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Null Controllability of Hilfer Fractional Stochastic Differential Inclusions

Hamdy M. Ahmed [1,*], Mahmoud M. El-Borai [2], Wagdy El-Sayed [2] and Alaa Elbadrawi [2]

1 Higher Institute of Engineering, El-Shorouk Academy, El-Shorouk City 11837, Egypt
2 Department of Mathematics, Faculty of Science, Alexandria University, Alexandria 21544, Egypt
* Correspondence: hamdy_17eg@yahoo.com

Abstract: This paper gives the null controllability for nonlocal stochastic differential inclusion with the Hilfer fractional derivative and Clarke subdifferential. Sufficient conditions for null controllability of nonlocal Hilfer fractional stochastic differential inclusion are established by using the fixed-point approach with the proof that the corresponding linear system is controllable. Finally, the theoretical results are verified with an example.

Keywords: Hilfer fractional derivative; stochastic differential inclusions; null controllability; Clarke subdifferential

MSC: 34A08; 39A50; 34K35

1. Introduction

The controllability of various deterministic and stochastic control systems has been investigated in many works (see [1–9]). It should be emphasized that there are many different notions of controllability for fractiona-evolution systems—for example, approximate controllability, complete controllability, null controllability, and so on. However, most work on controllability has focused on deterministic models rather than stochastic models. However, deterministic models often fluctuate due to the presence of environmental noise. It is reasonable and practical to import stochastic effects into investigations with deterministic models. In recent years, fractional stochastic differential equations and fractional stochastic inclusions have attracted the attention of many researchers and have become increasingly popular due to their practical applications in various fields of science and engineering (see [1–24]). Moreover, Hilfer proposed a generalized Riemann–Liouville fractional derivative—for brevity, this is called the Hilfer fractional derivative—which includes the Riemann–Liouville fractional derivative and Caputo fractional derivative (see [25,26]). Subsequently, a few authors have studied the controllability of fractional stochastic differential inclusions involving Hilfer fractional derivatives—for example, Yang and Wang [27] studied the approximate controllability of Hilfer fractional differential inclusions with nonlocal conditions. Dineshkumar et al. [28] discussed the approximate controllability of Sobolev-type Hilfer neutral fractional stochastic differential inclusions. However, no work has been reported in the literature regarding the null controllability of nonlocal stochastic differential inclusion with the Hilfer fractional derivative and Clarke subdifferential. In order to complete this part, in the present work, we analyze the following system:

$$\begin{cases} D_{0+}^{\hbar,\varrho}U(\varsigma) \in \mho U(\varsigma) + \gamma(\varsigma, U(\varsigma)) + \mathbb{A}\mathbb{X}(\varsigma) \\ +\vartheta(\varsigma, U(\varsigma))\frac{d\varpi(\varsigma)}{d\varsigma} + \partial\Xi(\varsigma, U(\varsigma)),\ \varsigma \in \mathbb{T} = (0, q], \\ I_{0+}^{(1-\varrho)(1-\hbar)}U(0) + \mu(U) = U_0, \end{cases} \tag{1}$$

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where $D_{0+}^{\hbar,\varrho}$ is the Hilfer fractional derivative, $0 \leq \hbar \leq 1$, $\frac{1}{2} < \varrho < 1$, and $\mho$ is the infinitesimal generator of a compact semigroup $\{\aleph(\varsigma), \varsigma \geq 0\}$ in Hilbert space Λ, where $\sup_{\varsigma\in\mathbb{T}} \|\aleph(\varsigma)\| \leq \Pi$, $\Pi > 1$.

In this paper, there exists a separable Hilbert space $\Im$ with norm $\|\cdot\|_{\Im}$ and inner product $\langle\cdot,\cdot\rangle_{\Im}$. Assume that $\{\varpi(\varsigma)\}_{\varsigma\geq 0}$ is an $\Im$-valued Wiener process with a finite trace nuclear covariance operator $\Phi \geq 0$, $L(\Psi, \Lambda)$ is the space of all bounded linear operators from Ψ into Λ, and Clarke's subdifferential of $\Xi(\varsigma, U(\varsigma))$ is $\partial\Xi(\varsigma, U(\varsigma))$ (see [21]). $U(\cdot)$ takes values in Λ, the function of the control is $\mathbb{X}(\cdot) \in L^2(\mathbb{T}, F)$ for the Hilbert space of admissible control functions, where F is a Hilbert space, and $\mathbb{A}$ is a bounded linear operator from F into Λ. $\gamma : \mathbb{T} \times \Lambda \to 2^{\Lambda}$ is a non-empty, bounded, closed, convex, and multivalued map, $\vartheta : \mathbb{T} \times \Lambda \to L_{\Phi}(\Im, \Lambda)$, and $\mu : C(\mathbb{T}, \Lambda) \to \Lambda$. In the current paper, the space of all Φ-Hilbert–Schmidt operators from Ψ to Λ is $L_{\Phi}(\Psi, \Lambda)$.

2. Preliminaries

Definition 1 ([25,26])**.** *The Hilfer fractional derivative of order $0 \leq \aleph \leq 1$ and $0 < \varrho < 1$ is defined as*

$$D_{0+}^{\hbar,\varrho} U(\varsigma) = I_{0+}^{\hbar(1-\varrho)} \frac{d}{dt} I_{0+}^{(1-\hbar)(1-\varrho)} U(\varsigma),$$

where

$$I^{\varrho} U = \frac{1}{\Gamma(\varrho)} \int_0^{\varsigma} \frac{U(\kappa)}{(\varsigma-\kappa)^{1-\varrho}} d\kappa, \ \ \varsigma > 0$$

and $\Gamma(\cdot)$ is the Gamma function.

Let $(\Omega, \mathbb{Y}, \{\mathbb{Y}_{\varsigma}\}_{\varsigma\geq 0}, K)$ be a complete probability space and let $\mathbb{D} := \mathbb{C}(\mathbb{T}, L^2(\mathbb{Y}, \Lambda))$ be a Banach space with norm $\|U\|_{\mathbb{D}} = \sup_{\varsigma\in\mathbb{T}} E\|\varsigma^{(1-\hbar)(1-\varrho)} U(\varsigma)\|^2)^{1/2}$, where $L^2(\mathbb{Y}, \Lambda) = L^2(\Omega, \mathbb{Y}, K, \Lambda)$. Let $L^2_{\mathbb{Y}}(\mathbb{T}, \Lambda)$ be the Hilbert space of all random processes that are $\mathbb{Y}_{\varsigma}$-adapted measurable as defined on $\mathbb{T}$ with values in Λ and the norm $\|U\|_{L^2_{\mathbb{Y}}(\mathbb{T},\Lambda)} = (\int_0^q E\|U(\varsigma)\|^2_{\Lambda})^{1/2} < \infty$.

