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Solvability of Nonlinear Impulsive Generalized Fractional Differential Equations with (p,q)-Laplacian Operator via Critical Point Theory

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Abstract: In this paper, we consider the nonlinear impulsive generalized fractional differential equations with (p,q)-Laplacian operator for $1 < p \leq q < \infty$, in which the nonlinearity f contains two fractional derivatives with respect to another function. Since the complexity of the nonlinear term and the impulses exist in generalized fractional calculus, it is difficult to find the corresponding variational functional of the problem. The existence of nontrivial solutions for the problem is established by the mountain pass theorem and iterative technique under some appropriate assumptions. Furthermore, our main result is demonstrated by an illustrative example to show its feasibility and effectiveness. Due to the employment of a generalized fractional operator, our results extend some existing research findings.

Keywords: (p,q)-Laplacian operator; generalized fractional differential operator; mountain pass theorem; impulse

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1. Introduction

Fractional calculus generalizes the definition of integer derivative and integral to arbitrary orders. In recent years, due to the memory and hereditary properties in fractional systems, fractional boundary value and initial value problems have been widely studied, for example, one can see the literature [1–9]. However, unlike classical calculus, the fractional derivative and integral can be defined in various non-equivalent ways, such as the Riemann–Liouville type calculus, the Caputo type calculus, the Hadamard type calculus, the Erdélyi–Kober type calculus and so on. For many practical models in the engineering field, in order to overcome the problem of selecting the best fractional calculus operators, an effective method is to consider the more general definitions of fractional calculus. Based on the above analysis, we will consider the fractional derivatives with respect to another function in this paper. In particular, by choosing suitable φ, we will obtain some well-known fractional differential operators.

Recently, more and more innovative results for fractional calculus regarding another function have been obtained in the literature [5,10–15] and the references therein. Moreover, as known to all, impulsive fractional differential equations are fundamental models for studying dynamic processes with sudden changes, and numerous researchers have obtained many interesting conclusions by different methods, please refer to articles [16–18] for more information. To the best of our knowledge, however, no researcher has used the variational method to study the generalized fractional differential equations with impulsive terms. This is one of the issues to be solved in this paper.

In [19], Long and Chen studied the many solutions for a class of p-Laplacian type fractional Dirichlet problem with instantaneous and non-instantaneous impulses by using variational methods and critical point theory. In [20], Li et al. investigated the existence of solutions for an impulsive fractional coupled system of (p,q)-Laplacian type without the Ambrosetti–Rabinowitz condition. Li et al. in [21] dealt with the nonlinear impulsive fractional differential equations with (p,q)-Laplacian operator and obtained the existence of nontrivial solutions.

In this paper, motivated by the above-mentioned works, we will research the following nonlinear impulsive generalized fractional differential equations with (p,q)-Laplacian operator

$$\begin{cases} {}_tD_T^{\alpha;\varphi}\Phi_p({}_0D_t^{\alpha;\varphi}u(t)) + |u(t)|^{p-2}u(t) + {}_tD_T^{\beta;\varphi}\Phi_q({}_0D_t^{\beta;\varphi}u(t)) + |u(t)|^{q-2}u(t) \\ \qquad = f\big(t, u(t), {}_0D_t^{\alpha;\varphi}u(t), {}_0D_t^{\beta;\varphi}u(t)\big), \quad t \neq t_j, \text{ a.e. } t \in [0,T], \\ \Delta\big({}_tD_T^{\alpha-1;\varphi}\Phi_p({}_0^CD_t^{\alpha;\varphi}u) + {}_tD_T^{\beta-1;\varphi}\Phi_q({}_0^CD_t^{\beta;\varphi}u)\big)(t_j) = I_j(u(t_j)), \quad j = 1,2,\ldots,m, \\ u(0) = u(T) = 0, \quad \text{a.e. } t \in [0,T], \end{cases} \tag{1}$$

where $\frac{1}{p} < \alpha \leq 1$, $\frac{1}{q} < \beta \leq 1$, $1 < p \leq q < \infty$, $\Phi_k(s) = |s|^{k-2}s(s \neq 0)$ with $\Phi_k(0) = 0$, ${}_0D_t^{\alpha;\varphi}$, ${}_tD_T^{\alpha;\varphi}$ and ${}_0^CD_t^{\alpha;\varphi}$ denote the left and right generalized Riemann–Liouville fractional derivatives and the left φ-Caputo fractional derivative, respectively, $f : [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function with respect to t for all $(x,y,z) \in \mathbb{R}^3$ and continuously differentiable with respect to x,y,z for a.e. $t \in [0,T]$, i.e., $f(\cdot,x,y,z) \in C([0,T],\mathbb{R})$ and $f(t,\cdot,\cdot,\cdot) \in C^1(\mathbb{R}^3,\mathbb{R})$, $I_j \in C(\mathbb{R},\mathbb{R})$ for $j = 1,2,\ldots,m$, $0 = t_0 < t_1 < \cdots < t_{m+1} = T$ and the operator Δ is defined by

$$\begin{aligned} &\Delta\big({}_tD_T^{\alpha-1;\varphi}\Phi_p({}_0^CD_t^{\alpha;\varphi}u) + {}_tD_T^{\beta-1;\varphi}\Phi_q({}_0^CD_t^{\beta;\varphi}u)\big)(t_j) \\ =&{}_tD_T^{\alpha-1;\varphi}\Phi_p({}_0^CD_t^{\alpha;\varphi}u)(t_j^+) - {}_tD_T^{\alpha-1;\varphi}\Phi_p({}_0^CD_t^{\alpha;\varphi}u)(t_j^-) \\ &+ {}_tD_T^{\beta-1;\varphi}\Phi_q({}_0^CD_t^{\beta;\varphi}u)(t_j^+) - {}_tD_T^{\beta-1;\varphi}\Phi_q({}_0^CD_t^{\beta;\varphi}u)(t_j^-), \end{aligned}$$

where

$$\begin{aligned} {}_tD_T^{\alpha-1;\varphi}\Phi_p({}_0^CD_t^{\alpha;\varphi}u)(t_j^+) &= \lim_{t \to t_j^+} {}_tD_T^{\alpha-1;\varphi}\Phi_p({}_0^CD_t^{\alpha;\varphi}u)(t), \\ {}_tD_T^{\alpha-1;\varphi}\Phi_p({}_0^CD_t^{\alpha;\varphi}u)(t_j^-) &= \lim_{t \to t_j^-} {}_tD_T^{\alpha-1;\varphi}\Phi_p({}_0^CD_t^{\alpha;\varphi}u)(t), \\ {}_tD_T^{\beta-1;\varphi}\Phi_q({}_0^CD_t^{\beta;\varphi}u)(t_j^+) &= \lim_{t \to t_j^+} {}_tD_T^{\beta-1;\varphi}\Phi_q({}_0^CD_t^{\beta;\varphi}u)(t), \\ {}_tD_T^{\beta-1;\varphi}\Phi_q({}_0^CD_t^{\beta;\varphi}u)(t_j^-) &= \lim_{t \to t_j^-} {}_tD_T^{\beta-1;\varphi}\Phi_q({}_0^CD_t^{\beta;\varphi}u)(t). \end{aligned}$$

Generally speaking, due to the presence of the generalized fractional operators, the results in this paper extend some existing results. If $\varphi(t) = t$, the existence of nontrivial solutions of problem (1) is researched in [21]. In addition, as far as we know, there are few researchers who deal with fractional differential equations with respect to another function by applying the variational method. This provides a reference to study the existence of solutions of the generalized fractional system with respect to another function. Therefore, the conclusions in this paper are of great significance.

The remaining parts of the article are organized as follows. In Section 2, we give some relevant definitions and lemmas. We present the variational structure and verify that the corresponding energy functional satisfies the mountain pass geometry and then obtain the nontrivial solutions of problem (1) by iterative technique in Section 3. In Section 4, an example is given to illustrate the effectiveness of our results.

2. Preliminaries and Statements

In this section, we outline some basic notations and related facts regarding the nonlinear impulsive generalized fractional differential equations which will be used later.

Let $\|u\|_{L^p} = (\int_0^T |u|^p dt)^{\frac{1}{p}}$ be the usual norm of space $L^p([0,T])$, $C([0,T],\mathbb{R})$ is the space of continuous functions with norm $\|u\| = \max_{0\leq t\leq T}|u(t)|$. Furthermore, define the equivalent norm of continuous function space with $\|u\|_\infty = \max_{0\leq t\leq T}|\varphi'(t)u(t)|$ and the equivalent norm of p-th Lebesgue measurable function space with $\|u\|_{p;\varphi} = (\int_0^T \varphi'(t)|u(t)|^p dt)^{\frac{1}{p}}$.

Definition 1. *([22,23]). Let $[a,b]$ be a finite or infinite interval of the real line $\mathbb{R}$ and $\varphi(t)$ be an increasing and positive monotone function on $[a,b]$ with the continuous derivative $\varphi'(t)$ on $[a,b]$. The left and right fractional integrals of function u with respect to another function φ on $[a,b]$ of order α are defined by*

$$ {}_aI_t^{\alpha;\varphi}u(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (\varphi(t)-\varphi(\tau))^{\alpha-1}\varphi'(\tau)u(\tau)d\tau, \quad \mathcal{R}(\alpha)>0, $$

$$ {}_tI_b^{\alpha;\varphi}u(t) = \frac{1}{\Gamma(\alpha)}\int_t^b (\varphi(\tau)-\varphi(t))^{\alpha-1}\varphi'(\tau)u(\tau)d\tau, \quad \mathcal{R}(\alpha)>0. $$

Remark 1. *Notice that when $\varphi(t)=t$, Definition 1 reduces to the definition of classical Riemann–Liouville fractional integrals. When $\varphi(t)=\ln t$, Definition 1 reduces to the definition of Hadamard fractional integrals as shown in the literature [24].*

Definition 2. *([22,23]). Let $n=[\mathcal{R}(\alpha)]+1$ with $\mathcal{R}(\alpha)\geq 0$ and $\varphi\in C^n([0,T])$ such that $\varphi^i\neq 0, i=1,2,\dots,n$, the left and right Riemann–Liouville fractional derivatives of function u with order α with respect to φ are given by*

$$ \begin{aligned} {}_0D_t^{\alpha;\varphi}u(t) =& (\frac{1}{\varphi'(t)}\frac{d}{dt})^n({}_0I_t^{n-\alpha;\varphi}u)(t) \\ =& \frac{1}{\Gamma(n-\alpha)}(\frac{1}{\varphi'(t)}\frac{d}{dt})^n\int_0^t(\varphi(t)-\varphi(\tau))^{n-\alpha-1}\varphi'(\tau)u(\tau)d\tau, \\ {}_tD_T^{\alpha;\varphi}u(t) =& (\frac{1}{\varphi'(t)}\frac{d}{dt})^n({}_tI_T^{n-\alpha;\varphi}u)(t) \\ =& \frac{(-1)^n}{\Gamma(n-\alpha)}(\frac{1}{\varphi'(t)}\frac{d}{dt})^n\int_t^T(\varphi(\tau)-\varphi(t))^{n-\alpha-1}\varphi'(\tau)u(\tau)d\tau. \end{aligned} $$

Remark 2. *Apparently, if $\varphi(t)=t$, Definition 2 becomes the definition of classical Riemann–Liouville fractional derivatives. If $\varphi(t)=\ln t$, Definition 2 simplifies to the definition of Hadamard fractional derivatives, more details can be found in the literature [24] and the references therein.*

Definition 3. *([25]). Let $n=[\mathcal{R}(\alpha)]+1$, $\varphi\in C^n([0,T])$ such that $\varphi^i\neq 0$, $i=1,2,\dots,n$ and $u\in AC^n_\varphi([0,T],\mathbb{R})$. Then, the left and right φ-Caputo fractional derivatives are depicted by*

$$ \begin{aligned} {}_0^CD_t^{\alpha;\varphi}u(t) =& {}_0I_t^{n-\alpha;\varphi}(\frac{1}{\varphi'(t)}\frac{d}{dt})^n u(t) \\ =& \frac{1}{\Gamma(n-\alpha)}\int_0^t(\varphi(t)-\varphi(\tau))^{n-\alpha-1}\varphi'(\tau)(\frac{1}{\varphi'(\tau)}\frac{d}{d\tau})^n u(\tau)d\tau, \\ {}_t^CD_T^{\alpha;\varphi}u(t) =& {}_tI_T^{n-\alpha;\varphi}(\frac{1}{\varphi'(t)}\frac{d}{dt})^n u(t) \\ =& \frac{(-1)^n}{\Gamma(n-\alpha)}\int_t^T(\varphi(\tau)-\varphi(t))^{n-\alpha-1}\varphi'(\tau)(\frac{1}{\varphi'(\tau)}\frac{d}{d\tau})^n u(\tau)d\tau. \end{aligned} $$

According to definitions of generalized fractional calculus and integration by parts formula, we obtain the following lemma.

Lemma 1. *Let $\alpha > 0, p \geq 0, q \geq 0, \frac{1}{p} + \frac{1}{q} \leq 1 + \alpha$ and $\varphi \in C^1[0,T]$.*
(1) If $f \in L^p([0,T], \mathbb{R})$ and $g \in L^q([0,T], \mathbb{R})$, then

$$\int_0^T \varphi'(t) f(t) {}_0I_t^{\alpha;\varphi} g(t) dt = \int_0^T \varphi'(t) g(t) {}_tI_T^{\alpha;\varphi} f(t) dt.$$

(2) If $f \in {}_0I_t^{\alpha;\varphi}(L^p[0,T], \mathbb{R})$ and $g \in {}_tI_T^{\alpha;\varphi}(L^q[0,T], \mathbb{R})$, then

$$\int_0^T \varphi'(t) f(t) {}_0D_t^{\alpha;\varphi} g(t) dt = \int_0^T \varphi'(t) g(t) {}_tD_T^{\alpha;\varphi} f(t) dt.$$

By Lemma 1, we give the following remark about the impulse item.

Remark 3. *For each $u \in E_0^{\alpha,p;\varphi}$, we have*

$$\begin{aligned} & \int_0^T \varphi'(t) \Phi_p({}_0D_t^{\alpha;\varphi} u(t)) {}_0D_t^{\alpha;\varphi} v(t) dt \\ = & \int_0^T \varphi'(t) {}_tD_T^{\alpha-1;\varphi} (\Phi_p({}_0D_t^{\alpha;\varphi} u(t))) v'_\varphi(t) dt \\ = & \sum_{j=0}^{m} {}_tD_T^{\alpha-1;\varphi} (\Phi_p({}_0D_t^{\alpha;\varphi} u(t))) v(t) \Big|_{t_j^+}^{t_{j+1}^-} - \sum_{j=0}^{m} \int_{t_j}^{t_{j+1}} v d({}_tD_T^{\alpha-1;\varphi} \Phi_p({}_0D_t^{\alpha;\varphi} u(t))) \\ = & -\sum_{j=0}^{m} \Delta\big({}_tD_T^{\alpha-1;\varphi} \Phi_p({}_0D_t^{\alpha;\varphi} u(t_j))\big) v(t_j) + \sum_{j=0}^{m} \int_{t_j}^{t_{j+1}} v\varphi'(t) {}_tD_T^{\alpha;\varphi} (\Phi_p({}_0D_t^{\alpha;\varphi} u(t))) dt \\ = & -\sum_{j=0}^{m} \Delta\big({}_tD_T^{\alpha-1;\varphi} \Phi_p({}_0D_t^{\alpha;\varphi} u(t_j))\big) v(t_j) + \int_0^T v\varphi'(t) {}_tD_T^{\alpha;\varphi} (\Phi_p({}_0D_t^{\alpha;\varphi} u(t))) dt. \end{aligned}$$

In order to investigate the existence of nontrivial solutions for problem (1), we present the workspace below.

