_{RLk} \\
& + \lambda_{RLa}\frac{\partial \phi_{RLa}}{\partial p_{RLk}^{(\alpha,\beta)}} \cdot \eta_{RLk}^{(\alpha,\beta)} + q_{RLk}(t_2)\xi_{RL0}(t_2) \cdot \frac{(1-\gamma)p_{RLk}^{(\alpha,\beta)}}{\Gamma(1-\beta)}\frac{\mathrm{d}}{\mathrm{d}t}(t_2 - t)^{-\beta} \\
& - q_{RLk}(t_1)\xi_{RL0}(t_1)\frac{\gamma p_{RLk}^{(\alpha,\beta)}}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}(t - t_1)^{-\alpha}\Big]\mathrm{d}t,
\end{aligned} \tag{54}$$

where $\xi_{RL0}(t_1) = \xi_{RL0}\left(t_1, \boldsymbol{q_{RL}}(t_1), \boldsymbol{p_{RL}}(t_1), \boldsymbol{p_{RL}^{(\alpha,\beta)}}(t_1)\right)$ and $\xi_{RL0}(t_2) = \xi_{RL0}\left(t_2, \boldsymbol{q_{RL}}(t_2), \boldsymbol{p_{RL}}(t_2), \boldsymbol{p_{RL}^{(\alpha,\beta)}}(t_2)\right)$.

Let $\Delta I_{RL} = 0$; Equation (54) gives

$$\begin{aligned}
& p_{RLk}^{(\alpha,\beta)} \cdot {}^{RL}D_{\gamma}^{\alpha,\beta}(\xi_{RLk} - \dot{q}_{RLk}\xi_{RL0}) + \left(p_{RLk}^{(\alpha,\beta)} \cdot {}^{RL}D_{\gamma}^{\alpha,\beta}q_{RLk} - H_{RL}\right) \cdot \dot{\xi}_{RL0} \\
& + p_{RLk}\dot{\xi}_{RLk} - \frac{\partial H_{RL}}{\partial q_{RLk}}\xi_{RLk} + \left(p_{RLk}^{(\alpha,\beta)}\frac{\mathrm{d}}{\mathrm{d}t}{}^{RL}D_{\gamma}^{\alpha,\beta}q_{RLk} - \frac{\partial H_{RL}}{\partial t}\right)\xi_{RL0} \\
& - q_{RLk}(t_1) \cdot \xi_{RL0}(t_1) \cdot \frac{\gamma p_{RLk}^{(\alpha,\beta)}}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}(t - t_1)^{-\alpha} + \lambda_{RLa}\eta_{RLk}^{(\alpha,\beta)}\frac{\partial \phi_{RLa}}{\partial p_{RLk}^{(\alpha,\beta)}} \\
& + \lambda_{RLa}\frac{\partial \phi_{RLa}}{\partial p_{RLk}}\eta_{RLk} + q_{RLk}(t_2)\xi_{RL0}(t_2) \cdot \frac{(1-\gamma)p_{RLk}^{(\alpha,\beta)}}{\Gamma(1-\beta)}\frac{\mathrm{d}}{\mathrm{d}t}(t_2 - t)^{-\beta} = 0.
\end{aligned} \tag{55}$$

Equation (55) is called the fractional Noether identity within ICRL.

If the infinitesimal generators ξ_{RL0}, ξ_{RLk}, η_{RLk}, and $\eta_{RLk}^{(\alpha,\beta)}$ satisfy Equation (55), then the corresponding infinitesimal transformations are called Noether symmetric transformations in terms of ICRL, which determine the Noether symmetry. Therefore, we have the following:

Theorem 1. *For the fractional constrained Hamiltonian system within ICRL (Equation (46)), if the infinitesimal generators ξ_{RL0}, ξ_{RLk}, η_{RLk},* and *$\eta_{RLk}^{(\alpha,\beta)}$ satisfy Equation (55), then there exists a conserved quantity:*

$$\begin{aligned}
C_{RL} = & \left(p_{RLk}^{(\alpha,\beta)} \cdot {}^{RL}D_{\gamma}^{\alpha,\beta}q_{RLk} - H_{RL}\right)\xi_{RL0} + \int_{t_1}^{t}\Big\{p_{RLk}^{(\alpha,\beta)} \cdot {}^{RL}D_{\gamma}^{\alpha,\beta}(\xi_{RLk} - \dot{q}_{RLk}\xi_{RL0}) \\
& + (\xi_{RLk} - \dot{q}_{RLk}\xi_{RL0})\Big[{}^{C}D_{1-\gamma}^{\beta,\alpha}p_{RLk}^{(\alpha,\beta)} - \frac{\gamma(t_2-\tau)^{-\alpha}}{\Gamma(1-\alpha)}p_{RLk}^{(\alpha,\beta)}(t_2) + \frac{(1-\gamma)(\tau-t_1)^{-\beta}}{\Gamma(1-\beta)} \\
& \times p_{RLk}^{(\alpha,\beta)}(t_1)\Big]\Big\}\mathrm{d}\tau - q_{RLk}(t_1)\frac{\gamma\xi_{RL0}(t_1)}{\Gamma(1-\alpha)}\int_{t_1}^{t} p_{RLk}^{(\alpha,\beta)}\frac{\mathrm{d}}{\mathrm{d}\tau}(\tau - t_1)^{-\alpha}\mathrm{d}\tau + p_{RLk}\xi_{RLk} \\
& + q_{RLk}(t_2)\xi_{RL0}(t_2)\frac{1-\gamma}{\Gamma(1-\beta)}\int_{t_1}^{t} p_{RLk}^{(\alpha,\beta)}\frac{\mathrm{d}}{\mathrm{d}\tau}(t_2 - \tau)^{-\beta}\mathrm{d}\tau
\end{aligned} \tag{56}$$