Definition 2 ([28])**.** *Let $\mathbb{J}$ be a Banach space with the dual spaces $\mathbb{J}^*$ and $\mathbb{G} : \mathbb{J} \to \mathbb{R}$, which is a locally Lipschitz functional on $\mathbb{J}$. Clarke's generalized directional derivative of $\mathbb{G}$ at the point $\beta \in \mathbb{J}$ in the direction $x \in \mathbb{J}$ is defined by*

$$\mathbb{G}^0(\beta; x) = \limsup_{\nu\to 0^+ \ \tau\to\beta} \frac{\mathbb{G}(\tau + \nu x) - \mathbb{G}(\tau)}{\nu}.$$

Clarke's generalized gradient of $\mathbb{G}$ at $\beta \in \mathbb{J}$ is given by

$$\partial\mathbb{G}(\beta) = \{\beta^* \in \mathbb{G}^* : \mathbb{G}^0(\beta; x) \geq \langle\beta^*, x\rangle, \ \forall\, x \in \mathbb{J}\}.$$

Definition 3 ([26,29])**.** *A $\mathbb{Y}_{\varsigma}$ stochastic process $U \in \mathbb{D}$ is a mild solution of (1) if $U(0) = U_0 - \mu(U) \in \Lambda$ and $\mathbb{B}(\varsigma) \in L^2_{\mathbb{Y}}(\mathbb{T}, \Lambda)$ such that $\mathbb{B}(\varsigma) \in \partial\Xi(\varsigma, U(\varsigma))$ for $\varsigma \in \mathbb{T}$ and*

$$\begin{aligned} U(\varsigma) \ \ = \ \ & \aleph_{\hbar,\varrho}(\varsigma)[U_0 - \mu(U)] + \int_0^{\varsigma} P_{\varrho}(\varsigma-\kappa)\gamma(\kappa, U(\kappa))d\kappa + \int_0^{\varsigma} P_{\varrho}(\varsigma-\kappa)\mathbb{A}\mathbb{X}(\kappa)d\kappa \\ & + \int_0^{\varsigma} P_{\varrho}(\varsigma-\kappa)\mathbb{B}(\kappa)d\kappa + \int_0^{\varsigma} P_{\varrho}(\varsigma-\kappa)\vartheta(\kappa, U(\kappa))d\varpi(\kappa), \ \ \varsigma \in \mathbb{T} \end{aligned} \tag{2}$$

where

$$\aleph_{\hbar,\varrho}(\varsigma) = I_{0+}^{\hbar(1-\varrho)} P_{\varrho}(\varsigma), \quad P_{\varrho}(\varsigma) = \varsigma^{\varrho-1} T_{\varrho}(\varsigma), \quad T_{\varrho}(\varsigma) = \int_0^{\infty} \varrho\theta\Psi_{\varrho}(\theta)\aleph(\varsigma^{\varrho}\theta)d\theta,$$

where

$$\Psi_{\varrho}(\theta) = \sum_{n=1}^{\infty} \frac{(-\theta)^{n-1}}{(n-1)!\Gamma(1-n\varrho)}, \theta \in (0, \infty).$$

Lemma 1 ([26])**.** *The operators $\aleph_{\hbar,\varrho}$ and P_{ϱ} have the following properties:*

(i) $\{P_\varrho(\varsigma) : \varsigma > 0\}$ *is continuous in the uniform operator topology.*

(ii) *For any fixed* $\varsigma > 0$, $\aleph_{\hbar,\varrho}(\varsigma)$ *and* $P_\varrho(\varsigma)$ *are linear and bounded operators, and*

$$\|P_\varrho(\varsigma)U\| \leq \frac{\Pi\varsigma^{\varrho-1}}{\Gamma(\varrho)}\|U\|,\ \|\aleph_{\hbar,\varrho}(\varsigma)U\| \leq \frac{\Pi\varsigma^{(\hbar-1)(1-\varrho)}}{\Gamma(\hbar(1-\varrho)+\varrho)}\|U\|.$$

(iii) $\{P_\varrho(\varsigma) : \varsigma > 0\}$ *and* $\{\aleph_{\hbar,\varrho}(\varsigma) : \varsigma > 0\}$ *are strongly continuous.*

Some assumptions are considered to establish the null controllability criteria for the nonlinear system (1)*:*

$(A1)$ *Let* $E\|\mathbb{A}\mathbb{X}(\varsigma)\|^2 \leq \Pi_{\mathbb{A}}E\|\mathbb{X}(\varsigma)\|^2$ *for all* $\mathbb{X}(\varsigma) \in F$ *on* $\mathbb{T}$ *where* $\Pi_{\mathbb{A}} > 0$.

$(A2)$ $\gamma : \mathbb{T} \times \Lambda \to 2^\Lambda$ *is locally Lipschitz continuous for all* $\varsigma \in \mathbb{T}$, z, U_1, $U_2 \in \Lambda$, $\exists\, C_2 > 0$ *such that*

$$E\|\gamma(\varsigma, U_1) - \gamma(\varsigma, U_2)\|^2 \leq C_2(E\|U_1 - U_2\|^2,\ E\|\gamma(\varsigma, U)\|^2 \leq C_2(1 + E\|U\|^2).$$

$(A3)$ $\vartheta : \times\Lambda \to L_\Phi(\Psi, \Lambda)$ *is locally Lipschitz continuous for all* $\varsigma \in \mathbb{T}$, U, U_1, $U_2 \in \Lambda$, *and there exist constants* $C_3 > 0$ *such that*

$$E\|\vartheta(\varsigma, U_1) - \vartheta(\varsigma, U_2)\|^2_\Phi \leq C_3(E\|U_1 - U_2\|^2,\ E\|\vartheta(\varsigma, U)\|^2_\Phi \leq C_3(1 + E\|U\|^2).$$

$(A4)$ $\Xi : \mathbb{T} \times \Lambda \to \mathbb{R}$ *such that:*

- (I) $\Xi(\cdot, U) : \mathbb{T} \to \mathbb{R}$ *is measurable* $\forall\, U \in \Lambda$,
- (II) $\Xi(\varsigma, \cdot) : \Lambda \to \mathbb{R}$ *is locally Lipschitz continuous for* $\varsigma \in \mathbb{T}$,
- (III) $\exists$ *a function* $\zeta \in L^1(\mathbb{T}, \mathbb{R}^+)$, $C_4 > 0$, *which satisfies*

$$E\|\partial\Xi(\varsigma, U)\|^2 = \sup\{E\|\mathbb{B}(\varsigma)\|^2 : \mathbb{B}(\varsigma) \in \partial\Xi(\varsigma, U)\} \leq \zeta(\varsigma) + C_4E\|U\|^2,$$

$\forall\, U \in \Lambda$ *a.e.* $\varsigma \in \mathbb{T}$ *and* $U \in \Lambda$.

$(A5)$ $\mu : C(\mathbb{T}, \Lambda) \to \Lambda$ *is continuous, for any* U, U_1, $U_2 \in C(I, \Lambda)$ $\exists\, C_5 > 0$, *such that*

$$E\|\mu(U_1) - \mu(U_2)\|^2 \leq C_5E\|U_1 - U_2\|^2,\ \ E\|\mu(U)\|^2 \leq C_5(1 + E\|U\|^2).$$

Now, we define an operator $\Theta : L^2_{\mathbb{Y}}(\mathbb{T}, \Lambda) \to 2^{L^2_{\mathbb{Y}}(\mathbb{T},\Lambda)}$ *as follows:* $\Theta(U) = \{\mathbb{B} \in L^2_{\mathbb{Y}}(\mathbb{T}, \Lambda) : \mathbb{B}(\varsigma) \in \partial\Xi(\varsigma, U(\varsigma))$ *a.e.* $\varsigma \in \mathbb{T}$ *for* $U \in L^2_{\mathbb{Y}}(\mathbb{T}, \Lambda)\}$.

