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Boundary Value Problem for Multi-Term Nonlinear Delay Generalized Proportional Caputo Fractional Differential Equations

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Abstract: A nonlocal boundary value problem for a couple of two scalar nonlinear differential equations with several generalized proportional Caputo fractional derivatives and a delay is studied. The exact solution of the scalar nonlinear differential equation with several generalized proportional Caputo fractional derivatives with different orders is obtained. A mild solution of the boundary value problem for the multi-term nonlinear couple of the given fractional equations is defined. The connection between the mild solution and the solution of the studied problem is discussed. As a partial case, several results for the nonlocal boundary value problem for the linear and non-linear multi-term Caputo fractional differential equations are provided. The results generalize several known results in the literature.

Keywords: generalized proportional Caputo fractional derivatives; boundary value problem; delay; integral presentation; existence

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1. Introduction

The fractional derivatives are intensively applied to model the dynamics of real-world processes and phenomena when the current state depends on the past behavior. The main properties of the fractional derivatives connected with their memory as well as their parameters (the fractional order) give us the opportunity to adjust the the fractional derivative to the real data and to create more adequate and realistic models. Some real life models by fractional derivatives in engineering systems are provided in the book [1]; biological systems are given in [2], and epidemiological systems are studied in [3]. In connection with this, several different types of fractional derivatives have been defined and studied such as the Hilfer operator [4,5], derivatives depending on another function [6,7], or involving arbitrary kernels [8,9].

Recently, starting from the definition of tempered fractional derivative, the generalized proportional derivative has been defined in [10,11]. Despite being a very recent idea, already several excellent works are available, for example, for some fundamental properties see [12,13], for stability properties see [14–16], and for stochastic differential equations see [17].

In this paper, a couple of nonlinear differential equations with a special type of delay and several generalized proportional Caputo type derivatives is considered. We begin with the linear scalar differential equation with several generalized proportional Caputo-type derivatives, and for the nonlocal boundary value problem an integral representation of the solution is obtained. Based on this representation, the mild solution of the couple of nonlinear delay multi-term delay equations is defined. Then the connection between the mild solution and the solution is discussed. Additionally, a partial case of initial value problems is investigated. The obtained results are generalizations of the recently studied works in the literature for multi-term differential equations with Caputo fractional derivatives. The proved results can be applied for investigating the qualitative properties such as Ulam-type stability of the couple of nonlinear differential equations with a special type of delay and several generalized proportional Caputo-type derivatives, and various types of boundary or initial conditions.

The paper is organized as follows. In Section 2, the basic definitions of generalized proportional fractional integrals and derivatives are given. Additionally, some basic properties are provided. The statement of the problem is set in Section 3. Additionally, the mild solution is defined and its existence, uniqueness and relation with the solution of the given problem is studied. The explicit solution to the linear problem is also obtained. In Section 4, the obtained results of the previous sections are extended to the multi-term Caputo fractional differential equations and compared with the existing results in the literature. In Section 5, a discussion about the proved results is provided and some possible future works are mentioned.

2. Notes on Fractional Calculus

We recall that the generalized proportional fractional integral and the generalized Caputo proportional fractional derivative of a function $u : [a,\infty) \to \mathbb{R}$ are defined, respectively, by (as long as all integrals are well defined, see [10,11])

$$({}_a\mathcal{I}^{\alpha,\rho}u)(t) = \frac{1}{\rho^{\alpha}\Gamma(\alpha)}\int_a^t e^{\frac{\rho-1}{\rho}(t-s)}(t-s)^{\alpha-1}u(s)\,ds, \quad t \in (a,b],\ \ \alpha \geq 0,\ \rho \in (0,1],$$

and

$$({}_a^C\mathcal{D}^{\alpha,\rho}u)(t) = \frac{1}{\rho^{1-q}\Gamma(1-\alpha)}\int_a^t e^{\frac{\rho-1}{\rho}(t-s)}(t-s)^{-\alpha}\Big(\mathcal{D}^{1,\rho}u\Big)(s)\,ds,$$
$$\text{for } \ t \in (a,b],\ \ \alpha \in (0,1),\ \rho \in (0,1],$$

where $(\mathcal{D}^{1,\rho}u)(t) = (1-\rho)u(t) + \rho u'(t)$.

Remark 1. *The generalized proportional Caputo fractional derivative is a generalization of the classical Caputo fractional derivative of order $\alpha \in (0,1)$: ${}_a^C\mathcal{D}^{\alpha}u(t)$ [18] in the case $\rho = 1$.*

Remark 2. *(see Remark 3.2 [10]) If $\alpha \in (0,1)$ and $\rho \in (0,1]$ then the relations $({}_a^C\mathcal{D}^{\alpha,\rho}e^{\frac{\rho-1}{\rho}(.)})(t) = 0$ for $t > a$ and $({}_a^C\mathcal{D}^{\alpha,\rho}K)(t) \neq 0$ for $K \in \mathbb{R},\ K \neq 0$ hold.*

We introduce the following classes of functions

$$C^{\alpha,\rho}[a,b] = \{u \in C^1([a,b],\mathbb{R}) :\ \ ({}_a^C\mathcal{D}^{\alpha,\rho}u)(t) \text{ exists for } t \in (a,b]\},$$
$$I^{\alpha,\rho}[a,b] = \{u \in C([a,b],\mathbb{R}) :\ \ ({}_a\mathcal{I}^{\alpha,\rho}u)(t) \text{ exists for } t \in (a,b]\}.$$

Note that if $u \in C^{\alpha,\rho}[a,b]$ then $\mathcal{D}^{1,\rho}u(.) \in I^{1-\alpha,\rho}[a,b]$.

We recall some results about generalized proportional Caputo fractional derivatives and their applications in differential equations, which will be applied in the main result in the paper.

Lemma 1 (Proposition 3.7 [10])**.** *For $\rho \in (0,1]$, $\alpha,\beta > 0$ we have*

$${}_a\mathcal{I}^{\alpha,\rho}(e^{\frac{\rho-1}{\rho}t}(t-a)^{\beta-1}) = \frac{\Gamma(\beta)}{\rho^{\alpha}\Gamma(\beta+\alpha)}e^{\frac{\rho-1}{\rho}t}(t-a)^{\alpha+\beta-1}.$$

Corollary 1. ${}_a\mathcal{I}^{\alpha,\rho}(e^{\frac{\rho-1}{\rho}t}) = \frac{1}{\rho^{\alpha}\Gamma(1+\alpha)}e^{\frac{\rho-1}{\rho}t}(t-a)^{\alpha}$.

Lemma 2 (Proposition 5.2 [10]). *For $\rho \in (0,1]$, $\alpha, \beta > 0, \beta \neq 1$ we have*

$$ {}_a^C\mathcal{D}^{\alpha,\rho}(e^{\frac{\rho-1}{\rho}t}(t-a)^{\beta-1}) = \frac{\rho^\alpha \Gamma(\beta)}{\Gamma(\beta-\alpha)} e^{\frac{\rho-1}{\rho}t}(t-a)^{\beta-\alpha-1}. \tag{1} $$

Lemma 3 (Theorem 3.8 [10]). *For $\rho \in (0,1]$, $\alpha, \beta > 0$ and $u \in C([a,b], \mathbb{R})$ the we have*

$$ \left({}_a\mathcal{I}^{\alpha,\rho}\left({}_a\mathcal{I}^{\beta,\rho}u\right)\right)(t) = ({}_a\mathcal{I}^{\alpha+\beta,\rho}u)(t), \ \ t \in (a,b]. \tag{2} $$

We will use the following result, which is a partial case of Theorem 5.3 [10] for $\alpha \in (0,1)$.

Lemma 4. *For $\rho \in (0,1]$, $\alpha \in (0,1]$ and $u \in C_a^{\alpha,\rho}[a,b]$, ${}_a^C\mathcal{D}^{\alpha,\rho}u(.) \in I_a^{\alpha,\rho}[a,b]$ we have*

$$ \left({}_a\mathcal{I}^{\alpha,\rho}\left({}_a^C\mathcal{D}^{\alpha,\rho}u\right)\right)(t) = u(t) - u(a)e^{\frac{\rho-1}{\rho}(t-a)}, \ \ t \in (a,b]. $$

Corollary 2 ([10]). *Let $\alpha \in (0,1)$, $\rho \in (0,1]$ and $u \in I^{\alpha,\rho}[a,b]$, ${}_a\mathcal{I}^{\alpha,\rho}u(.) \in C^{\alpha,\rho}[a,b]$. Then*

$$ ({}_a^C\mathcal{D}^{\alpha,\rho}({}_a\mathcal{I}^{\alpha,\rho}u))(t) = u(t), \quad t \in (a,b]. $$

3. Multi-Term Differential Equations with Generalized Proportional Caputo Fractional Derivatives

Let the sequences of numbers $1 > \alpha_1 > \alpha_2 > \cdots > \alpha_n > 0$ and $1 > \beta_1 > \beta_2 > \cdots > \beta_N > 0$ be given.

