fractal and fractional

MDPI

Article

Some Novel Inequalities for LR-(k,h-m)-p Convex Interval Valued Functions by Means of Pseudo Order Relation

Vuk Stojiljković [1,*,†], Rajagopalan Ramaswamy [2,†], Ola A. Ashour Abdelnaby [2,†] and Stojan Radenović [3]

[1] Faculty of Science, University of Novi Sad, Trg Dositeja Obradovića 3, 21000 Novi Sad, Serbia
[2] Department of Mathematics, College of Science and Humanities, Prince Sattam bin Abdulaziz Univeristy, Al-Kharj 16278, Saudi Arabia
[3] Faculty of Mechanical Engineering, University of Belgrade, Kraljice Marije 16, 11120 Beograd, Serbia
* Correspondence: vuk.stojiljkovic999@gmail.com; Tel.: +381-640123911
† These authors contributed equally to this work.

Abstract: In this paper, a new type of convexity is defined, namely, the left–right-(k,h-m)-p IVM (set-valued function) convexity. Utilizing the definition of this new convexity, we prove the Hadamard inequalities for noninteger Katugampola integrals. These inequalities generalize the noninteger Hadamard inequalities for a convex IVM, (p,h)-convex IVM, p-convex IVM, h-convex, s-convex in the second sense and many other related well-known classes of functions implicitly. An apt number of numerical examples are provided as supplements to the derived results.

Keywords: convex set-valued functions; left–right-(k,h-m)-p-convex set-valued functions; Katugampola noninteger integral operators; Hermite–Hadamard inequality

MSC: 26D10; 26A33

check for updates

Citation: Stojiljković, V.; Ramaswamy, R.; Abdelnaby, O.A.A. Radenović, S. Some Novel Inequalities for LR-(k,h-m)-p Convex Interval Valued Functions by Means of Pseudo Order Relation. *Fractal Fract.* **2022**, *6*, 726. https://doi.org/10.3390/fractalfract6120726

Academic Editors: Ivanka Stamova, Eze Nwaeze and Seth Kermausuor

Received: 28 October 2022
Accepted: 6 December 2022
Published: 9 December 2022

1. Introduction

From the time when Gauss, Cauchy, Chebyshev, to mention the most important, gave a theoretical background on the approximative methods, the big theory of inequalities started to develop. At the end of the 19th century and the beginning of the 20th century, a large number of inequalities were proven, and some of them became the classics we know today, while the rest of them remained isolated results. The first book to connect all the inequalities and make them formally as the field we know today is the book Inequalities, written by Hardy et al. [1]. The book Inequalities was the first of its kind to be dedicated solely to inequalities and therefore was an instrumental book to the field. This paper concerns itself with convex inequalities, the ones using the notion of convexity introduced by Jensen. Since Jensen discovered the first convex inequality [2], various inequalities have been discovered as a consequence of Jensen's inequality [3]. A variety of applications of convex inequalities exist in, for example, the fields of numerical analysis, physics and optimization problems. The following books can be referred to for more information [4–12].

Hadamard [13] gave the following:

Let $\mathfrak{L} : \mathbb{I} \to \mathbb{R}$ be a convex function on $\mathbb{I}$ in $\mathbb{R}$ and $\omega_1, \omega_2 \in \mathbb{I}$ such that $\omega_1 < \omega_2$, then

$$\mathfrak{L}\left(\frac{\omega_1+\omega_2}{2}\right) \leqslant \frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2} \mathfrak{L}(t)dt \leqslant \frac{\mathfrak{L}(\omega_1)+\mathfrak{L}(\omega_2)}{2}.$$

Various generalizations have been reported over the years [14–16]. The Hermite–Hadamard inequality has been obtained using many different convex generalizations of the Jensen's inequality, see [17–19]. We apply the set-valued function setting (IVM) in tandem with convexity properties along with noninteger integral operators.

In 1695, l'Hopital sent a letter to Leibniz. In his message, an important question about the order of the derivative emerged: What might be a derivative of order $\frac{1}{2}$? That letter sparked the interest of many upcoming mathematicians to investigate further into the matter of noninteger derivatives. Then, in 1822, Fourier suggested an integral representation in order to define the derivative, and his version can be considered as the first definition of the derivative of an arbitrary positive order. Abel in 1826 solved an integral equation associated with a tautochrone problem, which was the first application of FC (noninteger calculus). After Abel, many mathematicians proceeded to work in the field such as Riemann, Grünwald and Letnikov, Hadamard, Weyl and many more. In the late half of the 20th century, Caputo formulated a definition, more restrictive than the Riemann–Liouville one but more appropriate to discuss problems involving noninteger differential equations with initial conditions. Noninteger calculus was found to be useful in physics as well; for example, Whatcraft and Meerschaert (2008) described a noninteger conservation of mass, acoustic wave equations for complex media and many others. Different types of noninteger integrals and derivatives have been defined throughout the years; we refer the interested reader to the following books [20,21] for more information on the matter. Generalizations and the usage of the noninteger calculus in the field of inequalities is also widespread, see [22–28] for more information.

One of the highly influential papers in the last year was the paper written by Khan et al. [29], which brought the notion of fuzzy convex inequalities and as such is worth to mention. The notion itself is a broad field which can be investigated further on. See the cited paper for more information thereon.

The motivation behind this paper is to introduce a new class of IVM, namely left–right-(k,h-m)-p-convex inequalities. The defined IVM inequality generalizes previously defined IVM convex inequalities. Namely, it contains in itself a previously defined p,h-convex IVM. For more information about noninteger calculus, see the following [20,21,30,31].

2. Preliminaries

We require the following definitions and monograph in the sequel:
The following notion of IVM is used, which contains sets in itself .

$$\mathbb{K}_c = \{[\mathfrak{S}_*, \mathfrak{S}^*] : \mathfrak{S}_*, \mathfrak{S}^* \in \mathbb{R} \textit{ and } \mathfrak{S}_* \leqslant \mathfrak{S}^*\}.$$

The range $[\mathfrak{S}_*, \mathfrak{S}^*]$ is a positive range if $\mathfrak{S}_* \geqslant 0$ and is given as follows

$$\mathbb{K}_c^+ = \{[\mathfrak{S}_*, \mathfrak{S}^*] : \mathfrak{S}_*, \mathfrak{S}^* \in \mathbb{R} \textit{ and } \mathfrak{S}_* \geqslant 0\}.$$

The elementary operations for $[\mathfrak{T}_*, \mathfrak{T}^*], [\mathfrak{S}_*, \mathfrak{S}^*] \in K_c$ and $\omega_6 \in \mathbb{R}$ are defined as follows:

$$[\mathfrak{T}_*, \mathfrak{T}^*] + [\mathfrak{S}_*, \mathfrak{S}^*] = [\mathfrak{T}_* + \mathfrak{S}_*, \mathfrak{T}^* + \mathfrak{S}^*],$$

$$[\mathfrak{T}_*, \mathfrak{T}^*] \cdot [\mathfrak{S}_*, \mathfrak{S}^*] = [\min\{\mathfrak{T}_*\mathfrak{S}_*, \mathfrak{T}^*\mathfrak{S}_*, \mathfrak{T}_*\mathfrak{S}^*, \mathfrak{T}^*\mathfrak{S}^*\}, \max\{\mathfrak{T}_*, \mathfrak{S}_*, \mathfrak{T}^*\mathfrak{S}_*, \mathfrak{T}_*\mathfrak{S}^*, \mathfrak{T}^*\mathfrak{S}^*\}]$$

and

$$\omega_6 \cdot [\mathfrak{T}_*, \mathfrak{T}^*] = \begin{cases} [\omega_6\mathfrak{T}_*, \omega_6\mathfrak{T}^*], (\omega_6 > 0), \\ \{0\}, (\omega_6 = 0), \\ [\omega_6\mathfrak{T}^*, \omega_6\mathfrak{T}_*], (\omega_6 < 0) \end{cases},$$

respectively, and the difference is given by

$$\mathfrak{T} - \mathfrak{S} = [\mathfrak{T}_*, \mathfrak{T}^*] - [\mathfrak{S}_*, \mathfrak{S}^*] = [\mathfrak{T}_* - \mathfrak{S}^*, \mathfrak{T}^* - \mathfrak{S}_*].$$

The mathematical notion $\mathfrak{S} \supseteq \mathfrak{T}$ gives us

$$\mathfrak{S} \supseteq \mathfrak{T} \textit{ if and only if} [\mathfrak{S}_*, \mathfrak{S}^*] \supseteq [\mathfrak{T}_*, \mathfrak{T}^*] \textit{ if and only if} [\mathfrak{T}_* \geqslant \mathfrak{S}_*, \mathfrak{S}^* \geqslant \mathfrak{T}^*]$$

Remark 1 ([32])**.** *The relation "$\leqslant_p$" defined on K_c by*

$$[\mathfrak{T}_*, \mathfrak{T}^*] \leqslant_p [\mathfrak{S}_*, \mathfrak{S}^*]$$

$\iff \mathfrak{T}_* \leqslant \mathfrak{S}_*, \mathfrak{T}^* \leqslant \mathfrak{S}^*$ *for all* $[\mathfrak{T}_*, \mathfrak{T}^*], [\mathfrak{S}_*, \mathfrak{S}^*] \in \mathbb{R}$ *is a pseudo-order relation; for more details, see [32].*

The integral given by Moore [32] is introduced as:

Theorem 1 ([32])**.** *Given* $\mathfrak{L} : [\rho_1, \rho_2] \subset \mathbb{R} \to K_c$ *a set-valued function such that*

$$\mathfrak{L}(\psi_5) = [\mathfrak{L}_*(\psi_5), \mathfrak{L}^*(\psi_5)].$$

Then, $\mathfrak{L}$ *is Riemann integrable over* $[\rho_1, \rho_2] \iff \mathfrak{L}_*$ *and* $\mathfrak{L}^*$ *are both Riemann integrable over* $[\rho_1, \rho_2]$*.*

$$(IR)\int_{\rho_1}^{\rho_2} \mathfrak{L}(\psi_5)d\psi_5 = \left[(R)\int_{\rho_1}^{\rho_2} \mathfrak{L}_*(\psi_5)d\psi_5, (R)\int_{\rho_1}^{\rho_2} \mathfrak{L}^*(\psi_5)d\psi_5\right]$$