Definition 4. *Let $1 < p < \infty$, $\frac{1}{p} < \alpha \leq 1$. Define φ-Caputo fractional derivative space $E_0^{\alpha,p;\varphi}$ by the closure of $C_0^\infty([0,T], \mathbb{R})$ endowed with the norm*

$$\|u\|_{\alpha,p;\varphi} = \Big(\int_0^T \varphi'(t) |u(t)|^p dt + \int_0^T \varphi'(t) |{}_0^C D_t^{\alpha;\varphi} u(t)|^p dt \Big)^{\frac{1}{p}}.$$

Remark 4. *According to [26], we can see that $E_0^{\alpha,p;\varphi}$ is a reflexive and separable Banach space.*

Lemma 2. *([10]) If $u(0) = u(T) = 0$, then the following relationships hold,*

$$\begin{aligned} {}_0D_t^{\alpha;\varphi} u(t) &= {}_0^C D_t^{\alpha;\varphi} u(t), \\ {}_0D_t^{\alpha;\varphi} {}_0I_t^{\alpha;\varphi} u(t) &= u(t), \\ {}_0I_t^{\alpha;\varphi} {}_0D_t^{\alpha;\varphi} u(t) &= u(t). \end{aligned}$$

The fractional space $E_0^{\alpha,p;\varphi}$ has the following important properties.

Lemma 3. *([27]) Let $1 \leq p < \infty$ and $0 < \alpha \leq 1$, then*

$$\|{}_0I_\xi^{\alpha;\varphi} u\|_{L^p([0,T])} \leq \frac{M_{\varphi'} [\varphi(t)]^\alpha}{\Gamma(\alpha+1)} \|u\|_{L^p([0,T])},$$

for $u \in L^p([0,T], \mathbb{R})$, $t \in [0,T]$, $\xi \in [0,T]$ and $M_{\varphi'} = \max\limits_{0 \leq t \leq T} |\varphi'(t)|$.

Lemma 4. *Let $1 \leq p < \infty$ and $0 < \alpha \leq 1$. If $\alpha > \frac{1}{p}$, then*

$$\|\varphi'(t)^{\frac{1}{p}} u(t)\|_{L^p} \leq \frac{M_{\varphi'}[\varphi(T)]^{\alpha}}{\Gamma(\alpha+1)} \|\varphi'(t)^{\frac{1}{p}} {}_0D_t^{\alpha;\varphi} u(t)\|_{L^p}, \ u \in E_0^{\alpha,p;\varphi}. \tag{2}$$

Moreover, if $\frac{1}{p} + \frac{1}{p'} = 1$ and $\alpha > \frac{1}{p}$, then

$$\|\varphi'(t)u(t)\| \leq \frac{M_{\varphi'}[\varphi(T)]^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)((\alpha-1)p'+1)^{\frac{1}{p'}}} \|\varphi'(t)^{\frac{1}{p}} {}_0D_t^{\alpha;\varphi} u(t)\|_{L^p}, \ u \in E_0^{\alpha,p;\varphi}. \tag{3}$$

Proof. In terms of Lemmas 2 and 3, (2) is obvious.

Now, we prove inequality (3). By using the Hölder inequality and Lemma 2, for all $t \in [0,T]$, we obtain

$$\begin{aligned}
|\varphi'(t)u(t)| =& |\varphi'(t) {}_0I_t^{\alpha;\varphi} {}_0D_t^{\alpha;\varphi} u(t)| \\
=& |\frac{\varphi'(t)}{\Gamma(\alpha)} \int_0^t (\varphi(t)-\varphi(\tau))^{\alpha-1} \varphi'(\tau) {}_0D_\tau^{\alpha;\varphi} u(\tau) d\tau| \\
\leq& |\frac{\varphi'(t)}{\Gamma(\alpha)} (\int_0^t [(\varphi(t)-\varphi(\tau))^{\alpha-1} (\varphi'(\tau))^{\frac{1}{p'}}]^{p'} d\tau)^{\frac{1}{p'}} | (\int_0^t |(\varphi'(\tau))^{\frac{1}{p}}] {}_0D_\tau^{\alpha;\varphi} u(\tau)|^p d\tau)^{\frac{1}{p}} \\
=& |\frac{\varphi'(t)}{\Gamma(\alpha)} (\int_0^t [(\varphi(t)-\varphi(\tau))^{(\alpha-1)p'} (\varphi'(\tau)) d\tau)^{\frac{1}{p'}} | \|\varphi'(t)^{\frac{1}{p}} {}_0D_t^{\alpha;\varphi} u(t)\|_{L^p} \\
\leq& \frac{M_{\varphi'}[\varphi(T)]^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)((\alpha-1)p'+1)^{\frac{1}{p'}}} \|\varphi'(t)^{\frac{1}{p}} {}_0D_t^{\alpha;\varphi} u(t)\|_{L^p}.
\end{aligned}$$

This completes the proof. □

In this paper, define the fractional derivative space $E = E_0^{\alpha,p;\varphi} \bigcap E_0^{\beta,q;\varphi}$ endowed with the norm

$$\begin{aligned}
\|u\|_E =& \|u\|_{\alpha,p;\varphi} + \|u\|_{\beta,q;\varphi} \\
=& \left(\int_0^T \varphi'(t)|u(t)|^p dt + \int_0^T \varphi'(t) |{}_0^C D^{\alpha;\varphi} u(t)|^p dt \right)^{\frac{1}{p}} \\
&+ \left(\int_0^T \varphi'(t)|u(t)|^q dt + \int_0^T \varphi'(t) |{}_0^C D^{\beta;\varphi} u(t)|^q dt \right)^{\frac{1}{q}},
\end{aligned}$$

for any $u \in E$. According to articles [26,28], E is a reflexive Banach space.

By Lemma 4, $\|u\|_E$ is equivalent to

$$\|u\|_E = (\int_0^T \varphi'(t) |{}_0D^{\alpha;\varphi} u(t)|^p dt)^{\frac{1}{p}} + (\int_0^T \varphi'(t) |{}_0D^{\beta;\varphi} u(t)|^q dt)^{\frac{1}{q}}.$$

Lemma 5. *([27–29]) Let $\frac{1}{p} < \alpha \leq 1$ and $\frac{1}{q} < \beta \leq 1$. Then, the space E is compactly embedded in $C([0,T], \mathbb{R})$.*

Definition 5. *A function $u \in E$ is called a weak solution of problem (1), if the following equation holds*

$$\begin{aligned}
&\int_0^T \varphi'(t)|{}_0D_t^{\alpha;\varphi}u(t)|^{p-2}{}_0D_t^{\alpha;\varphi}u(t){}_0D_t^{\alpha;\varphi}v(t) + \varphi'(t)|u(t)|^{p-2}u(t)v(t) \\
&+\varphi'(t)|{}_0D_t^{\beta;\varphi}u(t)|^{q-2}{}_0D_t^{\beta;\varphi}u(t){}_0D_t^{\beta;\varphi}v(t) + \varphi'(t)|u(t)|^{q-2}u(t)v(t)dt \\
&+\sum_{j=1}^{m} I_j(u(t_j))v(t_j) \\
=\ &\int_0^T \varphi'(t)f(t,u(t),{}_0D_t^{\alpha;\varphi}(u(t)),{}_0D_t^{\beta;\varphi}(u(t)))v(t)dt,
\end{aligned}$$

for any $v \in E$. Furthermore, $u \in E$ is a classical solution of Equation (1) if and only if u satisfies Equation (1).

Lemma 6. *The weak solution of Equation (1) is the classical solution of Equation (1).*

Proof. The proof is similar to literature [30], so we omit it here. □

Theorem 1. *(Mountain pass theorem) ([31]) Let H be a real Banach space and $I \in C^1(H,\mathbb{R})$ satisfy Palais–Smale condition. Suppose that I satisfies the following conditions:*

(i) $I(0) = 0$,

(ii) there exist two constants $\rho, \beta > 0$ such that $I|_{\partial B_\rho(0)} \geq \beta$,

(iii) there exists $e \in H$ such that $I(e) \leq 0$.

Then, I possesses a critical value $c \geq \beta$ given by

$$c = \inf_{g\in\Gamma}\max_{s\in[0,1]} I(g(s)),$$

where $B_\rho(0)$ is an open ball in H of radius ρ centered at 0 and

$$\Gamma = \{g \in C([0,1],H) : g(0) = 0, g(1) = e\}.$$

3. Variational Setting and Main Results

In this section, in order to apply the variational method, we make a variational structure and give some basic assumptions which will be used in the proofs of our main results. What is more, the existence of nontrivial solutions for problem (1) is illustrated by the Mountain pass theorem and iteration method.

For a certain fixed $\eta \in E$, define the functional $\Psi_\eta : E \to \mathbb{R}$ as follows

$$\begin{aligned}
\Psi_\eta(u) = &\int_0^T \frac{\varphi'(t)}{p}(|{}_0D_t^{\alpha;\varphi}u(t)|^p + |u(t)|^p) + \frac{\varphi'(t)}{q}(|{}_0D_t^{\beta;\varphi}u(t)|^q + |u(t)|^q)dt \\
&+\sum_{j=1}^{m}\int_0^{u(t_j)} I_j(s)ds - \int_0^T \varphi'(t)F(t,u(t),{}_0D_t^{\alpha;\varphi}(\eta(t)),{}_0D_t^{\beta;\varphi}(\eta(t)))dt,
\end{aligned} \tag{4}$$

where $F(t,x,y,z) = \int_0^x f(t,s,y,z)ds$ for any $x,y,z \in \mathbb{R}$. It is obvious that $\Psi_\eta \in C^1(E,\mathbb{R})$ and for any $u,v \in E$, we have

$$\begin{aligned}
\langle \Psi'_\eta(u), v\rangle = &\int_0^T \varphi'(t)|{}_0D_t^{\alpha;\varphi}u(t)|^{p-2}{}_0D_t^{\alpha;\varphi}u(t){}_0D_t^{\alpha;\varphi}v(t) + \varphi'(t)|u(t)|^{p-2}u(t)v(t) \\
&+\varphi'(t)|{}_0D_t^{\beta;\varphi}u(t)|^{q-2}{}_0D_t^{\beta;\varphi}u(t){}_0D_t^{\beta;\varphi}v(t) + \varphi'(t)|u(t)|^{q-2}u(t)v(t)dt \\
&+\sum_{j=1}^{m} I_j(u(t_j))v(t_j) - \int_0^T \varphi'(t)f(t,u(t),{}_0D_t^{\alpha;\varphi}(\eta(t)),{}_0D_t^{\beta;\varphi}(\eta(t)))v(t)dt.
\end{aligned} \tag{5}$$

Remark 5. *By Definition 5, u is a weak solution of problem (1) if and only if u satisfies $\langle \Psi'_u(u), v\rangle = 0$ for all $v \in E$.*

Lemma 7. *Assume that there exist positive constants* $a_1, a_2, a_3 \geq 0$, $\tau > q$, $0 < \gamma_i < p(i{=}1{,}2{,}3)$, $0 < \sigma_j < p$ *and functions* $n_j \in L^1([0,T], \mathbb{R}^+)$ $(j = 1, 2, \cdots, m)$, $m \in L^1([0,T], \mathbb{R}^+)$ *such that* (S_1) *for any* $x, y, z \in \mathbb{R}$,

$$\tau F(t,x,y,z) - f(t,x,y,z)x \leq a_1|x|^{\gamma_1} + a_2|y|^{\gamma_2} + a_3|z|^{\gamma_3} + m(t), \text{ a.e. } t \in [0,T],$$

and

$$I_j(x)x - \tau \int_0^x I_j(s)ds \leq n_j(t)|x|^{\sigma_j}.$$

Then, the functional Ψ_η *satisfies the Palais–Smale condition.*

Proof. Firstly, we will claim that any Palais–Smale sequence of the functional Ψ_η is bounded. Assume that $\{u_n\} \subset E$ is a Palais–Smale sequence for the energy functional Ψ_η, i.e., $\Psi_\eta(u_n) \to c$ and $\Psi_\eta{}'(u_n) \to 0$. Then, using (4), (5) and (S_1), we have

$$\begin{aligned}
& \tau\Psi_\eta(u_n(t)) - \Psi'_\eta(u_n(t))u_n(t) \\
= & \ (\frac{\tau}{p} - 1)\|u_n\|^p_{\alpha,p;\varphi} + (\frac{\tau}{q} - 1)\|u_n\|^q_{\beta,q;\varphi} + \sum_{j=1}^m \int_0^{u_n(t_j)} \tau I_j(s)ds - \sum_{j=1}^m I_j(u_n(t_j))u_n(t_j) \\
& - \int_0^T \varphi'(t)(\tau F(t, u_n(t), {}_0D_t^{\alpha;\varphi}\eta(t), {}_0D_t^{\beta;\varphi}\eta(t)) - f(t, u_n(t), {}_0D_t^{\alpha;\varphi}\eta(t), {}_0D_t^{\beta;\varphi}\eta(t))u_n(t))dt \\
\geq & \ (\frac{\tau}{p} - 1)\|u_n\|^p_{\alpha,p;\varphi} + (\frac{\tau}{q} - 1)\|u_n\|^q_{\beta,q;\varphi} - \sum_{j=1}^m n_j(t)|u_n(t_j)|^{\sigma_j} \\
& - \int_0^T a_1\varphi'(t)|u_n(t)|^{\gamma_1} + a_2\varphi'(t)|{}_0D_t^{\alpha;\varphi}\eta(t)|^{\gamma_2} + a_3\varphi'(t)|{}_0D_t^{\beta;\varphi}\eta(t)|^{\gamma_3} + m(t)\varphi'(t)dt \\
\geq & \ (\frac{\tau}{p} - 1)\|u_n\|^p_{\alpha,p;\varphi} + (\frac{\tau}{q} - 1)\|u_n\|^q_{\beta,q;\varphi} - \sum_{j=1}^m n_j(t)\|u_n(t_j)\|^{\sigma_j} \\
& -a_1\|\varphi'(t)^{\frac{1}{\gamma_1}}u(t)\|^{\gamma_1}_{L^{\gamma_1}} - a_2\|\eta(t)\|^{\gamma_2}_{\alpha,p;\varphi} - a_3\|\eta(t)\|^{\gamma_3}_{\beta,q;\varphi} - T\|m(t)\|_\infty.
\end{aligned}$$

Due to $\tau > q \geq p$, $0 < \gamma_i < p(i = 1,2,3)$, $0 < \sigma_j < p$, $(j = 1, 2, \ldots, m)$ and $\Psi_\eta(u_n) \to c$, $\Psi'_\eta(u_n) \to 0$. Therefore, $\|u_n\|_{\alpha,p;\varphi}$ and $\|u_n\|_{\beta,q;\varphi}$ are bounded, i.e., $\{u_n\}$ is bounded in E. According to the reflexivity of the space E, there exists a subsequence, without loss of generality, still denoted $\{u_n\}$ such that $u_n \rightharpoonup u_0$ in E as $n \to \infty$. In view of $\lim\limits_{n\to\infty} \Psi'_\eta(u_n) = 0$, we obtain

$$\begin{aligned}
|\langle\Psi'_\eta(u_n) - \Psi'_\eta(u_0), u_n - u_0\rangle| & \leq |\Psi'_\eta(u_n)(u_n - u_0)| + |\Psi'_\eta(u_0)(u_n - u_0)| \\
& \leq \|\Psi'_\eta(u_n)\|_{E^*}\|u_n - u_0\| + |\Psi'_\eta(u_0)(u_n - u_0)| \to 0,
\end{aligned}$$

as $n \to \infty$, where E^* is the dual of E.