Proof. Using Equations (37), (46), and (55), we have

$$\begin{aligned}
&\tfrac{\mathrm{d}C_{RL}}{\mathrm{d}t} = \left(p_{RLk}^{(\alpha,\beta)} \cdot {}^{RL}D_{\gamma}^{\alpha,\beta} q_{RLk} - H_{RL}\right)\dot{\xi}_{RL0} + \xi_{RL0}\left(\dot{p}_{RLk}^{(\alpha,\beta)} \cdot {}^{RL}D_{\gamma}^{\alpha,\beta} q_{RLk} + p_{RLk}^{(\alpha,\beta)}\right.\\
&\left.\times \tfrac{\mathrm{d}}{\mathrm{d}t} {}^{RL}D_{\gamma}^{\alpha,\beta} q_{RLk} - \tfrac{\partial H_{RL}}{\partial t} - \tfrac{\partial H_{RL}}{\partial q_{RLk}}\dot{q}_{RLk} - \tfrac{\partial H_{RL}}{\partial p_{RLk}}\dot{p}_{RLk} - \tfrac{\partial H_{RL}}{\partial p_{RLk}^{(\alpha,\beta)}}\dot{p}_{RLk}^{(\alpha,\beta)}\right) + p_{RLk}^{(\alpha,\beta)}\\
&\times {}^{RL}D_{\gamma}^{\alpha,\beta}\left(\xi_{RLk} - \dot{q}_{RLk}\xi_{RL0}\right) - q_{RLk}(t_1)\tfrac{\gamma\xi_{RL0}(t_1)}{\Gamma(1-\alpha)} p_{RLk}^{(\alpha,\beta)} \tfrac{\mathrm{d}}{\mathrm{d}\tau}(t-t_1)^{-\alpha} + (\xi_{RLk}\\
&-\dot{q}_{RLk}\xi_{RL0})\left[{}^{C}D_{1-\gamma}^{\beta,\alpha} p_{RLk}^{(\alpha,\beta)} - \tfrac{\gamma(t_2-t)^{-\alpha}}{\Gamma(1-\alpha)} p_{RLk}^{(\alpha,\beta)}(t_2) + \tfrac{(1-\gamma)(t-t_1)^{-\beta}}{\Gamma(1-\beta)} p_{RLk}^{(\alpha,\beta)}(t_1)\right]\\
&+\dot{p}_{RLk}\xi_{RLk} + p_{RLk}\dot{\xi}_{RLk} + q_{RLk}(t_2)\xi_{RL0}(t_2)\tfrac{1-\gamma}{\Gamma(1-\beta)} p_{RLk}^{(\alpha,\beta)} \tfrac{\mathrm{d}}{\mathrm{d}\tau}(t_2-t)^{-\beta}\\
&= -\lambda_{RLa}\eta_{RLk}^{(\alpha,\beta)}\tfrac{\partial \phi_{RLa}}{\partial p_{RLk}^{(\alpha,\beta)}} - \lambda_{RLa}\eta_{RLk}\tfrac{\partial \phi_{RLa}}{\partial p_{RLk}} + \xi_{RL0}\left(-\dot{q}_{RLk}\dot{p}_{RLk} + \lambda_{RLa}\dot{p}_{RLk}\tfrac{\partial \phi_{RLa}}{\partial p_{RLk}}\right)\\
&+\xi_{RL0}\lambda_{RLa}\dot{p}_{RLk}^{(\alpha,\beta)}\tfrac{\partial \phi_{RLa}}{\partial p_{RLk}^{(\alpha,\beta)}} + \left(\xi_{RLk} - \dot{q}_{RLk}\xi_{RL0}\right)\left(-\dot{p}_{RLk} - \lambda_{RLa}\tfrac{\partial \phi_{RLa}}{\partial q_{RLk}}\right) + \dot{p}_{RLk}\xi_{RLk}\\
&= -\lambda_{RLa}\tfrac{\partial \phi_{RLa}}{\partial p_{RLk}^{(\alpha,\beta)}} \cdot \delta p_{RLk}^{(\alpha,\beta)} - \lambda_{RLa}\tfrac{\partial \phi_{RLa}}{\partial q_{RLk}} \cdot \delta q_{RLk} - \lambda_{RLa}\tfrac{\partial \phi_{RLa}}{\partial p_{RLk}} \cdot \delta p_{RLk}\\
&= -\lambda_{RLa} \cdot \delta\phi_{RLa} = 0
\end{aligned}$$

□

6.2. Noether Symmetry and Conserved Quantity within ICC

Hamilton action with ICC is defined as

$$I_C = \int_{t_1}^{t_2}\left[p_{Ck}\dot{q}_{Ck} + p_{Ck}^{(\alpha,\beta)} \cdot {}^{C}D_{\gamma}^{\alpha,\beta} q_{Ck} - H_C\left(t, \boldsymbol{q_C}, \boldsymbol{p_C}, \boldsymbol{p_C^{(\alpha,\beta)}}\right)\right]\mathrm{d}t \tag{57}$$

The infinitesimal transformations are given as

$$\begin{gathered}
\bar{t} = t + \Delta t,\ \overline{q}_{Ck}(\bar{t}) = q_{Ck}(t) + \Delta q_{Ck},\ \overline{p}_{Ck}(\bar{t}) = p_{Ck}(t) + \Delta p_{Ck},\\
\overline{p}_{Ck}^{(\alpha,\beta)}(\bar{t}) = p_{Ck}^{(\alpha,\beta)}(t) + \Delta p_{Ck}^{(\alpha,\beta)},\ k = 1, 2, \cdots, n,
\end{gathered} \tag{58}$$

and the expanded expression of Equation (58) is

$$\begin{aligned}
\bar{t} &= t + \theta_C\xi_{C0}\left(t, \boldsymbol{q_C}, \boldsymbol{p_C}, \boldsymbol{p_C^{(\alpha,\beta)}}\right) + o(\theta_C)\\
\overline{q}_{Ck}(\bar{t}) &= q_{Ck}(t) + \theta_C\xi_{Ck}\left(t, \boldsymbol{q_C}, \boldsymbol{p_C}, \boldsymbol{p_C^{(\alpha,\beta)}}\right) + o(\theta_C)\\
\overline{p}_{Ck}(\bar{t}) &= p_{Ck}(t) + \theta_C\eta_{Ck}\left(t, \boldsymbol{q_C}, \boldsymbol{p_C}, \boldsymbol{p_C^{(\alpha,\beta)}}\right) + o(\theta_C)\\
\overline{p}_{Ck}^{(\alpha,\beta)}(\bar{t}) &= p_{Ck}^{(\alpha,\beta)}(t) + \theta_C\eta_{Ck}^{(\alpha,\beta)}\left(t, \boldsymbol{q_C}, \boldsymbol{p_C}, \boldsymbol{p_C^{(\alpha,\beta)}}\right) + o(\theta_C)
\end{aligned} \tag{59}$$

where θ_C is a small parameter, ξ_{C0}, ξ_{Ck}, η_{Ck}, and $\eta_{Ck}^{(\alpha,\beta)}$ are called infinitesimal generators within ICC, and $o(\theta_C)$ means the higher order of θ_C.

The Hamilton action within ICC (Equation (57)) changes from I_C to $\overline{I}_C$ under the infinitesimal transformations; denoting as $\Delta I_C = \overline{I}_C - I_C$, without considering the higher order of θ_C, we have

$$\begin{aligned}
&\Delta I_C = \theta_C\int_{t_1}^{t_2}\left[p_{Ck}^{(\alpha,\beta)} \cdot {}^{C}D_{\gamma}^{\alpha,\beta}\left(\xi_{Ck} - \dot{q}_{Ck}\xi_{C0}\right) + \left(p_{Ck}^{(\alpha,\beta)} \cdot {}^{C}D_{\gamma}^{\alpha,\beta} q_{Ck} - H_C\right)\dot{\xi}_{C0}\right.\\
&+p_{Ck}\dot{\xi}_{Ck} + \left(p_{Ck}^{(\alpha,\beta)}\tfrac{\mathrm{d}}{\mathrm{d}t}{}^{C}D_{\gamma}^{\alpha,\beta} q_{Ck} - \tfrac{\partial H_C}{\partial t}\right)\xi_{C0} - \tfrac{\partial H_C}{\partial q_{Ck}}\xi_{Ck} + \lambda_{Ca}\tfrac{\partial \phi_{Ca}}{\partial p_{Ck}}\eta_{Ck}\\
&-\dot{q}_{Ck}(t_1)\xi_{C0}(t_1)\tfrac{\gamma p_{Ck}^{(\alpha,\beta)}}{\Gamma(1-\alpha)}(t-t_1)^{-\alpha} + \lambda_{Ca}\tfrac{\partial \phi_{Ca}}{\partial p_{Ck}^{(\alpha,\beta)}} \cdot \eta_{Ck}^{(\alpha,\beta)}\\
&\left.+\dot{q}_{Ck}(t_2)\xi_{C0}(t_2) \cdot \tfrac{(1-\gamma)p_{Ck}^{(\alpha,\beta)}}{\Gamma(1-\beta)}(t_2-t)^{-\beta}\right]\mathrm{d}t
\end{aligned} \tag{60}$$