Consider the couple of *delay differential equations with several generalized proportional Caputo fractional derivatives*, or so-called multi-term generalized proportional fractional delay differential equations

$$ \begin{aligned} &\sum_{i=1}^{n} A_i({}_0^C\mathcal{D}^{\alpha_i,\rho}x)(t) = f(t, x(t), x(\lambda_1 t), y(t)), \text{ for } \ t \in (0,1], \\ &\sum_{i=1}^{N} B_i({}_0^C\mathcal{D}^{\beta_i,\rho}y)(t) = g(t, y(t), y(\lambda_2 t), x(t)), \text{ for } \ t \in (0,1], \end{aligned} \tag{3} $$

with the nonlocal boundary value conditions

$$ \gamma_1 x(0) + \eta_1 x(\xi_1) + \mu_1 x(1) = \Phi_1(\xi_1), \qquad \gamma_2 y(0) + \eta_2 y(\xi_2) + \mu_2 y(1) = \Phi_2(\xi_2), \tag{4} $$

where $\lambda_1, \lambda_2 \in (0,1)$, $\xi_1, \xi_2 \in (0,1)$ are arbitrary points, the numbers $A_i, i = 1,2,\ldots,n$, B_i, $i = 1,2,\ldots,N$, γ_i, η_i, μ_i, $i = 1,2$ are such that $A_1 \neq 0$, $B_1 \neq 0, \gamma_i + \eta_i + \mu_i \neq 0$, $i = 1,2$, the functions $f, g : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$, $\Phi_i : (0,1) \to \mathbb{R}$, $i = 1,2$.

3.1. *Explicit Solution of the Multi-Term Linear Problem with Generalized Proportional Caputo Fractional Derivatives*

Consider the following linear multi-term generalized proportional fractional differential equation:

$$ \sum_{i=1}^{N} C_i({}_0^C\mathcal{D}^{p_i,\rho}z)(t) = F(t), \text{ for } \ t \in (0,1], \tag{5} $$

with the nonlocal boundary value condition

$$ \gamma z(0) + \beta z(\xi) + \mu z(1) = \Psi(\xi), \tag{6} $$

where $1 > p_1 > p_2 > \cdots > p_N > 0, C_i, i = 1, 2, \ldots, N: \ C_1 \neq 0$ are constants, $F : [0, 1] \to \mathbb{R}$, $\xi \in (0, 1)$ is an arbitrary point, γ, β, μ are arbitrary numbers, $\Psi \in C((0, 1), \mathbb{R})$ and

$$\gamma + \beta e^{\frac{\rho-1}{\rho}\xi} \sum_{k=1}^{N} \frac{C_k}{C_1 \rho^{p_1-p_k} \Gamma(1 + p_1 - p_k)} \xi^{p_1-p_k} + \mu e^{\frac{\rho-1}{\rho}} \sum_{k=1}^{N} \frac{C_k}{C_1 \rho^{p_1-p_k} \Gamma(1 + p_1 - p_k)} \neq 0. \tag{7}$$

Lemma 5. *Let $F \in I^{p_1,\rho}[0, 1]$, and inequality (7) holds. Then, the boundary value problem for the linear multi-term generalized proportional fractional differential Equations (5) and (6) has a unique solution given by*

$$\begin{aligned} z(t) &= \frac{P(\xi)}{K} e^{\frac{\rho-1}{\rho}t} \sum_{k=1}^{N} \frac{C_k}{C_1 \rho^{p_1-p_k} \Gamma(1 + p_1 - p_k)} t^{p_1-p_k} \\ &- \sum_{k=2}^{N} \frac{C_k}{C_1 \rho^{p_1-p_k} \Gamma(p_1 - p_k)} \int_0^t e^{\frac{\rho-1}{\rho}(t-s)} \frac{z(s)}{(t-s)^{1-p_1+p_k}} ds \\ &+ \frac{1}{C_1 \rho^{p_1} \Gamma(p_1)} \int_0^t e^{\frac{\rho-1}{\rho}(t-s)} \frac{F(s)}{(t-s)^{1-p_1}} ds, \quad t \in [0, 1], \end{aligned} \tag{8}$$

where

$$\begin{aligned} K &= \gamma + \beta e^{\frac{\rho-1}{\rho}\xi} \sum_{k=1}^{N} \frac{C_k}{C_1 \rho^{p_1-p_k} \Gamma(1 + p_1 - p_k)} \xi^{p_1-p_k} \\ &\quad + \mu e^{\frac{\rho-1}{\rho}} \sum_{k=1}^{N} \frac{C_k}{C_1 \rho^{p_1-p_k} \Gamma(1 + p_1 - p_k)} \neq 0, \\ P(\xi) &= \Psi(\xi) + \sum_{k=2}^{N} \frac{C_k}{C_1 \rho^{p_1-p_k} \Gamma(p_1 - p_k)} \Big(\beta \int_0^\xi e^{\frac{\rho-1}{\rho}(\xi-s)} \frac{z(s)}{(\xi-s)^{1-p_1+p_k}} ds \\ &\quad + \mu \int_0^1 e^{\frac{\rho-1}{\rho}(1-s)} \frac{z(s)}{(1-s)^{1-p_1+p_k}} ds \Big) \\ &\quad - \frac{1}{C_1 \rho^{p_1} \Gamma(p_1)} \left(\beta \int_0^\xi e^{\frac{\rho-1}{\rho}(\xi-s)} \frac{F(s)}{(\xi-s)^{1-p_1}} ds + \mu \int_0^1 e^{\frac{\rho-1}{\rho}(1-s)} \frac{F(s)}{(1-s)^{1-p_1}} ds \right). \end{aligned} \tag{9}$$

Proof. Since $p_1 > p_i, i = 2, 3, \ldots, N$ we take a generalized proportional fractional integral $({}_0\mathcal{I}^{p_1,\rho} z)(t)$ from both sides of (5), use Lemma 1 with $\beta = 1, \alpha = p_1 - p_k$, Lemmas 3, 4 and

$$\begin{aligned} &\left({}_0\mathcal{I}^{p_1,\rho} \left({}_0^C\mathcal{D}^{p_k,\rho} z \right) \right)(t) = \left({}_0\mathcal{I}^{p_1-p_k,\rho} \left({}_0\mathcal{I}^{p_k,\rho} \left({}_0^C\mathcal{D}^{p_k,\rho} z \right) \right) \right)(t) \\ &= \left({}_0\mathcal{I}^{p_1-p_k,\rho} \left(z(t) - z(0) e^{\frac{\rho-1}{\rho}t} \right) \right) \\ &= ({}_0\mathcal{I}^{p_1-p_k,\rho} z)(t) - z(0)({}_0\mathcal{I}^{p_1-p_k,\rho} e^{\frac{\rho-1}{\rho}t}) \\ &= ({}_0\mathcal{I}^{p_1-p_k,\rho} z)(t) - z(0) \frac{e^{\frac{\rho-1}{\rho}t}}{\rho^{p_1-p_k} \Gamma(1 + p_1 - p_k)} t^{p_1-p_k} \end{aligned}$$

to obtain

$$
\begin{aligned}
C_1 z(t) &= C_1 z(0) e^{\frac{\rho-1}{\rho}t} - \sum_{k=2}^{N} C_k \left(({}_0\mathcal{I}^{p_1-p_k,\rho} z)(t) - z(0)({}_0\mathcal{I}^{p_1-p_k,\rho} e^{\frac{\rho-1}{\rho}t} \right) \\
&\quad + ({}_0\mathcal{I}^{p_1,\rho} F)(t) \\
&= z(0) e^{\frac{\rho-1}{\rho}t} \left(C_1 + \sum_{k=2}^{N} \frac{C_k}{\rho^{p_1-p_k}\Gamma(1+p_1-p_k)} t^{p_1-p_k} \right) \\
&\quad - \sum_{k=2}^{N} C_k ({}_0\mathcal{I}^{p_1-p_k,\rho} z)(t) + ({}_0\mathcal{I}^{p_1,\rho} F)(t) \\
&= z(0) e^{\frac{\rho-1}{\rho}t} \sum_{k=1}^{N} \frac{C_k}{\rho^{p_1-p_k}\Gamma(1+p_1-p_k)} t^{p_1-p_k} \\
&\quad - \sum_{k=2}^{N} C_k ({}_0\mathcal{I}^{p_1-p_k,\rho} z)(t) + ({}_0\mathcal{I}^{p_1,\rho} F)(t) \\
&= z(0) e^{\frac{\rho-1}{\rho}t} \sum_{k=1}^{N} \frac{C_k}{\rho^{p_1-p_k}\Gamma(1+p_1-p_k)} t^{p_1-p_k} \\
&\quad - \sum_{k=2}^{N} C_k \frac{1}{\rho^{p_1-p_k}\Gamma(p_1-p_k)} \int_0^t e^{\frac{\rho-1}{\rho}(t-s)} \frac{z(s)}{(t-s)^{1-p_1+p_k}} ds \\
&\quad + \frac{1}{\rho^{p_1}\Gamma(p_1)} \int_0^t e^{\frac{\rho-1}{\rho}(t-s)} \frac{F(s)}{(t-s)^{1-p_1}} ds.
\end{aligned}
\tag{10}
$$

Then

$$
\begin{aligned}
z(\xi) &= z(0) e^{\frac{\rho-1}{\rho}\xi} \sum_{k=1}^{N} \frac{C_k}{C_1\rho^{p_1-p_k}\Gamma(1+p_1-p_k)} \xi^{p_1-p_k} \\
&\quad - \sum_{k=2}^{N} \frac{C_k}{C_1\rho^{p_1-p_k}\Gamma(p_1-p_k)} \int_0^\xi e^{\frac{\rho-1}{\rho}(\xi-s)} \frac{z(s)}{(\xi-s)^{1-p_1+p_k}} ds \\
&\quad + \frac{1}{C_1\rho^{p_1}\Gamma(p_1)} \int_0^\xi e^{\frac{\rho-1}{\rho}(\xi-s)} \frac{F(s)}{(\xi-s)^{1-p_1}} ds
\end{aligned}
\tag{11}
$$

and

$$
\begin{aligned}
z(1) &= z(0) e^{\frac{\rho-1}{\rho}} \sum_{k=1}^{N} \frac{C_k}{C_1\rho^{p_1-p_k}\Gamma(1+p_1-p_k)} \\
&\quad - \sum_{k=2}^{N} \frac{C_k}{C_1\rho^{p_1-p_k}\Gamma(p_1-p_k)} \int_0^1 e^{\frac{\rho-1}{\rho}(1-s)} \frac{z(s)}{(1-s)^{1-p_1+p_k}} ds \\
&\quad + \frac{1}{C_1\rho^{p_1}\Gamma(p_1)} \int_0^1 e^{\frac{\rho-1}{\rho}(1-s)} \frac{F(s)}{(1-s)^{1-p_1}} ds.
\end{aligned}
\tag{12}
$$