Definition 1 ([33])**.** *For the set* $\Upsilon^\star$ *in* $\mathbb{R}$*, a function* $\mathfrak{F} : \Upsilon^\star \to \mathbb{R}$ *is convex on* $\Upsilon^\star$ *if*

$$\mathfrak{F}(\omega_6\psi_5 + (1-\omega_6)\psi_6) \leqslant \omega_6\mathfrak{F}(\psi_5) + (1-\omega_6)f(\psi_6)$$

for all $\psi_5, \psi_6 \in \Upsilon^\star$ *and* $\omega_6 \in [0,1]$ *holds and is a concave function if the inequality is of the opposite sign.*

Khan et al. [34] proposed the following:

Definition 2. *The set-valued function* $\mathscr{L} : \mathscr{W}^\star \to K_c^+$ *is left–right-convex set-valued on a convex set* $\mathscr{W}^\star$ *in all cases*$\rho_1, \rho_2 \in \mathscr{W}^\star$ *and* $\omega_6 \in [0,1]$*, we have*

$$\mathscr{L}(\omega_6\rho_1 + (1-\omega_6)\rho_2) \leqslant_p \omega_6\mathscr{L}(\rho_1) + (1-\omega_6)\mathscr{L}(\rho_2).$$

If the inequality is of the opposite sign, then $\mathscr{L}$ *is left–right-concave on* $\mathscr{W}^\star$*. Moreover,* $\mathscr{L}$ *is affine on* $\mathscr{W}^\star \iff$ *it is both left–right-convex and left–right-concave on* $\mathscr{W}^\star$*.*

Now, we introduce the concept of the Katugampola noninteger integral operator for a set-valued function.

Definition 3. *Let* $q \geqslant 1$*,* $c \in \mathbb{R}$ *and* $\chi_c^q(u,v)$ *be the set of all complex-valued Lebesgue integrable set-valued functions* Q *on* $[\psi_5, \psi_6]$ *for which the norm* $||Q||\chi_c^q$ *is introduced by*

$$||Q||\chi_c^q = \left(\int_u^v |n^c Q(n)|^q \frac{dn}{n}\right)^{\frac{1}{q}} < +\infty$$

for $1 \leqslant q < +\infty$ *and*

$$||Q||\chi_c^{+\infty} = ess \sup_{u \leqslant n \leqslant v} n^c|Q(n)|.$$

Katugampola [35] presented a new noninteger integral to generalize the Riemann–Liouville and Hadamard noninteger integrals under certain conditions.

Let $p, \xi_\star > 0$ *and* $\mathfrak{F} \in L[\psi_5, \psi_6]$ *be the collection of all complex-valued Lebesgue integrable IVMs on* $[\psi_5, \psi_6]$*. Then, the set of left and right Katugampola noninteger integrals of* $\mathfrak{F} \in L[\psi_5, \psi_6]$ *with order* $\xi_\star > 0$ *are introduced by*

$$J_{\psi_5^+}^{p,\xi_\star}\mathfrak{F}(x) = \frac{p^{1-\xi_\star}}{\Gamma(\xi_\star)}\int_{\psi_5}^{x}(x^p - \mu^p)^{\xi_\star - 1}\mu^{p-1}\mathfrak{F}(\mu)d\mu \quad (x > \psi_5),$$

$$J_{\psi_6^-}^{p,\xi_\star}\mathfrak{F}(x) = \frac{p^{1-\xi_\star}}{\Gamma(\xi_\star)}\int_{x}^{\psi_6}(\mu^p - x^p)^{\xi_\star - 1}\mu^{p-1}\mathfrak{F}(\mu)d\mu \quad (x < \psi_6)$$

where $\Gamma(\xi_\star) = \int_0^{+\infty} e^{-x}x^{\xi_\star - 1}dx$ *is the Euler gamma function [36]*

Zhang and Wang [37] established the concept of p-convex functions given below.

Definition 4. *Let $p \in \mathbb{R}$ with $p \neq 0$. Then, the set $\Upsilon^\star$ is p-convex if*

$$[\rho x^p + (1-\rho)y^p]^{\frac{1}{p}} \in \Upsilon^\star$$

for all $x, y \in \Upsilon^\star, \rho \in [0,1]$, where $p = 2n+1$ and $n \in \mathcal{N}$ or p is an odd number.

Definition 5. *Let $\mathbb{I}$ be a p-convex set. A function $\mathfrak{F} : \mathbb{I} \to \mathbb{R}$ is a p-convex function or belongs to the class PC(I), if*

$$\mathfrak{F}\left((\rho x^p + (1-\rho)y^p)^{\frac{1}{p}}\right) \leqslant \rho\mathfrak{F}(x) + (1-\rho)f(y)$$

for all $x, y \in [\psi_5, \psi_6]$, $\rho \in [0,1]$. If the inequality is of the opposite sign, then $\mathfrak{F}$ is called a p-concave function.

The following definition is utilized by Khan et al. to produce generalizations of the HH inequality [38–40].

Definition 6. *The IVM $\mathfrak{L} : [\psi_5, \psi_6] \to K_c^+$ is a left–right-p-convex IVM in all cases $\tau_2, \tau_3 \in [\psi_5, \psi_6]$ and $\rho \in [0,1]$ and we have*

$$\mathfrak{L}\left([\rho\tau_2^p + (1-\rho)\tau_3^p]^{\frac{1}{p}}\right) \leqslant_p \rho\mathfrak{L}(\tau_2) + (1-\rho)\mathfrak{L}(\tau_3).$$

If the inequality is of the opposite sign, then $\mathfrak{L}$ is left–right-p-concave on $[\psi_5, \psi_6]$. The set of all left–right-p-convex (left–right-p-concave) IVMs is denoted by

$$Left - RightSX([\psi_5, \psi_6], K_c^+, p) \quad (Left - RightSV([\psi_5, \psi_6], K_c^+, p)).$$

Definition 7. *Let $J \subset \mathbb{R}$ be a set having in itself $(0,1)$ and let $\tau : J \to \mathbb{R}$ be a positive function, including zero. Let $\mathbb{I} \subset (0, +\infty)$ be a set and $p \in \mathbb{R} - 0$. A function $\mathscr{L} : \mathbb{I} \to \mathbb{R}$ is $(k, h\text{-}m)$-p convex, if*

$$\mathscr{L}\left([x^p\xi_\star + m(1-\xi_\star)y^p]^{\frac{1}{p}}\right) \leqslant \tau(\xi_\star^k)\mathscr{L}(x) + m\tau(1-\xi_\star^k)\mathscr{L}(y)$$

holds provided $[x^p\xi_\star + m(1-\xi_\star)y^p]^{\frac{1}{p}} \in \mathbb{I}$ for all $\xi_\star \in [0,1]$ and $(k,m) \in [0,1]$. If the inequality is of the opposite sign, then $\mathscr{L}$ is said to be $(k, h\text{-}m)$-p-concave. The set of all $(k, h\text{-}m)$-p-convex (concave) functions is denoted by

$$KHMPX(\mathbb{I}, p)\,,\, KHMPV(\mathbb{I}, p),$$

respectively. The set of all $(k, h\text{-}m)$-p-convex (concave) functions defined on closed, positive and bounded sets of $\mathbb{R}$ is given, respectively, by

$$KHMPX([\psi_5, \psi_6], K_c^+, p), \;\; KHMPV([\psi_5, \psi_6], K_c^+, p).$$

The motivation behind defining the $(k, h\text{-}m)$-p-convex IVM is the definition given by [41] above.

Now we define a new type, namely a $(k, h\text{-}m)$-*p-convex* I-V-F.

Definition 8. *Let $J \subset \mathbb{R}$ be a set containing $(0,1)$ and let $\tau : J \to \mathbb{R}$ be a positive function including zero. Let $\mathbb{I}$ be a positive subset of $\mathbb{R}$ and $p \in \mathbb{R} - 0$. The IVM $\mathscr{F} : [\psi_5, \psi_6] \to K_c^+$ is a left–right-$(k, h\text{-}m)$-p-convex IVM if*

$$\mathscr{F}\left([u^p\xi_\star + m(1-\xi_\star)v^p]^{\frac{1}{p}}\right) \leqslant_p \tau(\xi_\star^k)\mathscr{F}(u) + m\tau(1-\xi_\star^k)\mathscr{F}(v)$$

holds, provided $[u^p\xi_\star + m(1-\xi_\star)v^p]^{\frac{1}{p}} \in \mathbb{I}$ *for all* $\xi_\star \in [0,1]$ *and* $(k,m) \in [0,1]$. *If the inequality is of the opposite sign, then* $\mathscr{F}$ *is a left–right-*$(k, h\text{-}m)$*-p-concave IVM. The set of all left–right-*$(k, h\text{-}m)$*-p-convex (left–right-*$(k, h\text{-}m)$*-p-concave) IVMs is denoted by*

$$Left - RightSKHMPX([\psi_5, \psi_6], K_c^+, p) \quad (Left - RightSKHMPV([\psi_5, \psi_6], K_c^+, p)).$$

$\mathscr{F}$ *is left–right-(k,h-m)-p-affine* $\iff$ *it is both left–right-(k,h-m)-p-convex and left–right(k,h-m)-p-concave.*

The set of all left–right-(k,h-m)-p-affine IVMs is denoted by $Left\text{-}RightSKHMPA$ $([\psi_5, \psi_6], K_c^+, p))$.

Remark 2.