where $\xi_{C0}(t_1) = \xi_{C0}\left(t_1, \boldsymbol{q_C}(t_1), \boldsymbol{p_C}(t_1), \boldsymbol{p}_{\mathbf{C}}^{(\boldsymbol{\alpha},\boldsymbol{\beta})}(t_1)\right)$, $\xi_{C0}(t_2) = \xi_{C0}\left(t_2, \boldsymbol{q_C}(t_2), \boldsymbol{p_C}(t_2), \boldsymbol{p}_{\mathbf{C}}^{(\boldsymbol{\alpha},\boldsymbol{\beta})}(t_2)\right)$. Let $\Delta I_C = 0$; Equation (60) gives

$$\begin{aligned}&p_{Ck}\dot{\xi}_{Ck} + p_{Ck}^{(\alpha,\beta)} \cdot {}^{C}D_{\gamma}^{\alpha,\beta}(\xi_{Ck} - \dot{q}_{Ck}\xi_{C0}) + \left(p_{Ck}^{(\alpha,\beta)} \cdot {}^{C}D_{\gamma}^{\alpha,\beta}q_{Ck} - H_C\right)\dot{\xi}_{C0}\\&-\frac{\partial H_C}{\partial q_{Ck}}\xi_{Ck} + \lambda_{Ca}\frac{\partial \phi_{Ca}}{\partial p_{Ck}}\eta_{Ck} + \left(p_{Ck}^{(\alpha,\beta)}\frac{\mathrm{d}}{\mathrm{d}t}{}^{C}D_{\gamma}^{\alpha,\beta}q_{Ck} - \frac{\partial H_C}{\partial t}\right)\xi_{C0}\\&-\dot{q}_{Ck}(t_1)\cdot\xi_{C0}(t_1)\cdot\frac{\gamma p_{Ck}^{(\alpha,\beta)}}{\Gamma(1-\alpha)}(t-t_1)^{-\alpha} + \lambda_{Ca}\frac{\partial \phi_{Ca}}{\partial p_{Ck}^{(\alpha,\beta)}}\cdot\eta_{Ck}^{(\alpha,\beta)}\\&+\dot{q}_{Ck}(t_2)\xi_{C0}(t_2)\cdot\frac{(1-\gamma)p_{Ck}^{(\alpha,\beta)}}{\Gamma(1-\beta)}(t_2-t)^{-\beta} = 0.\end{aligned} \tag{61}$$

Equation (61) is called the fractional Noether identity within ICC.

If the infinitesimal generators ξ_{C0}, ξ_{Ck}, η_{Ck}, and $\eta_{Ck}^{(\alpha,\beta)}$ satisfy Equation (61), then the corresponding infinitesimal transformations are called Noether symmetric transformations in terms of ICC, which determine the Noether symmetry. Then we have the following:

Theorem 2. *For the fractional constrained Hamiltonian system within ICC (Equation (47)), if the infinitesimal generators ξ_{C0}, ξ_{Ck}, η_{Ck},* and *$\eta_{Ck}^{(\alpha,\beta)}$ satisfy Equation (61), then there exists a conserved quantity:*

$$\begin{aligned}&C_C = p_{Ck}\xi_{Ck} + \left(p_{Ck}^{(\alpha,\beta)} \cdot {}^{C}D_{\gamma}^{\alpha,\beta}q_{Ck} - H_C\right)\xi_{C0} + \int_{t_1}^{t}\left[p_{Ck}^{(\alpha,\beta)} \cdot {}^{C}D_{\gamma}^{\alpha,\beta}(\xi_{Ck} - \dot{q}_{Ck}\xi_{C0})\right.\\&\left.+(\xi_{Ck} - \dot{q}_{Ck}\xi_{C0})\cdot {}^{RL}D_{1-\gamma}^{\beta,\alpha}p_{Ck}^{(\alpha,\beta)}\right]\mathrm{d}\tau - \dot{q}_{Ck}(t_1)\frac{\gamma\xi_{C0}(t_1)}{\Gamma(1-\alpha)}\int_{t_1}^{t}p_{Ck}^{(\alpha,\beta)}(\tau-t_1)^{-\alpha}\mathrm{d}\tau\\&+\dot{q}_{Ck}(t_2)\xi_{C0}(t_2)\frac{1-\gamma}{\Gamma(1-\beta)}\int_{t_1}^{t}p_{Ck}^{(\alpha,\beta)}(t_2-\tau)^{-\beta}\mathrm{d}\tau\end{aligned} \tag{62}$$

Proof. Using Equations (40), (47), and (61), it is easy to obtain

$$\begin{aligned}\frac{\mathrm{d}C_C}{\mathrm{d}t} &= \dot{p}_{Ck}\xi_{Ck} + p_{Ck}\dot{\xi}_{Ck} + \left(p_{Ck}^{(\alpha,\beta)} \cdot {}^{C}D_{\gamma}^{\alpha,\beta}q_{Ck} - H_C\right)\dot{\xi}_{C0} + \xi_{C0}\left(-\frac{\partial H_C}{\partial t} - \frac{\partial H_C}{\partial q_{Ck}}\dot{q}_{Ck}\right.\\&\left.+\dot{p}_{Ck}^{(\alpha,\beta)} \cdot {}^{C}D_{\gamma}^{\alpha,\beta}q_{Ck} + p_{Ck}^{(\alpha,\beta)}\cdot\frac{\mathrm{d}}{\mathrm{d}t}{}^{C}D_{\gamma}^{\alpha,\beta}q_{Ck} - \frac{\partial H_C}{\partial p_{Ck}}\dot{p}_{Ck} - \frac{\partial H_C}{\partial p_{Ck}^{(\alpha,\beta)}}\dot{p}_{Ck}^{(\alpha,\beta)}\right) + p_{Ck}^{(\alpha,\beta)}\\&\times{}^{C}D_{\gamma}^{\alpha,\beta}(\xi_{Ck} - \dot{q}_{Ck}\xi_{C0}) - \dot{q}_{Ck}(t_1)\frac{\gamma\xi_{C0}(t_1)}{\Gamma(1-\alpha)}p_{Ck}^{(\alpha,\beta)}(t-t_1)^{-\alpha} + (\xi_{Ck} - \dot{q}_{Ck}\xi_{C0})\\&\times{}^{RL}D_{1-\gamma}^{\beta,\alpha}p_{Ck}^{(\alpha,\beta)} + \dot{q}_{Ck}(t_2)\xi_{C0}(t_2)\frac{1-\gamma}{\Gamma(1-\beta)}p_{Ck}^{(\alpha,\beta)}(t_2-t)^{-\beta}\\&= \dot{p}_{Ck}\xi_{Ck} + \xi_{C0}\dot{p}_{Ck}^{(\alpha,\beta)}\cdot\lambda_{Ca}\frac{\partial\phi_{Ca}}{\partial p_{Ck}^{(\alpha,\beta)}} + \xi_{C0}\dot{p}_{Ck}\left(-\dot{q}_{Ck} + \lambda_{Ca}\frac{\partial\phi_{Ca}}{\partial p_{Ck}}\right)\\&+(\xi_{Ck} - \dot{q}_{Ck}\xi_{C0})\left(-\dot{p}_{Ck} - \lambda_{Ca}\frac{\partial\phi_{Ca}}{\partial q_{Ck}}\right) - \lambda_{Ca}\frac{\partial\phi_{Ca}}{\partial p_{Ck}}\eta_{Ck} - \lambda_{Ca}\frac{\partial\phi_{Ca}}{\partial p_{Ck}^{(\alpha,\beta)}}\eta_{Ck}^{(\alpha,\beta)}\\&= -\lambda_{Ca}\frac{\partial\phi_{Ca}}{\partial p_{Ck}^{(\alpha,\beta)}}\cdot\delta p_{Ck}^{(\alpha,\beta)} - \lambda_{Ca}\frac{\partial\phi_{Ca}}{\partial p_{Ck}}\cdot\delta p_{Ck} - \lambda_{Ca}\frac{\partial\phi_{Ca}}{\partial q_{Ck}}\cdot\delta q_{Ck} = -\lambda_{Ca}\cdot\delta\phi_{Ca} = 0\end{aligned}$$