From (11), (12) and boundary condition (6), we find

$$
\begin{aligned}
z(0) &= \frac{1}{K} \Bigg\{ \Psi(\xi) \\
&+ \sum_{k=2}^{N} \frac{C_k}{C_1\rho^{p_1-p_k}\Gamma(p_1-p_k)} \left(\beta \int_0^\xi e^{\frac{\rho-1}{\rho}(\xi-s)} \frac{z(s)}{(\xi-s)^{1-p_1+p_k}} ds \right. \\
&\quad \left. + \mu \int_0^1 e^{\frac{\rho-1}{\rho}(1-s)} \frac{z(s)}{(1-s)^{1-p_1+p_k}} ds \right) \\
&- \frac{1}{C_1\rho^{p_1}\Gamma(p_1)} \left(\beta \int_0^\xi e^{\frac{\rho-1}{\rho}(\xi-s)} \frac{F(s)}{(\xi-s)^{1-p_1}} ds + \mu \int_0^1 e^{\frac{\rho-1}{\rho}(1-s)} \frac{F(s)}{(1-s)^{1-p_1}} ds \right) \Bigg\}.
\end{aligned}
\tag{13}
$$

Substitute equality (13) in (10) and obtain (8). □

Remark 3. *In the partial case $N = 1, C_1 = 1, \gamma = 1, \beta = \mu = 0$, $\Psi(s) \equiv a =$ constant the boundary value problem (5) and (6) is reduced to the initial value problem for scalar linear generalized proportional Caputo fractional differential equation of order $p \in (0, 1)$ and the Formula (8) is reduced to*

$$z(t) = ae^{\frac{\rho-1}{\rho}t} + \frac{1}{\rho^p\Gamma(p)}\int_0^t e^{\frac{\rho-1}{\rho}(t-s)}\frac{F(s)}{(t-s)^{1-p}}ds$$

(see, Example 5.7 [10] with $\lambda = 0$).

3.2. Mild Solution of the Boundary Value Problem for the Couple of Nonlinear Equations

We introduce the following condition:

A1. The following hold

$$\begin{aligned} K_1 &= \gamma_1 + \eta_1 e^{\frac{\rho-1}{\rho}\xi_1}\sum_{k=1}^{n}\frac{A_k}{A_1\rho^{\alpha_1-\alpha_k}\Gamma(1+\alpha_1-\alpha_k)}\xi_1^{\alpha_1-\alpha_k} \\ &\quad + \mu_1 e^{\frac{\rho-1}{\rho}}\sum_{k=1}^{n}\frac{A_k}{A_1\rho^{\alpha_1-\alpha_k}\Gamma(1+\alpha_1-\alpha_k)} \neq 0, \\ K_2 &= \gamma_2 + \eta_2 e^{\frac{\rho-1}{\rho}\xi_2}\sum_{k=1}^{N}\frac{B_k}{B_1\rho^{\beta_1-\beta_k}\Gamma(1+\beta_1-\beta_k)}\xi_2^{\beta_1-\beta_k} \\ &\quad + \mu_2 e^{\frac{\rho-1}{\rho}}\sum_{k=1}^{N}\frac{B_k}{B_1\rho^{\beta_1-\beta_k}\Gamma(1+\beta_1-\beta_k)} \neq 0. \end{aligned} \tag{14}$$

Following the integral representation (8), we will define a mild solution of the boundary value problem for the nonlinear delay differential equation with several generalized proportional Caputo fractional derivatives (3) and (4).

Definition 1. *The couple of functions $(x(t), y(t))$: $x \in I^{\alpha_1-\alpha_k,\rho}[0,1], k = 2, 3, \ldots, n$, $y \in I^{\beta_1-\beta_k,\rho}[0,1], k = 2, 3, \ldots, N$, is called a mild solution of the boundary value problem for multi-term generalized proportional Caputo fractional differential Equations (3) and (4) if they satisfy the integral equations*

$$\begin{aligned} x(t) &= \frac{P(\xi_1, x, y)}{K_1}e^{\frac{\rho-1}{\rho}t}\sum_{k=1}^{n}\frac{A_k}{A_1\rho^{\alpha_1-\alpha_k}\Gamma(1+\alpha_1-\alpha_k)}t^{\alpha_1-\alpha_k} \\ &\quad - \sum_{k=2}^{n}\frac{A_k}{A_1\rho^{\alpha_1-\alpha_k}\Gamma(\alpha_1-\alpha_k)}\int_0^t e^{\frac{\rho-1}{\rho}(t-s)}\frac{x(s)}{(t-s)^{1-\alpha_1+\alpha_k}}ds \\ &\quad + \frac{1}{A_1\rho^{\alpha_1}\Gamma(\alpha_1)}\int_0^t e^{\frac{\rho-1}{\rho}(t-s)}\frac{f(s, x(s), x(\lambda_1 s), y(s))}{(t-s)^{1-\alpha_1}}ds, \quad t \in [0,1], \\ y(t) &= \frac{Q(\xi_2, x, y)}{K_2}e^{\frac{\rho-1}{\rho}t}\sum_{k=1}^{N}\frac{B_k}{B_1\rho^{\beta_1-\beta_k}\Gamma(1+\beta_1-\beta_k)}t^{\beta_1-\beta_k} \\ &\quad - \sum_{k=2}^{N}\frac{B_k}{B_1\rho^{\beta_1-\beta_k}\Gamma(\beta_1-\beta_k)}\int_0^t e^{\frac{\rho-1}{\rho}(t-s)}\frac{y(s)}{(t-s)^{1-\beta_1+\beta_k}}ds \\ &\quad + \frac{1}{B_1\rho^{\beta_1}\Gamma(\beta_1)}\int_0^t e^{\frac{\rho-1}{\rho}(t-s)}\frac{g(s, y(s), y(\lambda_2 s), x(s))}{(t-s)^{1-\beta_1}}ds, \quad t \in [0,1], \end{aligned} \tag{15}$$

where K_1 and K_2 are defined by (14), and

$$
\begin{aligned}
P(\xi_1,x,y) &= \Phi_1(\xi_1) + \sum_{k=2}^{n} \frac{A_k}{A_1\rho^{\alpha_1-\alpha_k}\Gamma(\alpha_1-\alpha_k)} \Big(\eta_1 \int_0^{\xi_1} e^{\frac{\rho-1}{\rho}(\xi_1-s)} \frac{x(s)}{(\xi_1-s)^{1-\alpha_1+\alpha_k}} ds \\
&\quad + \mu_1 \int_0^1 e^{\frac{\rho-1}{\rho}(1-s)} \frac{x(s)}{(1-s)^{1-\alpha_1+\alpha_k}} ds \Big) \\
&\quad - \frac{1}{A_1\rho^{\alpha_1}\Gamma(\alpha_1)} \Big(\eta_1 \int_0^{\xi_1} e^{\frac{\rho-1}{\rho}(\xi_1-s)} \frac{f(s,x(s),x(\lambda_1 s),y(s))}{(\xi_1-s)^{1-\alpha_1}} ds \\
&\quad + \mu_1 \int_0^1 e^{\frac{\rho-1}{\rho}(1-s)} \frac{f(s,x(s),x(\lambda_1 s),y(s))}{(1-s)^{1-\alpha_1}} ds \Big), \\
Q(\xi_2,x,y) &= \Phi_2(\xi_2) + \sum_{k=2}^{N} \frac{B_k}{B_1\rho^{\beta_1-\beta_k}\Gamma(\beta_1-\beta_k)} \Big(\eta_2 \int_0^{\xi_2} e^{\frac{\rho-1}{\rho}(\xi_2-s)} \frac{y(s)}{(\xi_2-s)^{1-\beta_1+\beta_k}} ds \\
&\quad + \mu_2 \int_0^1 e^{\frac{\rho-1}{\rho}(1-s)} \frac{y(s)}{(1-s)^{1-\beta_1+\beta_k}} ds \Big) \\
&\quad - \frac{1}{B_1\rho^{\beta_1}\Gamma(\beta_1)} \Big(\eta_2 \int_0^{\xi_2} e^{\frac{\rho-1}{\rho}(\xi_2-s)} \frac{g(s,y(s),y(\lambda_2 s),x(s))}{(\xi_2-s)^{1-\beta_1}} ds \\
&\quad + \mu_2 \int_0^1 e^{\frac{\rho-1}{\rho}(1-s)} \frac{g(s.y(s),y(\lambda_2 s),x(s))}{(1-s)^{1-\beta_1}} ds \Big).
\end{aligned} \tag{16}
$$