Lemma 2 ([28])**.** *The set* $\Theta(U)$ *has nonempty, convex, and weakly compact values for* $U \in L^2_{\mathbb{Y}}(\mathbb{T}, \Lambda)$ *and if* $(A4)$ *holds.*

Lemma 3 ([30])**.** *If* $(A4)$ *holds, the operator* $\mathbb{Y}$ *satisfies the following: If* $U_n \to U$ *in* $L^2_{\mathbb{Y}}(\mathbb{T}, \Lambda)$, $\xi_n \to \xi$ *is weakly in* $L^2_{\mathbb{Y}}(\mathbb{T}, \Lambda)$ *and* $\xi_n \in \Theta(U_n)$, *then* $\xi \in \Theta(U)$.

3. Main Result

To prove the null controllability criteria for the nonlinear system (1), we present the linear Hilfer fractional stochastic differential equation as follows:

$$\begin{cases} D^{\hbar,\varrho}_{0+}w(\varsigma) = \mho w(\varsigma) + \gamma(\varsigma) + \mathbb{A}\mathbb{X}(\varsigma) + \vartheta(\varsigma)\frac{d\varpi(\varsigma)}{d\varsigma}, & \varsigma \in \mathbb{T} = (0, q], \\ I^{(1-\varrho)(1-\hbar)}_{0+}w(0) = w_0, \end{cases} \tag{3}$$

which is associated with the system (1)

Consider

$$L^q_0\mathbb{X} = \int_0^q P_\varrho(q - \kappa)\mathbb{A}\mathbb{X}(\kappa)d\kappa\ : L_2(\mathbb{T}, F) \to \Lambda,$$

where $L^q_0\mathbb{X}$ has a bounded inverse operator $(L_0)^{-1}$ with values in $L_2(\mathbb{T}, F)/ker(L^q_0)$, and

$$N_0^q(w,\gamma,\vartheta) = \aleph_{\hbar,\varrho}(\varsigma)w + \int_0^q P_\varrho(q-\kappa)\gamma(\kappa)d\kappa + \int_0^q P_\varrho(q-\kappa)\vartheta(\kappa)d\varpi(\kappa) : \Lambda \times L_2(\mathbb{T},F) \to \Lambda.$$

Definition 4 ([31])**.** *The linear system* (3) *is exactly null controllable on* $\mathbb{T}$ *if* $\exists\, \alpha > 0$ *such that* $\|(L_0^q)^* w\|^2 \geq \alpha \|(N_0^q)^* w\|^2\ \forall\, w \in \Lambda$.

Lemma 4 ([32])**.** *Let* (3) *be exactly null controllable on* $\mathbb{T}$*; then, the linear operator* $(L_0)^{-1}N_0^q : \Lambda \times L_2(\mathbb{T},\Lambda) \to L_2(\mathbb{T},F)$ *is bounded and the control*

$$\mathbb{X}(\varsigma) = -(L_0)^{-1}\left[\aleph_{\hbar,\varrho}(\varsigma)w_0 + \int_0^q P_\varrho(q-\kappa)\gamma(\kappa)d\kappa + \int_0^b P_\varrho(q-\kappa)\vartheta(\kappa)d\varpi(\kappa)\right](\varsigma)$$

transfers the system (3) *from* w_0 *to* 0.

Definition 5 ([30])**.** *The system* (1) *is exact null controllable on* $\mathbb{T}$ *if* $\exists$ *a stochastic control* $\mathbb{X} \in L_2(\mathbb{T},F)$ *such that the solution* $U(\varsigma)$ *of* (1) *satisfies* $U(q) = 0$.

To establish the null controllability, we add the following assumption:

$(A6)$ *The fractional linear system* (3) *is exactly null controllable on* $\mathbb{T}$.

Definition 6. *If the assumptions* $(A1)$–$(A6)$ *hold, then the system* (1) *is exactly null controllable on* $\mathbb{T}$ *provided that*

$$\begin{aligned}\Re_2 &= \left\{\frac{25C_5\Pi^2 q^{2(\hbar-1)(1-\varrho)}}{\Gamma^2(\hbar(1-\varrho)+\varrho)} + \frac{25\Pi^2 q^{2\varrho-1}}{(2\varrho-1)\Gamma^2(\varrho)}\left[(C_2 + Tr(\Phi)C_3) + C_4 q\right]\right\} \\ &\quad \times \left\{1 + \frac{25\Pi^2\Pi_{\mathbb{A}}\|(L_0)^{-1}\|^2 q^{2\varrho-1}}{(2\varrho-1)\Gamma^2(\varrho)}\right\} < 1.\end{aligned}$$

Proof. We define $\mathbb{W}_\ell : \mathbb{D} \to 2^{\mathbb{D}}$ as follows:

$$\mathbb{W}_\ell(U) = \left\{\begin{array}{l} Z \in \mathbb{D} : Z(\varsigma) = \aleph_{\hbar,\varrho}(\varsigma)[U_0 - \mu(U)] + \int_0^\varsigma P_\varrho(\varsigma-\kappa)\gamma(\kappa,U(\kappa))d\kappa \\ + \int_0^\varsigma P_\varrho(\varsigma-\kappa)\mathbb{A}\mathbb{X}(\kappa)d\kappa + \int_0^\varsigma P_\varrho(\varsigma-\kappa)\mathbb{B}(\kappa)d\kappa \\ + \int_0^\varsigma P_\varrho(\varsigma-\kappa)\vartheta(\kappa,U(\kappa))d\varpi(\kappa)\ , \mathbb{B} \in \Theta(U) \end{array}\right\}$$

where

$$\begin{aligned}\mathbb{X}(\varsigma) &= -(L_0)^{-1}\Big[\aleph_{\hbar,\varrho}(q)[U_0 - \mu(U)] + \int_0^q P_\varrho(\varsigma-\kappa)\gamma(\kappa,U(\kappa))d\kappa \\ &\quad + \int_0^q P_\varrho(\varsigma-\kappa)\mathbb{B}(\kappa)d\kappa + \int_0^q P_\varrho(\varsigma-\kappa)\vartheta(\kappa,U(\kappa))d\varpi(\kappa)\Big](\varsigma).\end{aligned}$$

□

In the following steps, we show that $\mathbb{W}_\ell$ has a fixed point:

Step 1: For each $U \in \mathbb{D}$, $\mathbb{W}_\ell(U)$ has nonempty, convex, and weakly compact values. According to Lemma 2.2, it is easy to see that $\mathbb{W}_\ell(U)$ has nonempty and weakly compact values. Moreover, as $\Theta(U)$ has convex values, if $\mathbb{B}_1,\ \mathbb{B}_2 \in \Theta(U)$, then $\delta\mathbb{B}_1 + (1-\delta)\mathbb{B}_2 \in \Theta(U)\ \forall\, \delta \in (0,1)$, which clearly implies that $\mathbb{W}_\ell(U)$ is convex.