Define

$$\begin{aligned}
X^{u,v}_{\alpha,p;\varphi} = \int_0^T & \varphi'(t)(\Phi_p({}_0D_t^{\alpha;\varphi}u(t)) - \Phi_p({}_0D_t^{\alpha;\varphi}v(t)))({}_0D_t^{\alpha;\varphi}u(t) - {}_0D_t^{\alpha;\varphi}v(t)) \\
& + \varphi'(t)(\Phi_p(u(t)) - \Phi_p(v(t)))(u(t) - v(t))dt.
\end{aligned} \tag{6}$$

Then, based on the definition of $\langle\Psi'_\eta(u), v\rangle$ and (6), we have

$$
\begin{aligned}
&\langle \Psi'_\eta(u_n) - \Psi'_\eta(u_0), u_n - u_0 \rangle \\
=&\int_0^T \varphi'(t)(\Phi_p({}_0D_t^{\alpha;\varphi}u_n(t)) - \Phi_p({}_0D_t^{\alpha;\varphi}u_0(t)))({}_0D_t^{\alpha;\varphi}u_n(t) - {}_0D_t^{\alpha;\varphi}u_0(t))dt \\
&+ \int_0^T \varphi'(t)(\Phi_p(u_n(t)) - \Phi_p(u_0(t)))(u_n(t) - u_0(t))dt \\
&+ \int_0^T \varphi'(t)(\Phi_q({}_0D_t^{\beta;\varphi}u_n(t)) - \Phi_q({}_0D_t^{\beta;\varphi}u_0(t)))({}_0D_t^{\beta;\varphi}u_n(t) - {}_0D_t^{\beta;\varphi}u_0(t))dt \\
&+ \int_0^T \varphi'(t)(\Phi_q(u_n(t)) - \Phi_q(u_0(t)))(u_n(t) - u_0(t))dt \\
&+ \sum_{j=1}^m [I_j(u_n(t_j)) - I_j(u_0(t_j))](u_n(t_j) - u_0(t_j)) \\
&+ \int_0^T [f(t, u_n(t), {}_0D_t^{\alpha;\varphi}\eta(t), {}_0D_t^{\beta;\varphi}\eta(t)) - f(t, u_0(t){}_0D_t^{\alpha;\varphi}\eta(t), {}_0D_t^{\beta;\varphi}\eta(t))]\varphi'(t)(u_n(t) - u_0(t))dt \\
=&X_{\alpha,p;\varphi}^{u_n,u_0} + X_{\beta,q;\varphi}^{u_n,u_0} + \sum_{j=1}^m [I_j(u_n(t_j)) - I_j(u_0(t_j))](u_n(t_j) - u_0(t_j)) \\
&+ \int_0^T [f(t, u_n(t), {}_0D_t^{\alpha;\varphi}\eta(t), {}_0D_t^{\beta;\varphi}\eta(t)) - f(t, u_0(t){}_0D_t^{\alpha;\varphi}\eta(t), {}_0D_t^{\beta;\varphi}\eta(t))]\varphi'(t)(u_n(t) - u_0(t))dt.
\end{aligned} \tag{7}
$$

Since E is a reflexive space, based on Lemma 5, $u_n \to u_0$ uniformly in $C([0,T], \mathbb{R})$, functions $I_j (j = 1, 2, \ldots, m)$, f and φ' are continuous. It is easy to check that

$$
\sum_{j=1}^m [I_j(u_n(t_j)) - I_j(u_0(t_j))](u_n(t_j) - u_0(t_j)) \to 0,
$$

and

$$
\int_0^T \left[f\left(t, u_n(t), {}_0D_t^{\alpha;\varphi}\eta(t), {}_0D_t^{\beta;\varphi}\eta(t)\right) - f\left(t, u_0(t), {}_0D_t^{\alpha;\varphi}\eta(t), {}_0D_t^{\beta;\varphi}\eta(t)\right)\right]\varphi'(t)(u_n(t) - u_0(t))dt \to 0,
$$

as $n \to \infty$.

Next, we will declare that $\{u_n\}$ converges to u_0 in E. By the same reasoning as in [20], we have the following inequality

$$
(|x|^{r-2}x - |y|^{r-2}y)(x - y) \geq \begin{cases} |x - y|^r, & r \geq 2, \\ \frac{|x-y|^2}{(|x|+|y|)^{2-r}}, & 1 < r \leq 2, \end{cases}
$$

for any $x, y \in \mathbb{R}^N$. Let

$$
\begin{aligned}
\Theta(\alpha, p; \varphi) &= \int_0^T \varphi'(t)\left(\Phi_p({}_0D_t^{\alpha;\varphi}u_n(t)) - \Phi_p({}_0D_t^{\alpha;\varphi}u_0(t))\right)({}_0D_t^{\alpha;\varphi}u_n(t) - {}_0D_t^{\alpha;\varphi}u_0(t))dt \\
&\geq \begin{cases} \displaystyle\int_0^T \varphi'(t)|{}_0D_t^{\alpha;\varphi}u_n(t) - {}_0D_t^{\alpha;\varphi}u_0(t)|^p dt, & p \geq 2, \\ \displaystyle\int_0^T \varphi'(t)\frac{|{}_0D_t^{\alpha;\varphi}u_n(t) - {}_0D_t^{\alpha;\varphi}u_0(t)|^2}{(|{}_0D_t^{\alpha;\varphi}u_n(t)| + |{}_0D_t^{\alpha;\varphi}u_0(t)|)^{2-p}}dt, & 1 < p < 2, \end{cases}
\end{aligned} \tag{8}
$$

$$\Theta(\beta,q;\varphi)=\int_0^T\varphi'(t)\big(\Phi_q({}_0D_t^{\beta;\varphi}u_n(t))-\Phi_q({}_0D_t^{\beta;\varphi}u_0(t))\big)({}_0D_t^{\alpha;\varphi}u_n(t)-{}_0D_t^{\beta;\varphi}u_0(t))dt$$
$$\geq\begin{cases}\int_0^T\varphi'(t)|{}_0D_t^{\beta;\varphi}u_n(t)-{}_0D_t^{\beta;\varphi}u_0(t)|^qdt, & q\geq 2,\\ \int_0^T\varphi'(t)\frac{|{}_0D_t^{\beta;\varphi}u_n(t)-{}_0D_t^{\beta;\varphi}u_0(t)|^2}{(|{}_0D_t^{\beta;\varphi}u_n(t)|+|{}_0D_t^{\beta;\varphi}u_0(t)|)^{2-q}}dt, & 1<q<2,\end{cases} \tag{9}$$

$$\Theta(p;\varphi)=\int_0^T\varphi'(t)\big(\Phi_p(u_n(t))-\Phi_p(u_0(t))\big)(u_n(t)-u_0(t))dt$$
$$\geq\begin{cases}\int_0^T\varphi'(t)|u_n(t)-u_0(t)|^pdt, & p\geq 2,\\ \int_0^T\varphi'(t)\frac{|u_n(t)-u_0(t)|^2}{(|u_n(t)|+|u_0(t)|)^{2-p}}dt, & 1<p<2,\end{cases} \tag{10}$$

$$\Theta(q;\varphi)=\int_0^T\varphi'(t)\big(\Phi_q(u_n(t))-\Phi_q(u_0(t))\big)(u_n(t)-u_0(t))dt$$
$$\geq\begin{cases}\int_0^T\varphi'(t)|u_n(t)-u_0(t)|^qdt, & q\geq 2,\\ \int_0^T\varphi'(t)\frac{|u_n(t)-u_0(t)|^2}{(|u_n(t)|+|u_0(t)|)^{2-q}}dt, & 1<q<2.\end{cases} \tag{11}$$

When $1<p<2$, with the help of Hölder inequality and $(a+b)^r\leq 2^r(a^r+b^r)$, we conclude

$$\begin{aligned}&\int_0^T\varphi'(t)|{}_0D_t^{\alpha;\varphi}u_n(t)-{}_0D_t^{\alpha;\varphi}u_0(t)|^pdt\\ \leq&\Big(\int_0^T\varphi'(t)\frac{|{}_0D_t^{\alpha;\varphi}u_n(t)-{}_0D_t^{\alpha;\varphi}u_0(t)|^2}{(|{}_0D_t^{\alpha;\varphi}u_n(t)|+|{}_0D_t^{\alpha;\varphi}u_0(t)|)^{2-p}}dt\Big)^{\frac{p}{2}}\Big(\int_0^T\varphi'(t)(|{}_0D_t^{\alpha;\varphi}u_n(t)|+|{}_0D_t^{\alpha;\varphi}u_0(t)|)^pdt\Big)^{\frac{2-p}{2}}\\ \leq&\Big(\int_0^T\varphi'(t)\frac{|{}_0D_t^{\alpha;\varphi}u_n(t)-{}_0D_t^{\alpha;\varphi}u_0(t)|^2}{(|{}_0D_t^{\alpha;\varphi}u_n(t)|+|{}_0D_t^{\alpha;\varphi}u_0(t)|)^{2-p}}dt\Big)^{\frac{p}{2}}\cdot 2^{\frac{p(2-p)}{2}}(\|u_n\|_{\alpha,p;\varphi}^p+\|u_0\|_{\alpha,p;\varphi}^p)^{\frac{2-p}{2}}.\end{aligned} \tag{12}$$

Then, we have

$$\begin{aligned}&\int_0^T\varphi'(t)\frac{|{}_0D_t^{\alpha;\varphi}u_n(t)-{}_0D_t^{\alpha;\varphi}u_0(t)|^2}{(|{}_0D_t^{\alpha;\varphi}u_n(t)|+|{}_0D_t^{\alpha;\varphi}u_0(t)|)^{2-p}}dt\\ \geq&2^{p-2}\Big(\int_0^T\varphi'(t)|{}_0D_t^{\alpha;\varphi}u_n(t)-{}_0D_t^{\alpha;\varphi}u_0(t)|^pdt\Big)^{\frac{2}{p}}(\|u_n\|_{\alpha,p;\varphi}^p+\|u_0\|_{\alpha,p;\varphi}^p)^{\frac{p-2}{p}}.\end{aligned} \tag{13}$$

Similarly, we can see that

$$\begin{aligned}&\int_0^T\varphi'(t)\frac{|u_n(t)-u_0(t)|^2}{(|u_n(t)|+|u_0(t)|)^{2-p}}dt\\ \geq&2^{p-2}\Big(\int_0^T\varphi'(t)|u_n(t)-u_0(t)|^pdt\Big)^{\frac{2}{p}}(\|\varphi'(t)^{\frac{1}{p}}u_n(t)\|_{L^p}^p+\|\varphi'(t)^{\frac{1}{p}}u_0(t)\|_{L^p}^p)^{\frac{p-2}{p}}.\end{aligned} \tag{14}$$

When $1<q<2$, we derive

$$\begin{aligned}&\int_0^T\varphi'(t)\frac{|{}_0D_t^{\beta;\varphi}u_n(t)-{}_0D_t^{\beta;\varphi}u_0(t)|^2}{(|{}_0D_t^{\beta;\varphi}u_n(t)|+|{}_0D_t^{\beta;\varphi}u_0(t)|)^{2-q}}dt\\ \geq&2^{q-2}\Big(\int_0^T\varphi'(t)|{}_0D_t^{\beta;\varphi}u_n(t)-{}_0D_t^{\beta;\varphi}u_0(t)|^qdt\Big)^{\frac{2}{q}}(\|u_n\|_{\beta,q;\varphi}^q+\|u_0\|_{\beta,q;\varphi}^q)^{\frac{q-2}{q}},\end{aligned} \tag{15}$$

and

$$\begin{aligned}&\int_0^T \varphi'(t)\frac{|u_n(t)-u_0(t)|^2}{(|u_n(t)|+|u_0(t)|)^{2-q}}dt\\ \geq&2^{q-2}\Big(\int_0^T \varphi'(t)|u_n(t)-u_0(t)|^q dt\Big)^{\frac{2}{q}}\big(\|\varphi'(t)^{\frac{1}{q}}u_n(t)\|_{L^q}^q+\|\varphi'(t)^{\frac{1}{q}}u_0(t)\|_{L^q}^q\big)^{\frac{q-2}{q}}.\end{aligned} \tag{16}$$

