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Article

Fractal Curves on Banach Algebras

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Abstract: Most of the fractal functions studied so far run through numerical values. Usually they are supported on sets of real numbers or in a complex field. This paper is devoted to the construction of fractal curves with values in abstract settings such as Banach spaces and algebras, with minimal conditions and structures, transcending in this way the numerical underlying scenario. This is performed via fixed point of an operator defined on a b-metric space of Banach-valued functions with domain on a real interval. The sets of images may provide uniparametric fractal collections of measures, operators or matrices, for instance. The defining operator is linked to a collection of maps (or iterated function system, and the conditions on these mappings determine the properties of the fractal function. In particular, it is possible to define continuous curves and fractal functions belonging to Bochner spaces of Banach-valued integrable functions. As residual result, we prove the existence of fractal functions coming from non-contractive operators as well. We provide new constructions of bases for Banach-valued maps, with a particular mention of spanning systems of functions valued on C*-algebras.

Keywords: fractals; fractal interpolation function; α-fractal function; iterated function system; Banach-valued mapping

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1. Introduction

Bolzano (1830) was one of the first scientists to propose a self-similar curve and prove its continuity and non-differentiability in a dense set of points of the interval where it is defined [1]. Thereafter, a collection of geometric "monsters" in terms of Poincaré began to appear (see for instance [2]). It is worth emphasizing the Weierstrass map [2,3], followed by those of Riemann [3] and Darboux [4], as pioneering the field of self-similar (nowadays called fractal) functions.

Mandelbrot proposed a variant of the Weierstrass map satisfying the self-affine equation: $f(x) = bf(ax)$, where a and b are real numbers [5]. That is to say f is a fixed point of the operator $Tf(x) = bf(ax)$.

From an analytical point of view, some procedures to define a fractal function (in approximate chronological order) are:

1. Definition of a functional series satisfying one of the following properties: its sum differentiated does not converge [3,4], some of its elements are discrete (that is to say, they range in a discrete set of numbers) [6], or the general term contains some random element [7].
2. The mapping is the solution of one or several functional equations (see, for instance, the De Rham's curve in [2]).
3. The graph of the map is an attractor of an iterated function system [8,9].

Wavelet expansions have now replaced the traditional trigonometric series of the first item and revealed a fundamental mathematical tool in both the theoretical approximation [10] and the applications (see, for instance, [11]).

In the third method quoted above, the fractal function is the outcome of an iterative procedure. However, in the seventies of the last century, the mappings acquired a new purpose, as modelers of evolutionary processes given by dynamical systems. Thus, we find the inverse process of (3), namely, a function providing an iterative system. The work of Julia and others at the beginning of the twentieth century is a clear referent [12]. Two remarkable examples are the Hénon [13] and Kaplan–Yorke maps [14].

Nowadays, the latest procedures (2) and (3) have gained an increasing importance with the definition of the so-called fractal interpolation functions, created in principle to join a collection of real data. They have been found useful in the enhancement and extension of the theory of functions, since they provide a much more general framework to define mappings with characteristics very different to those of the classical maps (continuity, differentiability, smoothness. . .). In this way, the field of Approximation Theory has been considerably enriched with different models. The contributions in the area of fractal interpolation are recent and numerous, both in theory as well as in applications (see, for instance, [15–33]).

One common feature of classical and modern fractal functions is the fact of underlying either an Euclidean space of reals or the field of complex numbers. Though the theory of iterated function systems is established in the framework of metric spaces, most of the contributions on fractal functions are supported on sets of numbers. However, the advent of an increasingly complex and changing world, and the advance of the science, entail the need for the sophistication of the theoretical and applied tools to be employed in order to undertake the solution of any scientific, social, biological, economic or technical problem.

One of the aims of this article is the definition of fractal curves in abstract scenarios, namely, Banach spaces and algebras, transcending in this way the numerical underlying structure. We describe a general method to obtain fractal curves whose values may be measures, operators or matrices, for instance. That is to say, we define uniparametric families of these type of elements. For then, we use the second procedure quoted above, that is to say, the proposed function is the solution of a functional equation defined in some abstract space of mappings, with minimal structures and conditions.

The paper is organized as follows. Section 2 describes the foundations of b-metric spaces. Afterwards, we generalize the concept of quasi-fixed point associated to a set of operators, introduced in the reference [34], to the framework of b-metric spaces and prove that the set of quasi-fixed points is a fractal set. We also provide the result of convergence of the iteration of operators (self-maps) acting on the space. Section 3 is devoted to the construction of fractal curves with values in abstract spaces such as Banach spaces and algebras, with minimal conditions. This is carried out via the fixed point of an operator defined on a b-metric space of functions with domain on a real interval and a partition of it. The defining operator is linked to a collection of maps (or iterated function system). The conditions on these mappings determine the properties of the fractal function. Thus, in Section 4, we construct continuous curves, and Section 5 studies fractal functions belonging to Bochner spaces of Banach-valued integrable functions. As residual result, we prove the existence of fractal functions coming from non-contractive operators as well. Section 6 extends the concepts of α-fractal function [27] and fractal convolution [35,36], studied in real interpolation, to abstract spaces. Section 7 is focused on new definitions of bases for Banach-valued maps, with a particular mention of spanning systems of functions valued on C*-algebras.

2. B-Metric Spaces and Convergence

In this section, we first recall the rudiments of the structure of b-metric space, and then we generalize the concept of the quasi-fixed point of a set of operators introduced in [34] and consider some of its properties of convergence and self-similarity.

Definition 1. *A b-metric space X is a set endowed with a mapping $d_b : X \times X \to \mathbb{R}^+$ with the following properties*

1. $d_b(x,y) \geq 0$, $d_b(x,y) = 0$ *if and only if* $x = y$.
2. $d_b(x,y) = d_b(y,x)$ *for any* $x, y \in X$.
3. *There exists* $s \geq 1$ *such that* $d_b(x,y) \leq s(d_b(x,z) + d_b(z,y))$ *for any* $x, y, z \in X$.

The constant s is the index of the b-metric space, and d_b is called a b-metric.

Example 1. *For $X = \mathbb{R}^n$, $x, y \in \mathbb{R}^n$, let us define*

$$d_b(x, y) = \sum_{i=1}^{n} |x_i - y_i|^2.$$

d_b is a b-metric with index $s = 2$ (see for instance [37]).

Example 2. *The Lebesgue space $\mathcal{L}^p(I)$ where I is a real bounded interval and $0 < p < 1$, with d_b defined as*

$$d_b(f, g) = (\int_I |f - g|^p dt)^{1/p},$$

is a b-metric space with index $s = 2^{\frac{1}{p}-1}$.