- *Setting* $k, m = 1$*, we get the p,h-convex IVM introduced by Khan et al. [42] given by*

$$\mathscr{F}\left([x^p\xi_\star + (1-\xi_\star)y^p]^{\frac{1}{p}}\right) \leqslant_p q(\xi_\star)\mathscr{F}(x) + q(1-\xi_\star)\mathscr{F}(y).$$

- *Setting* $k, m = 1, q(t) = t$*, we get a p-convex IVM.*

- *Setting* $p, m = 1, q(t) = t, k = 1$ *we get a convex IVM, namely we obtain*

$$\mathscr{F}(\omega_6\rho_1 + (1-\omega_6)\rho_2) \leqslant_p \omega_6\mathscr{F}(\rho_1) + (1-\omega_6)\mathscr{F}(\rho_2).$$

In the following, we obtain new HH type inequalities and as a consequence of the said generalization in the IVM sense, we obtain the results reported in the recent literature.

3. Main Results

Theorem 2. *Let* $J \subset \mathbb{R}$ *be a set containing* $(0,1)$ *and let* $\tau : J \to \mathbb{R}$ *be a positive function including zero. Let* $\mathbb{I} \subset (0, +\infty)$ *be a set,* $p \in \mathbb{R} - 0$ *and* $\mathscr{F} : [\psi_5, \psi_6] \to K_c^+$ *be an IVM introduced by* $\mathscr{F}(x) = [\mathscr{F}_*, \mathscr{F}^*]$*, for all* $x \in [\psi_5, \psi_6]$*. Then,* $\mathscr{F} \in Left - RightSKHMPX([\psi_5, \psi_6], K_c^+, p)$ $\iff$ *,* $\mathscr{F}_*, \mathscr{F}^* \in KHMPX([\psi_5, \psi_6], K_c^+, p)$*.*

Proof. Assume that $\mathscr{F}_*, \mathscr{F}^* \in KHMPX([\psi_5, \psi_6], K_c^+, p)$. Then, for all $x, y \in [\psi_5, \psi_6]$, $\xi_\star \in [0,1]$ and $(k,m) \in [0,1]$, we have

$$\mathscr{F}_*\left([x^p\xi_\star + m(1-\xi_\star)y^p]^{\frac{1}{p}}\right) \leqslant \tau(\xi_\star^k)\mathscr{F}_*(x) + m\tau(1-\xi_\star^k)\mathscr{F}_*(y),$$

and

$$\mathscr{F}^*\left([x^p\xi_\star + m(1-\xi_\star)y^p]^{\frac{1}{p}}\right) \leqslant \tau(\xi_\star^k)\mathscr{F}^*(x) + m\tau(1-\xi_\star^k)\mathscr{F}^*(y).$$

From the inequality defined in Definition 8 and the order relation $\leqslant_p$, we have

$$\left[\mathscr{F}_*\left([x^p\xi_\star + m(1-\xi_\star)y^p]^{\frac{1}{p}}\right), \mathscr{F}^*\left([x^p\xi_\star + m(1-\xi_\star)y^p]^{\frac{1}{p}}\right)\right]$$

$$\leqslant_p \left[\tau(\xi_\star^k)\mathscr{F}_*(x) + m\tau(1-\xi_\star^k)\mathscr{F}_*(y), \tau(\xi_\star^k)\mathscr{F}^*(x) + m\tau(1-\xi_\star^k)\mathscr{F}^*(y)\right],$$

$$= \tau(\xi_\star^k)[\mathscr{F}_*(x), \mathscr{F}^*(x)] + m\tau(1-\xi_\star^k)[\mathscr{F}_*(y), \mathscr{F}^*(y)],$$

that is

$$\mathscr{F}\left([x^p\xi_\star + m(1-\xi_\star)y^p]^{\frac{1}{p}}\right) \leqslant_p \tau(\xi_\star^k)\mathscr{F}(x) + m\tau(1-\xi_\star^k)\mathscr{F}(y), \ \textit{for all } x, y \in [\psi_5, \psi_6], \xi_\star \in [0,1].$$

Hence, $\mathscr{F} \in Left\text{-}RightSKHMPX([\psi_5, \psi_6], K_c^+, p)$.

Conversely, let $\mathscr{F} \in Left\text{-}RightSKHMPX([\psi_5, \psi_6], K_c^+, p)$. Then, for all $x, y \in [\psi_5, \psi_6]$, $\xi_\star \in [0,1]$ and $(k,m) \in [0,1]$, we have

$$\mathscr{F}\left([x^p\xi_\star + m(1-\xi_\star)y^p]^{\frac{1}{p}}\right) \leqslant_p \tau(\xi_\star^k)\mathscr{F}(x) + m\tau(1-\xi_\star^k)\mathscr{F}(y),$$

that is

$$\left[\mathscr{F}_*\left([x^p\xi_\star + m(1-\xi_\star)y^p]^{\frac{1}{p}}\right), \mathscr{F}^*\left([x^p\xi_\star + m(1-\xi_\star)y^p]^{\frac{1}{p}}\right)\right] \leqslant_p$$

$$\tau(\xi_\star^k)[\mathscr{F}_*(x), \mathscr{F}^*(y)] + m\tau(1-\xi_\star^k)[\mathscr{F}_*(y), \mathscr{F}^*(y)]$$

$$= \left[\tau(\xi_\star^k)\mathscr{F}_*(x) + m\tau(1-\xi_\star^k)\mathscr{F}_*(y), \tau(\xi_\star^k)\mathscr{F}^*(x) + m\tau(1-\xi_\star^k)\mathscr{F}^*(y)\right].$$

Hence, we have

$$\mathscr{F}_*\left([x^p\xi_\star + m(1-\xi_\star)y^p]^{\frac{1}{p}}\right) \leqslant_p \tau(\xi_\star^k)\mathscr{F}_*(x) + m\tau(1-\xi_\star^k)\mathscr{F}_*(y),$$

and

$$\mathscr{F}^*\left([x^p\xi_\star + m(1-\xi_\star)y^p]^{\frac{1}{p}}\right) \leqslant_p \tau(\xi_\star^k)\mathscr{F}^*(x) + m\tau(1-\xi_\star^k)\mathscr{F}^*(y).$$

□

Remark 3. *If $\mathscr{F}_* = \mathscr{F}^*$ then the left–right-(k,h-m)-p-convex function becomes a $(k, h\text{-}m)$-p-convex function.*

If $k, m = 1$, then the left–right-(k,h-m)-p IVM becomes a left–right-(p,h)-convex IVM [42].

If $\mathscr{F}_* = \mathscr{F}^*, m, p, k = 1$, $h(\xi_\star) = \xi_\star^s$ and $s \in (0,1]$, then the left–right-(k,h-m)-p-convex IVM becomes an s-convex function in the second sense, see [43].

If $m = 1, k = 1$ and $h(\xi_\star) = \xi_\star$ we get a p-convex IVM.

If $\mathscr{F}_* = \mathscr{F}^*$ with $m = 1, p = 1, \xi_\star = 1$, then the left–right-(k,h-m)-p-convex IVM reduces to an h-convex function, see [44].

If $\mathscr{F}_* = \mathscr{F}^*$ with $h(\xi_\star) = 1$ and $m = 1, p = 1$, then the left–right-(k,h-m)-p-convex IVM reduces to the a p-convex function, see [45].

If $\mathscr{F}_* = \mathscr{F}^*$ with $k, p, m = 1$ and $h(\xi_\star) = \xi_\star$, then left–right-(k,h-m)-p-convex IVM reduces to the classical convex function.

Theorem 3. *Let $\mathscr{F} : [a,b] \to \mathbb{K}_c^+$ be an IVM that is left–right-$(k, h\text{-}m)$-p-convex. Then, the inequality holds in one of the cases:*

1. $a > 0, b > a, 0 < m < \frac{a}{b}$
2. $a > 0, b < a, 0 < m \leqslant 1$

$$\mathscr{F}\left([\frac{a^p + m^p b^p}{2}]^{\frac{1}{p}}\right) \leqslant_p \frac{\Gamma(\beta+1)p^\beta}{(a^p - m^p b^p)^\beta}\left(\chi_{a^-}^{p,\beta}\mathscr{F}(mb)\tau\left(\frac{1}{2^k}\right) + m^{p\beta+p}\tau\left(1-\frac{1}{2^k}\right)\chi_{b^+}^{p,\beta}\mathscr{F}\left(\frac{a}{m}\right)\right)$$

$$\leqslant_p \beta p \cdot \left(\tau\left(\frac{1}{2^k}\right)\mathscr{F}(a) + m^p\tau\left(1-\frac{1}{2^k}\right)\mathscr{F}(b)\right)\int_0^1 t^{\beta p-1}\tau(t^{\xi_\star p})dt$$

$$+\beta p \cdot \left(\tau\left(\frac{1}{2^k}\right)m^p\mathscr{F}(b) + \tau\left(1-\frac{1}{2^k}\right)\mathscr{F}(a)\right)\int_0^1 t^{\beta p-1}\tau(1-t^{\xi_\star p})dt.$$

Proof. From the statement of a left–right-(k,h-m)-p-convex IVM, we have

$$\mathscr{F}\left([x^p\xi_\star + m(1-\xi_\star)y^p]^{\frac{1}{p}}\right) \leqslant_p \tau(\xi_\star^k)\mathscr{F}(x) + m\tau(1-\xi_\star^k)\mathscr{F}(y).$$

Setting $\xi_{\star} = \frac{1}{2}, m \to m^p$, we obtain

$$\mathscr{F}\left([\frac{x^p + m^p y^p}{2}]^{\frac{1}{p}}\right) \leqslant_p \tau\left(\frac{1}{2^k}\right)\mathscr{F}(x) + m^p \tau\left(1 - \frac{1}{2^k}\right)\mathscr{F}(y).$$