Theorem 1. *Let the condition A1 be satisfied and the couple $(x(t),y(t))$, $t \in [0,1]$, be a mild solution of the boundary value problem for multi-term generalized proportional Caputo fractional differential Equations (3) and (4) such that $x \in C^{\alpha_k,\rho}([0,1],\mathbb{R})$ for $k = 1,2,\dots,n$ and $y \in C^{\beta_k,\rho}([0,1],\mathbb{R})$ for $k = 1,2,\dots,N$. Then, the couple $(x(t),y(t))$ is a solution of the same problem.*

Proof. From Equation (15) it follows

$$
\begin{aligned}
&\gamma_1 x(0) + \eta_1 x(\xi_1) + \mu_1 x(1) \\
&= \frac{\gamma_1 P(\xi_1,x,y)}{K_1} + \eta_1 \Big\{ \frac{P(\xi_1,x,y)}{K_1} e^{\frac{\rho-1}{\rho}\xi_1} \sum_{k=1}^{n} \frac{A_k}{A_1\rho^{\alpha_1-\alpha_k}\Gamma(1+\alpha_1-\alpha_k)} {\xi_1}^{\alpha_1-\alpha_k} \\
&\quad - \sum_{k=2}^{n} \frac{A_k}{A_1\rho^{\alpha_1-\alpha_k}\Gamma(\alpha_1-\alpha_k)} \int_0^{\xi_1} e^{\frac{\rho-1}{\rho}(\xi_1-s)} \frac{x(s)}{(\xi_1-s)^{1-\alpha_1+\alpha_k}} ds \\
&\quad + \frac{1}{A_1\rho^{\alpha_1}\Gamma(\alpha_1)} \int_0^{\xi_1} e^{\frac{\rho-1}{\rho}(\xi_1-s)} \frac{f(s,x(s),x(\lambda_1 s),y(s))}{(\xi_1-s)^{1-\alpha_1}} ds \Big\} \\
&\quad + \mu_1 \Big\{ \frac{P(\xi_1)}{K_1} e^{\frac{\rho-1}{\rho}} \sum_{k=1}^{n} \frac{A_k}{A_1\rho^{\alpha_1-\alpha_k}\Gamma(1+\alpha_1-\alpha_k)} \\
&\quad - \sum_{k=2}^{n} \frac{A_k}{A_1\rho^{\alpha_1-\alpha_k}\Gamma(\alpha_1-\alpha_k)} \int_0^{1} e^{\frac{\rho-1}{\rho}(1-s)} \frac{x(s)}{(1-s)^{1-\alpha_1+\alpha_k}} ds \\
&\quad + \frac{1}{A_1\rho^{\alpha_1}\Gamma(\alpha_1)} \int_0^{1} e^{\frac{\rho-1}{\rho}(1-s)} \frac{f(s,x(s),x(\lambda_1 s),y(s))}{(1-s)^{1-\alpha_1}} ds \Big\} \\
&= \frac{P(\xi_1)}{K_1} \Big[\gamma_1 + \eta_1 e^{\frac{\rho-1}{\rho}\xi_1} \sum_{k=1}^{n} \frac{A_k}{A_1\rho^{\alpha_1-\alpha_k}\Gamma(1+\alpha_1-\alpha_k)} {\xi_1}^{\alpha_1-\alpha_k} \\
&\quad + \mu_1 e^{\frac{\rho-1}{\rho}} \sum_{k=1}^{n} \frac{A_k}{A_1\rho^{\alpha_1-\alpha_k}\Gamma(1+\alpha_1-\alpha_k)} \Big] \\
&\quad - \sum_{k=2}^{n} \frac{A_k}{A_1\rho^{\alpha_1-\alpha_k}\Gamma(\alpha_1-\alpha_k)} \Big[\eta_1 \int_0^{\xi_1} e^{\frac{\rho-1}{\rho}(\xi_1-s)} \frac{x(s)}{(\xi_1-s)^{1-\alpha_1+\alpha_k}} ds \\
&\quad\quad + \mu_1 \int_0^{1} e^{\frac{\rho-1}{\rho}(1-s)} \frac{x(s)}{(1-s)^{1-\alpha_1+\alpha_k}} ds \Big] \\
&\quad + \frac{1}{A_1\rho^{\alpha_1}\Gamma(\alpha_1)} \Big[\eta_1 \int_0^{\xi_1} e^{\frac{\rho-1}{\rho}(\xi_1-s)} \frac{f(s,x(s),x(\lambda_1 s),y(s))}{(\xi_1-s)^{1-\alpha_1}} ds \\
&\quad\quad + \mu_1 \int_0^{1} e^{\frac{\rho-1}{\rho}(1-s)} \frac{f(s,x(s),x(\lambda_1 s),y(s))}{(1-s)^{1-\alpha_1}} ds \Big] \\
&= \Phi(\xi_1).
\end{aligned} \tag{17}
$$

Equalities (17) show that the function $x(t)$ of the mild solution satisfies the boundary condition (4). Similarly, it can be proved about the function $y(t)$.

For any $k = 2, 3, \ldots, n$, we consider $w(t) = ({}_0\mathcal{I}^{\alpha_1-\alpha_k,\rho}x)(t)$. According to Corollary 2, the equalities $({}_0^C\mathcal{D}^{\alpha_1,\rho}\Big(({}_0\mathcal{I}^{\alpha_1-\alpha_k,\rho}x)(t)\Big) = ({}_0^C\mathcal{D}^{\alpha_1,\rho}w)(t) = ({}_0^C\mathcal{D}^{\alpha_1,\rho}\Big({}_0^C\mathcal{D}^{\alpha_k,\rho}\Big({}_0\mathcal{I}^{\alpha_k,\rho}w)\Big)\Big)(t)$ hold. Applying Lemma 2, we obtain

$$({}_0^C\mathcal{D}^{\alpha_1,\rho}\Big(({}_0\mathcal{I}^{\alpha_1-\alpha_k,\rho}x)(t)\Big) = ({}_0^C\mathcal{D}^{\alpha_k,\rho}\Big({}_0^C\mathcal{D}^{\alpha_1,\rho}\Big({}_0\mathcal{I}^{\alpha_1,\rho}x)\Big)\Big)(t) = ({}_0^C\mathcal{D}^{\alpha_k,\rho}x)(t). \tag{18}$$

In view of Lemma 1 with $\alpha = \alpha_1 - \alpha_k, \beta = 1$ we have $\frac{1}{\rho^{\alpha_1-\alpha_k}\Gamma(1+\alpha_1-\alpha_k)}e^{\frac{\rho-1}{\rho}t}t^{\alpha_1-\alpha_k} = {}_0\mathcal{I}^{\alpha_1-\alpha_k,\rho}e^{\frac{\rho-1}{\rho}t}$ and according to Lemma 4 with $\alpha = \alpha_k, u = x$ we obtain $x(t) - x(0)e^{\frac{\rho-1}{\rho}t} = {}_0\mathcal{I}^{\alpha_k,\rho}\Big({}_0^C\mathcal{D}^{\alpha_k,\rho}x\Big)(t)$. Then, using (15) for $t = 0$, we obtain $x(0) = \frac{P(\xi_1)}{K_1}$ and the first equation in (15) could be written in the form

$$\begin{aligned} x(t) = &-\sum_{k=2}^{n}\frac{A_k}{A_1}\,{}_0\mathcal{I}^{\alpha_1-\alpha_k,\rho}\Big({}_0\mathcal{I}^{\alpha_k,\rho}\Big({}_0^C\mathcal{D}^{\alpha_k,\rho}x\Big)\Big)(t) \\ &+\frac{1}{A_1}{}_0\mathcal{I}^{\alpha_1,\rho}f(t,x(t),x(\lambda_1 t),y(t)),\ \ t\in[0,1]. \end{aligned} \tag{19}$$

We take the generalized proportional Caputo fractional derivative ${}_0^C\mathcal{D}^{\alpha_1,\rho}$ of both sides of (19), apply Lemma 3 and Corollary 2 and obtain

$$A_1\,({}_0^C\mathcal{D}^{\alpha_1,\rho}x)(t) = -\sum_{k=2}^{n}A_k\,({}_0^C\mathcal{D}^{\alpha_k,\rho}x)(t) + f(t,x(t),x(\lambda_1 t),y(t)). \tag{20}$$

Equality (20) proves the function $x(t)$ satisfies the first equation of (3). Similarly, we can prove the function $y(t)$ satisfies the second equation of (3). □

Theorem 2. *Let the condition A1 be satisfied and the couple $(x(t),y(t))$ be a solution of the boundary value problem for multi-term generalized proportional Caputo fractional differential Equations (3) and (4) and the functions $F \in I_0^{\alpha_1,\rho}[0,1]$, $G \in I_0^{\beta_1,\rho}[0,1]$ where $F(t) = f(t,x(t),x(\lambda_1 t),y(t)$, $G(t) = g(t,y(t),y(\lambda_2 t),x(t)$. Then, the couple $(x(t),y(t))$ is a mild solution of the same problem.*

The proof of Theorem 2 is similar to the one of Lemma 5, and we omit it.

Now, we will study the existence of the mild solutions of (3) and (4) for $\rho \in (0,1)$. There are several approaches for studying the existence of a solution. Here, we use fixed point theorems and some results from functional analysis. Basically, we will prove the existence of a mild solution by the application of an appropriate operator equation.

Theorem 3. *Let the following conditions be satisfied:*

1. $\alpha_i, \beta_k \in (0,1),\ i = 1,2\ldots,n,\ k = 1,2\ldots,N, \rho \in (0,1)$ *and condition A1 be satisfied.*
2. *There exist constants $L_i, M_i,\ i = 1,2,3,$ such that for $t \in [0,1], x_i, z_i, y_i \in \mathbb{R},\ i = 1,2,$ the inequalities*

$$|f(t,x_1,z_1,y_1) - f(t,x_2,z_2,y_2)| \le L_1|x_1 - x_2) + L_2|z_1 - z_2| + L_3|y_1 - y_2|,$$
$$|g(t,x_1,z_1,y_1) - g(t,x_2,z_2,y_2)| \le M_1|x_1 - x_2) + M_2|z_1 - z_2| + M_3|y_1 - y_2|$$

hold.