Remark 1. *A metric space is a particular case of b-metric space with index $s = 1$.*

Let us consider a b-metric space X.

Definition 2. *A sequence $(x_n) \subseteq X$ is Cauchy if $d_b(x_m, x_n) \to 0$ as m, n tend to infinity.*

Definition 3. *A sequence $(x_n) \subseteq X$ is convergent if there exists $x \in X$ such that $d_b(x_n, x) \to 0$ as n tends to infinity.*

Definition 4. *A subset $A \subseteq X$ is complete if every Cauchy sequence is convergent.*

Definition 5. *A subset $A \subseteq X$ is bounded if*

$$diam(A) := \sup\{d_b(x, y) : x, y \in A\} < +\infty.$$

Definition 6. *A subset $A \subseteq X$ is compact if any sequence $(x_n) \subseteq A$ has a convergent subsequence.*

Though it may seem that the b-metric space (bms for short) structure is a plain generalization of that of metric space, this is not true. For instance, in a bms, the "open" balls may not be open! (see, for instance, [38]).

Definition 7. *For $A \subseteq X$, the closure $\bar{A}$ is defined as: $x \in \bar{A}$ if and only if there exists a sequence $(x_n) \subseteq A$ such that $\lim x_n = x$.*

In the reference [38] (Theorem 3.2), one can find the following generalization of the Cantor's intersection Theorem.

Theorem 1. *Let X be a bms. X is complete if and only if every nested sequence of non-empty closed subsets:*

$$A_1 \supseteq A_2 \supseteq \ldots \supseteq A_n \ldots,$$

$A_n \subseteq X$, such that $diam(A_n) \to 0$ satisfies the equality $\cap_{n=1}^{+\infty} A_n = \{x\}$, $x \in X$.

Definition 8. *A self-map $T : X \to X$, where X is a bms, is continuous if $x_n \to x$ implies $Tx_n \to Tx$.*

Banach's Theorem holds in bms also for some specific contractions (see, for instance, [38]).

Definition 9. *A self-map $T : X \to X$, where X is a bms with index s, is contractive if there exists $k \in \mathbb{R}$, $0 < k < s^{-1}$ such that*

$$d(Tx, Ty) \leq kd(x, y)$$

for all $x, y \in X$.

Theorem 2. *Let X be a complete bms and T contractive. Then, T has a unique fixed point $x^* \in X$ and $T^n x \to x^*$ for any $x \in X$, as n tends to infinity.*

In the following, we generalize the concept of quasi-fixed point, introduced in the reference [34], to the framework of bms.

Definition 10. *Let (T_n) be a sequence of self-maps $T_n : X \to X$, where X is a bms. An element $x^* \in X$ is a quasi-fixed point of (T_n) if*

$$\lim_{n \to +\infty} T_1 \circ T_2 \circ \ldots \circ T_n(x^*) = x^*.$$

In the next theorem, we generalize a result of convergence of non-autonomous discrete orbits to the context of bms. We can consider it a variant of Theorem 2 for a set of contractivities.

Theorem 3. *Let X be a complete and bounded bms with index s and (T_n) a sequence of contractivities with factors k_n, such that $k_n < s^{-1}$ for all n and $\prod_{i=1}^{n} k_i \to 0$ as n tends to infinity. Then, there is a unique quasi-fixed point $x^* \in X$ such that*

$$\lim_{n \to +\infty} T_1 \circ T_2 \circ \ldots \circ T_n(x) = x^*$$

for any $x \in X$. Moreover, x^* is the limit of the fixed points x_n of the composition $T_1 \circ T_2 \circ \ldots \circ T_n$.

Proof. Let us define the image sets of X: $M_n = T_1 \circ T_2 \circ \ldots \circ T_n(X)$. The set $(\bar{M}_n)$ is a nested sequence of closed sets of X:

$$\bar{M}_1 \supseteq \bar{M}_2 \supseteq \ldots \supseteq \bar{M}_n \ldots$$

Moreover,

$$diam(M_n) = \sup_{x,y \in X} \{d_b(T_1 \circ T_2 \circ \ldots \circ T_n(x), T_1 \circ T_2 \circ \ldots \circ T_n(y))\} \leq (\prod_{j=1}^{n} k_j) d(x, y).$$

According to Lemma 3.2 of the reference [38], if $K_n := \prod_{j=1}^{n} k_j$,

$$diam(\bar{M}_n) \leq s^2 diam(M_n) \leq s^2 K_n diam(X) \to 0$$

as n tends to infinity. Consequently, the sequence of the closed images satisfies the conditions of Cantor's intersection Theorem 1, and

$$\cap_{n=1}^{+\infty} \bar{M}_n = \{x^*\}.$$

Let us see that x^* is an attracting quasi-fixed point. For any $x \in X$, since $x^* \in \bar{M}_n$, we have

$$d_b(T_1 \circ T_2 \circ \ldots \circ T_n(x), x^*) \leq diam(\bar{M}_n) \to 0.$$

In particular, for $x = x^*$, the definition of quasi-fixed point is satisfied, and x^* is globally attracting.

Let x_n be the fixed point of $T_1 \circ T_2 \circ \ldots \circ T_n$. Then,

$$d_b(x_n, x^*) \leq s(d_b(T_1 \circ T_2 \circ \ldots \circ T_n(x_n), T_1 \circ T_2 \circ \ldots \circ T_n(x^*)) + d_b(T_1 \circ T_2 \circ \ldots \circ T_n(x^*), x^*))$$

and

$$d_b(x_n, x^*) \leq s(K_n d_b(x_n, x^*) + d_b(T_1 \circ T_2 \circ \ldots \circ T_n(x^*), x^*)).$$

Since the last term tends to zero by definition of quasi-fixed point, then $d_b(x_n, x^*) \to 0$ as n tends to infinity. □

In the next definition, we extend the concept of quasi-fixed point to a set of operators of any cardinal.

Definition 11. *Let $\mathcal{F} = \{T_i : X \to X, i \in J\}$ a set of self-maps on a bms X. An element $x^* \in X$ is a quasi-fixed point associated with $\mathcal{F}$ if there exists $\sigma \in J^{\mathbb{N}}$, $\sigma = (i_1 i_2 \ldots)$ such that*

$$\lim_{n \to +\infty} T_{i_1} \circ T_{i_2} \circ \ldots \circ T_{i_n}(x^*) = x^*.$$

The next result proves that if $\mathcal{F}$ is a set of contractivities with some conditions on the factors and the bms space X, the set $\mathcal{Q}$ of quasi-fixed points associated with $\mathcal{F}$ is a fractal set.