Setting $x^p = m^p(1-t^p)b^p + t^p a^p, y^p = (1-t^p)\frac{a^p}{m^p} + b^p t^p$, we obtain

$$\mathscr{F}\left([\frac{a^p + m^p b^p}{2}]^{\frac{1}{p}}\right)$$

$$\leqslant_p \tau\left(\frac{1}{2^k}\right)\mathscr{F}\left(\left[m^p(1-t^p)b^p + t^p a^p\right]^{\frac{1}{p}}\right) + m^p \tau\left(1 - \frac{1}{2^k}\right)\mathscr{F}\left(\left[(1-t^p)\frac{a^p}{m^p} + b^p t^p\right]^{\frac{1}{p}}\right).$$

It follows from the statement of the IVM

$$\mathscr{F}_*\left([\frac{a^p + m^p b^p}{2}]^{\frac{1}{p}}\right)$$

$$\leqslant \tau\left(\frac{1}{2^k}\right)\mathscr{F}_*\left(\left[m^p(1-t^p)b^p + t^p a^p\right]^{\frac{1}{p}}\right) + m^p \tau\left(1 - \frac{1}{2^k}\right)\mathscr{F}_*\left(\left[(1-t^p)\frac{a^p}{m^p} + b^p t^p\right]^{\frac{1}{p}}\right),$$

and

$$\mathscr{F}^*\left([\frac{a^p + m^p b^p}{2}]^{\frac{1}{p}}\right)$$

$$\leqslant \tau\left(\frac{1}{2^k}\right)\mathscr{F}^*\left(\left[m^p(1-t^p)b^p + t^p a^p\right]^{\frac{1}{p}}\right) + m^p \tau\left(1 - \frac{1}{2^k}\right)\mathscr{F}^*\left(\left[(1-t^p)\frac{a^p}{m^p} + b^p t^p\right]^{\frac{1}{p}}\right).$$

Multiplying the inequalities with $t^{\beta p-1}$ and integrating with respect to the variable used, we get

$$\frac{\mathscr{F}_*\left([\frac{a^p + m^p b^p}{2}]^{\frac{1}{p}}\right)}{\beta p} \leqslant$$

$$\int_0^1 t^{\beta p-1}\tau\left(\frac{1}{2^k}\right)\mathscr{F}_*\left(\left[m^p(1-t^p)b^p + t^p a^p\right]^{\frac{1}{p}}\right)dt$$

$$+\int_0^1 t^{\beta p-1} m^p \tau\left(1 - \frac{1}{2^k}\right)\mathscr{F}_*\left(\left[(1-t^p)\frac{a^p}{m^p} + b^p t^p\right]^{\frac{1}{p}}\right)dt,$$

and

$$\frac{\mathscr{F}^*\left([\frac{a^p + m^p b^p}{2}]^{\frac{1}{p}}\right)}{\beta p} \leqslant$$

$$\int_0^1 t^{\beta p-1}\tau\left(\frac{1}{2^k}\right)\mathscr{F}^*\left(\left[m^p(1-t^p)b^p + t^p a^p\right]^{\frac{1}{p}}\right)dt$$

$$+\int_0^1 t^{\beta p-1} m^p \tau\left(1 - \frac{1}{2^k}\right)\mathscr{F}^*\left(\left[(1-t^p)\frac{a^p}{m^p} + b^p t^p\right]^{\frac{1}{p}}\right)dt.$$

Focusing towards the right and setting $(1-t^p)m^p b^p + t^p a^p = k^p$ in the first integral and $(1-t^p)\frac{a^p}{m^p} + b^p t^p = k^p$ in the second integral, we obtain

$$= \frac{\tau\left(\frac{1}{2^k}\right)}{(a^p - m^p b^p)^\beta}\int_{mb}^{a}\mathscr{F}_*(k)(k^p - m^p b^p)^{\beta-1}k^{p-1}dk$$

$$+\frac{\tau\left(1 - \frac{1}{2^k}\right)m^{p+p\beta}}{(a^p - b^p m^p)^\beta}\int_b^{\frac{a}{m}}\mathscr{F}_*(k)\left(\frac{a^p}{m^p} - k^p\right)^{\beta-1}k^{p-1}dk,$$

and

$$= \frac{\tau\left(\frac{1}{2^k}\right)}{(a^p - m^p b^p)^\beta} \int_{mb}^{a} \mathscr{F}^*(k)(k^p - m^p b^p)^{\beta-1} k^{p-1} dk$$
$$+ \frac{\tau\left(1 - \frac{1}{2^k}\right) m^{p+p\beta}}{(a^p - b^p m^p)^\beta} \int_{b}^{\frac{a}{m}} \mathscr{F}^*(k)\left(\frac{a^p}{m^p} - k^p\right)^{\beta-1} k^{p-1} dk.$$

What we get when we recognize it in terms of a Katugampola integral is

$$= \frac{\Gamma(\beta)p^{\beta-1}}{(a^p - m^p b^p)^\beta}\left(\chi^{p,\beta}_{a^-}\mathscr{F}_*(mb)\tau\left(\frac{1}{2^k}\right) + m^{p\beta+p}\tau\left(1 - \frac{1}{2^k}\right)\chi^{p,\beta}_{b^+}\mathscr{F}_*\left(\frac{a}{m}\right)\right),$$

and

$$= \frac{\Gamma(\beta)p^{\beta-1}}{(a^p - m^p b^p)^\beta}\left(\chi^{p,\beta}_{a^-}\mathscr{F}^*(mb)\tau\left(\frac{1}{2^k}\right) + m^{p\beta+p}\tau\left(1 - \frac{1}{2^k}\right)\chi^{p,\beta}_{b^+}\mathscr{F}^*\left(\frac{a}{m}\right)\right),$$

which together yields

$$\left[\mathscr{F}_*\left([\frac{a^p + m^p b^p}{2}]^{\frac{1}{p}}\right), \mathscr{F}^*\left([\frac{a^p + m^p b^p}{2}]^{\frac{1}{p}}\right)\right] \leqslant_p$$
$$\left(\frac{\Gamma(\beta)p^{\beta-1}}{(a^p - m^p b^p)^\beta}\left(\chi^{p,\beta}_{a^-}\mathscr{F}_*(mb)\tau\left(\frac{1}{2^k}\right) + m^{p\beta+p}\tau\left(1 - \frac{1}{2^k}\right)\chi^{p,\beta}_{b^+}\mathscr{F}_*\left(\frac{a}{m}\right),\right)\right.$$
$$\left.\frac{\Gamma(\beta)p^{\beta-1}}{(a^p - m^p b^p)^\beta}\left(\chi^{p,\beta}_{a^-}\mathscr{F}^*(mb)\tau\left(\frac{1}{2^k}\right) + m^{p\beta+p}\tau\left(1 - \frac{1}{2^k}\right)\chi^{p,\beta}_{b^+}\mathscr{F}^*\left(\frac{a}{m}\right)\right)\right),$$

from which we get the original left part of the inequality.

Now to obtain the upper inequality, we use the left–right-(k,h-m)-p-convexity and apply the IVM property to the following expression

$$\tau\left(\frac{1}{2^k}\right)\mathscr{F}\left(\left[m^p(1 - t^p)b^p + t^p a^p\right]^{\frac{1}{p}}\right)$$
$$+ m^p\tau\left(1 - \frac{1}{2^k}\right)\mathscr{F}\left(\left[(1 - t^p)\frac{a^p}{m^p} + b^p t^p\right]^{\frac{1}{p}}\right)$$

and multiply it with $t^{\beta p-1}$, while integrating the expression; hence, we obtain

$$\frac{\Gamma(\beta)p^{\beta-1}}{(a^p - m^p b^p)^\beta}\left(\chi^{p,\beta}_{a^-}\mathscr{F}^*(mb)\tau\left(\frac{1}{2^k}\right) + m^{p\beta+p}\tau\left(1 - \frac{1}{2^k}\right)\chi^{p,\beta}_{b^+}\mathscr{F}^*\left(\frac{a}{m}\right)\right) \leqslant$$
$$\left(\tau\left(\frac{1}{2^k}\right)\mathscr{F}^*(a) + m^p\tau\left(1 - \frac{1}{2^k}\right)\mathscr{F}^*(b)\right)\int_0^1 t^{\beta p-1}\tau(t^{\xi_\star p})dt$$
$$+ \left(\tau\left(\frac{1}{2^k}\right)m^p\mathscr{F}^*(b) + \tau\left(1 - \frac{1}{2^k}\right)\mathscr{F}^*(a)\right)\int_0^1 t^{\beta p-1}h\left(1 - t^{\xi_\star p}\right)dt,$$

and

$$\frac{\Gamma(\beta)p^{\beta-1}}{(a^p - m^p b^p)^\beta}\left(\chi^{p,\beta}_{a^-}\mathscr{F}_*(mb)\tau\left(\frac{1}{2^k}\right) + m^{p\beta+p}\tau\left(1 - \frac{1}{2^k}\right)\chi^{p,\beta}_{b^+}\mathscr{F}_*\left(\frac{a}{m}\right)\right) \leqslant$$
$$\left(\tau\left(\frac{1}{2^k}\right)\mathscr{F}_*(a) + m^p\tau\left(1 - \frac{1}{2^k}\right)\mathscr{F}_*(b)\right)\int_0^1 t^{\beta p-1}\tau(t^{\xi_\star p})dt$$
$$+ \left(\tau\left(\frac{1}{2^k}\right)m^p\mathscr{F}_*(b) + \tau\left(1 - \frac{1}{2^k}\right)\mathscr{F}_*(a)\right)\int_0^1 t^{\beta p-1}h\left(1 - t^{\xi_\star p}\right)dt,$$

From which we get the original right-hand side inequality. Connecting the left- and right-hand side inequalities, we obtain the original inequality

$$\mathscr{F}\left([\frac{a^p+m^pb^p}{2}]^{\frac{1}{p}}\right) \leqslant_p \frac{\Gamma(\beta+1)p^\beta}{(a^p-m^pb^p)^\beta}\left(\chi^{p,\beta}_{a^-}\mathscr{F}(mb)\tau\left(\frac{1}{2^k}\right)+m^{p\beta+p}\tau\left(1-\frac{1}{2^k}\right)\chi^{p,\beta}_{b^+}\mathscr{F}\left(\frac{a}{m}\right)\right)$$
$$\leqslant_p \beta p\cdot\left(\tau\left(\frac{1}{2^k}\right)\mathscr{F}(a)+m^p\tau\left(1-\frac{1}{2^k}\right)\mathscr{F}(b)\right)\int_0^1 t^{\beta p-1}\tau(t^{\xi_\star p})dt$$
$$+\beta p\cdot\left(\tau\left(\frac{1}{2^k}\right)m^p\mathscr{F}(b)+\tau\left(1-\frac{1}{2^k}\right)\mathscr{F}(a)\right)\int_0^1 t^{\beta p-1}\tau(1-t^{\xi_\star p})dt.$$

□

Setting $p=3, k=\frac{1}{2}$ in the theorem, we obtain a new inequality of the left–right-(k,h-m)-p-convex type.