In general, when $1<p\leq q\leq 2$, from (6) and (8)–(16), we have

$$\begin{aligned}X_{\alpha,p;\varphi}^{u_n,u_0}+X_{\beta,q;\varphi}^{u_n,u_0}=&\Theta(\alpha,p;\varphi)+\Theta(\beta,q;\varphi)+\Theta(p;\varphi)+\Theta(q;\varphi)\\ \geq&C_1\Big[\Big(\int_0^T\varphi'(t)|{}_0D_t^{\alpha;\varphi}u_n(t)-{}_0D_t^{\alpha;\varphi}u_0(t)|^p dt\Big)^{\frac{2}{p}}+\Big(\int_0^T\varphi'(t)|u_n(t)-u_0(t)|^p dt\Big)^{\frac{2}{p}}\Big]\\ &+C_2\Big[\Big(\int_0^T\varphi'(t)|{}_0D_t^{\beta;\varphi}u_n(t)-{}_0D_t^{\beta;\varphi}u_0(t)|^q dt\Big)^{\frac{2}{q}}+\Big(\int_0^T\varphi'(t)|u_n(t)-u_0(t)|^q dt\Big)^{\frac{2}{q}}\Big]\\ \geq&C_3(\|u_n-u_0\|_{\alpha,p;\varphi}^2+\|u_n-u_0\|_{\beta,q;\varphi}^2),\end{aligned} \tag{17}$$

where

$$C_1=\min\big\{2^{p-2}(\|u_n\|_{\alpha,p;\varphi}^p+\|u_0\|_{\alpha,p;\varphi}^p)^{\frac{p-2}{p}},\ 2^{p-2}(\|\varphi'(t)^{\frac{1}{p}}u_n(t)\|_{L^p}^p+\|\varphi'(t)^{\frac{1}{p}}u_0(t)\|_{L^p}^p)^{\frac{p-2}{p}}\big\},$$

$$C_2=\min\big\{2^{q-2}(\|u_n\|_{\beta,q;\varphi}^q+\|u_0\|_{\beta,q;\varphi}^q)^{\frac{q-2}{q}},\ 2^{q-2}(\|\varphi'(t)^{\frac{1}{q}}u_n(t)\|_{L^q}^q+\|\varphi'(t)^{\frac{1}{q}}u_0(t)\|_{L^q}^q)^{\frac{q-2}{q}}\big\},$$

$$C_3=\min\big\{C_1 2^{-\frac{2}{p}},\ C_2 2^{-\frac{2}{q}}\big\}.$$

When $1<p<2\leq q$, we yield

$$\begin{aligned}X_{\alpha,p;\varphi}^{u_n,u_0}+X_{\beta,q;\varphi}^{u_n,u_0}=&\Theta(\alpha,p;\varphi)+\Theta(\beta,q;\varphi)+\Theta(p;\varphi)+\Theta(q;\varphi)\\ \geq&C_1 2^{-\frac{2}{p}}\|u_n-u_0\|_{\alpha,p;\varphi}^2+\|u_n-u_0\|_{\beta,q;\varphi}^q.\end{aligned} \tag{18}$$

When $2\leq p\leq q$, we obtain

$$\begin{aligned}X_{\alpha,p;\varphi}^{u_n,u_0}+X_{\beta,q;\varphi}^{u_n,u_0}=&\Theta(\alpha,p;\varphi)+\Theta(\beta,q;\varphi)+\Theta(p;\varphi)+\Theta(q;\varphi)\\ \geq&\int_0^T\varphi'(t)|{}_0D_t^{\alpha;\varphi}u_n(t)-{}_0D_t^{\alpha;\varphi}u_0(t)|^p dt+\int_0^T\varphi'(t)|u_n(t)-u_0(t)|^p dt\\ &+\int_0^T\varphi'(t)|{}_0D_t^{\beta;\varphi}u_n(t)-{}_0D_t^{\beta;\varphi}u_0(t)|^q dt+\int_0^T\varphi'(t)|u_n(t)-u_0(t)|^q dt\\ \geq&\|u_n-u_0\|_{\alpha,p;\varphi}^p+\|u_n-u_0\|_{\beta,q;\varphi}^q.\end{aligned} \tag{19}$$

Due to $X_{\alpha,p;\varphi}^{u_n,u_0}+X_{\beta,q;\varphi}^{u_n,u_0}\to 0$ as $n\to\infty$, by (17)–(19), we deduce $\|u_n-u_0\|_{\alpha,p;\varphi}\to 0$ and $\|u_n-u_0\|_{\beta,q;\varphi}\to 0$. Therefore, $\|u_n-u_0\|_E\to 0$ as $n\to\infty$. This completes the proof. □

Theorem 2. *Let* (S_1) *hold. If there exist some constants* $m_i,n_i\geq 0(i=1,2,3)$, $\delta,\epsilon>0$, $\lambda_1,\zeta_1>q-1$, $\lambda_2>p-1$, $\lambda_3>q-1$, $0<\zeta_2,\zeta_3\leq p-1$, $0<\xi_1<p$, $0<\xi_2<q$, $c_j,d_j\geq 0$, $\tau_j>q-1$ *and* $0<\mu_j\leq q-1$, $j=1,2,\dots,m$ *such that functions* f *and* $I_j(j=1,2,\dots,m)$ *satisfy*

(S_2) $f(t,x,y,z)\leq m_1|x|^{\lambda_1}+m_2|x|^{\lambda_2}|y|^{\xi_1}+m_3|x|^{\lambda_3}|z|^{\xi_2}$, $\forall x,y,z\in\mathbb{R}$, $|x|\leq\delta$, *a.e.* $t\in[0,T]$,

$f(t,x,y,z)>n_1|x|^{\zeta_1}-n_2|y|^{\zeta_2}-n_3|x|^{\zeta_3}-C$, $\forall x\geq 0$, $(y,z)\in\mathbb{R}\times\mathbb{R}$, *a.e.* $t\in[0,T]$;

(S_3) $I_j(s)\geq -c_j|s|^{\tau_j}$, $\forall|s|\leq\delta$,

$I_j(s)\leq d_j|s|^{\mu_j}$, $\forall|s|\geq\epsilon$, *a.e.* $s\in[0,T]$, $j=1,2,\dots,m$.

Then, for a fixed $\eta\in E$, Ψ_η *possesses a critical value in* E.

Proof. We are now in a position to check that I admits the mountain pass geometry. Based on Lemma 4, we have

$$\begin{aligned}\|u\|_E =&\|\varphi'(t)^{\frac{1}{p}}{}_0D_t^{\alpha;\varphi}u(t)\|_{L^p}+\|\varphi'(t)^{\frac{1}{q}}{}_0D_t^{\beta;\varphi}u(t)\|_{L^q}\\ \geq&\frac{\Gamma(\alpha)((\alpha-1)\frac{p}{p-1}+1)^{\frac{p-1}{p}})}{M_{\varphi'}[\varphi(T)]^{\alpha-\frac{1}{p}}}\|\varphi'(t)u(t)\|+\frac{\Gamma(\alpha)((\alpha-1)\frac{q}{q-1}+1)^{\frac{q-1}{q}})}{M_{\varphi'}[\varphi(T)]^{\beta-\frac{1}{q}}}\|\varphi'(t)u(t)\|\\ \geq&M_{p,q}\|\varphi'(t)u(t)\|,\end{aligned} \tag{20}$$

where $M_{p,q}=\min\{\frac{\Gamma(\alpha)((\alpha-1)\frac{p}{p-1}+1)^{\frac{p-1}{p}})}{M_{\varphi'}[\varphi(T)]^{\alpha-\frac{1}{p}}},\frac{\Gamma(\alpha)((\alpha-1)\frac{q}{q-1}+1)^{\frac{p-1}{p}})}{M_{\varphi'}[\varphi(T)]^{\beta-\frac{1}{q}}}\}$.

Let $\delta_0=M_{p,q}\delta$, if $u\in E$ and $\|u\|_E\leq\delta_0$, we can obtain $\|\varphi'(t)u(t)\|\leq\frac{1}{M_{p,q}}\|u\|_E\leq\delta$. It follows from (S_2) that

$$\begin{aligned}&F(t,u(t),{}_0D_t^{\alpha;\varphi}\eta(t),{}_0D_t^{\beta;\varphi}\eta(t))\\ =&\int_0^{u(t)}f(t,s,{}_0D_t^{\alpha;\varphi}\eta(t),{}_0D_t^{\beta;\varphi}\eta(t))ds\\ \leq&\int_0^{u(t)}m_1|s|^{\lambda_1}+m_2|s|^{\lambda_2}|{}_0D_t^{\alpha;\varphi}\eta(t)|^{\xi_1}+m_3|s|^{\lambda_3}|{}_0D_t^{\beta;\varphi}\eta(t)|^{\xi_2}ds\\ =&\frac{m_1}{\lambda_1+1}|u(t)|^{\lambda_1+1}+\frac{m_2}{\lambda_2+1}|u(t)|^{\lambda_2+1}|{}_0D_t^{\alpha;\varphi}\eta(t)|^{\xi_1}+\frac{m_3}{\lambda_3+1}|u(t)|^{\lambda_3+1}|{}_0D_t^{\beta;\varphi}\eta(t)|^{\xi_2}.\end{aligned}$$

Hence, for any $t\in[0,T]$, there exists a constant $K>0$ such that $\|\eta\|_E\leq K$. According to (20) and Hölder inequality, we obtain

$$\begin{aligned}&\int_0^T\varphi'(t)F(t,u(t),{}_0D_t^{\alpha;\varphi}\eta(t),{}_0D_t^{\beta;\varphi}\eta(t))dt\\ \leq&\int_0^T\varphi'(t)\frac{m_1}{\lambda_1+1}|u(t)|^{\lambda_1+1}\\ &+\frac{m_2}{\lambda_2+1}\varphi'(t)|u(t)|^{\lambda_2+1}|{}_0D_t^{\alpha;\varphi}\eta(t)|^{\xi_1}+\frac{m_3}{\lambda_3+1}\varphi'(t)|u(t)|^{\lambda_3+1}|{}_0D_t^{\beta;\varphi}\eta(t)|^{\xi_2}dt\\ \leq&\frac{m_1T}{\lambda_1+1}\|\varphi'(t)u(t)\|^{\lambda_1+1}+\frac{m_2T^{\frac{p-\xi_1}{p}}}{\lambda_2+1}\|\varphi'(t)u(t)\|^{\frac{p(\lambda_2+1)}{p-\xi_1}}\|\eta(t)\|_{\alpha,p,\varphi}^{\xi_1}\\ &+\frac{m_3T^{\frac{q-\xi_2}{q}}}{\lambda_3+1}\|\varphi'(t)u(t)\|^{\frac{q(\lambda_3+1)}{q-\xi_2}}\|\eta(t)\|_{\beta,q,\varphi}^{\xi_2}\\ \leq&\frac{m_1T}{(\lambda_1+1)M_{(p,q)}}(\|u(t)\|_{\alpha,p,\varphi}+\|u(t)\|_{\beta,q,\varphi})^{\lambda_1+1}+\frac{m_2T^{\frac{p-\xi_1}{p}}}{\lambda_2+1}K^{\xi_1}\|\varphi'(t)u(t)\|^{\frac{p(\lambda_2+1)}{p-\xi_1}}\\ &+\frac{m_3T^{\frac{q-\xi_2}{q}}}{\lambda_3+1}K^{\xi_2}\|\varphi'(t)u(t)\|^{\frac{q(\lambda_3+1)}{q-\xi_2}}\\ \leq&\overline{m}_1(\|u(t)\|_{\alpha,p,\varphi}^{\lambda_1+1}+\|u(t)\|_{\beta,q,\varphi}^{\lambda_1+1})+\overline{m}_2\|u(t)\|_{\alpha,p;\varphi}^{\frac{p(\lambda_2+1)}{p-\xi_1}}+\overline{m}_3\|u(t)\|_{\beta,q;\varphi}^{\frac{q(\lambda_3+1)}{q-\xi_2}},\end{aligned} \tag{21}$$

where

$$\overline{m}_1=\frac{m_1T}{(\lambda_1+1)M_{(p,q)}}2^{\lambda_1+1},\quad \overline{m}_2=\frac{m_2T^{\frac{p-\xi_1}{p}}}{\lambda_2+1}K^{\xi_1}(\frac{M_{\varphi'}[\varphi(T)]^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)((\alpha-1)\frac{p}{p-1}+1)^{\frac{p-1}{p}}})^{\frac{p(\lambda_2+1)}{p-\xi_1}},$$

$$\overline{m}_3=\frac{m_3T^{\frac{q-\xi_2}{q}}}{\lambda_3+1}K^{\xi_2}(\frac{M_{\varphi'}[\varphi(T)]^{\beta-\frac{1}{q}}}{\Gamma(\beta)((\beta-1)\frac{q}{q-1}+1)^{\frac{q-1}{q}}})^{\frac{q(\lambda_3+1)}{q-\xi_2}}.$$

By (S_3) and Lemma 4, one has

$$\begin{aligned}\int_0^{u(t_j)} I_j(s)ds &\geq -c_j\int_0^{u(t_j)}|s|^{\tau_j}ds \\ &= -\frac{c_j}{\tau_j+1}|u(t_j)|^{\tau_j+1} \\ &\geq -\frac{c_j}{\tau_j+1}\|u(t_j)\|^{\tau_j+1} \\ &\geq -\frac{c_j}{\tau_j+1}(\frac{1}{M_{p,q}})^{\tau_j+1}(\|u(t)\|_{\alpha,p,\varphi}+\|u(t)\|_{\beta,q,\varphi})^{\tau_j+1} \\ &\geq -\bar{c}_j(\|u(t)\|^{\tau_j+1}_{\alpha,p,\varphi}+\|u(t)\|^{\tau_j+1}_{\beta,q,\varphi}),\end{aligned} \tag{22}$$

where $\bar{c}_j = 2^{\tau_j+1}\frac{c_j}{\tau_j+1}(\frac{1}{M_{p,q}})^{\tau_j+1}$. Taking $\|u\|_{\alpha,p;\varphi}=\rho_1$, $\|u\|_{\beta,q;\varphi}=\rho_2$, $\|u\|_E=\rho_1+\rho_2=\rho$, by (4), (21) and (22), we have

$$\begin{aligned}\Psi_\eta(u) &\geq \frac{1}{p}\|u\|^p_{\alpha,p;\varphi}+\frac{1}{q}\|u\|^q_{\beta,q;\varphi}-\sum_{j=1}^m\bar{c}_j(\|u(t)\|^{\tau_j+1}_{\alpha,p,\varphi}+\|u(t)\|^{\tau_j+1}_{\beta,q,\varphi}) \\ &\quad -\overline{m}_1(\|u(t)\|^{\lambda_1+1}_{\alpha,p,\varphi}+\|u(t)\|^{\lambda_1+1}_{\beta,q,\varphi})-\overline{m}_2\|u(t)\|^{\frac{p(\lambda_2+1)}{p-\zeta_1}}_{\alpha,p;\varphi}-\overline{m}_3\|u(t)\|^{\frac{q(\lambda_3+1)}{q-\zeta_2}}_{\beta,q;\varphi} \\ &= \rho_1^p(\frac{1}{p}-\sum_{j=1}^m\bar{c}_j\rho_1^{\tau_j+1-p}-\overline{m}_1\rho_1^{\lambda_1+1-p}-\overline{m}_2\rho_1^{\frac{p(\lambda_2+1)}{p-\zeta_1}}) \\ &\quad +\rho_2^q(\frac{1}{q}-\sum_{j=1}^m\bar{c}_j\rho_2^{\tau_j+1-q}-\overline{m}_1\rho_2^{\lambda_1+1-q}-\overline{m}_2\rho_2^{\frac{q(\lambda_3+1)}{q-\zeta_2}}).\end{aligned}$$

For ρ_1,ρ_2 small enough, there exist $\sigma_1,\sigma_2>0$ such that

$$\frac{1}{p}-\sum_{j=1}^m\bar{c}_j\rho_1^{\tau_1+1-p}-\overline{m}_1\rho_1^{\lambda_1+1-p}-\overline{m}_2\rho_1^{\frac{p(\lambda_2+1)}{p-\zeta_1}}\geq\sigma_1,$$

$$\frac{1}{q}-\sum_{j=1}^m\bar{c}_j\rho_2^{\tau_1+1-q}-\overline{m}_1\rho_2^{\lambda_1+1-q}-\overline{m}_2\rho_2^{\frac{q(\lambda_3+1)}{q-\zeta_2}}\geq\sigma_2.$$