Theorem 4. *Let X be a complete and bounded bms with index s and $\mathcal{F}$ a set of contractivities whose factors k_i are such that $\sup_{i \in J} k_i < s^{-1}$. Then, the set $\mathcal{Q}$ of quasi-fixed points associated with $\mathcal{F}$ is self-similar in the sense that*

$$\mathcal{Q} = \cup_{i \in J} T_i(\mathcal{Q})$$

Proof. Let us check the equality proposed. If $x^* \in \mathcal{Q}$, according to Theorem 3, there exists $\sigma = (i_1 i_2 \ldots) \in J^{\mathbb{N}}$ such that

$$x^* = \lim_{n \to +\infty} T_{i_1} \circ T_{i_2} \circ \ldots \circ T_{i_n}(x) = T_{i_1}(\lim_{n \to +\infty} T_{i_2} \circ T_{i_3} \circ \ldots \circ T_{i_n}(x)) = T_{i_1}(\bar{x}),$$

where $\bar{x} \in \mathcal{Q}$. Consequently $x^* \in T_{i_1}(\mathcal{Q})$.

If $x' \in T_i(\mathcal{Q})$, then there exists $\sigma \in J^{\mathbb{N}}$ such that

$$x' = T_i(\lim_{n \to +\infty} T_{i_1} \circ T_{i_2} \circ \ldots \circ T_{i_n}(x)) = \lim_{n \to +\infty} T_i \circ T_{i_1} \circ T_{i_2} \circ \ldots \circ T_{i_n}(x) \in \mathcal{Q},$$

and the proof is completed. □

Consequently, the set $\mathcal{Q}$ is a fixed point of the operator:

$$\mathcal{T}(F) = \cup_{i \in J} T_i(F).$$

3. Fractal Interpolation in Banach Spaces and Algebras

In this section, we give very general conditions for the existence of a fractal curve with values on a Banach space. We use the term "curve" in a wide sense, representing any map $u : I \to \mathcal{A}$, where I is a real interval and $\mathcal{A}$ is a real Banach space or algebra.

Let us start defining general non-linear contractions for b-metric spaces.

Definition 12. *A function $\psi : [0, +\infty) \to [0, +\infty)$ is a fixed-point comparison function (or fp-comparison function for short) for a complete bms (E, d_b) if it is non-decreasing, and the inequality*

$$d_b(Tx, Ty) \leq \psi(d_b(x, y)), \tag{1}$$

for any $x, y \in E$ implies that $T : E \to E$ has a fixed point. Any T satisfying the inequality (1) for all x, y is a ψ-contraction.

Example 3. *The function $\psi(t) = kt$, where $0 < k < s^{-1}$, is a fp-comparison function for any complete b-metric space with index s, according to Theorem 2.*

Let us consider the real interval $I = [0, B]$ and a partition of it, $\Delta : t_0 = 0 < t_1 < t_2 < \ldots < t_N = B$. Let us denote $I_n = [t_{n-1}, t_n)$, for $i = 1, 2, \ldots, N-1$ and $I_N = [t_{N-1}, t_N]$ and define contractive homeomorphisms $L_n : I \to I_n$. For a real Banach space $\mathcal{A}$, define $F_n : I \times \mathcal{A} \to \mathcal{A}$, and $W_n(t, A) = (L_n(t), F_n(t, A))$ for $t \in I$ and $A \in \mathcal{A}$.

Theorem 5. *Let us consider the set $\mathcal{M}$ of curves in $\mathcal{A}$ with domain I, $\mathcal{M} = \{u : I \to \mathcal{A}\}$, and let a subset $E \subset \mathcal{M}$ be endowed with a structure of complete b-metric space (E, d_b). Consider the operator $T : E \to E$, defined as*

$$Tu(t) = F_n(L_n^{-1}(t), u \circ L_n^{-1}(t)), \tag{2}$$

for $t \in I_n$. Let T be a ψ-contraction, where ψ is an fp-comparison function for E, that is to say,

$$d_b(Tu, Tv) \le \psi(d_b(u, v)), \tag{3}$$

for any $u, v \in E$. Then, T admits a fixed point $\widetilde{u} : I \to \mathcal{A}$ whose graph is a self-similar set of $I \times \mathcal{A}$.

Proof. The existence of fixed point $\widetilde{u}$ is a consequence of the definition of the fp-comparison function. Moreover, if $\widetilde{u} : I \to \mathcal{A}$ is the fixed point of T:

$$graph(\widetilde{u}) = \cup_{n=1}^{N} \cup_{t \in I} \{(L_n(t), \widetilde{u} \circ L_n(t))\} = \cup_{n=1}^{N} \cup_{t \in I} \{(L_n(t), T\widetilde{u} \circ L_n(t))\}$$

$$graph(\widetilde{u}) = \cup_{n=1}^{N} \cup_{t \in I} \{(L_n(t), F_n(t, \widetilde{u}(t)))\} = \cup_{n=1}^{N} W_n(graph(\widetilde{u})).$$

Consequently, $graph(\widetilde{u})$ is an invariant of the operator

$$W(B) := \cup_{n=1}^{N} W_n(B),$$

and it owns a self-similar structure. □

Definition 13. *The mapping $\widetilde{u} : I \to \mathcal{A}$ is a Banach-valued fractal interpolation function.*

A different approach is given in the reference [26].

Remark 2. *Of course, the hypothesis that E is a b-metric space is not a sine qua non-condition. For instance, E could be an $\mathbb{F}$-complete $\mathbb{F}$-metric space and T a Banach contraction [39], as a type of $(\alpha, \beta)-$ admissible mapping [40]. Different metric structures can be read in the reference [41], for example. The election of a b-metric space is due to the fact that it well fits some spaces of integrable Banach-valued mappings (see Sections 5–7).*

4. Continuous Fractal Curves in Banach Spaces and Algebras

In this section, we consider conditions on the maps W_n in order to define continuous curves in $\mathcal{A}$.

We consider a set of data $\{(t_n, A_n)\}_{n=1}^{N} \subset I \times \mathcal{A}$ where $0 := t_0 < t_1 < t_2 < \ldots < t_N = B$ and assume that the mappings L_n, F_n satisfy the so-called "join-up" conditions

$$L_n(t_0) = t_{n-1}, \qquad L_n(t_N) = t_n \tag{4}$$

$$F_n(t_0, A_0) = A_{n-1}, \qquad F_n(t_N, A_N) = A_n \tag{5}$$

for $n = 1, 2, \ldots N$. In this case, the operator T defined in (2) maps the space $\mathcal{C}_0(I, \mathcal{A}) = \{u \in \mathcal{C}(I, \mathcal{A}); u(t_0) = A_0, u(t_N) = A_N\}$ into itself. The space $E = \mathcal{C}_0(I, \mathcal{A})$ is a complete b-metric space (with index 1) with respect to the distance $d_b(u, v) = ||u - v||_{sup}$, where $|| \cdot ||_{sup}$ denotes the supremum norm.