Corollary 1.

$$\mathscr{F}\left([\frac{a^3+m^3b^3}{2}]^{\frac{1}{3}}\right) \leqslant_p \frac{\Gamma(\beta+1)3^\beta}{(a^3-m^3b^3)^\beta}\left(\chi^{3,\beta}_{a^-}\mathscr{F}(mb)h\left(\frac{1}{2^{\frac{1}{2}}}\right)+m^{3\beta+3}h\left(1-\frac{1}{2^{\frac{1}{2}}}\right)\chi^{3,\beta}_{b^+}\mathscr{F}\left(\frac{a}{m}\right)\right)$$
$$\leqslant_p 3\beta\cdot\left(h\left(\frac{1}{2^{\frac{1}{2}}}\right)\mathscr{F}(a)+m^3h\left(1-\frac{1}{2^{\frac{1}{2}}}\right)\mathscr{F}(b)\right)\int_0^1 t^{\beta 3-1}h(t^{3\xi_\star})dt$$
$$+3\beta\cdot\left(h\left(\frac{1}{2^{\frac{1}{2}}}\right)m^3\mathscr{F}(b)+h\left(1-\frac{1}{2^{\frac{1}{2}}}\right)\mathscr{F}(a)\right)\int_0^1 t^{3\beta-1}h(1-t^{3\xi_\star})dt.$$

Theorem 4. *If the conditions are the same as in Theorem 3, then, the inequality holds in the following case with* $a\geqslant 0, b>a, \frac{a}{b}<m\leqslant 1$

$$\mathscr{F}\left([\frac{a^p+m^pb^p}{2}]^{\frac{1}{p}}\right) \leqslant_p \tau\left(\frac{1}{2^k}\right)\frac{\Gamma(\beta+1)2^\beta}{p^{-\beta}(m^pb^p-a^p)^\beta}\chi^{\beta,p}_{\left(\left(\frac{a^p}{2m^p}+\frac{b^p}{2}\right)^{\frac{1}{p}}\right)^-}\mathscr{F}\left(\frac{a}{m}\right)$$
$$+\tau\left(\frac{2^k-1}{2^k}\right)\frac{m^{p\beta+p}2^\beta\Gamma(\beta+1)}{p^{-\beta}(m^pb^p-a^p)^\beta}\chi^{\beta,p}_{\left(\left(\frac{a^p}{2}+\frac{b^pm^p}{2}\right)^{\frac{1}{p}}\right)^+}\mathscr{F}(mb)$$
$$\leqslant_p \beta p\left(\tau\left(\frac{1}{2^k}\right)\mathscr{F}(a)+m^p\tau\left(\frac{2^k-1}{2^k}\right)\mathscr{F}(b)\right)\int_0^1 h\left(\left(\frac{t^p}{2}\right)^{\xi_\star}\right)t^{\beta p-1}dt$$
$$+\beta p\left(\tau\left(\frac{1}{2^k}\right)m^p\mathscr{F}(b)+\tau\left(\frac{2^k-1}{2^k}\right)\mathscr{F}(a)\right)\int_0^1 h\left(1-\left(\frac{t^p}{2}\right)^{\xi_\star}\right)t^{\beta p-1}dt.$$

Proof. From the statement of the left–right-(k,h-m)-p-convex IVM, we have

$$\mathscr{F}\left([x^p\xi_\star+m(1-\xi_\star)y^p]^{\frac{1}{p}}\right)\leqslant_p \tau(\xi_\star^k)\mathscr{F}(x)+m\tau(1-\xi_\star^k)\mathscr{F}(y).$$

Setting $\xi_\star=\frac{1}{2}, m\to m^p$, we obtain

$$\mathscr{F}\left([\frac{x^p+m^py^p}{2}]^{\frac{1}{p}}\right)\leqslant_p \tau\left(\frac{1}{2^k}\right)\mathscr{F}(x)+m^p\tau\left(1-\frac{1}{2^k}\right)\mathscr{F}(y).$$

Setting $x^p=\frac{(at)^p}{2}+\frac{m^p(2-t^p)}{2}b^p, y^p=\frac{(bt)^p}{2}+\frac{(2-t^p)}{2}(\frac{a}{m})^p$ in the inequality, we get the following

$$\mathscr{F}\left([\frac{a^p+m^pb^p}{2}]^{\frac{1}{p}}\right)\leqslant_p \tau\left(\frac{1}{2^k}\right)\mathscr{F}\left([\frac{(at)^p}{2}+\frac{m^p(2-t^p)}{2}b^p]^{\frac{1}{p}}\right)$$

$$+m^p\tau\left(1-\frac{1}{2^k}\right)\mathscr{F}\left([\frac{(bt)^p}{2}+\frac{(2-t^p)}{2}\left(\frac{a}{m}\right)^p]^{\frac{1}{p}}\right).$$

It follows from the statement of the IVM that

$$\mathscr{F}_*\left([\frac{a^p+m^pb^p}{2}]^{\frac{1}{p}}\right)\leqslant\tau\left(\frac{1}{2^k}\right)\mathscr{F}_*\left([\frac{(at)^p}{2}+\frac{m^p(2-t^p)}{2}b^p]^{\frac{1}{p}}\right)$$
$$+m^p\tau\left(1-\frac{1}{2^k}\right)\mathscr{F}_*\left([\frac{(bt)^p}{2}+\frac{(2-t^p)}{2}\left(\frac{a}{m}\right)^p]^{\frac{1}{p}}\right),$$

and

$$\mathscr{F}^*\left([\frac{a^p+m^pb^p}{2}]^{\frac{1}{p}}\right)\leqslant\tau\left(\frac{1}{2^k}\right)\mathscr{F}^*\left([\frac{(at)^p}{2}+\frac{m^p(2-t^p)}{2}b^p]^{\frac{1}{p}}\right)$$
$$+m^p\tau\left(1-\frac{1}{2^k}\right)\mathscr{F}^*\left([\frac{(bt)^p}{2}+\frac{(2-t^p)}{2}\left(\frac{a}{m}\right)^p]^{\frac{1}{p}}\right).$$

Multiplying the inequalities with $t^{\beta p-1}$ and integrating with respect to the variable used, we get

$$\frac{\mathscr{F}_*\left([\frac{a^p+m^pb^p}{2}]^{\frac{1}{p}}\right)}{\beta p}\leqslant\int_0^1 t^{p\beta-1}\tau\left(\frac{1}{2^k}\right)\mathscr{F}_*\left([\frac{(at)^p}{2}+\frac{m^p(2-t^p)}{2}b^p]^{\frac{1}{p}}\right)dt$$
$$+\int_0^1 t^{\beta p-1}m^p\tau\left(1-\frac{1}{2^k}\right)\mathscr{F}_*\left([\frac{(bt)^p}{2}+\frac{(2-t^p)}{2}\left(\frac{a}{m}\right)^p]^{\frac{1}{p}}\right)dt,$$

and

$$\frac{\mathscr{F}^*\left([\frac{a^p+m^pb^p}{2}]^{\frac{1}{p}}\right)}{p\beta}\leqslant\int_0^1 t^{\beta p-1}\tau\left(\frac{1}{2^k}\right)\mathscr{F}^*\left([\frac{(at)^p}{2}+\frac{m^p(2-t^p)}{2}b^p]^{\frac{1}{p}}\right)dt$$
$$+\int_0^1 t^{\beta p-1}m^p\tau\left(1-\frac{1}{2^k}\right)\mathscr{F}^*\left([\frac{(bt)^p}{2}+\frac{(2-t^p)}{2}\left(\frac{a}{m}\right)^p]^{\frac{1}{p}}\right)dt.$$

Focusing on the lower end point function $\mathscr{F}_*$ and introducing the following substitution to the first integral $k^p=\frac{a^pt^p}{2}+\frac{m^p(2-t^p)}{2}b^p$ while nothing that $mb\geqslant a$, we get

$$\int_0^1 t^{p\beta-1}\tau\left(\frac{1}{2^k}\right)\mathscr{F}_*\left([\frac{(at)^p}{2}+\frac{m^p(2-t^p)}{2}b^p]^{\frac{1}{p}}\right)dt$$
$$=\frac{2^\beta}{(m^pb^p-a^p)^\beta}\int_{(\frac{a^p+b^pm^p}{2})^{\frac{1}{p}}}^{bm}\mathscr{F}_*(k)(m^pb^p-k^p)^{\beta-1}k^{p-1}dk$$
$$=\frac{2^\beta\Gamma(\beta)p^{\beta-1}}{(m^pb^p-a^p)^\beta}\chi^{p,\beta}_{((\frac{a^p+m^pb^p}{2})^{\frac{1}{p}})^+}\mathscr{F}_*(bm).$$