Step 2: The operator $\mathbb{W}_\ell$ is bounded on a bounded subset of $\mathbb{D}$.

Let us consider $\Delta_\wp = \{U \in \mathbb{D} : \|U\|^2_{\mathbb{D}} \leq \wp\}$, $\wp > 0$. It is obvious to conclude that $\Delta_\wp$ is a bounded, closed, and convex set of $\mathbb{D}$. We claim that there exists a constant $\rho > 0$ such that for each $\chi \in \mathbb{W}_\ell(U)$, $U \in \Delta_\wp$, $\|\chi\|^2_{\mathbb{D}} \leq \rho$.

If $\chi \in \mathbb{W}_\ell(U)$, then there exists a $\mathbb{B} \in \Theta(U)$ such that

$$\begin{aligned}\chi(\varsigma) &= \aleph_{\hbar,\varrho}(\varsigma)[U_0-\mu(U)]+\int_0^\varsigma P_\varrho(\varsigma-\kappa)\gamma(\kappa,U(\kappa))d\kappa \\ &\quad+\int_0^\varsigma P_\varrho(\varsigma-\kappa)\mathbb{A}\mathbb{X}(\kappa)d\kappa+\int_0^\varsigma P_\varrho(\varsigma-\kappa)\mathbb{B}(\kappa)d\kappa \\ &\quad+\int_0^\varsigma P_\varrho(\varsigma-\kappa)\vartheta(\kappa,U(\kappa))d\varpi(\kappa),\ \ \varsigma\in\mathbb{T}.\end{aligned} \tag{4}$$

From $(A1)$–$(A5)$ and Lemma 1, we get

$$\begin{aligned}\|\chi(\varsigma)\|_{\mathbb{D}}^2 \leq\ & 25\sup_{\varsigma\in\mathbb{T}}\varsigma^{2(1-\hbar)(1-\varrho)}\{E\|\aleph_{\hbar,\varrho}(\varsigma)[U_0-\mu(U)]\|^2 \\ &+E\|\int_0^\varsigma P_\varrho(\varsigma-\kappa)\gamma(\kappa,U(\kappa))d\kappa\|^2+E\|\int_0^\varsigma P_\varrho(\varsigma-\kappa)\mathbb{A}\mathbb{X}(\kappa)d\kappa\|^2 \\ &+E\|\int_0^\varsigma P_\varrho(\varsigma-\kappa)\mathbb{B}(\kappa)d\kappa\|^2+E\|\int_0^\varsigma P_\varrho(\varsigma-\kappa)\vartheta(\kappa,U(\kappa))d\varpi(\kappa)\|^2\} \\ \leq\ & 25\sup_{\varsigma\in\mathbb{T}}\varsigma^{2(1-\hbar)(1-\varrho)}\{E\|\aleph_{\hbar,\varrho}(\varsigma)[U_0-\mu(U)]\|^2+E\|\int_0^\varsigma P_\varrho(\varsigma-\kappa)\gamma(\kappa,U(\kappa))d\kappa\|^2 \\ &+\Pi_{\mathbb{A}}\|(L_0)^{-1}\|^2E\|\int_0^\varsigma P_\varrho(\varsigma-\kappa)\Big[\aleph_{\hbar,\varrho}(q)[U_0-\mu(U)]+\int_0^q P_\varrho(\varsigma-\kappa)\gamma(\kappa,U(\kappa))d\kappa \\ &+\int_0^q P_\varrho(\varsigma-\kappa)\mathbb{B}(\kappa)d\kappa+\int_0^q P_\varrho(\varsigma-\kappa)\vartheta(\kappa,U(\kappa))d\varpi(\kappa)\Big]d\kappa\|^2 \\ &+E\|\int_0^\varsigma P_\varrho(\varsigma-\kappa)\mathbb{B}(\kappa)d\kappa\|^2+E\|\int_0^\varsigma P_\varrho(\varsigma-\kappa)\vartheta(\kappa,U(\kappa))d\varpi(\kappa)\|^2\} \\ \leq\ & \left\{\frac{25\Pi^2}{\Gamma^2(\hbar(1-\varrho)+\varrho)}\Big[E\|U_0\|^2+C_5(1+\iota)\Big]\right. \\ &\left.+\frac{25\Pi^2q^{1-2\hbar(1-\varrho)}}{(2\varrho-1)\Gamma^2(\varrho)}\Big[(C_2+Tr(\Phi)C_3)(1+\iota)+\|\mathbb{B}\|_{L^1(I,R^+)}+C_4q\wp\Big]\right\} \\ &\times\left\{1+\frac{25\Pi^2\Pi_{\mathbb{A}}\|(L_0)^{-1}\|^2q^{2\varrho-1}}{(2\varrho-1)\Gamma^2(\varrho)}\right\}:=\rho.\end{aligned}$$

Thus, $\mathbb{W}_\ell(\Delta_\wp)$ is bounded in $\mathbb{D}$.

Step 3: The set $\{\mathbb{W}_\ell(U):U\in\Delta_\wp\}$ is equicontinuous.

For any $U\in\Delta_\wp$, $\chi\in\mathbb{W}_\ell(U)$, there exists a $\mathbb{B}\in\Theta(U)$ such that (4) holds for each $\varsigma\in\mathbb{T}$.