If F_n is a ψ_n-contraction in the second variable, that is to say,

$$||F_n(t, A) - F_n(t, A')|| \le \psi_n(||A - A'||),$$

for any $t \in I$ and $A, A' \in \mathcal{A}$, and $\psi(t) = \sup_n \psi_n(t)$ is an fp-comparison function for E, then

$$||Tu - Tv||_{sup} \leq \psi(||u - v||_{sup})$$

for any $u, v \in E$, and T admits a fixed point $\widetilde{u} \in \mathcal{C}_0(I, \mathcal{A})$, whose graph is a continuous fractal curve. Additionally, the map $\widetilde{u}$ passes through the points $\{(t_n, A_n)\}_{n=1}^N$ as in the real case.

If $\mathcal{A}$ is a real Banach algebra, let us consider

$$F_n(t, A) = A_n(t) \cdot A + q_n(t), \tag{6}$$

where $A_n, q_n : I \to \mathcal{A}$ are continuous functions satisfying the join-up conditions described. The dot $\cdot$ represents the product operation in the algebra $\mathcal{A}$. If

$$a := \sup\{||A_n(t)|| : t \in I, n = 1, 2, \dots N\} < 1, \tag{7}$$

we can consider the fp-comparison function $\psi(t) = at$ and T is a Banach contraction:

$$||Tu - Tv||_{sup} \leq a||u - v||_{sup}.$$

Consequently, it admits a globally attracting fixed point for the Picard iteration

$$u_{k+1} = Tu_k,$$

for $u_0 \in E$, $k \geq 0$.

Example 4. *If C is a Banach space, the set of linear and bounded operators $\mathcal{L}(C)$ is a Banach algebra. Let us consider the set of data $\{(t_n, S_n)\}_{n=1}^N$, where $t_n \in \mathbb{R}$ and $S_n \in \mathcal{L}(C)$. If F_n satisfies the conditions described, we obtain a continuous fractal curve of operators $\widetilde{u} : I \to \mathcal{L}(C)$ such that $\widetilde{u}(t_i) = S_i$.*

For instance, let us consider $C = \mathcal{C}[a, b]$, kernels $k_i \in \mathcal{C}[a, b]^2$, for $i = 1, 2, \dots, N$, and define the operators $S_i \in \mathcal{L}(C)$

$$S_i u(x) = \int_a^b k_i(x, y) u(y) dy,$$

for $u \in C$. The norm of S_i is

$$||S_i|| = \max_{t \in [a,b]} \int_a^b |k_i(x, y)| dy.$$

The image set of the fractal function $\widetilde{u}$ is a uniparametric family of linear and bounded operators $\{S_t\}_{t \in I}$ interpolating the given "points", that is to say, $S_{t_i} = S_i$, for all i.

5. Bochner Integrable Fractal Functions in Banach Algebras

In this section, we first remind the rudiments of integrability of Banach-valued mappings (see, for instance, [42,43]). We consider a real Banach space $\mathcal{A}$ with norm $|| \cdot ||$, and remind the definitions of the Bochner spaces of order p, $\mathcal{B}^p(I, \mathcal{A})$:

Definition 14. *Let the map $u : I \to \mathcal{A}$ be strongly measurable, then $u \in \mathcal{B}^p(I, \mathcal{A})$, for $0 < p < +\infty$ if the function $t \hookrightarrow ||u(t)||^p$ is Lebesgue integrable. In this case, we define:*

$$|u|_p = (\int_I ||u(t)||^p dt)^{1/p}.$$

The map u belongs to the class $\mathcal{B}^\infty(I, \mathcal{A})$, $u \in \mathcal{B}^\infty(I, \mathcal{A})$, if the function $t \hookrightarrow ||u(t)||$ is essentially bounded. Then,

$$|u|_\infty = esssup_{t \in I} ||u(t)||.$$

For $p \in [1, +\infty]$, $\mathcal{B}^p(I, \mathcal{A})$ is a real Banach space with respect to the norm $|\cdot|_p$. If $\mathcal{A}$ is a Hilbert space with inner product $< \cdot, \cdot >$, then $\mathcal{B}^2(I, \mathcal{A})$ is a Hilbert space with the inner product

$$< u, v >= \int_I < u(t), v(t) > dt,$$

for $u, v \in \mathcal{B}^2(I, \mathcal{A})$. Moreover,

$$\mathcal{B}^q(I, \mathcal{A}) \subset \mathcal{B}^p(I, \mathcal{A}),$$

for $1 \le p \le q \le +\infty$. For $0 < p < 1$,

$$|u|_p = (\int_I ||u(t)||^p dt)^{1/p}$$

is a quasi-norm with index $s = 2^{1/p-1}$. The mapping defined as $d_b(u, v) = |u - v|_p$ is a b-metric with index s, and $\mathcal{B}^p(I, \mathcal{A})$ is a complete b-metric space and a quasi-Banach space with respect to $|\cdot|_p$.

Let us recall the concept of quasi-norm:

Definition 15. *If B is a (real or complex) linear space, the mapping $|\cdot|_b : B \times B \to \mathbb{R}^+$ is a quasi-norm if:*

1. $|x|_b \ge 0$; $x = 0$ *if and only if* $|x|_b = 0$.
2. $|\lambda x|_b = |\lambda||x|_b$.
3. *There exists $s \ge 1$ such that $|x + y|_b \le s(|x|_b + |y|)_b$ for any $x, y \in B$.*

The "distance" associated with a quasi-norm: $d_b(u, v) := |u - v|_b$ is a b-metric since:

$$|u - v|_b = |u - w + w - v|_b \le s(|u - w|_b + |w - v|_b) \le s(d_b(u, w) + d_b(w, v)).$$

In this section, we define fractal integrable functions of type $u : I \to \mathcal{A}$, where $\mathcal{A}$ is a Banach space or algebra.

We consider the space E of Section 3, defined as $E = \mathcal{B}^p(I, \mathcal{A})$. Let the maps L_n be defined as $L_n(t) = a_n t + b_n$, satisfying the conditions (4) and F_n be Bochner p-integrable such that $F_n(t, \mathcal{A}) \in \mathcal{A}$.