Therefore

$$\Psi_\eta(u)>\rho_1^p\sigma_1+\rho_2^q\sigma_2:=\sigma>0,\quad \forall u\in E, \|u\|_E=\rho.$$

On the other hand, owing to (S_2) and (S_3), we obtain

$$\begin{aligned}&F(t,u(t),{}_0D_t^{\alpha;\varphi}\eta(t),{}_0D_t^{\beta;\varphi}\eta(t)) \\ =&\int_0^{u(t)}f(t,s,{}_0D_t^{\alpha;\varphi}\eta(t),{}_0D_t^{\beta;\varphi}\eta(t))ds \\ \geq&\frac{n_1}{\zeta_1+1}|u|^{\zeta_1+1}-n_2u(t)|{}_0D_t^{\alpha;\varphi}\eta(t)|^{\zeta_2}-n_3u(t)|{}_0D_t^{\beta;\varphi}\eta(t)|^{\zeta_3}-Cu(t),\end{aligned} \tag{23}$$

and

$$\int_0^{u(t)}I_j(s)ds\leq\int_0^{u(t)}d_j|s|^{\mu_j}ds=\frac{d_j}{\mu_j+1}|u|^{\mu_j+1}. \tag{24}$$

Taking $x > 0$, $u \in E \backslash \{0\}$, combining (23) and (24), we infer

$$
\begin{aligned}
\Psi_\eta(xu) =& \int_0^T x^p \frac{\varphi'(t)}{p}(|{}_0D_t^{\alpha;\varphi}u(t)|^p + |u(t)|^p) + x^q \frac{\varphi'(t)}{q}(|{}_0D_t^{\beta;\varphi}u(t)|^q + |u(t)|^q)dt \\
&+ \sum_{j=1}^m \int_0^{xu(t_j)} I_j(s)ds - \int_0^T \varphi'(t)F(t, xu(t), {}_0D_t^{\alpha;\varphi}(\eta(t)), {}_0D_t^{\beta;\varphi}(\eta(t)))dt \\
\leq& \frac{x^p}{p}\|u\|^p_{\alpha,p;\varphi} + \frac{x^q}{q}\|u\|^q_{\beta,q;\varphi} + \sum_{j=1}^m \frac{d_j}{\mu_j+1} x^{\mu_j+1}|u|^{\mu_j+1} \\
&- \int_0^T \varphi'(t)x^{\zeta_1+1}\frac{n_1}{\zeta_1+1}|u|^{\zeta_1+1}dt + \int_0^T n_2 x\varphi'(t)u(t)|{}_0D_t^{\alpha;\varphi}\eta(t)|^{\zeta_2}dt \\
&+ \int_0^T n_3 x\varphi'(t)u(t)|{}_0D_t^{\beta;\varphi}\eta(t)|^{\zeta_3}dt + \int_0^T Cx\varphi'(t)u(t)dt \qquad (25) \\
\leq& \frac{x^p}{p}\|u\|^p_{\alpha,p;\varphi} + \frac{x^q}{q}\|u\|^q_{\beta,q;\varphi} + \sum_{j=1}^m \frac{d_j}{\mu_j+1} x^{\mu_j+1}\|u\|^{\mu_j+1} \\
&- \frac{n_1 x^{\zeta_1+1}}{\zeta_1+1}\|\varphi'(t)^{\frac{1}{\zeta_1+1}}u(t)\|^{\zeta_1+1}_{L^{\zeta_1+1}} + n_2 x\|\varphi'(t)u(t)\|\|{}_0D_t^{\alpha;\varphi}\eta(t)\|^{\zeta_2}_{L^{\zeta_2}} \\
&+ n_3 x\|\varphi'(t)u(t)\|\|{}_0D_t^{\beta;\varphi}\eta(t)\|^{\zeta_3}_{L^{\zeta_3}} + CTx\|\varphi'(t)u(t)\| \to -\infty,
\end{aligned}
$$

as $x \to \infty$. Obviously, $\Psi_\eta(0) = 0$. Therefore, combining Mountain pass theorem with Lemma 7, Ψ_η has a weak solution on E. □

Theorem 3. *Let $\frac{1}{p} < \alpha \leq 1$, $\frac{1}{q} < \beta \leq 1$, $1 < p \leq q < \infty$. Assume that $(S_1) - (S_3)$ hold and there exist constants $K > 0$, $b_i \geq 0 (i = 1, 2, 3)$, $e_j > 0 (j = 1, 2, \dots, m)$ such that*

(S_4) $|f(t, x_1, y_1, z_1) - f(t, x_2, y_2, z_2)| \leq b_1|x_1 - x_2| + b_2|y_1 - y_2| + b_3|z_1 - z_2|$,
$|I_j(x_1) - I_j(x_2)| \leq e_j|x_1 - x_2|$, $\forall x_1, x_2 \in [-K, K]$, $y_1, y_2, z_1, z_2 \in \mathbb{R}$.

Then the nonlinear problem (1) admits at least one nontrivial solution.

Proof. We will take advantage of the iterative method to illustrate it. We divide the proof into two steps.

Step 1: At first, we consider for any $u_1 \in E$, $\|u_1\|_{\alpha,p;\varphi} \leq K_1$ and $\|u_1\|_{\beta,q;\varphi} \leq K_2$ with $K_1 + K_2 \leq K$. From Theorem 2, we know that there exists a critical point of Ψ_{u_1}, without loss of generality, we denote the critical point as $u_2 \in E$, i.e., $\Psi'_{u_1}(u_2) = 0$. Now, we assert that $\|u_2\|_E \leq K$.

Let $\overline{u} = \frac{u}{\|u\|_E}$, then $\|\overline{u}\|_E = 1$. Similar to (25), we derive

$$
\begin{aligned}
\Psi_{u_1}(u_2) \leq& \max_{0\leq x<\infty} \Psi_{u_1}(x\overline{u}) \\
\leq& \max_{0\leq x<\infty}\Bigg(\frac{x^p}{p} + \frac{x^q}{q} + \sum_{j=1}^m \frac{d_j}{\mu_j+1}x^{\mu_j+1}|\overline{u}|^{\mu_j+1} - \int_0^T \varphi'(t)x^{\zeta_1+1}\frac{n_1}{\zeta_1+1}|\overline{u}|^{\zeta_1+1}dt \\
&+ \int_0^T n_2 x\varphi'(t)\overline{u}(t)|{}_0D_t^{\alpha;\varphi}u_1(t)|^{\zeta_2}dt \\
&+ \int_0^T n_3 x\varphi'(t)\overline{u}(t)|{}_0D_t^{\beta;\varphi}u_1(t)|^{\zeta_3}dt + \int_0^T Cx\varphi'(t)\overline{u}(t)dt\Bigg) \\
\leq& \max_{0\leq x<\infty}\Bigg(\frac{x^p}{p} + \frac{x^q}{q} + \sum_{j=1}^m \overline{d}_j x^{\mu_j+1} - \overline{n}_1 x^{\zeta_1+1} + \overline{n}_2 K_1^{\zeta_2}x + \overline{n}_3 K_2^{\zeta_3}x + \overline{C}x\Bigg),
\end{aligned}
$$

where $\overline{n}_1 = \frac{n_1}{\zeta_1+1}\|\varphi'(t)^{\frac{1}{\zeta_1+1}}u(t)\|^{\zeta_1+1}_{L^{\zeta_1+1}}$, $\overline{d}_j = \frac{d_j}{\mu_j+1}\|u\|^{\mu_j+1}$, $\overline{n}_2 = n_2\|\varphi'(t)^{\frac{p-\zeta_2}{p}}\overline{u}\|_{L^{\frac{p}{p-\zeta_2}}}$, $\overline{n}_3 = n_3\|\varphi'(t)^{\frac{q-\zeta_3}{q}}\overline{u}\|_{L^{\frac{q}{q-\zeta_2}}}$, $\overline{C} = C\|\varphi'(t)\overline{u}(t)\|_{L^1}$. Taking $\epsilon_1 = (\frac{\tau-p}{2^{5+p}\tau\zeta_2})^{\frac{\zeta_2}{p}}$, by Young inequality, we gain

$$\begin{aligned}\overline{n}_2K_1^{\zeta_2}x &\le \frac{p-\zeta_2}{p}(\frac{1}{\epsilon_1}\overline{n}_2x)^{\frac{p}{p-\zeta_2}} + \frac{\zeta_2}{p}(\epsilon_1K_1^{\zeta_2})^{\frac{p}{\zeta_2}}\\ &= \overline{N}_2x^{\frac{p}{p-\zeta_2}} + \frac{\zeta_2}{p}\frac{\tau-p}{2^{5+p}\tau\zeta_2}K_1^p\\ &= \overline{N}_2x^{\frac{p}{p-\zeta_2}} + \frac{\tau-p}{2^{5+p}\tau p}K_1^p,\end{aligned}$$

where $\overline{N}_2 = \frac{p-\zeta_2}{p}(\frac{\overline{n}_2}{\epsilon_1})^{\frac{p}{p-\zeta_2}}$.

Similarly, taking $\epsilon_2 = (\frac{\tau-p}{2^{4+p}\tau\zeta_2})^{\frac{\zeta_3}{p}}$, we obtain

$$\begin{aligned}\overline{n}_3K_2^{\zeta_3}x &\le \frac{p-\zeta_3}{p}(\frac{1}{\epsilon_2}\overline{n}_3x)^{\frac{p}{p-\zeta_3}} + \frac{\zeta_3}{p}(\epsilon_2K_2^{\zeta_3})^{\frac{p}{\zeta_3}}\\ &= \overline{N}_3x^{\frac{p}{p-\zeta_3}} + \frac{\zeta_3}{p}\frac{\tau-p}{2^{4+p}\tau\zeta_3}K_2^p\\ &= \overline{N}_3x^{\frac{p}{p-\zeta_3}} + \frac{\tau-p}{2^{4+p}\tau p}K_2^p,\end{aligned}$$

where $\overline{N}_3 = \frac{p-\zeta_3}{p}(\frac{\overline{n}_3}{\epsilon_2})^{\frac{p}{p-\zeta_3}}$. Define

$$Q_p(x) = \frac{x^p}{p} + \frac{x^q}{q} + \sum_{j=1}^{m}\overline{d}_jx^{\mu_j+1} - \overline{n}_1x^{\zeta_1+1} + \overline{N}_2x^{\frac{p}{p-\zeta_2}} + \overline{N}_3x^{\frac{p}{p-\zeta_3}} + \overline{C}x.$$

Then

$$\Psi_{u_1}(u_2) \le \max_{0\le x<\infty} Q_p(x) + \frac{\tau-p}{2^{5+p}p\tau}K_1^p + \frac{\tau-p}{2^{4+p}p\tau}K_2^p.$$

If $0 \le x < 1$, then

$$Q_p(x) \le \frac{1}{p} + \frac{1}{q} + \sum_{j=1}^{m}\overline{d}_j + \overline{N}_2 + \overline{N}_3 + \overline{C} := C_{1,p}.$$

If $1 \le x < \infty$, due to $\zeta_1 > q-1$, $0 < \zeta_2, \zeta_3 \le p-1$, $0 < \mu_j \le q-1$, $j = 1, 2, \ldots, m$, we deduce

$$Q_p(x) \le (\frac{1}{p} + \frac{1}{q} + \sum_{j=1}^{m}\overline{d}_j + \overline{N}_2 + \overline{N}_3 + \overline{C})x^q - \overline{n}_1x^{\zeta_1+1} := \overline{Q}_p(x).$$

Let $\overline{Q}_p{}'(x) = 0$, then

$$\overline{x} = (\frac{q(\frac{1}{p} + \frac{1}{q} + \sum_{j=1}^{m}\overline{d}_j + \overline{N}_2 + \overline{N}_3 + \overline{C})}{(\zeta_1+1)\overline{n}_1})^{\frac{1}{\zeta_1-q+1}}.$$

Obviously

$$Q_p(x) \le \overline{Q}_p(x) \le \overline{Q}_p(\overline{x}) := C_{2,p},$$

which implies that

$$Q_p(x) \le \max\{C_{1,p}, C_{2,p}\} := C_p.$$

Therefore

$$\Psi_{u_1}(u_2(t)) \le C_p + \frac{\tau-p}{2^{5+p}p\tau}K_1^p + \frac{\tau-p}{2^{4+p}p\tau}K_2^p. \tag{26}$$

With the help of (26) and $\Psi'_{u_1}(u_2) = 0$, we get

$$\begin{aligned}
&\tau\Psi_{u_1}(u_2(t)) - \langle \Psi'_{u_1}(u_2), u_2\rangle \\
= \ &(\frac{\tau}{p} - 1)\|u_2\|^p_{\alpha,p;\varphi} + (\frac{\tau}{q} - 1)\|u_2\|^q_{\beta,q;\varphi} + \sum_{j=1}^{m}\int_0^{u_2(t_j)} \tau I_j(s)ds - \sum_{j=1}^{m} I_j(u_2(t_j))u_n(t_j) \\
&- \int_0^T \varphi'(t)(\tau F(t, u_2(t), {}_0D_t^{\alpha;\varphi}u_1(t){}_0D_t^{\beta;\varphi}u_1(t)) - f(t, u_2(t), {}_0D_t^{\alpha;\varphi}u_1(t), {}_0D_t^{\beta;\varphi}u_1(t))u_2(t))dt \\
\leq \ &\tau C_p + \frac{\tau - p}{2^{5+p}p}K_1^p + \frac{\tau - p}{2^{4+p}p}K_2^p.
\end{aligned}$$