For $0<\varsigma_1<\varsigma_2<q$, we get

$$\begin{aligned}&\|\chi(\varsigma_2)-\chi(\varsigma_1)\|_{\mathbb{D}}^2 \\ &\leq 25\|\Big(\aleph_{\hbar,\varrho}(\varsigma_2)-\aleph_{\hbar,\varrho}(\varsigma_1)\Big)[U_0-\mu(U)]\|_{\mathbb{D}}^2 \\ &+25\|\int_0^{\varsigma_2}P_\varrho(\varsigma_2-\kappa)\gamma(\kappa,U(\kappa))d\kappa-\int_0^{\varsigma_1}P_\varrho(\varsigma_1-\kappa)\gamma(\kappa,U(\kappa))d\kappa\|_{\mathbb{D}}^2 \\ &+25\|\int_0^{\varsigma_2}P_\varrho(\varsigma_2-\kappa)\vartheta(\kappa,U(\kappa))d\varpi(\kappa)-\int_0^{\varsigma_1}P_\varrho(\varsigma_1-\kappa)\vartheta(\kappa,U(\kappa))d\varpi(\kappa)\|_{\mathbb{D}}^2 \\ &+25\|\int_0^{\varsigma_2}P_\varrho(\varsigma_2-\kappa)\mathbb{B}(\kappa)d\kappa-\int_0^{\varsigma_1}P_\varrho(\varsigma_1-\kappa)\mathbb{B}(\kappa)d\kappa\|_{\mathbb{D}}^2 \\ &+25\|\int_0^{\varsigma_2}P_\varrho(\varsigma_2-\kappa)\mathbb{A}\mathbb{X}(\kappa)d\kappa-\int_0^{\varsigma_1}P_\varrho(\varsigma_1-\kappa)\mathbb{A}\mathbb{X}(\kappa)d\kappa\|_{\mathbb{D}}^2 \\ &=25\|\Big(\aleph_{\hbar,\varrho}(\varsigma_2)-\aleph_{\hbar,\varrho}(\varsigma_1)\Big)[U_0-\mu(U)]\|_{\mathbb{D}}^2 \\ &+25\|\int_{\varsigma_1}^{\varsigma_2}P_\varrho(\varsigma_2-\kappa)\gamma(\kappa,U(\kappa))d\kappa\|_{\mathbb{D}}^2 \\ &+25\|\int_0^{\varsigma_1}\Big[P_\varrho(\varsigma_2-\kappa)-P_\varrho(\varsigma_1-\kappa)\Big]\gamma(\kappa,U(\kappa))d\kappa\|_{\mathbb{D}}^2 \\ &+25\|\int_{\varsigma_1}^{\varsigma_2}P_\varrho(\varsigma_2-\kappa)\vartheta(\kappa,U(\kappa))d\varpi(\kappa)\|_{\mathbb{D}}^2 \\ &+25\|\int_0^{\varsigma_1}\Big[P_\varrho(\varsigma_2-\kappa)-P_\varrho(\varsigma_1-\kappa)\Big]\vartheta(\kappa,U(\kappa))d\varpi(\kappa)\|_{\mathbb{D}}^2 \\ &+25\|\int_{\varsigma_1}^{\varsigma_2}P_\varrho(\varsigma_2-\kappa)\mathbb{B}(\kappa)d\kappa\|_{\mathbb{D}}^2 \\ &+25\|\int_0^{\varsigma_1}\Big[P_\varrho(\varsigma_2-\kappa)-P_\varrho(\varsigma_1-\kappa)\Big]\mathbb{B}(\kappa)d\kappa\|_{\mathbb{D}}^2 \\ &+25\|\int_{\varsigma_1}^{\varsigma_2}P_\varrho(\varsigma_2-\kappa)\mathbb{A}\mathbb{X}(\kappa)d\kappa\|_{\mathbb{D}}^2 \\ &+25\|\int_0^{\varsigma_1}\Big[P_\varrho(\varsigma_2-\kappa)-P_\varrho(\varsigma_1-\kappa)\Big]\mathbb{A}\mathbb{X}(\kappa)d\kappa\|_{\mathbb{D}}^2.\end{aligned} \tag{5}$$

Since $\aleph(\varsigma)(\varsigma > 0)$ is compact, then, the right-hand side of (5) tends to zero as $\varsigma_2 \to \varsigma_1$. Thus, $\mathbb{W}_\ell(U)(\varsigma)$ is continuous in $(0, q]$. In addition, for $\varsigma_1 = 0$ and $0 < \varsigma_2 \leq q$, $E\|\chi(\varsigma_2) - \chi(0)\|^2_{\mathbb{D}} \to 0$ with respect to $U \in \Delta_\wp$ as $\varsigma_2 \to 0$.
Hence, we conclude that $\{\mathbb{W}_\ell(U)(\varsigma) : U \in \Delta_\wp\}$ is equicontinuous in $\mathbb{D}$.

Step 4: $\mathbb{W}_\ell$ is completely continuous.
We show that $\forall\ \varsigma \in \mathbb{T}$, $\wp > 0$, the set $\mathbb{H}_\wp(\varsigma) = \{\chi(\varsigma) : \chi \in \mathbb{W}_\ell(\Delta_\wp)\}$ is relatively compact in Λ. Clearly, $\mathbb{H}_\wp(0)$ is compact. Let $0 < \varsigma \leq q$ be fixed, $0 < \epsilon < \varsigma$, for $U \in \Delta_\wp$, and we define

$$\begin{aligned}
\chi^{\epsilon,\jmath}(\varsigma) \quad = \quad & \frac{\varrho}{\Gamma(\hbar(1-\varrho))}\int_0^{\varsigma-\epsilon}\int_\jmath^\infty \theta(\varsigma-\kappa)^{\hbar(1-\varrho)-1}\kappa^{\varrho-1}\Psi_\varrho(\theta)\aleph(\kappa^\varrho\theta)[U_0-\mu(U)]d\theta d\kappa \\
& +\varrho\int_0^{\varsigma-\epsilon}\int_\jmath^\infty \theta(\varsigma-\kappa)^{\varrho-1}\Psi_\varrho(\theta)\aleph((\varsigma-\kappa)^\varrho\theta)\gamma(\kappa, U(\kappa))d\theta d\kappa \\
& +\varrho\int_0^{\varsigma-\epsilon}\int_\jmath^\infty \theta(\varsigma-\kappa)^{\varrho-1}\Psi_\varrho(\theta)\aleph((\varsigma-\kappa)^\varrho\theta)\mathbb{A}\mathbb{X}(\kappa)d\theta d\kappa \\
& +\varrho\int_0^{\varsigma-\epsilon}\int_\jmath^\infty \theta(\varsigma-\kappa)^{\varrho-1}\Psi_\varrho(\theta)\aleph((\varsigma-\kappa)^\varrho\theta)\mathbb{B}(\kappa)d\theta d\kappa \\
& +\varrho\int_0^{\varsigma-\epsilon}\int_\jmath^\infty \theta(\varsigma-\kappa)^{\varrho-1}\Psi_\varrho(\theta)\aleph((\varsigma-\kappa)^\varrho\theta)\vartheta(\kappa, U(\kappa))d\theta d\kappa \\
= \quad & \frac{\varrho\aleph(\epsilon^\varrho\jmath)}{\Gamma(\hbar(1-\varrho))}\int_0^{\varsigma-\epsilon}\int_\jmath^\infty \theta(\varsigma-\kappa)^{\hbar(1-\varrho)-1}\kappa^{\varrho-1}\Psi_\varrho(\theta)\aleph(\kappa^\varrho\theta-\epsilon^\varrho\jmath)[U_0-\mu(U)]d\theta d\kappa \\
& +\varrho\aleph(\epsilon^\varrho\jmath)\int_0^{\varsigma-\epsilon}\int_\jmath^\infty \theta(\varsigma-\kappa)^{\varrho-1}\Psi_\varrho(\theta)\aleph((\varsigma-\kappa)^\varrho\theta-\epsilon^\varrho\jmath)\gamma(\kappa, U(\kappa))d\theta d\kappa \\
& +\varrho\aleph(\epsilon^\varrho\jmath)\int_0^{\varsigma-\epsilon}\int_\jmath^\infty \theta(\varsigma-\kappa)^{\varrho-1}\Psi_\varrho(\theta)\aleph((\varsigma-\kappa)^\varrho\theta-\epsilon^\varrho\jmath)\mathbb{A}\mathbb{X}(\kappa)d\theta d\kappa \\
& +\varrho\aleph(\epsilon^\varrho\jmath)\int_0^{\varsigma-\epsilon}\int_\jmath^\infty \theta(\varsigma-\kappa)^{\varrho-1}\Psi_\varrho(\theta)\aleph((\varsigma-\kappa)^\varrho\theta-\epsilon^\varrho\jmath)\mathbb{B}(\kappa)d\theta d\kappa \\
& +\varrho\aleph(\epsilon^\varrho\jmath)\int_0^{\varsigma-\epsilon}\int_\jmath^\infty \theta(\varsigma-\kappa)^{\varrho-1}\Psi_\varrho(\theta)\aleph((\varsigma-\kappa)^\varrho\theta-\epsilon^\varrho\jmath)\vartheta(\kappa, U(\kappa))d\theta d\kappa.
\end{aligned}$$