3. *The inequalities*

$$\begin{aligned}
\mathcal{P}_1 &= \mathcal{L}\Big[1+\frac{(|\eta_1|+|\mu_1|)}{|K_1|}\sum_{k=1}^{n}\frac{|A_k|}{|A_1|\rho^{\alpha_1-\alpha_k}\Gamma(1+\alpha_1-\alpha_k)}\Big] \\
&\quad\times\Big(\sum_{k=2}^{n}\frac{|A_k|(\Gamma(\alpha_1-\alpha_k)-\Gamma(\alpha_1-\alpha_k,\frac{1-\rho}{\rho}))}{|A_1|(1-\rho)^{\alpha_1-\alpha_k}\Gamma(\alpha_1-\alpha_k)}+\frac{\Gamma(\alpha_1)-\Gamma(\alpha_1,\frac{1-\rho}{\rho})}{|A_1|(1-\rho)^{\alpha_1}\Gamma(\alpha_1)}\Big)<1, \\
\mathcal{P}_2 &= \mathcal{M}\Big[1+\frac{(|\eta_2|+|\mu_2|)}{|K_2|}\sum_{k=1}^{N}\frac{|B_k|}{|B_1|\rho^{\beta_1-\beta_k}\Gamma(1+\beta_1-\beta_k)}\Big] \\
&\quad\times\Big(\sum_{k=2}^{N}\frac{|B_k|(\Gamma(\beta_1-\beta_k)-\Gamma(\beta_1-\beta_k,\frac{1-\rho}{\rho}))}{|B_1|(1-\rho)^{\beta_1-\beta_k}\Gamma(\beta_1-\beta_k)}+\frac{\Gamma(\beta_1)-\Gamma(\beta_1,\frac{1-\rho}{\rho})}{|B_1|(1-\rho)^{\beta_1}\Gamma(\beta_1)}\Big)<1,
\end{aligned} \tag{21}$$

hold where $\mathcal{L}=\max\{1,L_1+L_2,L_3\}$, $\mathcal{M}=\max\{1,M_1+M_2,M_3\}$.

Then, the boundary value problem for multi-term generalized proportional Caputo fractional differential Equations (3) and (4) has a unique mild solution.

Proof. Denote $||x||=\max_{t\in[0,1]}|x(t)|$ for any $x\in C([0,1],\mathbb{R})$ and define the set $\mathcal{W}=C([0,1],\mathbb{R})\times C([0,1],\mathbb{R})$. The set $\mathcal{W}$ is a Banach space with the norm $||u||_{\mathcal{W}}=\max\{||u_1||, ||u_2||\}:\ u=(u_1,u_2)\in\mathcal{W}$. Let $x,y\in C([0,1],\mathbb{R})$ and define the operator $\Omega=(\Omega_1,\Omega_2):\mathcal{W}\to\mathbb{R}^2$ by the equalities

$$\begin{aligned}
\Omega_1x(t) &= \frac{P(\xi_1,x,y)}{K_1}e^{\frac{\rho-1}{\rho}t}\sum_{k=1}^{n}\frac{A_k}{A_1\rho^{\alpha_1-\alpha_k}\Gamma(1+\alpha_1-\alpha_k)}t^{\alpha_1-\alpha_k} \\
&\quad-\sum_{k=2}^{n}\frac{A_k}{A_1\rho^{\alpha_1-\alpha_k}\Gamma(\alpha_1-\alpha_k)}\int_0^t e^{\frac{\rho-1}{\rho}(t-s)}\frac{x(s)}{(t-s)^{1-\alpha_1+\alpha_k}}ds \\
&\quad+\frac{1}{A_1\rho^{\alpha_1}\Gamma(\alpha_1)}\int_0^t e^{\frac{\rho-1}{\rho}(t-s)}\frac{f(s,x(s),x(\lambda_1 s),y(s))}{(t-s)^{1-\alpha_1}}ds, \\
\Omega_2y(t) &= \frac{Q(\xi_2,x,y)}{K_2}e^{\frac{\rho-1}{\rho}t}\sum_{k=1}^{N}\frac{B_k}{B_1\rho^{\beta_1-\beta_k}\Gamma(1+\beta_1-\beta_k)}t^{\beta_1-\beta_k} \\
&\quad-\sum_{k=2}^{N}\frac{B_k}{B_1\rho^{\beta_1-\beta_k}\Gamma(\beta_1-\beta_k)}\int_0^t e^{\frac{\rho-1}{\rho}(t-s)}\frac{y(s)}{(t-s)^{1-\beta_1+\beta_k}}ds \\
&\quad+\frac{1}{B_1\rho^{\beta_1}\Gamma(\beta_1)}\int_0^t e^{\frac{\rho-1}{\rho}(t-s)}\frac{g(s,y(s),y(\lambda_2 s),x(s))}{(t-s)^{1-\beta_1}}ds, \\
&\quad\text{for } t\in[0,1],\ x,y\in C[0,1],
\end{aligned} \tag{22}$$

where K_1, K_2, $P(\xi_1,x,y)$, and $Q(\xi_2,x,y)$ are defined by (14) and (16).

The fixed point of the operator Ω (if any) is a mild solution of (3) and (4).

Let $x_i,y_i\in C([0,1],\mathbb{R}),\ i=1,2$. Then, we have $\int_0^t\frac{e^{\frac{\rho-1}{\rho}(t-s)}}{(t-s)^{1-\alpha}}ds=\frac{\rho^\alpha}{(1-\rho)^\alpha}(\Gamma(\alpha)-\Gamma(\alpha,\frac{1-\rho}{\rho}t))$, where $\Gamma(a,z)=\int_z^\infty t^{a-1}e^{-t}dt$ is the incomplete gamma function, $\Gamma(\alpha,ct)$ is a decreasing function for $t\in[0,1],\alpha\in(0,1)$ and $c>0$. From the first equality of (16), we obtain

$$
\begin{aligned}
&|P_1(\xi_1,x_1,y_1)-P_1(\xi_1,x_2,y_2)| \\
&\le \sum_{k=2}^{n}\frac{|A_k|\,||x_1-x_2||}{|A_1|(1-\rho)^{\alpha_1-\alpha_k}\Gamma(\alpha_1-\alpha_k)}(|\eta_1|+|\mu_1|)\Big(\Gamma(\alpha_1-\alpha_k)-\Gamma(\alpha_1-\alpha_k,\frac{1-\rho}{\rho}\xi_1)\Big) \\
&\quad+\frac{(L_1+L_2)||x_1-x_2||+L_3||y_1-y_2||}{|A_1|(1-\rho)^{\alpha_1}\Gamma(\alpha_1)}(|\eta_1|+|\mu_1|)\Big(\Gamma(\alpha_1)-\Gamma(\alpha_1,\frac{1-\rho}{\rho}\xi_1)\Big) \\
&\le \mathcal{L}\max\{||x_1-x_2||,||y_1-y_2||\}(|\eta_1|+|\mu_1|) \\
&\quad\times\Big(\sum_{k=2}^{n}\frac{|A_k|}{|A_1|(1-\rho)^{\alpha_1-\alpha_k}\Gamma(\alpha_1-\alpha_k)}\Big(\Gamma(\alpha_1-\alpha_k)-\Gamma(\alpha_1-\alpha_k,\frac{1-\rho}{\rho}\xi_1)\Big) \\
&\quad+\frac{1}{|A_1|(1-\rho)^{\alpha_1}\Gamma(\alpha_1)}\Big(\Gamma(\alpha_1)-\Gamma(\alpha_1,\frac{1-\rho}{\rho}\xi_1)\Big)\Big),
\end{aligned}
\tag{23}
$$