□

7. Lie Symmetry and Conserved Quantity

7.1. Lie Symmetry and Conserved Quantity within ICRL

Lie symmetry means the invariance of the differential equations of motion under the infinitesimal transformations of time and coordinates. We begin with the fractional constrained Hamilton equation within ICRL.

We write the fractional constrained Hamilton equation within ICRL (Equation (46)) in another form:

$$\dot{q}_{RLk} = s_{RLk}\left(t, \boldsymbol{q_{RL}}, \boldsymbol{p_{RL}}, \boldsymbol{p}_{\mathbf{RL}}^{(\boldsymbol{\alpha},\boldsymbol{\beta})}\right),\ k = 1, 2, \cdots, n, \tag{63}$$

$${}^{RL}D_{\gamma}^{\alpha,\beta}q_{RLk} = h_{RLk}\left(t, \boldsymbol{q_{RL}}, \boldsymbol{p_{RL}}, \boldsymbol{p}_{\mathbf{RL}}^{(\boldsymbol{\alpha},\boldsymbol{\beta})}\right),\ k = 1, 2, \cdots, n, \tag{64}$$

$$\dot{p}_{RLk} = -{}^{C}D^{\beta,\alpha}_{1-\gamma}p^{(\alpha,\beta)}_{RLk} + f_{RLk}\left(t, \boldsymbol{q_{RL}}, \boldsymbol{p_{RL}}, \boldsymbol{p^{(\alpha,\beta)}_{RL}}\right),\ k = 1, 2, \cdots, n. \tag{65}$$

Then we study Equations (63)–(65) under the infinitesimal transformations (Equation (53)). For Equation (63), we have

$$\begin{aligned} \dot{\bar{q}}_{RLk} - s_{RLk}\left(\bar{t}, \bar{\boldsymbol{q}}_{\boldsymbol{RL}}, \bar{\boldsymbol{p}}_{\boldsymbol{RL}}, \bar{\boldsymbol{p}}^{(\boldsymbol{\alpha},\boldsymbol{\beta})}_{\boldsymbol{RL}}\right) = \dot{q}_{RLk} - s_{RLk}\left(t, \boldsymbol{q_{RL}}, \boldsymbol{p_{RL}}, \boldsymbol{p^{(\alpha,\beta)}_{RL}}\right) \\ +\theta_{RL}\left[\dot{\xi}_{RLk} - \dot{q}_{RLk}\dot{\xi}_{RL0} - X^{(0)}_{RL}(s_{RLk})\right] \end{aligned} \tag{66}$$

where $X^{(0)}_{RL} = \xi_{RL0}\frac{\partial}{\partial t} + \xi_{RLi}\frac{\partial}{\partial q_{RLi}} + \eta_{RLi}\frac{\partial}{\partial p_{RLi}} + \eta^{(\alpha,\beta)}_{RLi}\frac{\partial}{\partial p^{(\alpha,\beta)}_{RLi}}$, $i = 1, 2, \cdots, n$. For Equation (64), we have

$$\begin{aligned} {}^{RL}D^{\alpha,\beta}_{\gamma}\bar{q}_{RLk} - h_{RLk}\left(t, \bar{\boldsymbol{q}}_{\boldsymbol{RL}}, \bar{\boldsymbol{p}}_{\boldsymbol{RL}}, \bar{\boldsymbol{p}}^{(\boldsymbol{\alpha},\boldsymbol{\beta})}_{\boldsymbol{RL}}\right) = {}^{RL}D^{\alpha,\beta}_{\gamma}q_{RLk} - h_{RLk}\left(t, \boldsymbol{q_{RL}}, \boldsymbol{p_{RL}}, \boldsymbol{p^{(\alpha,\beta)}_{RL}}\right) \\ +\theta_{RL}\left[{}^{RL}D^{\alpha,\beta}_{\gamma}(\xi_{RLk} - \dot{q}_{RLk}\xi_{RL0}) + \xi_{RL0}\frac{\mathrm{d}}{\mathrm{d}t}{}^{RL}D^{\alpha,\beta}_{\gamma}q_{RLk} - X^{(0)}_{RL}(h_{RLk}) \right. \\ \left. -q_{RLk}(t_1)\frac{\gamma\xi_{RL0}(t_1)}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}(t-t_1)^{-\alpha} + q_{RLk}(t_2)\xi_{RL0}(t_2)\frac{1-\gamma}{\Gamma(1-\beta)}\frac{\mathrm{d}}{\mathrm{d}t}(t_2-t)^{-\beta}\right] \end{aligned} \tag{67}$$

For Equation (65), we have

$$\begin{aligned} \dot{\bar{p}}_{RLk} + {}^{C}D^{\beta,\alpha}_{1-\gamma}\bar{p}^{(\alpha,\beta)}_{RLk} - f_{RLk}\left(\bar{t}, \bar{\boldsymbol{q}}_{\boldsymbol{RL}}, \bar{\boldsymbol{p}}_{\boldsymbol{RL}}, \bar{\boldsymbol{p}}^{(\boldsymbol{\alpha},\boldsymbol{\beta})}_{\boldsymbol{RL}}\right) = \dot{p}_{RLk} + {}^{C}D^{\beta,\alpha}_{1-\gamma}p^{(\alpha,\beta)}_{RLk} \\ -f_{RLk}\left(t, \boldsymbol{q_{RL}}, \boldsymbol{p_{RL}}, \boldsymbol{p^{(\alpha,\beta)}_{RL}}\right) + \theta_{RL}\left[\dot{\eta}_{RLk} - \dot{p}_{RLk}\dot{\xi}_{RL0} + {}^{C}D^{\beta,\alpha}_{1-\gamma}\left(\eta^{(\alpha,\beta)}_{RLk} - \dot{p}^{(\alpha,\beta)}_{RLk}\xi_{RL0}\right)\right. \\ -\frac{1-\gamma}{\Gamma(1-\beta)}(t-t_1)^{-\beta}\dot{p}^{(\alpha,\beta)}_{RLk}(t_1)\xi_{RL0}(t_1) + \xi_{RL0}\frac{\mathrm{d}}{\mathrm{d}t}{}^{C}D^{\beta,\alpha}_{1-\gamma}p^{(\alpha,\beta)}_{RLk} \\ \left. +\frac{\gamma}{\Gamma(1-\alpha)}(t_2-t)^{-\alpha}\dot{p}^{(\alpha,\beta)}_{RLk}(t_2)\xi_{RL0}(t_2) - X^{(0)}_{RL}(f_{RLk})\right] \end{aligned} \tag{68}$$