Since $\aleph(\epsilon^\varrho\jmath)$, $\epsilon^\varrho\jmath > 0$ is a compact operator, the set $\mathbb{H}^{\epsilon,\jmath}_\wp(\varsigma) = \{\chi^{\epsilon,\jmath}(\varsigma) : \chi^{\epsilon,\jmath} \in \mathbb{W}_\ell(\Delta_\wp)\}$ is relatively compact in Λ. In addition, we have

$$\begin{aligned}
& E\|\chi(\varsigma)-\chi^{\epsilon,\jmath}(\varsigma)\|^2_{\mathbb{D}} = \sup_{\varsigma\in\mathbb{T}}\varsigma^{2(1-\hbar)(1-\varrho)}E\|\chi(\varsigma)-\chi^{\epsilon,\jmath}(\varsigma)\|^2 \\
& \leq \tfrac{25\varrho^2}{\Gamma^2(\hbar(1-\varrho))}\sup_{\varsigma\in\mathbb{T}}\varsigma^{2(1-\hbar)(1-\varrho)} \\
& \times E\|\textstyle\int_0^\varsigma\int_0^\jmath \theta(\varsigma-\kappa)^{\hbar(1-\varrho)-1}\kappa^{\varrho-1}\Psi_\varrho(\theta)\aleph(\kappa^\varrho\theta)[U_0-\mu(U)]d\theta d\kappa\|^2 \\
& +\tfrac{25\varrho^2}{\Gamma^2(\hbar(1-\varrho))}\sup_{\varsigma\in\mathbb{T}}\varsigma^{2(1-\hbar)(1-\varrho)} \\
& \times E\|\textstyle\int_{\varsigma-\epsilon}^\varsigma\int_\jmath^\infty \theta(\varsigma-\kappa)^{\hbar(1-\varrho)-1}\kappa^{\varrho-1}\Psi_\varrho(\theta)\aleph(\kappa^\varrho\theta)[U_0-\mu(U)]d\theta d\kappa\|^2 \\
& +25\varrho^2\sup_{\varsigma\in\mathbb{T}}\varsigma^{2(1-\hbar)(1-\varrho)}E\|\textstyle\int_0^\varsigma\int_0^\jmath \theta(\varsigma-\kappa)^{\varrho-1}\Psi_\varrho(\theta)\aleph((\varsigma-\kappa)^\varrho\theta)\gamma(\kappa, U(\kappa))d\theta d\kappa\|^2 \\
& +25\varrho^2\sup_{\varsigma\in\mathbb{T}}\varsigma^{2(1-\hbar)(1-\varrho)}E\|\textstyle\int_{\varsigma-\epsilon}^\varsigma\int_\jmath^\infty \theta(\varsigma-\kappa)^{\varrho-1}\Psi_\varrho(\theta)\aleph((\varsigma-\kappa)^\varrho\theta)\gamma(\kappa, U(\kappa))d\theta d\kappa\|^2 \\
& +25\varrho^2\sup_{\varsigma\in\mathbb{T}}\varsigma^{2(1-\hbar)(1-\varrho)}E\|\textstyle\int_0^\varsigma\int_0^\jmath \theta(\varsigma-\kappa)^{\varrho-1}\Psi_\varrho(\theta)\aleph((\varsigma-\kappa)^\varrho\theta)\mathbb{A}\mathbb{X}(\kappa)d\theta d\kappa\|^2 \\
& +25\varrho^2\sup_{\varsigma\in\mathbb{T}}\varsigma^{2(1-\hbar)(1-\varrho)}E\|\textstyle\int_{\varsigma-\epsilon}^\varsigma\int_\jmath^\infty \theta(\varsigma-\kappa)^{\varrho-1}\Psi_\varrho(\theta)\aleph((\varsigma-\kappa)^\varrho\theta)\mathbb{A}\mathbb{X}(\kappa)d\theta d\kappa\|^2 \\
& +25\varrho^2\sup_{\varsigma\in\mathbb{T}}\varsigma^{2(1-\hbar)(1-\varrho)}E\|\textstyle\int_0^\varsigma\int_0^\jmath \theta(\varsigma-\kappa)^{\varrho-1}\Psi_\varrho(\theta)\aleph((\varsigma-\kappa)^\varrho\theta)\mathbb{B}(\kappa)d\theta d\kappa\|^2 \\
& +25\varrho^2\sup_{\varsigma\in\mathbb{T}}\varsigma^{2(1-\hbar)(1-\varrho)}E\|\textstyle\int_{\varsigma-\epsilon}^\varsigma\int_\jmath^\infty \theta(\varsigma-\kappa)^{\varrho-1}\Psi_\varrho(\theta)\aleph((\varsigma-\kappa)^\varrho\theta)\mathbb{B}(\kappa)d\theta d\kappa\|^2 \\
& +25\varrho^2\sup_{\varsigma\in\mathbb{T}}\varsigma^{2(1-\hbar)(1-\varrho)}E\|\textstyle\int_0^\varsigma\int_0^\jmath \theta(\varsigma-\kappa)^{\varrho-1}\Psi_\varrho(\theta)\aleph((\varsigma-\kappa)^\varrho\theta)\vartheta(\kappa, U(\kappa))d\theta d\kappa\|^2 \\
& +25\varrho^2\sup_{\varsigma\in\mathbb{T}}\varsigma^{2(1-\hbar)(1-\varrho)}E\|\textstyle\int_{\varsigma-\epsilon}^\varsigma\int_\jmath^\infty \theta(\varsigma-\kappa)^{\varrho-1}\Psi_\varrho(\theta)\aleph((\varsigma-\kappa)^\varrho\theta)\vartheta(\kappa, U(\kappa))d\theta d\kappa\|^2.
\end{aligned} \tag{6}$$

We see that when $\epsilon \to 0^+$ and $\jmath \to 0^+$, the inequality (6) tends to zero. Thus, the set $\mathbb{H}_\wp(\varsigma)$ is relatively compact in Λ. Hence, from Step 3 and the Arzela–Ascoli theorem, $\mathbb{W}_\ell$ is completely continuous.