Introducing a substitution to the second integral, namely, $k^p=\frac{(bt)^p}{2}+\frac{(2-t^p)}{2}\left(\frac{a}{m}\right)^p$, we get in a similar manner

$$\int_0^1 t^{\beta p-1}m^p\tau\left(1-\frac{1}{2^k}\right)\mathscr{F}_*\left([\frac{(bt)^p}{2}+\frac{(2-t^p)}{2}\left(\frac{a}{m}\right)^p]^{\frac{1}{p}}\right)dt$$
$$=\int_{\frac{a}{m}}^{\left(\frac{b^p}{2}+\frac{a^p}{2m^p}\right)^{\frac{1}{p}}}\mathscr{F}_*(k)\left(k^p-\frac{a^p}{m^p}\right)^{\beta-1}k^{p-1}dk\cdot\frac{1}{(\frac{b^p}{2}-\frac{a^p}{2m^p})^\beta}$$
$$=\frac{m^{p\beta}\Gamma(\beta)2^\beta}{p^{1-\beta}(m^pb^p-a^p)^\beta}\chi^{\beta,p}_{\left(\left(\frac{a^p}{2m^p}+\frac{b^p}{2}\right)^{\frac{1}{p}}\right)^-}\mathscr{F}_*\left(\frac{a}{m}\right).$$

We obtain similar equalities with the upper end function, namely,

$$\int_0^1 t^{p\beta-1}\tau\left(\frac{1}{2^k}\right)\mathscr{F}^*\left(\left[\frac{(at)^p}{2}+\frac{m^p(2-t^p)}{2}b^p\right]^{\frac{1}{p}}\right)dt$$

$$=\frac{2^\beta}{(m^pb^p-a^p)^\beta}\int_{\left(\frac{a^p+b^pm^p}{2}\right)^{\frac{1}{p}}}^{bm}\mathscr{F}^*(k)(m^pb^p-k^p)^{\beta-1}k^{p-1}dk$$

$$=\frac{2^\beta\Gamma(\beta)p^{\beta-1}}{(m^pb^p-a^p)^\beta}\chi^{p,\beta}_{\left(\left(\frac{a^p+m^pb^p}{2}\right)^{\frac{1}{p}}\right)+}\mathscr{F}^*(bm).$$

$$\int_0^1 t^{\beta p-1}m^p\tau\left(1-\frac{1}{2^k}\right)\mathscr{F}^*\left(\left[\frac{(bt)^p}{2}+\frac{(2-t^p)}{2}\left(\frac{a}{m}\right)^p\right]^{\frac{1}{p}}\right)dt$$

$$=\int_{\frac{a}{m}}^{\left(\frac{b^p}{2}+\frac{a^p}{2m^p}\right)^{\frac{1}{p}}}\mathscr{F}^*(k)\left(k^p-\frac{a^p}{m^p}\right)^{\beta-1}k^{p-1}dk\cdot\frac{1}{\left(\frac{b^p}{2}-\frac{a^p}{2m^p}\right)^\theta}$$

$$=\frac{m^{p\beta}\Gamma(\beta)2^\beta}{p^{1-\beta}(m^pb^p-a^p)^\beta}\chi^{\beta,p}_{\left(\left(\frac{a^p}{2m^p}+\frac{b^p}{2}\right)^{\frac{1}{p}}\right)^-}\mathscr{F}^*\left(\frac{a}{m}\right).$$

Hence, we obtain

$$\left(\frac{\mathscr{F}_*\left(\left[\frac{a^p+m^pb^p}{2}\right]^{\frac{1}{p}}\right)}{\beta p},\frac{\mathscr{F}^*\left(\left[\frac{a^p+m^pb^p}{2}\right]^{\frac{1}{p}}\right)}{\beta p}\right)\leqslant$$

$$\left(\tau\left(\frac{1}{2^k}\right)\frac{p^{\beta-1}\Gamma(\beta)2^\beta}{p^{-\beta+1}(m^pb^p-a^p)^\beta}\chi^{\beta,p}_{\left(\left(\frac{a^p}{2m^p}+\frac{b^p}{2}\right)^{\frac{1}{p}}\right)^-}\mathscr{F}_*\left(\frac{a}{m}\right)\right.$$

$$+\tau\left(\frac{2^k-1}{2^k}\right)\frac{m^{p\beta+p}2^\beta\Gamma(\beta)}{p^{-\beta+1}(m^pb^p-a^p)^\beta}\chi^{\beta,p}_{\left(\left(\frac{a^p}{2}+\frac{b^pm^p}{2}\right)^{\frac{1}{p}}\right)^+}\mathscr{F}_*(mb),$$

$$\tau\left(\frac{1}{2^k}\right)\frac{p^{\beta-1}\Gamma(\beta)2^\beta}{p^{-\beta+1}(m^pb^p-a^p)^\beta}\chi^{\beta,p}_{\left(\left(\frac{a^p}{2m^p}+\frac{b^p}{2}\right)^{\frac{1}{p}}\right)^-}\mathscr{F}^*\left(\frac{a}{m}\right)$$

$$\left.+\tau\left(\frac{2^k-1}{2^k}\right)\frac{m^{p\beta+p}2^\beta\Gamma(\beta)}{p^{-\beta+1}(m^pb^p-a^p)^\beta}\chi^{\beta,p}_{\left(\left(\frac{a^p}{2}+\frac{b^pm^p}{2}\right)^{\frac{1}{p}}\right)^+}\mathscr{F}^*(mb)\right)$$

and the left inequality follows.

Using the definition of the left–right-(k,h-m)-p-convex function towards the right

$$\tau\left(\frac{1}{2^k}\right)\mathscr{F}\left(\left[\frac{(at)^p}{2}+\frac{m^p(2-t^p)}{2}b^p\right]^{\frac{1}{p}}\right)$$

$$+m^p\tau\left(1-\frac{1}{2^k}\right)\mathscr{F}\left(\left[\frac{(bt)^p}{2}+\frac{(2-t^p)}{2}\left(\frac{a}{m}\right)^p\right]^{\frac{1}{p}}\right),$$

while using the definition of the IVM property, multiplying with $t^{p\beta-1}$ and integrating with respect to the variable used, we obtain

$$\tau\left(\frac{1}{2^k}\right)\frac{\Gamma(\beta)2^\beta}{p^{-\beta+1}(m^pb^p-a^p)^\beta}\chi^{\beta,p}_{\left(\left(\frac{a^p}{2m^p}+\frac{b^p}{2}\right)^{\frac{1}{p}}\right)^-}\mathscr{F}\left(\frac{a}{m}\right)$$

$$+\tau\left(\frac{2^k-1}{2^k}\right)\frac{2^{\beta}m^{p\beta+p}\Gamma(\beta)}{p^{-\beta+1}(m^pb^p-a^p)^{\beta}}\chi^{\beta,p}_{\left(\left(\frac{a^p}{2}+\frac{b^pm^p}{2}\right)^{\frac{1}{p}}\right)^+}\mathscr{F}(mb)$$

$$\leqslant\left(\tau\left(\frac{1}{2^k}\right)\mathscr{F}(a)+m^p\tau\left(\frac{2^k-1}{2^k}\right)\mathscr{F}(b)\right)\int_0^1 h\left(\left(\frac{t^p}{2}\right)^{\xi_\star}\right)t^{\beta p-1}dt$$

$$+\left(\tau\left(\frac{1}{2^k}\right)m^p\mathscr{F}(b)+\tau\left(\frac{2^k-1}{2^k}\right)\mathscr{F}(a)\right)\int_0^1 h\left(1-\left(\frac{t^p}{2}\right)^{\xi_\star}\right)t^{\beta p-1}dt.$$

Now, connecting the left- and right-hand sides, we obtain the original inequality. □

Setting $p=3, k=\frac{1}{2}$, we get a new IVM noninteger inequality.

Corollary 2.

$$\mathscr{F}\left([\frac{a^3+m^3b^3}{2}]^{\frac{1}{3}}\right)\leqslant_p h\left(\frac{1}{2^{\frac{1}{2}}}\right)\frac{\Gamma(\beta+1)2^{\beta}}{3^{-\beta}(m^3b^3-a^3)^{\beta}}\chi^{\beta,3}_{\left(\left(\frac{a^3}{2m^3}+\frac{b^3}{2}\right)^{\frac{1}{3}}\right)^-}\mathscr{F}\left(\frac{a}{m}\right)$$

$$+h\left(\frac{2^{\frac{1}{2}}-1}{2^{\frac{1}{2}}}\right)\frac{m^{3\beta+3}2^{\beta}\Gamma(\beta+1)}{3^{-\beta}(m^3b^3-a^3)^{\beta}}\chi^{\beta,3}_{\left(\left(\frac{a^3}{2}+\frac{b^3m^3}{2}\right)^{\frac{1}{3}}\right)^+}\mathscr{F}(mb)$$

$$\leqslant_p 3\beta\left(h\left(\frac{1}{2^{\frac{1}{2}}}\right)\mathscr{F}(a)+m^3h\left(\frac{2^{\frac{1}{2}}-1}{2^{\frac{1}{2}}}\right)\mathscr{F}(b)\right)\int_0^1 h\left(\left(\frac{t^3}{2}\right)^{\xi_\star}\right)t^{3\beta-1}dt$$

$$+3\beta\left(h\left(\frac{1}{2^{\frac{1}{2}}}\right)m^3\mathscr{F}(b)+h\left(\frac{2^{\frac{1}{2}}-1}{2^{\frac{1}{2}}}\right)\mathscr{F}(a)\right)\int_0^1 h\left(1-\left(\frac{t^3}{2}\right)^{\xi_\star}\right)t^{3\beta-1}dt.$$