For the fractional primary constraint (Equation (37)), we have

$$\phi_{RLa}\left(t, \bar{\boldsymbol{q}}_{\boldsymbol{RL}}, \bar{\boldsymbol{p}}_{\boldsymbol{RL}}, \bar{\boldsymbol{p}}^{(\boldsymbol{\alpha},\boldsymbol{\beta})}_{\boldsymbol{RL}}\right) = \phi_{RLa}\left(t, \boldsymbol{q_{RL}}, \boldsymbol{p_{RL}}, \boldsymbol{p^{(\alpha,\beta)}_{RL}}\right) + \theta_{RL}X^{(0)}_{RL}(\phi_{RLa}). \tag{69}$$

From the definition of Lie symmetry, we obtain

$$\dot{\xi}_{RLk} - \dot{q}_{RLk}\dot{\xi}_{RL0} - X^{(0)}_{RL}(s_{RLk}) = 0, \tag{70}$$

$$\begin{aligned} {}^{RL}D^{\alpha,\beta}_{\gamma}(\xi_{RLk} - \dot{q}_{RLk}\xi_{RL0}) + \xi_{RL0}\frac{\mathrm{d}}{\mathrm{d}t}{}^{RL}D^{\alpha,\beta}_{\gamma}q_{RLk} - X^{(0)}_{RL}(h_{RLk}) \\ -q_{RLk}(t_1)\frac{\gamma\xi_{RL0}(t_1)}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}(t-t_1)^{-\alpha} + q_{RLk}(t_2)\xi_{RL0}(t_2)\frac{1-\gamma}{\Gamma(1-\beta)}\frac{\mathrm{d}}{\mathrm{d}t}(t_2-t)^{-\beta} = 0, \end{aligned} \tag{71}$$

$$\begin{aligned} {}^{C}D^{\beta,\alpha}_{1-\gamma}\left(\eta^{(\alpha,\beta)}_{RLk} - \dot{p}^{(\alpha,\beta)}_{RLk}\xi_{RL0}\right) + \xi_{RL0}\frac{\mathrm{d}}{\mathrm{d}t}{}^{C}D^{\beta,\alpha}_{1-\gamma}p^{(\alpha,\beta)}_{RLk} - X^{(0)}_{RL}(f_{RLk}) \\ +\dot{\eta}_{RLk} - \dot{p}_{RLk}\dot{\xi}_{RL0} - \frac{1-\gamma}{\Gamma(1-\beta)}(t-t_1)^{-\beta}\dot{p}^{(\alpha,\beta)}_{RLk}(t_1)\xi_{RL0}(t_1) \\ +\frac{\gamma}{\Gamma(1-\alpha)}(t_2-t)^{-\alpha}\dot{p}^{(\alpha,\beta)}_{RLk}(t_2)\xi_{RL0}(t_2) = 0, \end{aligned} \tag{72}$$

and

$$X^{(0)}_{RL}(\phi_{RLa}) = 0. \tag{73}$$

Equations (70)–(72) are called determined equations within ICRL, and Equation (73) is called the limited equation within ICRL.

However, if we consider the deduction process of the fractional constrained Hamilton equation (Equation (46)), an extra additional limited equation,

$$\frac{\partial\phi_{RLa}}{\partial q_{RLi}}(\xi_{RLi} - \dot{q}_{RLi}\xi_{RL0}) + \frac{\partial\phi_{RLa}}{\partial p_{RLi}}(\eta_{RLi} - \dot{p}_{RLi}\xi_{RL0}) + \frac{\partial\phi_{RLa}}{\partial p^{(\alpha,\beta)}_{RLi}}\left(\eta^{(\alpha,\beta)}_{RLi} - \dot{p}^{(\alpha,\beta)}_{RLi}\xi_{RL0}\right) = 0, \tag{74}$$

needs to be exposed on the infinitesimal generators.

Definition 2. *For the fractional constrained Hamiltonian system, if the infinitesimal generators satisfy the determined equations, then the corresponding symmetry is called Lie symmetry.*

Definition 3. *For the fractional constrained Hamiltonian system, if the infinitesimal generators satisfy both the determined equations and the limited equation, then the corresponding symmetry is called weak Lie symmetry.*

Definition 4. *For the fractional constrained Hamiltonian system, if the infinitesimal generators satisfy the determined equations, the limited equation, and the additional limited equation, then the corresponding symmetry is called strong Lie symmetry.*

Lie symmetry can lead to a conserved quantity under some conditions.

Theorem 3. *For the fractional constrained Hamiltonian system within ICRL (Equation (46)), if the infinitesimal generators ξ_{RL0}, ξ_{RLk}, η_{RLk}, and $\eta_{RLk}^{(\alpha,\beta)}$ satisfy the determined equations (Equations (70)–(72)) and the structure equation (Equation (55)), then there exists a Lie symmetry conserved quantity (Equation (56)).*

Theorem 4. *For the fractional constrained Hamiltonian system within ICRL (Equation (46)), if the infinitesimal generators ξ_{RL0}, ξ_{RLk}, η_{RLk}, and $\eta_{RLk}^{(\alpha,\beta)}$ satisfy the determined equations (Equations (70)–(72)), the limited equation (Equation (73)) and the structure equation (Equation (55)), then there exists a weak Lie symmetry conserved quantity (Equation (56)).*

Theorem 5. *For the fractional constrained Hamiltonian system within ICRL (Equation (46)), if the infinitesimal generators ξ_{RL0}, ξ_{RLk}, η_{RLk}, and $\eta_{RLk}^{(\alpha,\beta)}$ satisfy the determined equations (Equations (70)–(72)), the limited equation (Equation (73)), the additional limited equation (Equation (74)), and the structure equation (Equation (55)), then there exists a strong Lie symmetry conserved quantity (Equation (56)).*

7.2. Lie Symmetry and Conserved Quantity within ICC

We write the fractional constrained Hamilton equation within ICC (Equation (47)) in another form

$$\dot{q}_{Ck} = s_{Ck}\left(t, \boldsymbol{q}_C, \boldsymbol{p}_C, \boldsymbol{p}_C^{(\alpha,\beta)}\right),\ k = 1, 2, \cdots, n, \tag{75}$$

$${}^{C}D_{\gamma}^{\alpha,\beta} q_{Ck} = h_{Ck}\left(t, \boldsymbol{q}_C, \boldsymbol{p}_C, \boldsymbol{p}_C^{(\alpha,\beta)}\right),\ k = 1, 2, \cdots, n, \tag{76}$$

$$\dot{p}_{Ck} = -{}^{RL}D_{1-\gamma}^{\beta,\alpha} p_{Ck}^{(\alpha,\beta)} + f_{Ck}\left(t, \boldsymbol{q}_C, \boldsymbol{p}_C, \boldsymbol{p}_C^{(\alpha,\beta)}\right),\ k = 1, 2, \cdots, n. \tag{77}$$