Step 5: $\mathbb{W}_\ell$ has a closed graph.
Let us consider $U_n \to U_*$ in $\mathbb{D}$, $\chi_n \in \mathbb{W}_\ell(U_n)$, and $\chi_n \to \chi_*$ in $\mathbb{D}$. We will prove that $\chi_* \in \mathbb{W}_\ell(U_*)$.
Actually, $\chi_n \in \mathbb{W}_\ell(U_n)$ shows that there exists a $\mathbb{B}_n \in \Theta(U_n)$ such that

$$\begin{aligned}\chi_n(\varsigma) \quad = \quad & \aleph_{\hbar,\varrho}(\varsigma)[U_0 - \mu(U_n)] + \int_0^\varsigma P_\varrho(\varsigma-\kappa)\gamma(\kappa, U_n(\kappa))d\kappa \\ & + \int_0^\varsigma P_\varrho(\varsigma-\kappa)\mathbb{A}\mathbb{X}(\kappa)d\kappa + \int_0^\varsigma P_\varrho(\varsigma-\kappa)\mathbb{B}_n(\kappa)d\kappa \\ & + \int_0^\varsigma P_\varrho(\varsigma-\kappa)\vartheta(\kappa, U_n(\kappa))d\varpi(\kappa),\ \varsigma \in \mathbb{T}\end{aligned} \tag{7}$$

From $(A2)$–$(A5)$, it is easy to verify that $\{\mu(U_n), \gamma(\cdot, U_n), \mathbb{B}_n, \vartheta(\cdot, U_n)\}_{n\geq 1} \subseteq \Lambda \times \Lambda \times L^2_\Upsilon(I, \Lambda) \times L_\Phi$ is bounded. Then, we get

$$(\mu(U_n), \gamma(\cdot, U_n), \mathbb{B}_n, \vartheta(\cdot, U_n)) \to (\mu(U_*), \gamma(\cdot, U_*), \mathbb{B}_*, \vartheta(\cdot, U_*)) weakly in \Lambda \times \Lambda \times L^2_\Upsilon(I, \Lambda) \times L_\Phi. \tag{8}$$

According to (7), (8), and the compactness of the operator $\aleph(\varsigma)$, we have that

$$\begin{aligned}\chi_n(\varsigma) \quad \to \quad & \aleph_{\hbar,\varrho}(\varsigma)[U_0 - \mu(U_*)] + \int_0^\varsigma P_\varrho(\varsigma-\kappa)\gamma(\kappa, U_*(\kappa))d\kappa \\ & + \int_0^\varsigma P_\varrho(\varsigma-\kappa)\mathbb{A}\mathbb{X}(\kappa)d\kappa + \int_0^\varsigma P_\varrho(\varsigma-\kappa)\mathbb{B}_*(\kappa)d\kappa \\ & + \int_0^\varsigma P_\varrho(\varsigma-\kappa)\vartheta(\kappa, U_*(\kappa))d\varpi(\kappa).\end{aligned} \tag{9}$$

Concentrating on $\chi_n \to \chi_*$ in $\mathbb{D}$ and $\mathbb{B}_n \in \Theta(U_n)$, from (9) and Lemma 3, we can get $\mathbb{B}_* \in \Theta(U_*)$. Therefore, we can show that $\chi_* \in \mathbb{W}_\ell(U_*)$, which shows that $\mathbb{W}_\ell$ has a closed graph. By using Proposition 3.3.12(2) of [30], we find the conclusion that $\mathbb{W}_\ell$ is upper semicontinuous.

Step 6: An a priori estimate.
From previous steps, we found that $\mathbb{W}_\ell$ is compact convex valued and upper semicontinuous and that $\mathbb{W}_\ell(\Delta_\wp)$ is relatively compact. By Theorem 2.10 from [29], we can prove that the set $\Im = \{U \in \mathbb{D} : \eta U \in \mathbb{W}_\ell,\ \ \eta > 1\}$ is bounded.
Let us consider $U \in \Im$ and assume that there occurs a $\mathbb{B} \in \Theta(U)$ such that

$$\begin{aligned}U(\varsigma) \quad = \quad & \eta^{-1}\aleph_{\hbar,\varrho}(\varsigma)[U_0 - \mu(U)] + \eta^{-1}\int_0^\varsigma P_\varrho(\varsigma-\kappa)\gamma(\kappa, U(\kappa))d\kappa \\ & + \eta^{-1}\int_0^\varsigma P_\varrho(\varsigma-\kappa)\mathbb{A}\mathbb{X}(\kappa)d\kappa + \eta^{-1}\int_0^\varsigma P_\varrho(\varsigma-\kappa)\mathbb{B}(\kappa)d\kappa \\ & + \eta^{-1}\int_0^\varsigma P_\varrho(\varsigma-\kappa)\vartheta(\kappa, U(\kappa))d\varpi(\kappa).\end{aligned} \tag{10}$$

From $(A1)$–$(A5)$ and Lemma 1, we can get

$$\begin{aligned}\|U(\varsigma)\|^2 \quad \leq \quad & 25|\eta^{-1}|^2\{E\|\aleph_{\hbar,\varrho}(\varsigma)[U_0 - \mu(U)]\|^2 \\ & + E\|\int_0^\varsigma P_\varrho(\varsigma-\kappa)\gamma(\kappa, U(\kappa))d\kappa\|^2 + E\|\int_0^\varsigma P_\varrho(\varsigma-\kappa)\mathbb{A}\mathbb{X}(\kappa)d\kappa\|^2 \\ & + E\|\int_0^\varsigma P_\varrho(\varsigma-\kappa)\mathbb{B}(\kappa)d\kappa\|^2 + E\|\int_0^\varsigma P_\varrho(\varsigma-\kappa)\vartheta(\kappa, U(\kappa))d\varpi(\kappa)\|^2\} \\ \leq \quad & 25\{E\|\aleph_{\hbar,\varrho}(\varsigma)[U_0 - \mu(U)]\|^2 + E\|\int_0^\varsigma P_\varrho(\varsigma-\kappa)\gamma(\kappa, U(\kappa))d\kappa\|^2 \\ & + \Pi_{\mathbb{A}}\|(L_0)^{-1}\|^2 E\|\int_0^\varsigma P_\varrho(\varsigma-\kappa)\left[\aleph_{\hbar,\varrho}(q)[U_0 - \mu(U)] + \int_0^q P_\varrho(\varsigma-\kappa)\gamma(\kappa, U(\kappa))d\kappa\right. \\ & \left. + \int_0^q P_\varrho(\varsigma-\kappa)\mathbb{B}(\kappa)d\kappa + \int_0^q P_\varrho(\varsigma-\kappa)\vartheta(\kappa, U(\kappa))d\varpi(\kappa)\right]d\kappa\|^2 \\ & + E\|\int_0^\varsigma P_\varrho(\varsigma-\kappa)\mathbb{B}(\kappa)d\kappa\|^2 + E\|\int_0^\varsigma P_\varrho(\varsigma-\kappa)\vartheta(\kappa, U(\kappa))d\varpi(\kappa)\|^2\} \\ \leq \quad & \left\{\frac{25\Pi^2 q^{2(\hbar-1)(1-\varrho)}}{\Gamma^2(\hbar(1-\varrho)+\varrho)}\left[(E\|U_0\|^2 + C_5(1 + E\|U(\varsigma)\|^2)\right]\right. \\ & \left. + \frac{25\Pi^2 q^{2\varrho-1}}{(2\varrho-1)\Gamma^2(\varrho)}\left[(C_2 + Tr(\Phi)C_3)(1 + E\|U(\varsigma)\|^2) + \|\mathbb{B}\|_{L^1(\mathbb{T},R^+)} + C_4 qE\|U(\varsigma)\|^2\right]\right\} \\ & \times\left\{1 + \frac{25\Pi^2\Pi_{\mathbb{A}}\|(L_0)^{-1}\|^2 q^{2\varrho-1}}{(2\varrho-1)\Gamma^2(\varrho)}\right\} \\ & \leq \Re_1 + \Re_2 E\|U(\varsigma)\|^2\end{aligned} \tag{11}$$