It follows from (S_1), Lemma 4 and Hölder inequality that

$$\begin{aligned}
&(\frac{\tau}{p} - 1)\|u_2\|^p_{\alpha,p;\varphi} \\
\leq \ &\tau C_p + \frac{\tau - p}{2^{5+p}p}K_1^p + \frac{\tau - p}{2^{4+p}p}K_2^p - \sum_{j=1}^{m}\int_0^{u_2(t_j)} \tau I_j(s)ds \\
&+ \sum_{j=1}^{m} I_j(u_2(t_j))u_2(t_j) + \int_0^T \varphi'(t)\big(\tau F(t, u_2(t), {}_0D_t^{\alpha;\varphi}u_1(t), {}_0D_t^{\beta;\varphi}u_1(t)) \\
&- f(t, u_2(t), {}_0D_t^{\alpha;\varphi}u_1(t), {}_0D_t^{\beta;\varphi}u_1(t))u_2(t)\big)dt \\
\leq \ &\tau C_p + \frac{\tau - p}{2^{5+p}p}K_1^p + \frac{\tau - p}{2^{4+p}p}K_2^p + \int_0^T a_1\varphi'(t)|u_2(t)|^{\gamma_1} + a_2\varphi'(t)|{}_0D_t^{\alpha;\varphi}u_1(t)|^{\gamma_2} \qquad (27) \\
&+ a_3\varphi'(t)|{}_0D_t^{\beta;\varphi}u_1(t)|^{\gamma_3} + m(t)\varphi'(t)dt + \sum_{j=1}^{m} n_j(t)|(u_2(t_j))|^{\sigma_j} \\
\leq \ &\tau C_p + \frac{\tau - p}{2^{5+p}p}K_1^p + \frac{\tau - p}{2^{4+p}p}K_2^p + a_1 M_{\varphi'}^{\frac{p-\gamma_1}{p}}(\frac{M_{\varphi'}[\varphi(T)]^{\alpha}}{\Gamma(\alpha+1)})^{\gamma_1}\|u_2(t)\|^{\gamma_1}_{\alpha,p;\varphi} \\
&+ a_2 M_{\varphi'}^{\frac{p-\gamma_2}{p}}\|u_1\|^{\gamma_2}_{\alpha,p;\varphi} + a_3 M_{\varphi'}^{\frac{q-\gamma_3}{q}}\|u_1\|^{\gamma_3}_{\beta,q;\varphi} + \|m(t)\varphi'(t)\|_{L^1} \\
&+ \sum_{j=1}^{m} n_j(t)(\frac{[\varphi(T)]^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)((\alpha-1)\frac{p}{p-1}+1)^{\frac{p-1}{p}}})^{\sigma_j}\|u_2\|^{\sigma_j}_{\alpha,p;\varphi}.
\end{aligned}$$

Taking $\epsilon_3 = (\frac{2\gamma_1}{\tau-p})^{\frac{\gamma_1}{p}}$, $\epsilon_4 = (\frac{2^{p+5}\gamma_2}{\tau-p})^{\frac{\gamma_2}{p}}$, $\epsilon_5 = (\frac{2^{p+4}\gamma_3}{\tau-p})^{\frac{\gamma_3}{p}}$ and $\epsilon_6 = (\frac{4m\sigma_j}{\tau-p})^{\frac{\sigma_j}{p}}$, by Young inequality, we obtain

$$\begin{aligned}
&a_1 M_{\varphi'}^{\frac{p-\gamma_1}{p}}(\frac{M_{\varphi'}[\varphi(T)]^{\alpha}}{\Gamma(\alpha+1)})^{\gamma_1}\|u_2(t)\|^{\gamma_1}_{\alpha,p;\varphi} \\
\leq \ &\frac{p-\gamma_1}{p}((\frac{2\gamma_1}{\tau-p})^{\frac{\gamma_1}{p}} a_1 M_{\varphi'}^{\frac{p-\gamma_1}{p}}(\frac{M_{\varphi'}[\varphi(T)]^{\alpha}}{\Gamma(\alpha+1)})^{\gamma_1})^{\frac{p}{p-\gamma_1}} + \frac{\gamma_1}{p}((\frac{\tau-p}{2\gamma_1})^{\frac{\gamma_1}{p}}\|u_2(t)\|^{\gamma_1}_{\alpha,p;\varphi})^{\frac{p}{\gamma_1}} \qquad (28) \\
:= \ &C_1 + \frac{\tau-p}{2p}\|u_2(t)\|^p_{\alpha,p;\varphi},
\end{aligned}$$

$$\begin{aligned}
a_2 M_{\varphi'}^{\frac{p-\gamma_2}{p}}\|u_1\|^{\gamma_2}_{\alpha,p;\varphi} \leq &\frac{p-\gamma_2}{p}(\epsilon_4 a_2 M_{\varphi'}^{\frac{p-\gamma_2}{p}})^{\frac{p}{p-\gamma_2}} + \frac{\gamma_2}{p}(\frac{1}{\epsilon_4}\|u_1\|^{\gamma_2}_{\alpha,p;\varphi})^{\frac{p}{\gamma_2}} \\
:= &C_2 + \frac{\tau-p}{2^{p+5}p}K_1^p,
\end{aligned} \qquad (29)$$

$$a_3 M_{\varphi'}^{\frac{q-\gamma_3}{q}} \|u_1\|_{\beta,q;\varphi}^{\gamma_3} \leq \frac{p-\gamma_3}{p}(\epsilon_5 a_3 M_{\varphi'}^{\frac{q-\gamma_3}{q}})^{\frac{p}{p-\gamma_3}} + \frac{\gamma_3}{p}(\frac{1}{\epsilon_5}\|u_1\|_{\alpha,p;\varphi}^{\gamma_2})^{\frac{p}{\gamma_2}} \\ := C_3 + \frac{\tau-p}{2^{p+4}p}K_2^p, \tag{30}$$

$$\begin{aligned} & n_j(t)(\frac{[\varphi(T)]^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)((\alpha-1)\frac{p}{p-1}+1)^{\frac{p-1}{p}}})^{\sigma_j}\|u_2\|_{\alpha,p;\varphi}^{\sigma_j} \\ \leq \; & \frac{p-\sigma_j}{p}(\epsilon_6 n_j(t)(\frac{[\varphi(T)]^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)((\alpha-1)\frac{p}{p-1}+1)^{\frac{p-1}{p}}})^{\sigma_j})^{\frac{p}{p-\sigma_j}} + \frac{\sigma_j}{p}(\frac{1}{\epsilon_6}\|u_2\|_{\alpha,p;\varphi}^{\sigma_j})^{\frac{p}{\sigma_j}} \\ := \; & C_j' + \frac{\tau-p}{4mp}\|u_2\|_{\alpha,p;\varphi}^p. \end{aligned} \tag{31}$$

Combining (28) and (31), (27) is transformed into

$$\begin{aligned} & (\frac{\tau}{p}-1)\|u_2\|_{\alpha,p;\varphi}^p \\ \leq & \tau C_p + \frac{\tau-p}{2^{5+p}p}K_1^p + \frac{\tau-p}{2^{4+p}p}K_2^p + \|m(t)\varphi'(t)\|_{L^1} + C_1 + \frac{\tau-p}{2p}\|u_2(t)\|_{\alpha,p;\varphi}^p \\ & + C_2 + \frac{\tau-p}{2^{p+5}p}K_1^p + C_3 + \frac{\tau-p}{2^{p+4}p}K_2^p + \sum_{j=1}^m C_j' + \frac{\tau-p}{4p}\|u_2\|_{\alpha,p;\varphi}^p \\ = & \tau C_p + C_1 + C_2 + C_3 + \sum_{j=1}^m C_j' + \frac{\tau-p}{2^{4+p}p}K_1^p + \frac{\tau-p}{2^{3+p}p}K_2^p + \frac{3(\tau-p)}{4p}\|u_2\|_{\alpha,p;\varphi}^p + \|m(t)\varphi'(t)\|_{L^1}. \end{aligned} \tag{32}$$

Therefore

$$\|u_2\|_{\alpha,p;\varphi}^p \leq \frac{4p}{\tau-p}(\tau C_p + C_1 + C_2 + C_3 + \sum_{j=1}^m C_j' + \|m(t)\varphi'(t)\|_{L^1}) + \frac{K_1^p}{2^{p+2}} + + \frac{K_2^p}{2^{p+1}}. \tag{33}$$

Define

$$K_1 = (\frac{2^{p+4}p}{\tau-p}(\tau C_p + C_1 + C_2 + C_3 + \sum_{j=1}^m C_j' + \|m(t)\varphi'(t)\|_{L^1}))^{\frac{1}{p}}.$$

Then

$$\|u_2\|_{\alpha,p;\varphi}^p \leq (\frac{1}{2^{p+1}}K_1^p + \frac{1}{2^{p+1}}K_2^p)^{\frac{1}{p}} \leq \frac{1}{2}(K_1 + K_2).$$

Similar to (27)–(33), taking $\epsilon_1' = (\frac{\tau-q}{2^{4+q}\tau\zeta_2})^{\frac{\zeta_2}{q}}$ and $\epsilon_2' = (\frac{\tau-q}{2^{q+5}\tau\zeta_3})^{\frac{\zeta_3}{q}}$, by Young inequality, we conclude

$$\overline{n}_2 K_1^{\zeta_2} x \leq \frac{q-\zeta_2}{q}(\frac{1}{\epsilon_1'}\overline{n}_2 x)^{\frac{q}{q-\zeta_2}} + \frac{\zeta_2}{q}(\epsilon_1' K_1^{\zeta_2})^{\frac{q}{\zeta_2}} = \overline{N}_2' x^{\frac{q}{q-\zeta_2}} + \frac{\tau-q}{2^{4+q}\tau q}K_1^q,$$

$$\overline{n}_3 K_2^{\zeta_3} x \leq \frac{q-\zeta_3}{q}(\frac{1}{\epsilon_2'}\overline{n}_3 x)^{\frac{q}{q-\zeta_3}} + \frac{\zeta_3}{q}(\epsilon_2' K_2^{\zeta_3})^{\frac{q}{\zeta_3}} = \overline{N}_3' x^{\frac{q}{q-\zeta_3}} + \frac{\tau-q}{2^{5+q}\tau q}K_2^q,$$

where $\overline{N}_2' = \frac{q-\zeta_2}{q}(\frac{\overline{n}_2}{\epsilon_1'})^{\frac{q}{q-\zeta_2}}$, $\overline{N}_3' = \frac{q-\zeta_3}{q}(\frac{\overline{n}_3}{\epsilon_2'})^{\frac{q}{q-\zeta_3}}$.

Let

$$Q_q(x) = \frac{x^p}{p} + \frac{x^q}{q} + \sum_{j=1}^m \overline{d}_j x^{\mu_j+1} - \overline{n}_1 x^{\zeta_1+1} + \overline{N}_2' x^{\frac{p}{p-\zeta_2}} + \overline{N}_3' x^{\frac{q}{q-\zeta_3}} + \overline{C}x.$$

Then

$$\Psi_{u_1}(u_2) \leq \max_{0 \leq x < \infty} Q_q(x) + \frac{\tau - q}{2^{q+4} q\tau} K_1^q + \frac{\tau - q}{2^{5+q} q\tau} K_2^q.$$

Similar to $Q_p(x)$, there exists a nonnegative constant C_q satisfying $Q_q(x) \leq C_q$ for all $0 \leq x < \infty$. Therefore

$$\Psi_{u_1}(u_2) \leq C_q + \frac{\tau - q}{2^{q+4} q\tau} K_1^q + \frac{\tau - q}{2^{5+q} q\tau} K_2^q.$$

In terms of (S_1), Lemma 4 and Hölder inequality, we yield

$$\begin{aligned}
(\frac{\tau}{q} - 1)\|u_2\|^q_{\beta,q;\varphi} \leq\ & \tau C_q + \frac{\tau - q}{2^{q+4} q} K_1^q + \frac{\tau - q}{2^{q+5} q} K_2^q - \sum_{j=1}^m \int_0^{u_2(t_j)} \tau I_j(s) ds \\
& + \sum_{j=1}^m I_j(u_2(t_j)) u_2(t_j) + \int_0^T \varphi'(t)(\tau F(t, u_2(t), {}_0D_t^{\alpha;\varphi} u_1(t), {}_0D_t^{\beta;\varphi} u_1(t)) \\
& - f(t, u_2(t), {}_0D_t^{\alpha;\varphi} u_1(t), {}_0D_t^{\beta;\varphi} u_1(t)) u_2(t)) dt \\
\leq\ & \tau C_q + \frac{\tau - q}{2^{q+4} q} K_1^q + \frac{\tau - q}{2^{q+5} q} K_2^q + a_1 M_{\varphi'}^{\frac{q-\gamma_1}{q}} (\frac{M_{\varphi'}[\varphi(T)]^\beta}{\Gamma(\beta+1)})^{\gamma_1} \|u_2(t)\|^{\gamma_1}_{\beta,q;\varphi} \\
& + a_2 M_{\varphi'}^{\frac{p-\gamma_2}{p}} K_1^{\gamma_2} + a_3 M_{\varphi'}^{\frac{q-\gamma_3}{q}} K_2^{\gamma_3} + \|m(t)\varphi'(t)\|_{L^1} \\
& + \sum_{j=1}^m n_j(t) (\frac{[\varphi(T)]^{\beta - \frac{1}{q}}}{\Gamma(\beta)((\beta-1)\frac{q}{q-1} + 1)^{\frac{q-1}{q}}})^{\sigma_j} \|u_2\|^{\sigma_j}_{\beta,q;\varphi}.
\end{aligned}$$

Taking $\epsilon_3{}' = (\frac{2\gamma_1}{\tau - q})^{\frac{\gamma_1}{q}}$, $\epsilon_4{}' = (\frac{2^{q+4}\gamma_2}{\tau - q})^{\frac{\gamma_2}{q}}$, $\epsilon_5{}' = (\frac{2^{q+5}\gamma_3}{\tau - q})^{\frac{\gamma_3}{q}}$ and $\epsilon_6{}' = (\frac{4m\sigma_j}{\tau - q})^{\frac{\sigma_j}{q}}$, by Young inequality, we obtain

$$\begin{aligned}
& a_1 M_{\varphi'}^{\frac{q-\gamma_1}{q}} (\frac{M_{\varphi'}[\varphi(T)]^\beta}{\Gamma(\beta+1)})^{\gamma_1} \|u_2(t)\|^{\gamma_1}_{\beta,q;\varphi} \\
\leq\ & \frac{q - \gamma_1}{q} ((\frac{2\gamma_1}{\tau - q})^{\frac{\gamma_1}{q}} a_1 M_{\varphi'}^{\frac{q-\gamma_1}{q}} (\frac{M_{\varphi'}[\varphi(T)]^\beta}{\Gamma(\beta+1)})^{\gamma_1})^{\frac{q}{q-\gamma_1}} + \frac{\gamma_1}{q} ((\frac{\tau - q}{2\gamma_1})^{\frac{\gamma_1}{q}} \|u_2(t)\|^{\gamma_1}_{\beta,q;;\varphi})^{\frac{q}{\gamma_1}} \\
:=\ & C_1{}' + \frac{\tau - q}{2q} \|u_2(t)\|^q_{\beta,q;\varphi},
\end{aligned}$$

$$\begin{aligned}
a_2 M_{\varphi'}^{\frac{p-\gamma_2}{p}} K_1^{\gamma_2} \leq\ & \frac{q - \gamma_2}{q} (\epsilon_4{}' a_2 M_{\varphi'}^{\frac{p-\gamma_2}{p}})^{\frac{q}{q-\gamma_2}} + \frac{\gamma_2}{q} (\frac{1}{\epsilon_4{}'} K_1^{\gamma_2})^{\frac{q}{\gamma_2}} \\
:=\ & C_2{}' + \frac{\tau - q}{2^{q+4} q} K_1^q,
\end{aligned}$$

$$\begin{aligned}
a_3 M_{\varphi'}^{\frac{q-\gamma_3}{q}} K_2^{\gamma_3} \leq\ & \frac{p - \gamma_3}{p} (\epsilon_5 a_3 M_{\varphi'}^{\frac{q-\gamma_3}{q}})^{\frac{p}{p-\gamma_3}} + \frac{\gamma_3}{p} (\frac{1}{\epsilon_5} K_2^{\gamma_2})^{\frac{p}{\gamma_2}} \\
:=\ & C_3' + \frac{\tau - p}{2^{p+4} p} K_2^p,
\end{aligned}$$

$$
\begin{aligned}
& n_j(t)\Big(\frac{[\varphi(T)]^{\beta-\frac{1}{q}}}{\Gamma(\beta)((\beta-1)\frac{q}{q-1}+1)^{\frac{q-1}{q}}}\Big)^{\sigma_j}\|u_2\|^{\sigma_j}_{\beta,q;\varphi} \\
\leq\ & \frac{q-\sigma_j}{q}\Big(\epsilon_6{}' n_j(t)\big(\frac{[\varphi(T)]^{\beta-\frac{1}{q}}}{\Gamma(\beta)((\beta-1)\frac{q}{q-1}+1)^{\frac{q-1}{q}}}\big)^{\sigma_j}\Big)^{\frac{q}{q-\sigma_j}}+\frac{\sigma_j}{q}\Big(\frac{1}{\epsilon_6{}'}\|u_2\|^{\sigma_j}_{\beta,q;\varphi}\Big)^{\frac{q}{\sigma_j}} \\
:=\ & C_j'' + \frac{\tau-q}{4mq}\|u_2\|^q_{\beta,q;\varphi}.
\end{aligned}
$$