Then, similarly, we can obtain the determined equations within ICC

$$\dot{\xi}_{Ck} - \dot{q}_{Ck}\dot{\xi}_{C0} - X_C^{(0)}(s_{Ck}) = 0, \tag{78}$$

$$\begin{gathered} {}^{C}D_{\gamma}^{\alpha,\beta}(\xi_{Ck} - \dot{q}_{Ck}\xi_{C0}) + \xi_{C0}\frac{\mathrm{d}}{\mathrm{d}t}{}^{C}D_{\gamma}^{\alpha,\beta} q_{Ck} - X_C^{(0)}(h_{Ck}) \\ -\dot{q}_{Ck}(t_1)\frac{\gamma\xi_{C0}(t_1)}{\Gamma(1-\alpha)}(t-t_1)^{-\alpha} + \dot{q}_{Ck}(t_2)\xi_{C0}(t_2)\frac{1-\gamma}{\Gamma(1-\beta)}(t_2-t)^{-\beta} = 0, \end{gathered} \tag{79}$$

$$\begin{gathered} \dot{\eta}_{Ck} - \dot{p}_{Ck}\dot{\xi}_{C0} + {}^{RL}D_{1-\gamma}^{\beta,\alpha}\left(\eta_{Ck}^{(\alpha,\beta)} - \dot{p}_{Ck}^{(\alpha,\beta)}\xi_{C0}\right) - X_C^{(0)}(f_{Ck}) \\ +\xi_{C0}\frac{\mathrm{d}}{\mathrm{d}t}{}^{RL}D_{1-\gamma}^{\beta,\alpha} p_{Ck}^{(\alpha,\beta)} - \frac{1-\gamma}{\Gamma(1-\beta)} p_{Ck}^{(\alpha,\beta)}(t_1)\xi_{C0}(t_1)\frac{\mathrm{d}}{\mathrm{d}t}(t-t_1)^{-\beta} \\ +\frac{\gamma}{\Gamma(1-\alpha)} p_{Ck}^{(\alpha,\beta)}(t_2)\xi_{C0}(t_2)\frac{\mathrm{d}}{\mathrm{d}t}(t_2-t)^{-\alpha} = 0, \end{gathered} \tag{80}$$

the limited equation within ICC

$$X_C^{(0)}(\phi_{Ca}) = 0, \tag{81}$$

and the additional limited equation within ICC

$$\frac{\partial \phi_{Ca}}{\partial q_{Ci}}\left(\xi_{Ci}-\dot{q}_{Ci}\xi_{C0}\right)+\frac{\partial \phi_{Ca}}{\partial p_{Ci}}\left(\eta_{Ci}-\dot{p}_{Ci}\xi_{C0}\right)+\frac{\partial \phi_{Ca}}{\partial p_{Ci}^{(\alpha,\beta)}}\left(\eta_{Ci}^{(\alpha,\beta)}-\dot{p}_{Ci}^{(\alpha,\beta)}\xi_{C0}\right)=0, \tag{82}$$

where $X_C^{(0)}=\xi_{C0}\frac{\partial}{\partial t}+\xi_{Ci}\frac{\partial}{\partial q_{Ci}}+\eta_{Ci}\frac{\partial}{\partial p_{Ci}}+\eta_{Ci}^{(\alpha,\beta)}\frac{\partial}{\partial p_{Ci}^{(\alpha,\beta)}}$, $i=1,2,\cdots,n$. Therefore, we have the following:

Theorem 6. *For the fractional constrained Hamiltonian system within ICC (Equation (47)), if the infinitesimal generators ξ_{C0}, ξ_{Ck}, η_{Ck}, and $\eta_{Ck}^{(\alpha,\beta)}$ satisfy the determined equations (Equations (78)–(80)) and the structure equation (Equation (61)), then there exists a Lie symmetry conserved quantity (Equation (62)).*

Theorem 7. *For the fractional constrained Hamiltonian system within ICC (Equation (47)), if the infinitesimal generators ξ_{C0}, ξ_{Ck}, η_{Ck}, and $\eta_{Ck}^{(\alpha,\beta)}$ satisfy the determined equations (Equations (78)–(80)), the limited equation (Equation (81)), and the structure equation (Equation (61)), then there exists a weak Lie symmetry conserved quantity (Equation (62)).*

Theorem 8. *For the fractional constrained Hamiltonian system within ICC (Equation (47)), if the infinitesimal generators ξ_{C0}, ξ_{Ck}, η_{Ck}, and $\eta_{Ck}^{(\alpha,\beta)}$ satisfy the determined equations (Equations (78)–(80)), the limited equation (Equation (81)), the additional limited equation (Equation (82)), and the structure equation (Equation (61)), then there exists a strong Lie symmetry conserved quantity (Equation (62)).*

8. An Example

The fractional singular system is

$$L_{RL}=\dot{q}_{RL1}q_{RL2}-q_{RL1}\dot{q}_{RL2}+q_{RL1}^2+q_{RL2}^2+\frac{1}{2}\left[\left({}^{RL}D_\gamma^{\alpha,\beta}q_{RL1}\right)^2+\left({}^{RL}D_\gamma^{\alpha,\beta}q_{RL2}\right)^2\right], \tag{83}$$

and we try to study its Noether symmetry and Lie symmetry.

From Equations (28) and (29), we have

$$\begin{aligned}&p_{RL1}=\frac{\partial L_{RL}}{\partial \dot{q}_{RL1}}=q_{RL2},\ p_{RL2}=\frac{\partial L_{RL}}{\partial \dot{q}_{RL2}}=-q_{RL1},\ p_{RL1}^{(\alpha,\beta)}=\frac{\partial L_{RL}}{\partial {}^{RL}D_\gamma^{\alpha,\beta}q_{RL1}}={}^{RL}D_\gamma^{\alpha,\beta}q_{RL1},\\&p_{RL2}^{(\alpha,\beta)}=\frac{\partial L}{\partial {}^{RL}D_\gamma^{\alpha,\beta}q_{RL2}}={}^{RL}D_\gamma^{\alpha,\beta}q_{RL2},\ H_{RL}=\frac{1}{2}\left[\left(p_{RL1}^{(\alpha,\beta)}\right)^2+\left(p_{RL2}^{(\alpha,\beta)}\right)^2\right]-q_{RL1}^2-q_{RL2}^2.\end{aligned} \tag{84}$$

Then Equation (30) gives

$$\begin{aligned}H_{RL11}=\frac{\partial^2 L_{RL}}{\partial \dot{q}_{RL1}\partial \dot{q}_{RL1}}=0,\ H_{RL12}=\frac{\partial^2 L_{RL}}{\partial \dot{q}_{RL1}\partial \dot{q}_{RL2}}=0,\\H_{RL21}=\frac{\partial^2 L_{RL}}{\partial \dot{q}_{RL2}\partial \dot{q}_{RL1}}=0,\ H_{RL22}=\frac{\partial^2 L_{RL}}{\partial \dot{q}_{RL2}\partial \dot{q}_{RL2}}=0.\end{aligned} \tag{85}$$