and

$$
\begin{aligned}
&|\Omega_1x_1(t)-\Omega_1x_2(t)|\le|\Phi_1(\xi_1,x_1,y_1)-\Phi_1(\xi_1,x_2,y_2)|\frac{1}{|K_1|}e^{\frac{\rho-1}{\rho}t}\sum_{k=1}^{n}\frac{|A_k|}{|A_1|\rho^{\alpha_1-\alpha_k}\Gamma(1+\alpha_1-\alpha_k)}t^{\alpha_1-\alpha_k} \\
&+\sum_{k=2}^{n}\frac{|A_k|}{|A_1|\rho^{\alpha_1-\alpha_k}\Gamma(\alpha_1-\alpha_k)}\int_0^t e^{\frac{\rho-1}{\rho}(t-s)}\frac{|x_1(s)-x_2(s)|}{(t-s)^{1-\alpha_1+\alpha_k}}ds \\
&+\frac{1}{|A_1|\rho^{\alpha_1}\Gamma(\alpha_1)}\int_0^t\frac{e^{\frac{\rho-1}{\rho}(t-s)}}{(t-s)^{1-\alpha_1}}\Big(L_1|x_1(s)-x_2(s)| \\
&\quad+L_2|x_1(\lambda_1s)-x_2(\lambda_1s)|+L_3|y_1(s)-y_2(s)|\Big)ds \\
&\le\mathcal{L}\max\{||x_1-x_2||,||y_1-y_2||\}\frac{(|\eta_1|+|\mu_1|)}{|K_1|}\sum_{k=1}^{n}\frac{|A_k|}{|A_1|\rho^{\alpha_1-\alpha_k}\Gamma(1+\alpha_1-\alpha_k)} \\
&\quad\times\Big(\sum_{k=2}^{n}\frac{|A_k|(\Gamma(\alpha_1-\alpha_k)-\Gamma(\alpha_1-\alpha_k,\frac{1-\rho}{\rho}))}{|A_1|(1-\rho)^{\alpha_1-\alpha_k}\Gamma(\alpha_1-\alpha_k)}+\frac{\Gamma(\alpha_1)-\Gamma(\alpha_1,\frac{1-\rho}{\rho}\xi_1)}{|A_1|(1-\rho)^{\alpha_1}\Gamma(\alpha_1)}\Big) \\
&\quad+||x_1-x_2||\sum_{k=2}^{n}\frac{|A_k|}{|A_1|\Gamma(\alpha_1-\alpha_k)(1-\rho)^{\alpha_1-\alpha_k}}(\Gamma(\alpha_1-\alpha_k)-\Gamma(\alpha_1-\alpha_k,\frac{1-\rho}{\rho})) \\
&\quad+\frac{(L_1+L_2)||x_1-x_2||+L_3||y_1-y_2||}{|A_1|\Gamma(\alpha_1)(1-\rho)^{\alpha_1}}(\Gamma(\alpha_1)-\Gamma(\alpha_1,\frac{1-\rho}{\rho})) \\
&\le\mathcal{L}\max\{||x_1-x_2||,||y_1-y_2||\}\Big[1+\frac{(|\eta_1|+|\mu_1|)}{|K_1|}\sum_{k=1}^{n}\frac{|A_k|}{|A_1|\rho^{\alpha_1-\alpha_k}\Gamma(1+\alpha_1-\alpha_k)}\Big] \\
&\quad\times\Big(\sum_{k=2}^{n}\frac{|A_k|(\Gamma(\alpha_1-\alpha_k)-\Gamma(\alpha_1-\alpha_k,\frac{1-\rho}{\rho}))}{|A_1|(1-\rho)^{\alpha_1-\alpha_k}\Gamma(\alpha_1-\alpha_k)}+\frac{\Gamma(\alpha_1)-\Gamma(\alpha_1,\frac{1-\rho}{\rho})}{|A_1|(1-\rho)^{\alpha_1}\Gamma(\alpha_1)}\Big).
\end{aligned}
\tag{24}
$$

Similarly, we get

$$
\begin{aligned}
&|\Omega_2y_1(t)-\Omega_2y_2(t)|\le\mathcal{M}\max\{||x_1-x_2||,||y_1-y_2||\} \\
&\quad\times\Big[1+\frac{(|\eta_1|+|\mu_1|)}{|K_2|}\sum_{k=1}^{N}\frac{|B_k|}{|B_1|\rho^{\beta_1-\beta_k}\Gamma(1+\beta_1-\beta_k)}\Big] \\
&\quad\times\Big(\sum_{k=2}^{N}\frac{|B_k|(\Gamma(\beta_1-\beta_k)-\Gamma(\beta_1-\beta_k,\frac{1-\rho}{\rho}))}{|B_1|(1-\rho)^{\beta_1-\beta_k}\Gamma(\beta_1-\beta_k)}+\frac{\Gamma(\beta_1)-\Gamma(\beta_1,\frac{1-\rho}{\rho})}{|B_1|(1-\rho)^{\beta_1}\Gamma(\beta_1)}\Big).
\end{aligned}
\tag{25}
$$

From inequalities (24) and (25) and condition 3 it follows that

$$||Q||_{\mathcal{W}}\le\max\{\mathcal{P}_1,\mathcal{P}_2\}\max\{||x_1-x_2||,||y_1-y_2||\},$$

i.e., the operator Q is a contraction operator. According to the Banach contraction principle the operator Ω has a unique fixed point $(x^*, y^*) \in \mathcal{W}$, which is a mild solution of (3) and (4). □

3.3. Mild Solution of the Initial Value Problem for the Couple of Nonlinear Equations

In the partial case $\gamma_1 = \gamma_2 = 1, \eta_1 = \eta_2 = \mu_1 = \mu_2 = 0$ and $\Phi(s) \equiv a, \Phi_2(s) \equiv b, a, b$ are real constants, the boundary value problem for the coupled nonlinear delay differential equation with several generalized proportional Caputo fractional derivatives (3) and (4) is reduced to the coupled nonlinear delay differential equation with several generalized proportional Caputo fractional derivatives (3) with the initial value conditions $x(0) = a$, $y(0) = b$.

Definition 2. *The couple of functions $(x(t), y(t))$ is called a mild solution of the coupled multi-term generalized proportional Caputo fractional delay differential Equation (3) with the initial conditions $x(0) = a,\ y(0) = b$ if they satisfy the integral equations*

$$\begin{aligned}
x(t) = {} & a e^{\frac{\rho-1}{\rho}t} \sum_{k=1}^{n} \frac{A_k}{A_1 \rho^{\alpha_1-\alpha_k} \Gamma(1+\alpha_1-\alpha_k)} t^{\alpha_1-\alpha_k} \\
& - \sum_{k=2}^{n} \frac{A_k}{A_1 \rho^{\alpha_1-\alpha_k} \Gamma(\alpha_1-\alpha_k)} \int_0^t e^{\frac{\rho-1}{\rho}(t-s)} \frac{x(s)}{(t-s)^{1-\alpha_1+\alpha_k}} ds \\
& + \frac{1}{A_1 \rho^{\alpha_1} \Gamma(\alpha_1)} \int_0^t e^{\frac{\rho-1}{\rho}(t-s)} \frac{f(s, x(s), x(\lambda_1 s), y(s))}{(t-s)^{1-\alpha_1}} ds, \quad t \in [0,1], \\
y(t) = {} & b e^{\frac{\rho-1}{\rho}t} \sum_{k=1}^{N} \frac{B_k}{B_1 \rho^{\beta_1-\beta_k} \Gamma(1+\beta_1-\beta_k)} t^{\beta_1-\beta_k} \\
& - \sum_{k=2}^{N} \frac{B_k}{B_1 \rho^{\beta_1-\beta_k} \Gamma(\beta_1-\beta_k)} \int_0^t e^{\frac{\rho-1}{\rho}(t-s)} \frac{y(s)}{(t-s)^{1-\beta_1+\beta_k}} ds \\
& + \frac{1}{B_1 \rho^{\beta_1} \Gamma(\beta_1)} \int_0^t e^{\frac{\rho-1}{\rho}(t-s)} \frac{g(s, y(s), y(\lambda_2 s), x(s))}{(t-s)^{1-\beta_1}} ds, \quad t \in [0,1].
\end{aligned} \tag{26}$$

As a corollary of Theorem 1 the following result follows:

Theorem 4. *Let the couple $(x(t), y(t)),\ t \in [0,1]$, be a mild solution of the initial value problem for coupled multi-term generalized proportional Caputo fractional differential Equation (3) with the initial conditions $x(0) = a,\ y(0) = b$ such that $x \in C^{\alpha_k,\rho}[0,1]$ for $k = 1, 2, \ldots, n$ and $y \in C^{\beta_k,\rho}[0,1]$ for $k = 1, 2, \ldots, N$. Then, the couple $(x(t), y(t))$ is a solution of (3) with the initial conditions $x(0) = a,\ y(0) = b$.*

As a partial case of Theorem 2, we have

Theorem 5. *Let the couple $(x(t), y(t))$ be a solution of the initial value problem for coupled multi-term generalized proportional Caputo fractional differential Equation (3) with the initial conditions $x(0) = a,\ y(0) = b$ and the functions $F(t) = f(t, x(t), x(\lambda_1 t), y(t)),\ F \in I_0^{\alpha_1,\rho}[0,1]$ and $G(t) = g(t, y(t), y(\lambda_2 t), x(t)),\ G \in I_0^{\beta_1,\rho}[0,1]$. Then, the couple $(x(t), y(t))$ is a mild solution of the same problem.*

The existence result for the initial value problem for coupled multi-term generalized proportional Caputo fractional differential Equation (3) with the initial conditions $x(0) = a$, $y(0) = b$ is a partial case of Theorem 3.

Theorem 6. *Let the following conditions be satisfied:*

1. *Let the conditions A1 and and 2 of Theorem 3 be satisfied.*

2. *The inequalities*

$$\mathcal{L}\Big(\sum_{k=2}^{n}\frac{|A_k|(\Gamma(\alpha_1-\alpha_k)-\Gamma(\alpha_1-\alpha_k,\frac{1-\rho}{\rho}))}{|A_1|(1-\rho)^{\alpha_1-\alpha_k}\Gamma(\alpha_1-\alpha_k)}+\frac{\Gamma(\alpha_1)-\Gamma(\alpha_1,\frac{1-\rho}{\rho})}{|A_1|(1-\rho)^{\alpha_1}\Gamma(\alpha_1)}\Big)<1,$$
$$\mathcal{M}\Big(\sum_{k=2}^{N}\frac{|B_k|(\Gamma(\beta_1-\beta_k)-\Gamma(\beta_1-\beta_k,\frac{1-\rho}{\rho}))}{|B_1|(1-\rho)^{\beta_1-\beta_k}\Gamma(\beta_1-\beta_k)}+\frac{\Gamma(\beta_1)-\Gamma(\beta_1,\frac{1-\rho}{\rho})}{|B_1|(1-\rho)^{\beta_1}\Gamma(\beta_1)}\Big)<1,$$

hold where $\mathcal{L}=\max\{1,L_1+L_2,L_3\}$, $\mathcal{M}=\max\{1,M_1+M_2,M_3\}$.