Let the operator T be defined as in (2), then

$$|Tu - Tv|_p^p = \sum_{n=1}^{N} \int_{I_n} ||F_n(L_n^{-1}(t), u \circ L_n^{-1}(t)) - F_n(L_n^{-1}(t), v \circ L_n^{-1}(t))||^p dt,$$

with the change $\tilde{t} = L_n^{-1}(t)$, we obtain

$$|Tu - Tv|_p^p = \sum_{n=1}^{N} a_n \int_I ||F_n(t, u(t)) - F_n(t, v(t))||^p dt. \tag{8}$$

Since $\sum_{n=1}^{N} a_n = 1$ due to the join-up conditions (4), if F_n is a ψ_n-contraction in the second variable,

$$|Tu - Tv|_p^p \le \int_I \psi^p(||u(t) - v(t)||) dt,$$

where $\psi(t) = \sup_n \psi_n(t)$.

In the case $\psi_n(t) = k_n t$, where $k_n < s^{-1}$, then $\psi(t) = kt$, with $k = \sup_n k_n$ and

$$|Tu - Tv|_p \le k|u - v|_p, \tag{9}$$

where $sk < 1$. Thus, T is a contraction on the b-metric space $E = \mathcal{B}^p(I, \mathcal{A})$, and it owns a fixed point $\widetilde{u} \in \mathcal{B}^p(I, \mathcal{A})$. The function $\widetilde{u}$ is a global attractor for the T-iterations.

This happens in particular when $F_n(t, A) = A_n(t) \cdot A + q_n(t)$, where $A_n \in \mathcal{B}^{+\infty}(I, \mathcal{A})$, $q_n \in \mathcal{B}^p(I, \mathcal{A})$ and the constant

$$k = a := \max\{|A_n|_\infty : n = 1, 2, \dots, N\}$$

satisfies the condition

$$as < 1.$$

Remark 3. *It is easy to check that the inequalities of type (9), proved for $p < +\infty$, are applicable to the case $p = +\infty$ as well.*

Non-Contractive Case

In this subsection, we consider the case where the operator T is non-expansive (not necessarily contractive), and $p = 2$.

Definition 16. *A self-map $T : E \to E$ on a normed space E is non-expansive if for any $x, y \in E$,*

$$||Tx - Ty|| \leq ||x - y||.$$

The following theorem, due to Browder, ensures the existence of a fixed point for non-expansive maps in Hilbert spaces (see [44]).

Theorem 6. *Let C be a closed bounded convex subset of a Hilbert space H, and $T : C \to C$ a non-expansive map, then T has a fixed point in C.*

On the basis of this theorem, we prove that there exist fractal interpolation functions defined by non-contractive operators.

Let us consider $I = [0, 1]$ and $W_n(t, A) = (L_n(t), F_n(t, A))$ defined in $I \times \mathcal{A}$, where $\mathcal{A}$ is a real Hilbert space or algebra.

Theorem 7. *Let us assume that $F_n(t, A)$ satisfies the inequality $||F_n(t, A)|| \leq 1$, for all $t \in I$, $A \in \mathcal{A}$ and $n = 1, 2, \dots N$, and be such that $||F_n(t, A) - F_n(t, A')|| \leq ||A - A'||$ for all n, t and $A, A' \in \mathcal{A}$. Then, the operator $T : \mathcal{B}^2(I, \mathcal{A}) \to \mathcal{B}^2(I, \mathcal{A})$, defined as*

$$Tu(t) = F_n(L_n^{-1}(t), u \circ L_n^{-1}(t))$$

for $t \in I_n$, is non-expansive in the closed ball $C = \bar{B}(0, 1) = \{u \in \mathcal{B}^2(I, \mathcal{A}) : |u|_2 \leq 1\}$. Consequently, T has a fixed point in C.

Proof. If F_n satisfies the conditions given, arguing as in (8),

$$|Tu|_2^2 = \sum_{n=1}^{N} a_n \int_I ||F_n(t, u(t))||^2 dt \leq 1,$$

since $\sum_{n=1}^{N} a_n = 1$ due to the join-up conditions (4).

Thus, $T(C) \subseteq C$. Moreover, by the expression (8) and the second condition on F_n,

$$|Tu - Tv|_2 \leq |u - v|_2$$

for any $u, v \in \mathcal{B}^2(I, \mathcal{A})$ and, in particular, for $u, v \in C$. Consequently, C and T satisfy the hypotheses of Browder's Theorem, and T has a fixed point $\bar{u}$ in C. □

The graph of the fixed point $\bar{u}$ of T has a self-similar structure for the reasons given in the proof of Theorem 5, but $\bar{u}$ may not be unique.

6. Generalized α-Fractal Functions

In this section, we consider again $E = \mathcal{B}^p(I, \mathcal{A})$, $0 < p \leq +\infty$, where $\mathcal{A}$ is a real Banach algebra, and the maps L_n, defined as affine homeomorphisms satisfying (4). Let us consider for $u, v \in \mathcal{B}^p(I, \mathcal{A})$ the maps

$$F_n(t, A) = A_n(t) \cdot (A - v(t)) + u \circ L_n(t), \tag{10}$$

for any $t \in I_n$, $A_n \in \mathcal{B}^\infty(I, \mathcal{A})$. The operator $T_{u,v} : \mathcal{B}^p(I, \mathcal{A}) \to \mathcal{B}^p(I, \mathcal{A})$, defined as

$$T_{u,v}(w)(t) = u(t) + A_n(t) \cdot ((w - v) \circ L_n^{-1}(t)) \tag{11}$$

for $t \in I_n$, is such that

$$|T_{u,v}(w) - T_{u,v}(w')|_p \leq a|w - w'|_p,$$

where

$$a := \sup_{t \in I}\{||A_n(t)|| : n = 1, 2, \ldots, N\}$$

Thus, $T_{u,v}$ is a bms contraction if $a < s^{-1}$. According to Theorem 2, it owns a fixed point $\widetilde{u} \in \mathcal{B}^p(I, \mathcal{A})$ (with the condition on a).

Definition 17. *The map $\widetilde{u} : I \to \mathcal{A}$ is the generalized α-fractal function of the operator $T_{u,v}$.*

Remark 4. *This concept extends the notion of α-fractal function defined in previous papers for real functions (see, for instance, [27]) to the setting of Banach-valued maps.*

As in the real case, we can define a binary internal operation in $\mathcal{B}^p(I, \mathcal{A})$:

$$u * v = \widetilde{u}.$$

It is called fractal convolution of Bochner maps. The algebraic properties of this operation are similar to those of the real case (see the references [35,36,45]):

- The fractal convolution is idempotent, that is to say, $u * u = u$ for any $u \in \mathcal{B}^p(I, \mathcal{A})$.
- The operator $\mathcal{P} : \mathcal{B}^p(I, \mathcal{A}) \times \mathcal{B}^p(I, \mathcal{A}) \to \mathcal{B}^p(I, \mathcal{A})$ is linear, namely,

$$(u_1 + u_2) * (v_1 + v_2) = (u_1 * v_1) + (u_2 * v_2),$$

$$\lambda(u * v) = (\lambda u) * (\lambda v).$$

for any $u, v \in \mathcal{B}^p(I, \mathcal{A})$ and $\lambda \in \mathbb{R}$.