Namely, $R=\operatorname{rank}\left[H_{RLij}\right]=0$; then Equation (37) gives two fractional primary constraints:

$$\phi_{RL1}=p_{RL1}-q_{RL2}=0,\ \phi_{RL2}=p_{RL2}+q_{RL1}=0 \tag{86}$$

From Equation (49), we obtain

$$\begin{aligned}\lambda_{RL1}&=-q_{RL2}+\frac{1}{2}\left[{}^{C}D_{1-\gamma}^{\beta,\alpha}p_{RL2}^{(\alpha,\beta)}-\frac{\gamma(t_2-t)^{-\alpha}}{\Gamma(1-\alpha)}p_{RL2}^{(\alpha,\beta)}(t_2)+\frac{(1-\gamma)(t-t_1)^{-\beta}}{\Gamma(1-\beta)}p_{RL2}^{(\alpha,\beta)}(t_1)\right],\\\lambda_{RL2}&=q_{RL1}-\frac{1}{2}\left[{}^{C}D_{1-\gamma}^{\beta,\alpha}p_{RL1}^{(\alpha,\beta)}-\frac{\gamma(t_2-t)^{-\alpha}}{\Gamma(1-\alpha)}p_{RL1}^{(\alpha,\beta)}(t_2)+\frac{(1-\gamma)(t-t_1)^{-\beta}}{\Gamma(1-\beta)}p_{RL1}^{(\alpha,\beta)}(t_1)\right].\end{aligned} \tag{87}$$

Then, making use of Equation (46), the fractional constrained Hamilton equation within ICRL can be obtained as

$$\begin{gathered}\dot{p}_{RL1}=q_{RL1}-\frac{1}{2}\left[{}^{C}D_{1-\gamma}^{\beta,\alpha}p_{RL1}^{(\alpha,\beta)}-\frac{\gamma(t_2-t)^{-\alpha}}{\Gamma(1-\alpha)}p_{RL1}^{(\alpha,\beta)}(t_2)+\frac{(1-\gamma)(t-t_1)^{-\beta}}{\Gamma(1-\beta)}p_{RL1}^{(\alpha,\beta)}(t_1)\right],\\ \dot{p}_{RL2}=q_{RL2}-\frac{1}{2}\left[{}^{C}D_{1-\gamma}^{\beta,\alpha}p_{RL2}^{(\alpha,\beta)}-\frac{\gamma(t_2-t)^{-\alpha}}{\Gamma(1-\alpha)}p_{RL2}^{(\alpha,\beta)}(t_2)+\frac{(1-\gamma)(t-t_1)^{-\beta}}{\Gamma(1-\beta)}p_{RL2}^{(\alpha,\beta)}(t_1)\right],\\ \dot{q}_{RL1}=-q_{RL2}+\frac{1}{2}\left[{}^{C}D_{1-\gamma}^{\beta,\alpha}p_{RL2}^{(\alpha,\beta)}-\frac{\gamma(t_2-t)^{-\alpha}}{\Gamma(1-\alpha)}p_{RL2}^{(\alpha,\beta)}(t_2)+\frac{(1-\gamma)(t-t_1)^{-\beta}}{\Gamma(1-\beta)}p_{RL2}^{(\alpha,\beta)}(t_1)\right],\\ \dot{q}_{RL2}=q_{RL1}-\frac{1}{2}\left[{}^{C}D_{1-\gamma}^{\beta,\alpha}p_{RL1}^{(\alpha,\beta)}-\frac{\gamma(t_2-t)^{-\alpha}}{\Gamma(1-\alpha)}p_{RL1}^{(\alpha,\beta)}(t_2)+\frac{(1-\gamma)(t-t_1)^{-\beta}}{\Gamma(1-\beta)}p_{RL1}^{(\alpha,\beta)}(t_1)\right],\\ {}^{RL}D_{\gamma}^{\alpha,\beta}q_{RL1}=p_{RL1}^{(\alpha,\beta)},\ {}^{RL}D_{\gamma}^{\alpha,\beta}q_{RL2}=p_{RL2}^{(\alpha,\beta)}\end{gathered}\tag{88}$$

The Noether identity (Equation (55)) gives

$$\begin{gathered}p_{RL1}^{(\alpha,\beta)}\cdot{}^{RL}D_{\gamma}^{\alpha,\beta}(\xi_{RL1}-\dot{q}_{RL1}\xi_{RL0})+p_{RL2}^{(\alpha,\beta)}\cdot{}^{RL}D_{\gamma}^{\alpha,\beta}(\xi_{RL2}-\dot{q}_{RL2}\xi_{RL0})\\ +\left\{\frac{1}{2}\left[\left(p_{RL1}^{(\alpha,\beta)}\right)^2+\left(p_{RL2}^{(\alpha,\beta)}\right)^2\right]+q_{RL1}^2+q_{RL2}^2\right\}\cdot\dot{\xi}_{RL0}+p_{RL1}\dot{\xi}_{RL1}+p_{RL2}\dot{\xi}_{RL2}\\ +q_{RL1}\xi_{RL1}+q_{RL2}\xi_{RL2}+\left(p_{RL1}^{(\alpha,\beta)}\frac{\mathrm{d}}{\mathrm{d}t}{}^{RL}D_{\gamma}^{\alpha,\beta}q_{RL1}+p_{RL2}^{(\alpha,\beta)}\frac{\mathrm{d}}{\mathrm{d}t}{}^{RL}D_{\gamma}^{\alpha,\beta}q_{RL2}\right)\xi_{RL0}\\ -q_{RL1}(t_1)\cdot\xi_{RL0}(t_1)\cdot\frac{\gamma p_{RL1}^{(\alpha,\beta)}}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}(t-t_1)^{-\alpha}-q_{RL2}(t_1)\cdot\xi_{RL0}(t_1)\cdot\frac{\gamma p_{RL2}^{(\alpha,\beta)}}{\Gamma(1-\alpha)}\\ \times\frac{\mathrm{d}}{\mathrm{d}t}(t-t_1)^{-\alpha}+\lambda_{RL1}\eta_{RL1}-\lambda_{RL1}\eta_{RL2}+q_{RL1}(t_2)\xi_{RL0}(t_2)\cdot\frac{(1-\gamma)p_{RL1}^{(\alpha,\beta)}}{\Gamma(1-\beta)}\\ \times\frac{\mathrm{d}}{\mathrm{d}t}(t_2-t)^{-\beta}+q_{RL2}(t_2)\xi_{RL0}(t_2)\cdot\frac{(1-\gamma)p_{RL2}^{(\alpha,\beta)}}{\Gamma(1-\beta)}\frac{\mathrm{d}}{\mathrm{d}t}(t_2-t)^{-\beta}=0\end{gathered}\tag{89}$$

Then we can verify that

$$\xi_{RL0}=-1,\ \xi_{RL1}=\xi_{RL2}=0,\ \eta_{RL1}=\eta_{RL2}=0,\ \text{and}\ \eta_{RL1}^{(\alpha,\beta)}=\eta_{RL2}^{(\alpha,\beta)}=0\tag{90}$$

satisfy Equation (89). Therefore, from Theorem 1, we obtain a conserved quantity:

$$\begin{gathered}C_{RL}=-\frac{1}{2}\left[\left(p_{RL1}^{(\alpha,\beta)}\right)^2+\left(p_{RL2}^{(\alpha,\beta)}\right)^2\right]-q_{RL1}^2-q_{RL2}^2+\int_{t_1}^{t}\left\{p_{RL1}^{(\alpha,\beta)}\frac{\mathrm{d}}{\mathrm{d}\tau}{}^{RL}D_{\gamma}^{\alpha,\beta}\dot{q}_{RL1}\right.\\ +\frac{\mathrm{d}}{\mathrm{d}\tau}{}^{RL}D_{\gamma}^{\alpha,\beta}\dot{q}_{RL2}\cdot p_{RL2}^{(\alpha,\beta)}+\dot{q}_{RL1}\cdot\left[{}^{C}D_{1-\gamma}^{\beta,\alpha}p_{RL1}^{(\alpha,\beta)}-\frac{\gamma(t_2-\tau)^{-\alpha}}{\Gamma(1-\alpha)}p_{RL1}^{(\alpha,\beta)}(t_2)\right.\\ \left.+\frac{(1-\gamma)(\tau-t_1)^{-\beta}}{\Gamma(1-\beta)}p_{RL1}^{(\alpha,\beta)}(t_1)\right]+\dot{q}_{RL2}\cdot\left[-\frac{\gamma(t_2-\tau)^{-\alpha}}{\Gamma(1-\alpha)}p_{RL2}^{(\alpha,\beta)}(t_2)\right.\\ \left.\left.+{}^{C}D_{1-\gamma}^{\beta,\alpha}p_{RL2}^{(\alpha,\beta)}+\frac{(1-\gamma)(\tau-t_1)^{-\beta}}{\Gamma(1-\beta)}p_{RL2}^{(\alpha,\beta)}(t_1)\right]\right\}\mathrm{d}\tau\end{gathered}\tag{91}$$

Equations (70)–(72) give the determined equations

$$\begin{gathered}\dot{\xi}_{RL1}-\dot{q}_{RL1}\dot{\xi}_{RL0}=-\xi_{RL0}\frac{\partial}{\partial t}\left[\frac{1}{2}p_{RL2}^{\alpha}(t_2)\frac{(t_2-t)^{-\alpha}}{\Gamma(1-\alpha)}\right]-\xi_{RL2},\\ \dot{\xi}_{RL2}-\dot{q}_{RL2}\dot{\xi}_{RL0}=\xi_{RL0}\frac{\partial}{\partial t}\left[\frac{1}{2}p_{RL1}^{\alpha}(t_2)\frac{(t_2-t)^{-\alpha}}{\Gamma(1-\alpha)}\right]+\xi_{RL1},\\ {}_{t_1}D_t^{\alpha}(\xi_{RL1}-\dot{q}_{RL1}\xi_{RL0})+\xi_{RL0}\frac{\mathrm{d}}{\mathrm{d}t}{}_{t_1}D_t^{\alpha}q_{RL1}-\frac{1}{\Gamma(1-\alpha)}q_{RL1}(t_1)\xi_{RL0}(t_1)\frac{\mathrm{d}}{\mathrm{d}t}(t-t_1)^{-\alpha}=\eta_{RL1}^{(\alpha,\beta)},\\ {}_{t_1}D_t^{\alpha}(\xi_{RL2}-\dot{q}_{RL2}\xi_{RL0})+\xi_{RL0}\frac{\mathrm{d}}{\mathrm{d}t}{}_{t_1}D_t^{\alpha}q_{RL2}-\frac{1}{\Gamma(1-\alpha)}q_{RL2}(t_1)\xi_{RL0}(t_1)\frac{\mathrm{d}}{\mathrm{d}t}(t-t_1)^{-\alpha}=\eta_{RL2}^{(\alpha,\beta)},\\ \dot{\eta}_{RL1}-\dot{p}_{RL1}\dot{\xi}_{RL0}-\xi_{RL0}\frac{\mathrm{d}}{\mathrm{d}t}{}_{t}^{\mathrm{C}}D_{t_2}^{\alpha}p_{RL1}^{(\alpha,\beta)}-{}_{t}^{\mathrm{C}}D_{t_2}^{\alpha}\left(\eta_{RL1}^{(\alpha,\beta)}-\dot{p}_{RL1}^{(\alpha,\beta)}\xi_{RL0}\right)\\ +\frac{1}{\Gamma(1-\alpha)}(t_2-t)^{-\alpha}\dot{p}_{RL1}^{(\alpha,\beta)}(t_2)\xi_{RL0}(t_2)=\xi_{RL0}\frac{\partial}{\partial t}\left[\frac{1}{2}p_{RL1}^{(\alpha,\beta)}(t_2)\frac{(t_2-t)^{-\alpha}}{\Gamma(1-\alpha)}\right]+\xi_{RL1},\\ \dot{\eta}_{RL2}-\dot{p}_{RL2}\dot{\xi}_{RL0}-\xi_{RL0}\frac{\mathrm{d}}{\mathrm{d}t}{}_{t}^{\mathrm{C}}D_{t_2}^{\alpha}p_{RL2}^{(\alpha,\beta)}-{}_{t}^{\mathrm{C}}D_{t_2}^{\alpha}\left(\eta_{RL2}^{(\alpha,\beta)}-\dot{p}_{RL2}^{(\alpha,\beta)}\xi_{RL0}\right)\\ +\frac{1}{\Gamma(1-\alpha)}(t_2-t)^{-\alpha}\dot{p}_{RL2}^{(\alpha,\beta)}(t_2)\xi_{RL0}(t_2)=\xi_{RL0}\frac{\partial}{\partial t}\left[\frac{1}{2}p_{RL2}^{(\alpha,\beta)}(t_2)\frac{(t_2-t)^{-\alpha}}{\Gamma(1-\alpha)}\right]+\xi_{RL2}.\end{gathered}\tag{92}$$

Equation (73) gives the limited equation

$$-\xi_{RL2} + \eta_{RL1} = 0,\ \xi_{RL1} + \eta_{RL2} = 0 \tag{93}$$

Equation (74) gives the additional limited equation

$$\xi_{RL2} - \dot{q}_{RL2}\xi_{RL0} + \eta_{RL1} - \dot{p}_{RL1}\xi_{RL0} = 0,\ \xi_{RL1} - \dot{q}_{RL1}\xi_{RL0} + \eta_{RL2} - \dot{p}_{RL2}\xi_{RL0} = 0 \tag{94}$$

Taking the calculation, we find that Equation (90) also meets the determined equation (Equation (92)) as well as the limited equation (Equation (93)) under the condition $p_{RL1}^{(\alpha,\beta)}(t_1) = p_{RL2}^{(\alpha,\beta)}(t_1) = p_{RL1}^{(\alpha,\beta)}(t_2) = p_{RL2}^{(\alpha,\beta)}(t_2) = 0$. It is noted that Equation (90) is not the solution to the additional limited equation (Equation (94)). Therefore, Equation (91) is also a Lie symmetry conserved quantity as well as a weak Lie symmetry conserved quantity, but not a strong Lie symmetry conserved quantity.

9. Results and Discussion

Based on ICRL and ICC, the fractional Lagrange equations (Equations (24) and (27)), the fractional primary constraints (Equations (37) and (40)), the fractional constrained Hamilton equations (Equations (46) and (47)), and the consistency conditions (Equations (49) and (50)) are presented. Noether symmetry and Lie symmetry are investigated, and the corresponding conserved quantities are achieved. Here only the Noether type conserved quantity is deduced from the Lie symmetry. It is significant if the Hojman type conserved quantity can be deduced from Lie symmetry in the future. Moreover, the Mei symmetry method is another important tool to find solutions to the differential equations of motion. Therefore, Lie symmetry and the corresponding Hojman type conserved quantity, Mei symmetry and the corresponding Mei type conserved quantity, as well as the perturbation to symmetry are the future research directions. As for the example, it is helpful and straightforward if a numerical calculation could be given to show that the obtained conservation law is a constant. Therefore, the use of simulation to illustrate obtained results is also an important research direction in the near future.

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