Then, (27) is changed into

$$
\begin{aligned}
(\frac{\tau}{q}-1)\|u_2\|^q_{\beta,q;\varphi} \leq\ & \tau C_q + C_1{}' + C_2{}' + C_3{}' + \sum_{j=1}^{m} C_j'' + \frac{\tau-q}{2^{q+3}q}K_1^q + \frac{\tau-q}{2^{q+4}q}K_2^q \\
& + \frac{\tau-q}{2q}\|u_2\|^q_{\beta,q;\varphi} + \frac{\tau-q}{4q}\|u_2\|^q_{\beta,q;\varphi} + \|m(t)\varphi'(t)\|_{L^1}.
\end{aligned}
$$

Define

$$
K_2 = \Big(\frac{2^{q+4}q}{\tau-q}\big(\tau C_p + C_1{}' + C_2{}' + C_3{}' + \sum_{j=1}^{m} C_j'' + \|m(t)\varphi'(t)\|_{L^1}\big)\Big)^{\frac{1}{q}}.
$$

Then

$$
\|u_2\|^q_{\beta,q;\varphi} \leq \Big(\frac{1}{2^{q+1}}K_1^q + \frac{1}{2^{q+1}}K_2^q\Big)^{\frac{1}{q}} \leq \frac{1}{2}(K_1+K_2).
$$

Therefore

$$
\|u_2\|_E \leq K_1 + K_2 \leq K.
$$

Suppose that $\|u_{n-1}\|^p_{\alpha,p;\varphi} \leq K_1$ and $\|u_{n-1}\|^q_{\beta,q;\varphi} \leq K_2$, by repeating the above discussion process, we can obtain that $\|u_n\|_E \leq K$ for all $n \in N$. Therefore, we have constructed a sequence of critical points, which is bounded. The reflexivity of the space E implies that $u_n \rightharpoonup u^*$ as $n \to \infty$.

Step 2: We need to show that $u_n \to u^*$ in E as $n \to \infty$ and u^* is a nontrivial solution of problem (1). For any $m, n \in N$, $m \neq n$, in view of (7), one has

$$
\begin{aligned}
& X^{u_m,u_n}_{\alpha,p;\varphi} + X^{u_m,u_n}_{\beta,q;\varphi} \\
=\ & (\Psi'_{u_{m-1}}(u_m(t)) - \Psi'_{u_{n-1}}(u_n(t)))(u_m(t)-u_n(t)) \\
& - \sum_{j=1}^{m}[I_j(u_m(t_j)) - I_j(u_n(t_j))](u_m(t_j)-u_n(t_j)) \\
& + \int_0^T f(t, u_m(t), {}_0D_t^{\alpha;\varphi}u_{m-1}(t), {}_0D_t^{\beta;\varphi}u_{m-1}(t))\varphi'(t)(u_m(t)-u_n(t))dt \\
& - \int_0^T f(t, u_n(t), {}_0D_t^{\alpha;\varphi}u_{n-1}(t), {}_0D_t^{\beta;\varphi}u_{n-1}(t))\varphi'(t)(u_m(t)-u_n(t))dt.
\end{aligned}
$$

Combining $\Psi'_{u_{m-1}}(u_m) = 0$, $\Psi'_{u_{n-1}}(u_n) = 0$ with (S_4), we obtain

$$\begin{aligned}
X^{u_m,u_n}_{\alpha,p;\varphi} + X^{u_m,u_n}_{\beta,q;\varphi} \leq & \int_0^T [b_1|u_m(t) - u_n(t)| + b_2|{}_0D_t^{\alpha;\varphi}u_{m-1}(t) - {}_0D_t^{\alpha;\varphi}u_{n-1}(t)| \\
& + b_3|{}_0D_t^{\beta;\varphi}u_{m-1}(t) - {}_0D_t^{\beta;\varphi}u_{n-1}(t)|]\varphi'(t)|u_m(t) - u_n(t)|dt \\
& + \sum_{j=1}^{m} \varphi'(t_j)e_j|u_m(t_j) - u_n(t_j)|^2 \\
\leq & (b_1 M_{\varphi'} T\|u_m - u_n\| + b_2 M_{\varphi'}^{\frac{p-1}{p}} \|u_{m-1} - u_{n-1}\|^p_{\alpha,p;\varphi} \\
& + b_3 M_{\varphi'}^{\frac{q-1}{q}} \|u_{m-1} - u_{n-1}\|^q_{\beta,q;\varphi})\|u_m - u_n\| + \sum_{j=1}^{m} \varphi'(t_j)e_j\|u_m - u_n\|^2 \\
= & \big(b_1 M_{\varphi'} T\|u_m - u_n\| + b_2 M_{\varphi'}^{\frac{p-1}{p}} \|u_{m-1} - u_{n-1}\|^p_{\alpha,p;\varphi} \\
& + b_3 M_{\varphi'}^{\frac{q-1}{q}} \|u_{m-1} - u_{n-1}\|^q_{\beta,q;\varphi} + \sum_{j=1}^{m} \varphi'(t_j)e_j\|u_m - u_n\|\big)\|u_m - u_n\|.
\end{aligned} \tag{34}$$

By Lemma 5, we know $\|u_m - u_n\| \to 0$. Therefore, (34) implies that

$$X^{u_m,u_n}_{\alpha,p;\varphi} + X^{u_m,u_n}_{\beta,q;\varphi} \to 0 \ (m, n \to \infty). \tag{35}$$

By (17), (18), (19) and (35), we can see that $\|u_m - u_n\|_{\alpha,p;\varphi} \to 0$ and $\|u_m - u_n\|_{\beta,q;\varphi} \to 0$ as $m, n \to \infty$. Hence, $\{u_n\}$ converges to $u^* \in E$ strongly.

Next, we will prove that $\Psi'_{u^*}(u^*) = 0$. As shown in [32], for any $s_1, s_2 \in \mathbb{R}^N$, there exist nonnegative constants r_1 r_2 such that

$$|\ |s_1|^{r-2}s_1 - |s_2|^{r-2}s_2\ | \leq \begin{cases} r_1|s_1 - s_2|(|s_1| + |s_2|)^{r-2}, & r \geq 2, \\ r_2|s_1 - s_2|^{r-1}, & 1 < r \leq 2. \end{cases} \tag{36}$$

Due to (36), for $p \geq 2$, there exist constants $r_1, r_2 > 0$ such that

$$\begin{aligned}
& \Big| |{}_0D_t^{\alpha;\varphi}u_n(t)|^{p-2}{}_0D_t^{\alpha;\varphi}u_n(t) - |{}_0D_t^{\alpha;\varphi}u^*(t)|^{p-2}{}_0D_t^{\alpha;\varphi}u^*(t) \Big| \\
& - \Big| |u_n(t)|^{p-2}u_n(t) - |u^*(t)|^{p-2}u^*(t) \Big| \\
\leq & r_1\Big|{}_0D_t^{\alpha;\varphi}(u_n(t) - u^*(t))\Big|\big(|{}_0D_t^{\alpha;\varphi}u_n(t)| + |{}_0D_t^{\alpha;\varphi}u^*(t)|\big)^{p-2} \\
& + r_2|u_n(t) - u^*(t)|\big(|u_n(t)| + |u^*(t)|\big)^{p-2}.
\end{aligned} \tag{37}$$

Based on (37), Lemma 4 and Hölder inequality, we obtain

$$\begin{aligned}
&\int_0^T \varphi'(t)(|{}_0D_t^{\alpha;\varphi}u_n(t)|^{p-2}{}_0D_t^{\alpha;\varphi}u_n(t) - |{}_0D_t^{\alpha;\varphi}u^*(t)|^{p-2}{}_0D_t^{\alpha;\varphi}u^*(t)){}_0D_t^{\alpha;\varphi}v(t) \\
&+ \varphi'(t)(|u_n(t)|^{p-2}u_n(t) - |u^*(t)|^{p-2}u^*(t))v(t)dt \\
\leq &\int_0^T r_1\varphi'(t)|{}_0D_t^{\alpha;\varphi}(u_n(t) - u^*(t))|(|{}_0D_t^{\alpha;\varphi}u_n(t)| + |{}_0D_t^{\alpha;\varphi}u^*(t)|)^{p-2}{}_0D_t^{\alpha;\varphi}v(t) \\
&+ r_2\varphi'(t)r_2|u_n(t) - u^*(t)|(|u_n(t)| + |u^*(t)|)^{p-2}{}_0D_t^{\alpha;\varphi}v(t)dt \\
\leq &r_1\|u_n(t) - u^*(t)\|_{\alpha,p;\varphi}\|\varphi'(t)^{\frac{p-2}{p}}(|{}_0D_t^{\alpha;\varphi}u_n(t)| + |{}_0D_t^{\alpha;\varphi}u^*(t)|)\|_{L^p}^{p-2}\|\varphi'(t)^{\frac{1}{p}}{}_0D_t^{\alpha;\varphi}v(t)\|_{L^p} \\
&+ r_2\|\varphi'(t)^{\frac{1}{p}}(u_n(t) - u^*(t))\|_{L^p}\|\varphi'(t)^{\frac{p-2}{p}}(|u_n(t)| + |u^*(t)|)\|_{L^p}^{p-2}\|\varphi'(t)^{\frac{1}{p}}v(t)\|_{L^p} \\
\leq &r_1\|u_n(t) - u^*(t)\|_{\alpha,p;\varphi}\|\varphi'(t)^{\frac{p-2}{p}}(|{}_0D_t^{\alpha;\varphi}u_n(t)| + |{}_0D_t^{\alpha;\varphi}u^*(t)|)\|_{L^p}^{p-2}\|\varphi'(t)^{\frac{1}{p}}{}_0D_t^{\alpha;\varphi}v(t)\|_{L^p} \\
&+ r_2\frac{M_{\varphi'}[\varphi(T)]^\alpha}{\Gamma(\alpha+1)}\|u_n(t) - u^*(t)\|_{\alpha,p;\varphi}\|\varphi'(t)^{\frac{p-2}{p}}(|u_n(t)| + |u^*(t)|)\|_{L^p}^{p-2}\|\varphi'(t)^{\frac{1}{p}}v(t)\|_{L^p},
\end{aligned}$$

for any $v \in E$. If $1 < p \leq 2$, there exist constants $r_3, r_4 > 0$ such that

$$\begin{aligned}
&\int_0^T \varphi'(t)(|{}_0D_t^{\alpha;\varphi}u_n(t)|^{p-2}{}_0D_t^{\alpha;\varphi}u_n(t) - |{}_0D_t^{\alpha;\varphi}u^*(t)|^{p-2}{}_0D_t^{\alpha;\varphi}u^*(t)){}_0D_t^{\alpha;\varphi}v(t) \\
&+ \varphi'(t)(|u_n(t)|^{p-2}u_n(t) - |u^*(t)|^{p-2}u^*(t))v(t)dt \\
\leq &\int_0^T r_3\varphi'(t)|{}_0D_t^{\alpha;\varphi}u_n(t) - {}_0D_t^{\alpha;\varphi}u^*(t)|^{p-1}{}_0D_t^{\alpha;\varphi}v(t) + r_4\varphi'(t)|u_n(t) - u^*(t)|^{p-1}v(t)dt \\
\leq &r_3\|u_n(t) - u^*(t)\|_{\alpha,p;\varphi}\|\varphi'(t)^{\frac{1}{p}}{}_0D_t^{\alpha;\varphi}v(t)\|_{L^p} + r_4\frac{M_{\varphi'}[\varphi(T)]^\alpha}{\Gamma(\alpha+1)}\|u_n(t) - u^*(t)\|_{\alpha,p;\varphi}\|\varphi'(t)^{\frac{1}{p}}v(t)\|_{L^p}.
\end{aligned} \tag{38}$$