According to the expression (11), the fractal convolution satisfies for any $t \in I_n$:

$$(u * v)(t) = u(t) + A_n(t) \cdot ((u * v) - v) \circ L_n^{-1}(t),$$

and arguing, as in previous sections, for $0 < p \leq +\infty$,

$$|u * v - u|_p \leq a|u * v - v|_p \leq as(|u * v - u|_p + |u - v|_p). \tag{12}$$

Then, we have

$$|u * v - u|_p \leq \frac{as}{1 - as}|u - v|_p \tag{13}$$

if $as < 1$, for any $u, v \in \mathcal{B}^p(I, \mathcal{A})$. Let us define the side operators $L_u, R_v : \mathcal{B}^p(I, \mathcal{A}) \to \mathcal{B}^p(I, \mathcal{A})$, for any $u, v \in \mathcal{B}^p(I, \mathcal{A})$, as

$$L_u(v) = u * v,$$

$$R_v(u) = u * v.$$

Due to the linearity of $\mathcal{P}$,

$$L_u(v) + R_u(v) = (u * v) + (v * u) = (u + v) * (u + v) = u + v = u + Id(v),$$

where Id represents the identity operator. Consequently:

1. $L_u + R_u = u + Id$.
2. $L_0 + R_0 = Id$,

where $u = 0$ represents the mapping $u(t) = 0 \in \mathcal{A}$ for any $t \in I$. The linearity of $\mathcal{P}$ implies that L_0, R_0 are linear operators. Let us see that they are also bounded. Using (13) for $u = 0$,

$$|L_0(v)|_p = |0 * v|_p = |0 * v - 0|_p \leq \frac{as}{1 - as}|v|_p.$$

Then,

$$|L_0|_p \leq \frac{as}{1 - as}.$$

From the second item:

$$|R_0|_p = |Id - L_0|_p \leq s(1 + |L_0|_p) \leq \frac{s}{1 - as}.$$

Thus, L_0, R_0 are also bounded.

Remark 5. *Let us notice that, for the sake of simplicity, we are using the same notation for the quasi-norm of elements of Bochner spaces and their operators, though obviously they are different concepts.*

Proposition 1. *The side operators L_u, R_v are Lipschitz for any $u, v \in \mathcal{B}^p(I, \mathcal{A})$, $0 < p \leq +\infty$.*

Proof. The linearity of the operator $\mathcal{P}$ implies that

$$|L_u v - L_u v'|_p \leq |L_0(v - v')|_p \leq \frac{as}{1 - as}|v - v'|_p$$

and

$$|R_v u - R_v u'|_p \leq |R_0(u - u')|_p \leq \frac{s}{1 - as}|u - u'|_p.$$

□

7. Fractal Bases of Bochner Spaces

In this section, we consider a new convolution of operators of functional spaces and describe a procedure to construct fractal bases of Bochner spaces.

Let us now consider, the case $1 \leq p \leq +\infty$, and the space of linear and bounded operators on the space of Bochner p-integrable mappings, $\mathcal{L}(\mathcal{B}^p(I, \mathcal{A}))$. This set is a Banach algebra, since $\mathcal{B}^p(I, \mathcal{A})$ is a Banach space.

The maps considered are: $L_n(t) = a_n t + b_n$, with the conditions (4). We define F_n as in the previous section (10) and

$$a := \sup_{t \in I}\{||A_n(t)|| : n = 1, 2, \ldots, N\} < 1.$$

For $S, T \in \mathcal{L}(\mathcal{B}^p(I, \mathcal{A}))$, let us define the convolution $S * T$ as

$$(S * T)(u) = S(u) * T(u),$$

for any $u \in \mathcal{B}^p(I, \mathcal{A})$.

Proposition 2. *The convolution of operators satisfies the following properties:*

- $S * S = S$ *for any* $S \in \mathcal{L}(\mathcal{B}^p(I, \mathcal{A}))$.
- $S * T \in \mathcal{L}(\mathcal{B}^p(I, \mathcal{A}))$ *if* $S, T \in \mathcal{L}(\mathcal{B}^p(I, \mathcal{A}))$ *and*

$$|S * T|_p \leq \frac{|S|_p + a|T|_p}{1 - a}.$$

- *For any* $S, T, S', T' \in \mathcal{L}(\mathcal{B}^p(I, \mathcal{A}))$,

$$|S * T - S * T'|_p \leq \frac{a}{1 - a}|T - T'|_p,$$

$$|S * T - S' * T|_p \leq \frac{1}{1 - a}|S - S'|_p$$

Proof. The first item is a consequence of the idempotency of the fractal convolution of maps.

Using the fixed point equation for $S(u)$, $T(u)$ (see, for instance, (12)), we obtain

$$|(S * T)(u) - Su|_p \leq a|(S * T)(u) - Tu|_p \leq a|(S * T)(u)|_p + a|T|_p|u|_p.$$

Then,

$$|S * T|_p \leq \frac{|S|_p + a|T|_p}{1 - a}.$$

For the third item, the Lipschitz property of the side operators (Proposition 1) implies that

$$|S(u) * T(u) - S(u) * T'(u)|_p \leq \frac{a}{1 - a}|T(u) - T'(u)|_p \leq \frac{a}{1 - a}|T - T'|_p|u|_p.$$

Then,

$$|S * T - S * T'|_p \leq \frac{a}{1 - a}|T - T'|_p.$$

The last inequality is deduced in a similar way. □

Corollary 1. *The convolution of operators satisfies all the conditions required to be a metric convolution in the metric space* $\mathcal{L}(\mathcal{B}^p(I, \mathcal{A}))$, *as defined in [46], and the properties deduced for the operation are applicable to it.*

Due to idempotency, the convolved operators fill the whole space:

$$\mathcal{L}(\mathcal{B}^p(I, \mathcal{A})) = \{S * T : S, T \in \mathcal{L}(\mathcal{B}^p(I, \mathcal{A}))\}$$

The following result can be read in ref. [47].