Corollary 3. *Setting $\mathscr{F}_*=\mathscr{F}^*, h(t)=t, m, k, \xi_\star=1$, we get a classical noninteger p-convex inequality, namely,*

$$\mathscr{F}\left([\frac{a^p+m^pb^p}{2}]^{\frac{1}{p}}\right)\leqslant\frac{\Gamma(\beta+1)2^{\beta}}{2p^{-\beta}(b^p-a^p)^{\beta}}\chi^{\beta,p}_{\left(\left(\frac{a^p}{2}+\frac{b^p}{2}\right)^{\frac{1}{p}}\right)^-}\mathscr{F}(a)$$

$$+\frac{2^{\beta-1}\Gamma(\beta+1)}{p^{-\beta}(b^p-a^p)^{\beta}}\chi^{\beta,p}_{\left(\left(\frac{a^p}{2}+\frac{b^pm^p}{2}\right)^{\frac{1}{p}}\right)^+}\mathscr{F}(b)$$

$$\leqslant\beta p\left(\frac{1}{2}\mathscr{F}(a)+\frac{1}{2}\mathscr{F}(b)\right)\int_0^1\left(\frac{t^p}{2}\right)t^{\beta p-1}dt$$

$$+\beta p\left(\frac{1}{2}\mathscr{F}(b)+\frac{1}{2}\mathscr{F}(a)\right)\int_0^1\left(1-\left(\frac{t^p}{2}\right)\right)t^{\beta p-1}dt.$$

Example 1. *Setting $k=1, h(t)=t, m=1$, we recover a left–right-p-convex IVM, which is also a left–right-(k,h-m)-p-convex IVM.*

Using a similar construction as in the paper [23] and setting $p=1, \beta=\frac{1}{3}, k, \xi_\star=1$:

$$\mathscr{F}\left([\frac{a^p+m^pb^p}{2}]^{\frac{1}{p}}\right)\leqslant_p\tau\left(\frac{1}{2^k}\right)\frac{\Gamma(\beta+1)2^{\beta}}{p^{-\beta}(m^pb^p-a^p)^{\beta}}\chi^{\beta,p}_{\left(\left(\frac{a^p}{2m^p}+\frac{b^p}{2}\right)^{\frac{1}{p}}\right)^-}\mathscr{F}\left(\frac{a}{m}\right)$$

$$+\tau\left(\frac{2^k-1}{2^k}\right)\frac{m^{p\beta+p}2^{\beta}\Gamma(\beta+1)}{p^{-\beta}(m^pb^p-a^p)^{\beta}}\chi^{\beta,p}_{\left(\left(\frac{a^p}{2}+\frac{b^pm^p}{2}\right)^{\frac{1}{p}}\right)^+}\mathscr{F}(mb)$$

$$\leqslant_p \beta p\left(\tau\left(\frac{1}{2^k}\right)\mathscr{F}(a)+m^p\tau\left(\frac{2^k-1}{2^k}\right)\mathscr{F}(b)\right)\int_0^1 h\left(\left(\frac{t^p}{2}\right)^{\xi_\star}\right)t^{\beta p-1}dt$$

$$+\beta p\left(\tau\left(\frac{1}{2^k}\right)m^p\mathscr{F}(b)+\tau\left(\frac{2^k-1}{2^k}\right)\mathscr{F}(a)\right)\int_0^1 h\left(1-\left(\frac{t^p}{2}\right)^{\xi_\star}\right)t^{\beta p-1}dt.$$

$$\mathscr{F}_*\left([\frac{a^p+m^pb^p}{2}]^{\frac{1}{p}}\right)=\frac{1}{2},$$

$$\tau\left(\frac{1}{2^k}\right)\frac{\Gamma(\beta+1)2^{\beta}}{p^{-\beta}(m^pb^p-a^p)^{\beta}}\chi^{\beta,p}_{\left(\left(\frac{a^p}{2m^p}+\frac{b^p}{2}\right)^{\frac{1}{p}}\right)^-}\mathscr{F}_*\left(\frac{a}{m}\right)$$

$$+\tau\left(\frac{2^k-1}{2^k}\right)\frac{m^{p\beta+p}2^{\beta}\Gamma(\beta+1)}{p^{-\beta}(m^pb^p-a^p)^{\beta}}\chi^{\beta,p}_{\left(\left(\frac{a^p}{2}+\frac{b^pm^p}{2}\right)^{\frac{1}{p}}\right)^+}\mathscr{F}_*(mb)=\frac{1}{2},$$

$$\beta p\left(\tau\left(\frac{1}{2^k}\right)\mathscr{F}_*(a)+m^p\tau\left(\frac{2^k-1}{2^k}\right)\mathscr{F}_*(b)\right)\int_0^1 h\left(\left(\frac{t^p}{2}\right)^{\xi_\star}\right)t^{\beta p-1}dt$$

$$+\beta p\left(\tau\left(\frac{1}{2^k}\right)m^p\mathscr{F}_*(b)+\tau\left(\frac{2^k-1}{2^k}\right)\mathscr{F}_*(a)\right)\int_0^1 h\left(1-\left(\frac{t^p}{2}\right)^{\xi_\star}\right)t^{\beta p-1}dt=\frac{1}{2}.$$

Therefore, we get

$$\frac{1}{2}\leqslant\frac{1}{2}\leqslant\frac{1}{2}.$$

Now evaluating the top point function, we achieve

$$\mathscr{F}\left([\frac{a^p+m^pb^p}{2}]^{\frac{1}{p}}\right)=1.648,$$

$$\tau\left(\frac{1}{2^k}\right)\frac{\Gamma(\beta+1)2^{\beta}}{p^{-\beta}(m^pb^p-a^p)^{\beta}}\chi^{\beta,p}_{\left(\left(\frac{a^p}{2m^p}+\frac{b^p}{2}\right)^{\frac{1}{p}}\right)^-}\mathscr{F}\left(\frac{a}{m}\right)$$

$$+\tau\left(\frac{2^k-1}{2^k}\right)\frac{m^{p\beta+p}2^{\beta}\Gamma(\beta+1)}{p^{-\beta}(m^pb^p-a^p)^{\beta}}\chi^{\beta,p}_{\left(\left(\frac{a^p}{2}+\frac{b^pm^p}{2}\right)^{\frac{1}{p}}\right)^+}\mathscr{F}(mb)=1.712,$$

$$\beta p\left(\tau\left(\frac{1}{2^k}\right)\mathscr{F}(a)+m^p\tau\left(\frac{2^k-1}{2^k}\right)\mathscr{F}(b)\right)\int_0^1 h\left(\left(\frac{t^p}{2}\right)^{\xi_\star}\right)t^{\beta p-1}dt$$

$$+\beta p\left(\tau\left(\frac{1}{2^k}\right)m^p\mathscr{F}(b)+\tau\left(\frac{2^k-1}{2^k}\right)\mathscr{F}(a)\right)\int_0^1 h\left(1-\left(\frac{t^p}{2}\right)^{\xi_\star}\right)t^{\beta p-1}dt=1.859.$$

Hence, we achieve

$$[\frac{1}{2},1.648]\leqslant_p[\frac{1}{2},1.712]\leqslant_p[\frac{1}{2},1.859],$$

which verifies our result.

Theorem 5. *Using the same conditions as in Theorem 3 and $a \geqslant 0, b > a, \frac{a}{b} < m \leqslant 1$ results in the following inequality:*

$$\mathscr{F}\left([\frac{a^p + m^p b^p}{2}]^{\frac{1}{p}}\right) \leqslant_p$$

$$\frac{\tau\left(\frac{1}{2^{\xi_\star}}\right)2^\theta p^\theta \Gamma(\theta+1)}{(b^p m^p - a^p)^\theta} \chi^{\theta,p}_{\left(\left(\frac{q}{2}(a^p - m^p b^p) + \frac{a^p}{2} + \frac{b^p m^p}{2}\right)^{\frac{1}{p}}\right)^+} \mathscr{F}\left(\left(\frac{q}{2}(a^p - b^p m^p) + m^p b^p\right)^{\frac{1}{p}}\right)$$

$$+ \frac{\tau\left(1 - \frac{1}{2^{\xi_\star}}\right) m^{p+\theta p} 2^\theta p^\theta \Gamma(\theta+1)}{(b^p m^p - a^p)^\theta} \chi^{p,\theta}_{\left(\left(\frac{q}{2}(b^p - \frac{a^p}{m^p}) + \frac{a^p}{2m^p} + \frac{b^p}{2}\right)^{\frac{1}{p}}\right)^-} \mathscr{F}\left(\left(\frac{q}{2}(b^p - \frac{a^p}{m^p}) + \frac{a^p}{m^p}\right)^{\frac{1}{p}}\right)$$

$$\leqslant_p \left(\left(\tau\left(\frac{1}{2^{\xi_\star}}\right)\mathscr{F}(a) + \tau\left(1 - \frac{1}{2^{\xi_\star}}\right) m^p \mathscr{F}(b)\right) \int_0^1 h\left(\left(\frac{q+t^p}{2}\right)^k\right) t^{\theta p - 1} dt\right)\theta p$$

$$+ \left(\left(\tau\left(\frac{1}{2^{\xi_\star}}\right)\mathscr{F}(b) m^p + \tau\left(1 - \frac{1}{2^{\xi_\star}}\right)\mathscr{F}(a)\right) \int_0^1 h\left(1 - \left(\frac{q+t^p}{2}\right)^k\right) t^{\theta p - 1} dt\right)\theta p.$$

Proof. Since $\mathscr{F}$ is left–right-(k ,h-m)-p-convex, we have the following inequality

$$\mathscr{F}\left((tx^p + m(1-t)y^p)^{\frac{1}{p}}\right) \leqslant_p h(t^{\xi_\star})\mathscr{F}(x) + mh(1 - t^{\xi_\star})\mathscr{F}(y).$$

Setting $t = \frac{1}{2}, m \to m^p$ and $x^p = \frac{q+t^p}{2}a^p + \left(1 - \frac{q+t^p}{2}\right)b^p m^p, y^p = \frac{q+t^p}{2}b^p + \left(1 - \frac{q+t^p}{2}\right)$ $\frac{a^p}{m^p}$ in the inequality, we achieve

$$\mathscr{F}\left([\frac{a^p + m^p b^p}{2}]^{\frac{1}{p}}\right) \leqslant_p \tau\left(\frac{1}{2^{\xi_\star}}\right)\mathscr{F}\left((\frac{q+t^p}{2}a^p + \left(1 - \frac{q+t^p}{2}\right)b^p m^p)^{\frac{1}{p}}\right)$$
$$+\tau\left(1 - \frac{1}{2^{\xi_\star}}\right) m^p \mathscr{F}\left((\frac{q+t^p}{2}b^p + \left(1 - \frac{q+t^p}{2}\right)\frac{a^p}{m^p})^{\frac{1}{p}}\right).$$