By the fact $\|u_n(t) - u^*(t)\|_E \to 0$ as $n \to \infty$, we know

$$\begin{aligned}
&\int_0^T \varphi'(t)\Phi_p({}_0D_t^{\alpha;\varphi}u_n(t)){}_0D_t^{\alpha;\varphi}v(t) + \varphi'(t)\Phi_p(u_n(t))v(t)dt \\
&\to \int_0^T \varphi'(t)\Phi_p({}_0D_t^{\alpha;\varphi}u^*(t)){}_0D_t^{\alpha;\varphi}v(t) + \varphi'(t)\Phi_p(u^*(t))v(t)dt,
\end{aligned} \tag{39}$$

as $n \to \infty$. Likewise, we acquire

$$\begin{aligned}
&\int_0^T \varphi'(t)\Phi_q({}_0D_t^{\beta;\varphi}u_n(t)){}_0D_t^{\beta;\varphi}v(t) + \varphi'(t)\Phi_q(u_n(t))v(t)dt \\
&\to \int_0^T \varphi'(t)\Phi_q({}_0D_t^{\beta;\varphi}u^*(t)){}_0D_t^{\beta;\varphi}v(t) + \varphi'(t)\Phi_q(u^*(t))v(t)dt,
\end{aligned} \tag{40}$$

as $n \to \infty$. Furthermore

$$\begin{aligned}
&\int_0^T \varphi'(t) f(t, u_n(t), {}_0D_t^{\alpha;\varphi}(u_{n-1}(t)), {}_0D_t^{\beta;\varphi}(u_{n-1}(t)))v(t)dt \\
&- \int_0^T \varphi'(t) f(t, u^*(t), {}_0D_t^{\alpha;\varphi}(u^*(t)), {}_0D_t^{\beta;\varphi}(u^*(t)))v(t)dt \\
\le &\int_0^T \varphi'(t)v(t)(b_1|u_n(t) - u^*(t)| + b_2\big|{}_0D_t^{\alpha;\varphi}(u_{n-1}(t)) - {}_0D_t^{\alpha;\varphi}(u^*(t))\big|)dt \\
&+ \int_0^T b_3\varphi'(t)v(t)\big|{}_0D_t^{\beta;\varphi}(u_{n-1}(t)) - {}_0D_t^{\beta;\varphi}(u^*(t))\big|dt \\
\le &b_1\|\varphi'(t)^{\frac{1}{p}}|u_n(t) - u^*(t)|\|_{L^p}\|\varphi'(t)^{\frac{p-1}{p}}|v(t)|\|_{L^{\frac{p}{p-1}}} \\
&+ b_2\|u_{n-1}(t) - u^*(t)\|_{\alpha,p;\varphi}\|\varphi'(t)^{\frac{p-1}{p}}|v(t)|\|_{L^{\frac{p}{p-1}}} \\
&+ b_3\|u_{n-1}(t) - u^*(t)\|_{\beta,q;\varphi}\|\varphi'(t)^{\frac{q-1}{q}}|v(t)|\|_{L^{\frac{q}{q-1}}} \\
\le &b_1\frac{M_{\varphi'}[\varphi(T)]^\alpha}{\Gamma(\alpha+1)}\|u_n(t) - u^*(t)\|_{\alpha,p;\varphi}\|\varphi'(t)^{\frac{p-1}{p}}|v(t)|\|_{L^{\frac{p}{p-1}}} + b_2\|u_{n-1}(t) - u^*(t)\|_{\alpha,p;\varphi} \\
&\times \|\varphi'(t)^{\frac{p-1}{p}}|v(t)|\|_{L^{\frac{p}{p-1}}} + b_3\|u_{n-1}(t) - u^*(t)\|_{\beta,q;\varphi}\|\varphi'(t)^{\frac{q-1}{q}}|v(t)|\|_{L^{\frac{q}{q-1}}},
\end{aligned} \tag{41}$$

which implies that

$$\begin{aligned}
&\int_0^T \varphi'(t) f(t, u_n(t), {}_0D_t^{\alpha;\varphi}(u_{n-1}(t)), {}_0D_t^{\beta;\varphi}(u_{n-1}(t)))v(t)dt \\
&\quad \to \int_0^T \varphi'(t) f(t, u^*(t), {}_0D_t^{\alpha;\varphi}(u^*(t)), {}_0D_t^{\beta;\varphi}(u^*(t)))v(t)dt,
\end{aligned} \tag{42}$$

as $n \to \infty$. Notice that $I_j \in C(\mathbb{R}, \mathbb{R})$, then we obtain

$$\sum_{j=1}^m I_j(u_n(t_j))v(t_j) \to \sum_{j=1}^m I_j(u^*(t_j))v(t_j), \tag{43}$$

as $n \to \infty$. From (38)–(43) and the fact $\langle \Psi'_{u_{n-1}}(u_n), v\rangle = 0$, we deduce

$$\begin{aligned}
&\int_0^T \varphi'(t)\Phi_p({}_0D_t^{\alpha;\varphi}u^*(t)){}_0D_t^{\alpha;\varphi}v(t) + \varphi'(t)\Phi_p(u^*(t))v(t)dt \\
&+ \int_0^T \varphi'(t)\Phi_q({}_0D_t^{\beta;\varphi}u^*(t)){}_0D_t^{\beta;\varphi}v(t) + \varphi'(t)\Phi_q(u^*(t))v(t)dt + \sum_{j=1}^m I_j(u^*(t_j))v(t_j) \\
= \; &\int_0^T \varphi'(t) f(t, u^*(t), {}_0D_t^{\alpha;\varphi}(u^*(t)), {}_0D_t^{\beta;\varphi}(u^*(t)))v(t)dt,
\end{aligned}$$

which demonstrates that $\langle \Psi'_{u^*}(u^*), v\rangle = 0$ for any $v \in E$, i.e., u^* is a weak solution of problem (1). According to Lemma 6, we know that u^* is also the classical solution of problem (1). Furthermore, we can see that $\Psi_{u^*}(u^*) \ge \sigma > 0$ by Theorem 1. Therefore, u^* is a nontrivial solution of problem (1). □

4. Example

In this section, we consider the following nonlinear impulsive generalized fractional differential equations with (p,q)-Laplacian operator

$$\begin{cases} {}_tD_T^{\frac{1}{2};t^2}\Phi_3({}_0D_t^{\frac{1}{2};t^2}u(t)) + |u(t)|u(t) + {}_tD_T^{\frac{1}{3};t^2}\Phi_4({}_0D_t^{\frac{1}{3};t^2}u(t)) + |u(t)|^2u(t), \\ \quad = f\big(t,u(t),{}_0D_t^{\frac{1}{2};t^2}u(t),{}_0D_t^{\frac{1}{3};t^2}u(t)\big), \quad t \neq t_j, \text{ a.e. } t \in [0,T], \\ \Delta\big({}_tD_T^{-\frac{1}{2};t^2}\Phi_3({}_0^CD_t^{\frac{1}{2};t^2}u) + {}_tD_T^{-\frac{1}{3};t^2}\Phi_4({}_0^CD_t^{\frac{1}{3};t^2}u)\big)(t_j) = I_j(u(t_j)), \quad j = 1,2,\ldots,m, \\ u(0) = u(T) = 0, \quad \text{a.e. } t \in [0,T], \end{cases} \tag{44}$$

where $f(t,u,v,w) = 5e^{-t}u^4 + 4tu^4\sin v + 4tu^4\cos w$, $I_j(t) = \frac{1}{2}t^2$. Then, we obtain that $F(t,u,v,w) = e^{-t}u^5 + \frac{4}{5}tu^5\sin v + \frac{4}{5}tu^5\cos w$.

Let $p = 3$, $q = 4$, $\tau = 5$. A simple calculation shows that

$$5F(t,u,v,w) - f(t,u,v,w)u \leq 0,$$

and

$$I_j(u)u - 5\int_0^u I_j(s)ds \leq 0,$$

which means (S_1) holds. By Lemma 7, the functional of problem (44) satisfies Palais–Smale condition.

Choose $\lambda_1 = \lambda_2 = \lambda_3 = 4$, $\xi_1 = 1$, $\xi_2 = 1$, $m_1 = 5$, $m_2 = m_3 = 4$, $\zeta_1 = 4$, $n_1 = \frac{5}{e^T}$, $n_2 = n_3 = 0$, one has

$$\begin{aligned} f(t,u,v,w) &= 5e^{-t}u^4 + 4tu^4\sin v + 4tu^4\cos w \leq 5|u|^4 + 4T|u|^4|v| + 4T|u|^4|w|, \\ f(t,u,v,w) &\geq \frac{5}{e^T}|u|^4. \end{aligned}$$

Choose $\tau_j = 4$, $\mu_j = 2(j = 1,2,...,m)$, we have

$$I_j(t) \geq -\frac{1}{2}t^4 \quad \text{and} \quad I_j(t) \leq 2t^2.$$

Then, the conditions (S_2) and (S_3) of Theorem 3 are satisfied. Furthermore, according to the mean value theorem, the condition (S_4) holds. As a consequence, the nonlinear impulsive generalized fractional differential equation (44) has at least a nontrivial solution by adopting Theorem 3.

5. Conclusions

By means of the mountain pass theorem and iterative technique, this paper derives some new solvability results for nonlinear impulsive generalized fractional differential equations with (p,q)-Laplacian operator in Banach space. In detail, since the nonlinearity f contains generalized fractional derivatives ${}_0D_t^{\alpha;\varphi}u$ and ${}_0D_t^{\beta;\varphi}u$, and the impulses exist in generalized fractional calculus, it is more difficult to prove the existence of a nontrivial solution to problem (1). By imposing constraints on the nonlinear f and impulses, we have proved that functional (7) satisfies the Palais–Smale condition the first time and verified that functional (7) has at least one critical point by the mountain pass theorem. Finally, a series of critical points is constructed and the iterative method is used to prove that the critical points sequence converges to a point, which is a non-trivial solution of problem (1).

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References

1. Zhang, W.; Ni, J. New multiple positive solutions for Hadamard-type fractional differential equations with nonlocal conditions on an infinite interval. *Appl. Math. Lett.* **2021**, *118*, 107165. [CrossRef]
2. Sun, H.R.; Zhang, Q.G. Existence of solutions for a fractional boundary value problem via the Mountain pass method and an iterative technique. *Comput. Math. Appl.* **2012**, *64*, 3436–3443. [CrossRef]
3. Jiao, F.; Zhou, Y. Existence of solutions for a class of fractional boundary value problems via critical point theory. *Comput. Math. Appl. Int. J.* **2011**, *62*, 1181–1199. [CrossRef]
4. Ahmad, B.; Zhou, Y.; Alsaedi, A.; Al-Hutami, H. Existence and uniqueness results for a nonlocal q-fractional integral boundary value problem of sequential orders. *J. Comput. Anal. Appl.* **2016**, *20*, 514–529.
5. Fahad, H.M.; Fernandez, A. Operational calculus for Caputo fractional calculus with respect to functions and the associated fractional differential equations. *Appl. Math. Comput.* **2021**, *409*, 126400. [CrossRef]
6. El-Hady, E.; Ogrekci, S. On Hyers-Ulam-Rassias stability of fractional differential equations with Caputo derivative. *J. Math. Comput. Sci.* **2021**, *22*, 325–332. [CrossRef]
7. Hattaf, K.; Mohsen, A.A.; Al-Husseiny, H. Gronwall inequality and existence of solutions for differential equations with generalized Hattaf fractional derivative. *J. Math. Comput. Sci.* **2022**, *27*, 18–27. [CrossRef]
8. Nikan, Q.; Avazzadeh, Z.; Tenreiro Machado, J. An efficient local meshless approach for solving nonlinear time-fractional fourth-order diffusion model. *J. King Saud Univ.—Sci.* **2021**, *33*, 101243. [CrossRef]
9. Phong, T.; Long, L. Well-posed results for nonlocal fractional parabolic equation involving Caputo-Fabrizio operator. *J. Math. Comput. Sci.* **2022**, *22*, 357–367. [CrossRef]
10. Almeida, R.; Malinowska, A.B.; Monteiro, T. Fractional differential equations with a Caputo derivative with respect to a kernel function and their applications. *Math. Methods Appl. Sci.* **2017**, *41*, 336–352. [CrossRef]
11. Restrepo, J.E.; Ruzhansky, M.; Suragan, D. Explicit solutions for linear variable-coefficient fractional differential equations with respect to functions. *Appl. Math. Comput.* **2021**, *403*, 126177. [CrossRef]
12. Sousa, J.V.C.; De Oliveira, E.C. On the ψ-Hilfer fractional derivative. *Commun. Nonlinear Sci. Numer. Simul.* **2018**, *60*, 72–91. [CrossRef]
13. Belmor, S.; Jarad, F.; Abdeljawad, T.; Alqudah, M.A. On fractional differential inclusion problems involving fractional order derivative with respect to another function. *Fractals* **2020**, *28*, 204002. [CrossRef]
14. Liu, K.; Fečkan, M.; Wang, J. Hyers-Ulam stability and existence of solutions to the generalized Liouville-Caputo fractional differential equations. *Symmetry* **2020**, *12*, 955. [CrossRef]
15. Lin, L.; Liu, Y.; Zhao, D. Controllability of impulsive ψ-Caputo fractional evolution equations with nonlocal conditions. *Mathematics* **2021**, *9*, 1358. [CrossRef]
16. Zhao, D. A study on controllability of a class of impulsive fractional nonlinear evolution equations with delay in Banach spaces. *Fractal Fract.* **2021**, *5*, 279. [CrossRef]
17. Alsarori, N.; Ghadle, K.; Sessa, S.; Saleh, H.; Alabiad, S. New study of the existence and dimension of the set of solutions for nonlocal impulsive differential inclusions with a sectorial operator. *Symmetry* **2021**, *13*, 491. [CrossRef]
18. Alharbi, A.; Guefaifia, R.; Boulaaras, S. A new proof of the existence of nonzero weak solutions of impulsive fractional boundary value problems. *Mathematics* **2020**, *8*, 856. [CrossRef]
19. Long, W.W.; Chen, G.P. Infinitely many solutions for a class of p-Laplacian type fractional Dirichlet problem with instantaneous and non-instantaneous impulses. *Appl. Math. Sci.* **2022**, *16*, 79–94. [CrossRef]
20. Li, D.; Chen, F.; An, Y. The existence of solutions for an impulsive fractional coupled system of (p, q)-Laplacian type without the Ambrosetti-Rabinowitz condition. *Math. Methods Appl. Sci.* **2019**, *42*, 1449–1464. [CrossRef]
21. Li, D.; Chen, F.; Wu, Y.; An, Y. Variational formulation for nonlinear impulsive fractional differential equations with (p,q)-Laplacian operator. *Math. Methods Appl. Sci.* **2022**, *45*, 515–531. [CrossRef]
22. Kilbas, A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractinal Differential Equations*; North-Holland Mathematics Studies: Amsterdam, The Netherlands, 2006; p. 204.
23. Jarad, F.; Abdeljawad, T. Generalized fractional derivatives and Laplace transform. *Discret. Contin. Dyn. Syst.-S* **2018**, *13*, 709–722. [CrossRef]
24. Samko, S.G.; Kilbas, A.A.; Marichev, O.I. *Fractional Integrals and Derivatives*; Gordon and Breach Science Publishers: Yverdon-les-Bains, Switzerland, 1993.
25. Yang, X.J.; Machado, J. A new fractional operator of variable order: Application in the description of anomalous diffusion. *Phys. A Stat. Mech. Appl.* **2017**, *481*, 276–283. [CrossRef]
26. Li, D.; Chen, F.; An, Y. Existence and multiplicity of nontrivial solutions for nonlinear fractional differential systems with p-Laplacian via critical point theory. *Math. Methods Appl. Sci.* **2018**, *41*, 3197–3212. [CrossRef]

27. Khaliq, A.; Rehman, M.U. Existence of weak solutions for Ψ-Caputo fractional boundary value problem via variational methods. *J. Appl. Anal. Comput.* **2021**, *11*, 1768–1778. [CrossRef]
28. Zhou, Y. *Basic Theory of Fractional Differential Equations*; World Scientific: Singapore, 2014; pp. 177–202.
29. Gao, D.; Li, J. Applications of variational methods to a impulsive fractional differential equation with a parameter. *Dyn. Syst. Appl.* **2018**, *27*, 973–987.
30. Zhao, Y.; Tang, L. Multiplicity results for impulsive fractional differential equations with p-Laplacian via variational methods. *Bound. Value Probl.* **2017**, *2017*, 123. [CrossRef]
31. Rabinowitz, P. *Minimax Methods in Critical Point Theory with Applications to Differential Equations*; American Mathematical Society: Providence, RI, USA, 1986.
32. Jia, M.; Liu, X. Multiplicity of solutions for integral boundary value problems of fractional differential equations with upper and lower solutions. *Appl. Math. Comput.* **2014**, *232*, 313–323. [CrossRef]