Proposition 3. *Let* $U, V : X \to Y$, *such that* U *is linear, bounded and invertible,* V *linear and* X, Y *Banach spaces. If there exist constants* $\lambda_1, \lambda_2 \in [0, 1)$ *such that*

$$\|Ux - Vx\| \leq \lambda_1\|Ux\| + \lambda_2\|Vx\|,$$

for any $x \in X$, *then* V *is invertible and*

$$\frac{1 - \lambda_1}{1 + \lambda_2}\|Ux\| \leq \|Vx\| \leq \frac{1 + \lambda_1}{1 - \lambda_2}\|Ux\|, \tag{14}$$

$$\frac{1 - \lambda_2}{1 + \lambda_1}\frac{1}{||U||}\|y\| \leq \|V^{-1}y\| \leq \frac{1 + \lambda_2}{1 - \lambda_1}||U^{-1}||\|y\|,$$

for any $x \in X$ *and* $y \in Y$.

Theorem 8. *Let $S \in \mathcal{L}(\mathcal{B}^p(I,\mathcal{A}))$ and $T : \mathcal{B}^p(I,\mathcal{A}) \to \mathcal{B}^p(I,\mathcal{A})$ linear. If S is invertible and $|Tu|_p \leq |Su|_p$ for any $u \in \mathcal{B}^p(I,\mathcal{A})$, then $S * T$ is invertible and*

$$|S * T|_p \leq \frac{1+a}{1-a}|S|_p, \tag{15}$$

$$|(S * T)^{-1}|_p \leq \frac{1+a}{1-a}|S^{-1}|_p.$$

The sequence (u_m) is a Schauder basis of $\mathcal{B}^p(I,\mathcal{A})$ if and only if $(S(u_m) * T(u_m))$ is a basis. If (u_m) is a normalized basis, then $(S(u_m) * T(u_m))$ is a bounded basis and

$$\frac{1-a}{1+a}|S^{-1}|_p^{-1} \leq |(S * T)(u_m)|_p \leq \frac{1+a}{1-a}|S|_p.$$

Proof. Arguing as in previous sections (12), one has

$$|(S * T)(u) - Su|_p \leq a|(S * T)(u) - Tu|_p \leq a|(S * T)(u)|_p + a|Su|_p.$$

The hypotheses of Proposition 3 are satisfied for $U = S$, $V = S * T$ and $\lambda_1 = \lambda_2 = a < 1$. Then, $S * T$ is invertible and

$$|S * T|_p \leq \frac{1+a}{1-a}|S|_p,$$

$$|(S * T)^{-1}|_p \leq \frac{1+a}{1-a}|S^{-1}|_p.$$

An automorphism preserves the bases, and thus $(S(u_m) * T(u_m))$ is a basis if and only if (u_m) is. Let (u_m) be a normalized sequence then, according to (14),

$$|(S * T)(u_m)|_p \leq \frac{1+a}{1-a}|S|_p$$

For the left bound, let us consider

$$1 = |u_m|_p \leq |S^{-1}|_p |Su_m|_p.$$

Applying (14),

$$|(S * T)(u_m)|_p \geq \frac{1-a}{1+a}|Su_m|_p \geq \frac{1-a}{1+a}|S^{-1}|_p^{-1}.$$

□

Remark 6. *As a particular case, we obtain bases of type $(u_m * 0) = ((Id * T_0)(u_m))$, where T_0 is the null operator and Id is the identity, considered, for instance, in the reference [36] in the real setting ($\mathcal{A} = \mathbb{R}$). With the basic hypothesis $a < 1$, according to Theorem 8, the system $(u_m * 0)$ is a Schauder basis if (u_m) is.*

In the next subsection, we consider bases related to $C*$-algebras.

Fractal Bases of Mappings Valued in $C*$-Algebras

Let us assume that $\mathcal{A}$ is a real $C*$-algebra. Thus, it is endowed with an involution, and each element $A \in \mathcal{A}$ has an adjoint denoted by A^+. For $u \in \mathcal{B}^p(I,\mathcal{A})$, let us define $u^+ \in \mathcal{B}^p(I,\mathcal{A})$ as

$$u^+(t) = (u(t))^+,$$

for any $t \in I$. Define the operator $C : \mathcal{B}^p(I,\mathcal{A}) \to \mathcal{B}^p(I,\mathcal{A})$

$$C(u) = u * u^+,$$

with the usual fractal convolution of maps. C is a convolution of operators since $C = Id * H$, where $H(u) = u^+$ for $u \in \mathcal{B}^p(I, \mathcal{A})$. Let $\mathcal{H}$ denote the set of self-adjoint elements of $\mathcal{A}$. The set of maps:

$$U_H = \{u : I \to \mathcal{H}; u \in \mathcal{B}^p(I, \mathcal{A})\}$$

is such that $U_H \subseteq Fix(C)$, according to the properties of the fractal convolution.
The norm of H is one, since

$$|Hu|_p = (\int_I ||u^+(t)||_p^p dt)^{1/p} = (\int_I ||u(t)||_p^p dt)^{1/p} = |u|_p.$$

Consequently, $C = Id * H$ is an automorphism of $\mathcal{B}^p(I, \mathcal{A})$ according to Theorem 8. The norms of C and C^{-1} own the same bound:

$$|C|_p \leq \frac{1+a}{1-a}, \tag{16}$$

$$|C^{-1}|_p \leq \frac{1+a}{1-a}. \tag{17}$$

Due to isomorphism, we have the following theorem:

Theorem 9. *(u_m) is a Schauder basis of $\mathcal{B}^p(I, \mathcal{A})$ if and only if $(u_m * u_m^+)$ is a Schauder basis of $\mathcal{B}^p(I, \mathcal{A})$.*

In the case where $p = 2$ and $\mathcal{A}$ is a Hilbert space, the following result concerning frames is obtained.