Then, the initial value problem for coupled multi-term generalized proportional Caputo fractional differential Equation (3) has a unique mild solution.

Remark 4. *In the partial case* $A_k=0$, $k=2,3,\dots,n, f(t,x,z,y)\equiv f(t,x)$, *and* $b_k=0$, $k=1,2,\dots,N$, *the coupled multi-term generalized proportional Caputo fractional delay differential Equation (3) with the initial conditions* $x(0)=a$, $y(0)=b$ *is reduced to a scalar generalized proportional Caputo fractional differential equation with the initial condition* $x(0)=a$, *and the integral Equation (26) is reduced to the integral Equation (2) [19].*

4. Multi-Term Caputo Fractional Differential Equations

According to Remark 1, the Caputo fractional derivative is a partial case of a generalized proportional Caputo fractional derivative with $\rho=1$. Thus, from the previous sections, we obtain the results for Caputo fractional differential equations—linear and nonlinear coupled multi-terms.

Now, we introduce the following classes of functions:

$$C^{\alpha}[a,b]=\{u\in C^1([a,b],\mathbb{R}):\ ({}_a^CD^{\alpha}u)(t)\ \text{ exists on } (a,b]\},$$

and

$$I^{\alpha}[a,b]=\{u\in C([a,b],\mathbb{R}):\ ({}_aI^{\alpha}u)(t)\ \text{ exists on } (a,b]\},$$

where $({}_a^CD^{\alpha}u)(t)$ is the Caputo fractional derivative and $({}_aI^{\alpha}u)(t)$ is the Riemann–Liouville fractional integral of order $\alpha\in(0,1)$ with the lower limit a.

4.1. Explicit Solution of the Multi-Term Linear Problem with Caputo Fractional Derivatives

Consider the following linear multi-term Caputo fractional differential equation

$$\sum_{i=1}^{N}C_i\,{}_0^C\mathcal{D}^{p_i}z(t)=F(t),\ \text{for }\ t\in(0,1],\tag{27}$$

with the nonlocal boundary value conditions

$$\gamma z(0)+\beta z(\xi)+\mu z(1)=\Psi(\xi),\tag{28}$$

where $1>p_1>p_2>\dots>p_N>0$, $C_i, i=1,2,\dots,N:\ C_1\neq 0$ are constants, $F:[0,1]\to\mathbb{R})$, $\xi\in(0,1)$ is an arbitrary point, γ,β,μ are arbitrary numbers, $\Psi\in C((0,1),\mathbb{R})$ and

$$\gamma+\beta\sum_{k=1}^{N}\frac{C_k}{C_1\rho^{p_1-p_k}\Gamma(1+p_1-p_k)}\xi^{p_1-p_k}+\mu\sum_{k=1}^{N}\frac{C_k}{C_1\rho^{p_1-p_k}\Gamma(1+p_1-p_k)}\neq 0.\tag{29}$$

As a particular case of Lemma 5 with $\rho=1$, we obtain:

Lemma 6. *Let $F \in I^{p_1}[0,1]$, and inequality (29) holds. Then, the boundary value problem for the linear multi-term Caputo fractional differential Equations (27) and (28) has a unique solution given by*

$$\begin{aligned} z(t) = &\frac{P(\xi)}{K}\sum_{k=1}^{N}\frac{C_k}{C_1\Gamma(1+p_1-p_k)}t^{p_1-p_k} - \sum_{k=2}^{N}\frac{C_k}{C_1\Gamma(p_1-p_k)}\int_0^t \frac{z(s)}{(t-s)^{1-p_1+p_k}}ds \\ &+ \frac{1}{C_1\Gamma(p_1)}\int_0^t \frac{F(s)}{(t-s)^{1-p_1}}ds, \ \ t \in [0,1], \end{aligned} \tag{30}$$

where

$$\begin{aligned} K &= \gamma + \beta\sum_{k=1}^{N}\frac{C_k}{C_1\Gamma(1+p_1-p_k)}\xi^{p_1-p_k} + \mu\sum_{k=1}^{N}\frac{C_k}{C_1\Gamma(1+p_1-p_k)} \neq 0, \\ P(\xi) &= \Psi(\xi) + \sum_{k=2}^{N}\frac{C_k}{C_1\Gamma(p_1-p_k)}\Big(\beta\int_0^{\xi}\frac{z(s)}{(\xi-s)^{1-p_1+p_k}}ds \\ &\quad + \mu\int_0^1 \frac{z(s)}{(1-s)^{1-p_1+p_k}}ds\Big) \\ &\quad - \frac{1}{C_1\Gamma(p_1)}\Big(\beta\int_0^{\xi}\frac{F(s)}{(\xi-s)^{1-p_1}}ds + \mu\int_0^1\frac{F(s)}{(1-s)^{1-p_1}}ds\Big). \end{aligned} \tag{31}$$

Remark 5. *Note the boundary value problems for the multi-term linear Caputo fractional differential equation of the type (27) and (28) were studied in [20–22], but their integral representations seem to have inaccuracies (see Theorem 3.1 [20], Theorem 3 [21], and Theorem 3.1 [22]). For example, in the proof of Theorem 3 [21], the second line of Equation (5) is not correct because $I^{\alpha_1}D^{\alpha_k}\mathcal{V}_1(t) = I^{\alpha_1-\alpha_i+\alpha_i}D^{\alpha_k}\mathcal{V}_1(t) = I^{\alpha_1-\alpha_i}\Big(I^{\alpha_i}D^{\alpha_k}\mathcal{V}_1(t)\Big) = I^{\alpha_1-\alpha_i}\Big(\mathcal{V}(t) - C_k\Big) = I^{\alpha_1-\alpha_i}\mathcal{V}_1(t) - I^{\alpha_1-\alpha_i}C_i \neq I^{\alpha_1-\alpha_i}\mathcal{V}_1(t)$ for $i = 2,\dots,n$ where C_i are constants.*

Remark 6. *It the particular case $N = 1, \gamma = 1, \beta = \mu = 0, \Psi(s) \equiv a = const$, we obtain from Lemma 6 and Equation (30) the classical formula $z(t) = a + \frac{1}{C_1\Gamma(p_1)}\int_0^t \frac{F(s)}{(t-s)^{1-p_1}}ds$ for the solution of the initial value problem of the scalar linear Caputo fractional differential equation.*

4.2. Mild Solution of the Boundary Value Problem for the Couple of Nonlinear Equations

In the particular case $\rho = 1$, i.e., the case where Caputo fractional derivatives are applied, we obtain the following couple of multi-term Caputo fractional delay differential equations

$$\begin{aligned} &\sum_{i=1}^{n} A_i\, {}_0^C D^{\alpha_i}x(t) = f(t, x(t), x(\lambda_1 t), y(t)), \text{ for } t \in (0,1], \\ &\sum_{i=1}^{m} B_i\, {}_0^C D^{\beta_i}y(t) = g(t, y(t), y(\lambda_2 t), x(t)), \text{ for } t \in (0,1]. \end{aligned} \tag{32}$$

Now we will define mild solutions of the nonlocal boundary value problem for the couple of nonlinear multi-term Caputo fractional delay differential equations.

For this we need to introduce the following condition:

A2. The following hold

$$
\begin{aligned}
K_1 &= \gamma_1 + \eta_1 e^{\frac{\rho-1}{\rho}\xi_1} \sum_{k=1}^{n} \frac{A_k}{A_1\Gamma(1+\alpha_1-\alpha_k)} \xi_1^{\alpha_1-\alpha_k} \\
&\quad + \mu_1 e^{\frac{\rho-1}{\rho}} \sum_{k=1}^{n} \frac{A_k}{A_1\Gamma(1+\alpha_1-\alpha_k)} \neq 0, \\
K_2 &= \gamma_2 + \eta_2 e^{\frac{\rho-1}{\rho}\xi_2} \sum_{k=1}^{N} \frac{B_k}{B_1\Gamma(1+\beta_1-\beta_k)} \xi_2^{\beta_1-\beta_k} \\
&\quad + \mu_2 e^{\frac{\rho-1}{\rho}} \sum_{k=1}^{N} \frac{B_k}{B_1\Gamma(1+\beta_1-\beta_k)} \neq 0.
\end{aligned} \tag{33}
$$

Following the integral representation (30), we will define a mild solution of the boundary value problem for the nonlinear delay differential equation with several Caputo fractional derivatives (4) and (32).