where

$$\Re_1 = \left\{ \frac{25\Pi^2 q^{2(\hbar-1)(1-\varrho)}(E\|U_0\|^2 + C_5)}{\Gamma^2(\hbar(1-\varrho)+\varrho)} + \frac{25\Pi^2 q^{2\varrho-1}}{(2\varrho-1)\Gamma^2(\varrho)}\left[(C_2 + Tr(\Phi)C_3) + \|\mathbb{B}\|_{L^1(I,R^+)}\right]\right\} \times \left\{1 + \frac{25\Pi^2\Pi_{\mathbb{A}}\|(L_0)^{-1}\|^2 q^{2\varrho-1}}{(2\varrho-1)\Gamma^2(\varrho)}\right\},$$

and

$$\Re_2 = \left\{ \frac{25C_5\Pi^2 q^{2(\hbar-1)(1-\varrho)}}{\Gamma^2(\hbar(1-\varrho)+\varrho)} + \frac{25\Pi^2 q^{2\varrho-1}}{(2\varrho-1)\Gamma^2(\varrho)}\left[(C_2 + Tr(\Phi)C_3) + C_4 q\right]\right\}\left\{1 + \frac{25\Pi^2\Pi_{\mathbb{A}}\|(L_0)^{-1}\|^2 q^{2\varrho-1}}{(2\varrho-1)\Gamma^2(\varrho)}\right\}.$$

Since $\Re_2 < 1$, from (3.9), we obtain

$$\|U\|^2_{\mathbb{D}} = \sup_{\varsigma\in\mathbb{T}} E\|\varsigma^{(1-\hbar)(1-\varrho)}U(\varsigma)\|^2 \le \Re_1 + \Re_2\|U\|^2_{\mathbb{D}}.$$

Then, $\|U\|^2_{\mathbb{D}} \le \frac{\Re_1}{1-\Re_2}$ implies that the set $\Im$ is bounded. According to Theorem 2.10 from [29], $\mathbb{W}_\ell$ has a fixed point. Any fixed point of $\mathbb{W}_\ell$ is a mild solution of the system (1) on $\mathbb{T}$. Hence, the system (1) is exact null controllable on $\mathbb{T}$.

4. Example

The following control system is described by Hilfer fractional stochastic partial differential inclusions with the Clarke subdifferential and nonlocal conditions:

$$\begin{cases} D^{\frac{1}{5},\frac{3}{4}}_{0+}U(\varsigma,z) \in \frac{\partial^2}{\partial z^2}(U(\varsigma,z)) \\ +\frac{1}{20}sin(U(\varsigma,z)) + \gamma(\varsigma,z) + \frac{1}{5}cos(U(\varsigma,z))\frac{d\varpi(\varsigma)}{d\varsigma} + \partial\Xi(\varsigma,U(\varsigma,z)),\ \varsigma\in\mathbb{T}=(0,1], \\ U(\varsigma,0) = U(\varsigma,2) = 0,\ \ \varsigma\in\mathbb{T}, \\ I^{\frac{3}{5}}_{0+}U(0,z) + \sum_{i=1}^{p} a_i\, U(\varsigma_i,z) = U_0(z),\ 0\le z\le 3, \end{cases} \tag{12}$$

where $D^{\frac{1}{5},\frac{3}{4}}_{0+}$ is the Hilfer fractional derivative of order $\hbar = \frac{1}{5}$, $\varrho = \frac{3}{4}$, $0 < \varsigma_0 < \varsigma_1 < ... < \varsigma_p < 1$, $U_0(z) \in \Lambda = L^2([0,3])$, and ϖ is a Wiener process. The functions are defined as $U(\varsigma)(z) = U(\varsigma,z)$, $\frac{1}{8}sin(U(\varsigma,z)) = \gamma(\varsigma,U(\varsigma,z))$, $\frac{1}{20}cos(U(\varsigma,z)) = \vartheta(\varsigma,U(\varsigma,z))$, $\Xi(\varsigma,U(\varsigma))(z) = \Xi(\varsigma,U(\varsigma,z))$, and $\mathbb{X}(\varsigma)(z) = \gamma(\varsigma,z)$. The bounded linear operator $\mathbb{A}$ is defined by $\mathbb{A}y = \varrho(\varsigma,z)$, $\varsigma\in\mathbb{T}$, $0\le z\le 2$, $y\in F$.

Let $\Lambda = \Psi = F = L^2([0,3])$, and let the operator $\mho : D(\mho)\subset\Lambda\to\Lambda$ be given by $\mho = \frac{\partial^2}{\partial z^2}$, with $D(\mho) = \{U\in\Lambda, z, \frac{\partial z}{\partial z}$ being absolutely continuous, $\frac{\partial^2 U}{\partial z^2}\in\Lambda,\ U(0) = U(2) = 0\}$.

Then, $\mho$ can be written as

$$\mho U = \sum_{n=1}^{\infty}(-n^2)(U,U_n)U_n,\quad z\in D(\mho),$$

where $U_n(s) = \sqrt{\frac{2}{\pi}}\sin ns,\ n = 1,2,...$ is the orthogonal set of eigenvectors of $\mho$.

Furthermore, for $U\in\Lambda$, we have

$$\aleph(\varsigma)z = \sum_{n=1}^{\infty} e^{\frac{-n^2\varsigma}{1+n^2}}(z,U_n)U_n.$$

The operator $\mho$ is the infinitesimal generator of a compact semigroup $\{\aleph(\varsigma)\}_{\varsigma\ge 0}$ in Λ. From the above choice, the system (12) can be written in the abstract form of (1), all assumptions of Theorem 4 are satisfied, and

$$\left\{ \frac{25C_5\Pi^2 q^{2(\hbar-1)(1-\varrho)}}{\Gamma^2(\hbar(1-\varrho)+\varrho)} + \frac{25\Pi^2 q^{2\varrho-1}}{(2\varrho-1)\Gamma^2(\varrho)}\Big[(C_2+Tr(\Phi)C_3)+C_4 q\Big]\right\}$$
$$\times\left\{1+\frac{25\Pi^2\Pi_{\mathbb{A}}\|(L_0)^{-1}\|^2 q^{2\varrho-1}}{(2\varrho-1)\Gamma^2(\varrho)}\right\}<1.$$

Thus, the system (12) is null controllable on $(0,1]$.

5. Conclusions

In this paper, a control system described by Hilfer fractional stochastic differential inclusions with the Clarke subdifferential and nonlocal conditions was presented. By using the fixed-point technique, fractional calculus, stochastic analysis, properties of the Clarke subdifferential, and non-smooth analysis, the null controllability of the considered system was investigated. Moreover, we provided an example in order to illustrate the applicability of the results.

In future work, we can present the boundary null controllability of non-instantaneous impulsive fractional stochastic evolution inclusions.

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