Definition 18. *A sequence $(x_m)_{m=0}^{+\infty} \subset X$, where X is a Hilbert space is a frame if there exist positive constants A, B, such that for any $x \in X$:*

$$A||x||^2 \leq \sum_{m=0}^{+\infty} | < x, x_m > |^2 \leq B||x||^2. \tag{18}$$

Theorem 10. *If $\mathcal{A}$ is a real Hilbert space and (u_m) is a frame of $\mathcal{B}^2(I, \mathcal{A})$ with bounds A, B, then $(u_m * u_m^+)$ is also a frame with bounds $A(\frac{1+a}{1-a})^{-2}$ and $B(\frac{1+a}{1-a})^2$.*

Proof. Since (u_m) is a frame with bounds A, B, if C^+ is the adjoint operator of C, the inequalities (18) applied to $x = C^+(u)$ take the form:

$$A|C^+(u)|_2^2 \leq \sum_{m=0}^{+\infty} | < C^+(u), u_m > |_2^2 \leq B|C^+(u)|_2^2 \leq B|C|_2^2|u|_2^2.$$

As

$$\sum_{m=0}^{+\infty} | < u, C(u_m) > |_2^2 = \sum_{m=0}^{+\infty} | < C^+(u), u_m > |_2^2,$$

then

$$A|C^+(u)|_2^2 \leq \sum_{m=0}^{+\infty} | < u, C(u_m) > |_2^2 \leq B|C|_2^2|u|_2^2 \tag{19}$$

obtaining the right inequality. For the left,

$$|u|_2^2 = |(C^+)^{-1} \circ C^+(u)|_2^2 \leq |(C^+)^{-1}|_2^2|C^+(u)|_2^2,$$

and

$$A|(C^+)^{-1}|_2^{-2}|u|_2^2 \leq A|C^+(u)|_2^2.$$

Bearing in mind that $|(C^+)^{-1}|_2 = |C^{-1}|_2$, from (19),

$$A|C^{-1}|_2^{-2}|u|_2^2 \leq \sum_{m=0}^{+\infty} |<u, C(u_m)>|_2^2 \leq B|C|_2^2|u|_2^2.$$

Since $C(u_m) = u_m * u_m^+$, the bounds (16) and (17) provide the result. □

Definition 19. *If (u_m) is a frame, the system $(u_m * u_m^+)$ is the Hermitian fractal frame associated with (u_m).*

In the following, we recall the operators linked to a frame and give a characterization of the frame operator of the Hermitian fractal frame.

Given a frame (u_m) of $\mathcal{B}^2(I, \mathcal{A})$, the analysis operator is defined as $\mathcal{T} : \mathcal{B}^2(I, \mathcal{A}) \to l^2(\mathbb{R})$

$$\mathcal{T}(u) = (<u, u_m>)$$

The synthesis operator is the adjoint of $\mathcal{T}$, expressed as $\mathcal{T}^+ : l^2(\mathbb{R}) \to \mathcal{B}^2(I, \mathcal{A})$

$$\mathcal{T}^+((c_m)) = \sum_{m=0}^{+\infty} c_m u_m.$$

The frame operator is $\mathcal{S} = \mathcal{T}^+ \circ \mathcal{T} : \mathcal{B}^2(I, \mathcal{A}) \to \mathcal{B}^2(I, \mathcal{A})$:

$$\mathcal{S}(u) = \sum_{m=0}^{+\infty} <u, u_m> u_m. \tag{20}$$

Let $\mathcal{T}_H$ and $\mathcal{S}_H$ be the analysis and frame operators associated with the Hermitian frame $(u_m * u_m^+)$. Then,

$$<\mathcal{S}_H u, u> = <\mathcal{T}_H^+ \circ \mathcal{T}_H u, u> = <\mathcal{T}_H u, \mathcal{T}_H u> = ||\mathcal{T}_H u||_2^2,$$

where $|| \cdot ||_2$ represents the 2-norm in $l^2(\mathbb{R})$. Using the frame inequalities according to Theorem 10

$$A|u|_2^2(\frac{1+a}{1-a})^{-2} \leq <\mathcal{S}_H u, u> \leq B(\frac{1+a}{1-a})^2|u|_2^2.$$

The frame operator allows the definition of the dual frame: $v_m = \mathcal{S}_H^{-1}(u_m * u_m^+)$ (see, for instance, [48]) and the expansion of an element in terms of the frame [47]:

$$u = \sum_{m=0}^{+\infty} <u, v_m> (u_m * u_m^+).$$

In the next Theorem, we provide the relation between the frame operators of original and fractal frame. The result proves that they are congruent.

Theorem 11. *$\mathcal{S}_H$ is the frame operator of the Hermitian frame $(u_m * u_m^+)$ if and only if $\mathcal{S}_H = C \circ \mathcal{S} \circ C^+$, where $\mathcal{S}$ is the frame operator of (u_m), $C = Id * H$ and $Hu = u^+$.*

Proof. C is defined as $Cu = u * u^+$. Then, according to (20),

$$(\mathcal{S} \circ C^+)u = \sum_{m=0}^{+\infty} <C^+u, u_m> u_m = \sum_{m=0}^{+\infty} <u, Cu_m> u_m.$$

Since C is linear and bounded

$$(C \circ \mathcal{S} \circ C^+)u = \sum_{m=0}^{+\infty} <u, Cu_m> Cu_m.$$

The last sum agrees with the fractal frame operator on u, and thus

$$\mathcal{S}_H = C \circ \mathcal{S} \circ C^+.$$

The argument can be followed in the inverse way as well. $\square$

8. Conclusions

The concept of quasi-fixed point, associated to a set of contractions on a metric space, can be generalized to b-metric spaces, if some conditions on the contraction ratios are imposed. On these hypotheses, the family of quasi-fixed points is a fractal set of the b-metric space where they are defined.

For a Banach space or algebra $\mathcal{A}$, we consider a set E of curves of type $u : I \to \mathcal{A}$, where I is a compact real interval and E is endowed with a structure of b-metric space. On E, we define a non-linear ψ-contraction $T : E \to E$. If ψ is a suitable comparison function, then T owns a fixed point $\widetilde{u} : I \to \mathcal{A}$, called Banach-valued fractal interpolation function. The operator T is linked to a partition of I and a collection of maps $\{W_n : I \times \mathcal{A} \to I \times \mathcal{A}\}$. Suitably choosing the properties of W_n, the fractal function $\widetilde{u}$ may be continuous on I. In this way, we can define curves of measures, operators or matrices, depending on the algebra or space considered.

The defined fractal functions may be integrable curves on $\mathcal{A}$, that is to say, mappings belonging to Bochner spaces of order p, where p ranges from zero to infinity. We proved that in some specific cases, the operator T is non-expansive (not necessarily contractive, even in the widest sense), though it owns fixed points whose graphs are self-similar. Thus, the existence of fractal interpolation functions coming from non-contractive operators is proved. This fact holds in the real case ($\mathcal{A} = \mathbb{R}$) in particular.

The concept of α-fractal function [27] has been generalized to Banach-valued maps. The fractal convolution [36,45] is extended to mappings and operators defined on Banach structures. These operations enable the definition of fractal bases for the Bochner spaces $\mathcal{B}^p(I, \mathcal{A})$, as perturbations of pre-existing bases in the same space. A particular case for $C*$-algebras using the involution operator of this type of structures was considered, thus obtaining specific fractal frames and bases for them. It was also proved that the frame operators of original and fractal frames are congruent in the Hilbert case.

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