Definition 3. *The couple of functions $(x(t), y(t))$: $x \in I^{\alpha_1-\alpha_k}[0,1], k = 2,3,\ldots,n$, $y \in I^{\beta_1-\beta_k}[0,1], k = 2,3,\ldots,m$, is called a mild solution of the boundary value problem for multi-term Caputo fractional differential Equations (4) and (32) if they satisfy the integral equations*

$$
\begin{aligned}
x(t) &= \frac{P(\xi_1, x, y)}{K_1} \sum_{k=1}^{n} \frac{A_k}{A_1\Gamma(1+\alpha_1-\alpha_k)} t^{\alpha_1-\alpha_k} \\
&\quad - \sum_{k=2}^{n} \frac{A_k}{A_1\Gamma(\alpha_1-\alpha_k)} \int_0^t \frac{x(s)}{(t-s)^{1-\alpha_1+\alpha_k}} ds \\
&\quad + \frac{1}{A_1\Gamma(\alpha_1)} \int_0^t \frac{f(s, x(s), x(\lambda_1 s), y(s))}{(t-s)^{1-\alpha_1}} ds, \quad t \in [0,1], \\
y(t) &= \frac{Q(\xi_2, x, y)}{K_2} \sum_{k=1}^{N} \frac{B_k}{B_1\Gamma(1+\beta_1-\beta_k)} t^{\beta_1-\beta_k} \\
&\quad - \sum_{k=2}^{N} \frac{B_k}{B_1\Gamma(\beta_1-\beta_k)} \int_0^t \frac{y(s)}{(t-s)^{1-\beta_1+\beta_k}} ds \\
&\quad + \frac{1}{B_1\Gamma(\beta_1)} \int_0^t \frac{g(s, y(s), y(\lambda_2 s), x(s))}{(t-s)^{1-\beta_1}} ds, \quad t \in [0,1],
\end{aligned} \tag{34}
$$

where K_1 and K_2 are defined by (33),

$$
\begin{aligned}
P(\xi_1, x, y) &= \Phi_1(\xi_1) + \sum_{k=2}^{n} \frac{A_k}{A_1\Gamma(\alpha_1-\alpha_k)} \Big(\eta_1 \int_0^{\xi_1} \frac{x(s)}{(\xi_1-s)^{1-\alpha_1+\alpha_k}} ds \\
&\quad + \mu_1 \int_0^1 \frac{x(s)}{(1-s)^{1-\alpha_1+\alpha_k}} ds \Big) \\
&\quad - \frac{1}{A_1\Gamma(\alpha_1)} \Big(\eta_1 \int_0^{\xi_1} \frac{f(s, x(s), x(\lambda_1 s), y(s))}{(\xi_1-s)^{1-\alpha_1}} ds + \mu_1 \int_0^1 \frac{f(s, x(s), x(\lambda_1 s), y(s))}{(1-s)^{1-\alpha_1}} ds \Big), \\
Q(\xi_2, x, y) &= \Phi_2(\xi_2) + \sum_{k=2}^{N} \frac{B_k}{B_1\Gamma(\beta_1-\beta_k)} \Big(\eta_2 \int_0^{\xi_2} \frac{y(s)}{(\xi_2-s)^{1-\beta_1+\beta_k}} ds \\
&\quad + \mu_2 \int_0^1 \frac{y(s)}{(1-s)^{1-\beta_1+\beta_k}} ds \Big) \\
&\quad - \frac{1}{B_1\rho^{\beta_1}\Gamma(\beta_1)} \Big(\eta_2 \int_0^{\xi_2} \frac{g(s, y(s), y(\lambda_2 s), x(s))}{(\xi_2-s)^{1-\beta_1}} ds + \mu_2 \int_0^1 \frac{g(s.y(s), y(\lambda_2 s), x(s))}{(1-s)^{1-\beta_1}} ds \Big).
\end{aligned}
$$

Remark 7. *The equivalence between the integral presentation (26) giving us the mild solution and the solution of (4) and (32) does not follow immediately from Lemma 6, as it is done in some papers (see [20,21]) since the linear case (27) is a partial case of the nonlinear (32) not conversely.*

As a particular case of Theorem 1 with $\rho = 1$, we obtain:

Theorem 7. *Let the condition A2 be satisfied and the couple* $(x(t), y(t)) : \quad x \in I^{\alpha_1 - \alpha_k}[0,1]$, $k = 2, 3, \ldots, n$, $y \in I^{\beta_1 - \beta_k}[0,1], k = 2, 3, \ldots, N$, *be a mild solution of the boundary value problem for multi-term Caputo fractional differential Equations* (4) *and* (32) *such that* $x \in C^{\alpha_k}([0,1], \mathbb{R})$ *for* $k = 1, 2, \ldots, n$ *and* $y \in C^{\beta_k}([0,1], \mathbb{R})$ *for* $k = 1, 2, \ldots, N$. *Then, the couple* $(x(t), y(t))$ *is a solution of the same problem.*

As a particular case of Theorem 2 with $\rho = 1$, we obtain the following result:

Theorem 8. *Let the condition A2 be satisfied and the couple* $(x(t), y(t))$ *be a solution of the boundary value problem for multi-term Caputo fractional differential Equations* (4) *and* (32) *and the functions* $F(t) = f(t, x(t), x(\lambda_1 t), y(t))$, $F \in I_0^{\alpha_1}[0,1]$ *and* $G(t) = g(t, y(t), y(\lambda_2 t), x(t))$, $G \in I_0^{\beta_1}[0,1]$. *Then, the couple* $(x(t), y(t))$ *is a mild solution of the same problem.*

The existence result for (4) and (32) is similar to Theorem 3.

Theorem 9. *Let the following conditions be satisfied:*

1. *Condition A2 is satisfied.*
2. *There exist constants* L_i, M_i, $i = 1, 2, 3$, *such that for* $t \in [0,1], x_i, z_i, y_i \in \mathbb{R}$, $i = 1, 2$, *the inequalities*

$$\begin{aligned} |f(t, x_1, z_1, y_1) - f(t, x_2, z_2, y_2)| &\leq L_1|x_1 - x_2| + L_2|z_1 - z_2| + L_3|y_1 - y_2|, \\ |g(t, x_1, z_1, y_1) - g(t, x_2, z_2, y_2)| &\leq M_1|x_1 - x_2| + M_2|z_1 - z_2| + M_3|y_1 - y_2| \end{aligned} \tag{35}$$

   hold.
3. *The inequalities*

$$\begin{aligned} &\mathcal{L}\Big[1 + \frac{(|\eta_1| + |\mu_1|)}{|K_1|} \sum_{k=1}^{n} \frac{|A_k|}{|A_1|\Gamma(1 + \alpha_1 - \alpha_k)}\Big] \\ &\quad \times \Big(\sum_{k=2}^{n} \frac{|A_k|}{|A_1|\Gamma(1 + \alpha_1 - \alpha_k)} + \frac{1}{|A_1|\Gamma(1 + \alpha_1)} \Big) < 1, \\ &\mathcal{M}\Big[1 + \frac{(|\eta_2| + |\mu_2|)}{|K_2|} \sum_{k=1}^{N} \frac{|B_k|}{|B_1|\Gamma(1 + \beta_1 - \beta_k)}\Big] \\ &\quad \times \Big(\sum_{k=2}^{N} \frac{|B_k|}{|B_1|\Gamma(1 + \beta_1 - \beta_k)} + \frac{1}{|B_1|\Gamma(1 + \beta_1)} \Big) < 1 \end{aligned} \tag{36}$$

   hold where $\mathcal{L} = \max\{1, L_1 + L_2, L_3\}$, $\mathcal{M} = \max\{1, M_1 + M_2, M_3\}$.

   Then, the boundary value problem for the coupled multi-term Caputo fractional differential Equations (4) *and* (32) *has a unique mild solution.*

The proof is similar to the one of Theorem 3, where we apply $\int_0^t \frac{ds}{(t-s)^{1-\alpha}} = \frac{t^\alpha}{\alpha}$, $\alpha \in (0,1)$, $\alpha\Gamma(\alpha) = \Gamma(1 + \alpha)$ and the inequality (24) is replaced by

$$
\begin{aligned}
|\Omega_1 x_1(t) - \Omega_1 x_2(t)| &\le \frac{|P(\xi_1, x_1, y_1) - P(\xi_1, x_2, y_2)|}{|K_1|} \sum_{k=1}^{n} \frac{|A_k|}{|A_1|\Gamma(1+\alpha_1-\alpha_k)} t^{\alpha_1-\alpha_k} \\
&+ \sum_{k=2}^{n} \frac{|A_k|}{|A_1|\Gamma(\alpha_1-\alpha_k)} \int_0^t \frac{|x_1(s)-x_2(s)|}{(t-s)^{1-\alpha_1+\alpha_k}} ds \\
&+ \frac{1}{|A_1|\Gamma(\alpha_1)} \int_0^t \frac{1}{(t-s)^{1-\alpha_1}} \Big(L_1|x_1(s)-x_2(s)| \\
&\quad + L_2|x_1(\lambda_1 s)-x_2(\lambda_1 s)| + L_3|y_1(s)-y_2(s)| \Big) ds \\
&\le \mathcal{L} \max\{||x_1-x_2||, ||y_1-y_2||\} \Big[1 + \frac{(|\eta_1|+|\mu_1|)}{|K_1|} \sum_{k=1}^{n} \frac{|A_k|}{|A_1|\Gamma(1+\alpha_1-\alpha_k)} \Big] \\
&\quad \times \Big(\sum_{k=2}^{n} \frac{|A_k|}{|A_1|\Gamma(1+\alpha_1-\alpha_k)} + \frac{1}{|A_1|\Gamma(1+\alpha_1)} \Big).
\end{aligned}
$$

Remark 8. *The unlocal boundary value problem for the coupled system of Caputo fractional differential Equation* (32) *is studied in* [21]*, but the study is based on a integral presentation with inaccuracies (see Remark 5).*

5. Conclusions

In this paper, an explicit solution of the linear fractional differential equation with several generalized proportional Caputo fractional derivatives and nonlocal boundary value condition is obtained. This explicit solution could be applied in the study of various qualitative properties, and in algorithms for construction approximate solutions such as monotone-iterative technique. The mild solution of the couple of nonlinear fractional differential equation with several generalized proportional Caputo fractional derivatives and delays is defined. The new formulas for the mild solutions of the boundary value problem for the nonlinear couple multi-term generalized proportional Caputo fractional differential equations could be used to study several qualitative properties such as Ulam-type stability of the given problem. Additionally, as a partial case, they could provide sufficient conditions for stability properties of fractional multi-term equations with Caputo fractional derivatives.

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