It follows from the statement of the IVM that

$$\mathscr{F}_*\left([\frac{a^p + m^p b^p}{2}]^{\frac{1}{p}}\right) \leqslant \tau\left(\frac{1}{2^{\xi_\star}}\right)\mathscr{F}_*\left((\frac{q+t^p}{2}a^p + \left(1 - \frac{q+t^p}{2}\right)b^p m^p)^{\frac{1}{p}}\right)$$
$$+\tau\left(1 - \frac{1}{2^{\xi_\star}}\right) m^p \mathscr{F}_*\left((\frac{q+t^p}{2}b^p + \left(1 - \frac{q+t^p}{2}\right)\frac{a^p}{m^p})^{\frac{1}{p}}\right),$$

and

$$\mathscr{F}^*\left([\frac{a^p + m^p b^p}{2}]^{\frac{1}{p}}\right) \leqslant \tau\left(\frac{1}{2^{\xi_\star}}\right)\mathscr{F}^*\left((\frac{q+t^p}{2}a^p + \left(1 - \frac{q+t^p}{2}\right)b^p m^p)^{\frac{1}{p}}\right)$$
$$+\tau\left(1 - \frac{1}{2^{\xi_\star}}\right) m^p \mathscr{F}^*\left((\frac{q+t^p}{2}b^p + \left(1 - \frac{q+t^p}{2}\right)\frac{a^p}{m^p})^{\frac{1}{p}}\right).$$

Multiplying the inequalities with $t^{\theta p - 1}$ and integrating with respect to the variable used, we get

$$\frac{\mathscr{F}_*\left([\frac{a^p + m^p b^p}{2}]^{\frac{1}{p}}\right)}{\theta p} \leqslant \int_0^1 t^{p\theta - 1}\tau\left(\frac{1}{2^{\xi_\star}}\right)\mathscr{F}_*\left((\frac{q+t^p}{2}a^p + \left(1 - \frac{q+t^p}{2}\right)b^p m^p)^{\frac{1}{p}}\right)dt$$
$$+ \int_0^1 t^{p\theta - 1}\tau\left(1 - \frac{1}{2^{\xi_\star}}\right) m^p \mathscr{F}_*\left((\frac{q+t^p}{2}b^p + \left(1 - \frac{q+t^p}{2}\right)\frac{a^p}{m^p})^{\frac{1}{p}}\right)dt,$$

and

$$\frac{\mathscr{F}^*\left([\frac{a^p+m^pb^p}{2}]^{\frac{1}{p}}\right)}{\theta p} \leqslant \int_0^1 t^{p\theta-1}\tau\left(\frac{1}{2\xi_\star}\right)\mathscr{F}^*\left((\frac{q+t^p}{2}a^p+\left(1-\frac{q+t^p}{2}\right)b^pm^p)^{\frac{1}{p}}\right)dt$$

$$+\int_0^1 t^{p\theta-1}\tau\left(1-\frac{1}{2\xi_\star}\right)m^p\mathscr{F}^*\left((\frac{q+t^p}{2}b^p+\left(1-\frac{q+t^p}{2}\right)\frac{a^p}{m^p})^{\frac{1}{p}}\right)dt.$$

Introducing the substitution $z^p = \frac{q+t^p}{2}a^p + (1-\frac{q+t^p}{2})b^pm^p$ in the first integral and $z^p = \frac{q+t^p}{2}b^p + (1-\frac{q+t^p}{2})a^pm^p$ in the second integral, we get for the first and second integrals, respectively,

$$\int_0^1 t^{p\beta-1}\tau\left(\frac{1}{2\xi_\star}\right)\mathscr{F}_*\left((\frac{q+t^p}{2}a^p+\left(1-\frac{q+t^p}{2}\right)b^pm^p)^{\frac{1}{p}}\right)dt =$$

$$\int_{\left(\frac{q}{2}(a^p-b^pm^p)+\frac{a^p}{2}+\frac{m^pb^p}{2}\right)^{\frac{1}{p}}}^{\left(\frac{q}{2}(a^p-m^pb^p)+b^pm^p\right)^{\frac{1}{p}}} \mathscr{F}_*(z)(b^pm^p+\frac{q}{2}(a^p-b^pm^p)-z^p)^{\frac{\theta}{k}-1}z^{p-1}dz\cdot\frac{2^\theta}{(b^pm^p-a^p)^\theta}$$

$$=\frac{\tau\left(\frac{1}{2\xi_\star}\right)2^\theta\Gamma(\theta)}{p^{1-\theta}(b^pm^p-a^p)^\theta}\chi^{\theta,p}_{\left(\left(\frac{q}{2}(a^p-m^pb^p)+\frac{a^p}{2}+\frac{b^pm^p}{2}\right)^{\frac{1}{p}}\right)^+}\mathscr{F}_*\left(\left(\frac{q}{2}(a^p-b^pm^p)+m^pb^p\right)^{\frac{1}{p}}\right),$$

and

$$\int_0^1 t^{p\beta-1}\tau\left(1-\frac{1}{2\xi_\star}\right)m^p\mathscr{F}_*\left((\frac{q+t^p}{2}b^p+\left(1-\frac{q+t^p}{2}\right)\frac{a^p}{m^p})^{\frac{1}{p}}\right)dt$$

$$=\int_{\left(\frac{q}{2}\left(b^p-\frac{a^p}{m^p}\right)+\frac{a^p}{m^p}\right)^{\frac{1}{p}}}^{\left(\frac{q}{2}\left(b^p-\frac{a^p}{m^p}\right)+\frac{a^p}{2m^p}+\frac{b^p}{2}\right)^{\frac{1}{p}}} \mathscr{F}_*(z)\left(z^p-\frac{q}{2}(b^p-\frac{a^p}{m^p})-\frac{a^p}{m^p}\right)^{\frac{\theta}{k}}\cdot\frac{2^\theta m^{\theta p}}{(m^pb^p-a^p)^\theta}$$

$$=\frac{2^\theta m^{p\theta}\Gamma(\theta)}{p^{1-\theta}(b^pm^p-a^p)^\theta}\chi^{\theta,p}_{\left(\left(\frac{q}{2}(b^p-\frac{a^p}{m^p})+\frac{a^p}{2m^p}+\frac{b^p}{2}\right)^{\frac{1}{p}}\right)^+}\mathscr{F}_*\left(\left(\frac{q}{2}(b^p-\frac{a^p}{m^p})+\frac{a^p}{m^p}\right)^{\frac{1}{p}}\right).$$

Using a similar technique leads us to the equalities for the upper end point functions,

$$\int_0^1 t^{p\beta-1}\tau\left(\frac{1}{2\xi_\star}\right)\mathscr{F}^*\left((\frac{q+t^p}{2}a^p+\left(1-\frac{q+t^p}{2}\right)b^pm^p)^{\frac{1}{p}}\right)dt$$

$$=\int_{\left(\frac{q}{2}(a^p-b^pm^p)+\frac{a^p}{2}+\frac{m^pb^p}{2}\right)^{\frac{1}{p}}}^{\left(\frac{q}{2}(a^p-m^pb^p)+b^pm^p\right)^{\frac{1}{p}}} \mathscr{F}^*(z)(b^pm^p+\frac{q}{2}(a^p-b^pm^p)-z^p)^{\frac{\theta}{k}-1}z^{p-1}dz\cdot\frac{2^\theta}{(b^pm^p-a^p)^\theta}$$

$$=\frac{\tau\left(\frac{1}{2\xi_\star}\right)2^\theta\Gamma(\theta)}{p^{1-\theta}(b^pm^p-a^p)^\theta}\chi^{\theta,p}_{\left(\left(\frac{q}{2}(a^p-m^pb^p)+\frac{a^p}{2}+\frac{b^pm^p}{2}\right)^{\frac{1}{p}}\right)^+}\mathscr{F}^*\left(\left(\frac{q}{2}(a^p-b^pm^p)+m^pb^p\right)^{\frac{1}{p}}\right),$$

and

$$\int_0^1 t^{p\beta-1}\tau\left(1-\frac{1}{2\xi_\star}\right)m^p\mathscr{F}^*\left((\frac{q+t^p}{2}b^p+\left(1-\frac{q+t^p}{2}\right)\frac{a^p}{m^p})^{\frac{1}{p}}\right)dt$$

$$=\int_{\left(\frac{q}{2}\left(b^p-\frac{a^p}{m^p}\right)+\frac{a^p}{m^p}\right)^{\frac{1}{p}}}^{\left(\frac{q}{2}\left(b^p-\frac{a^p}{m^p}\right)+\frac{a^p}{2m^p}+\frac{b^p}{2}\right)^{\frac{1}{p}}} \mathscr{F}^*(z)\left(z^p-\frac{q}{2}(b^p-\frac{a^p}{m^p})-\frac{a^p}{m^p}\right)^{\frac{\theta}{k}}\cdot\frac{2^\theta m^{\theta p}}{(m^pb^p-a^p)^\theta}$$

$$=\frac{2^\theta m^{p\theta}\Gamma(\theta)}{p^{1-\theta}(b^pm^p-a^p)^\theta}\chi^{\theta,p}_{\left(\left(\frac{q}{2}(b^p-\frac{a^p}{m^p})+\frac{a^p}{2m^p}+\frac{b^p}{2}\right)^{\frac{1}{p}}\right)^+}\mathscr{F}^*\left(\left(\frac{q}{2}(b^p-\frac{a^p}{m^p})+\frac{a^p}{m^p}\right)^{\frac{1}{p}}\right).$$

Hence, we achieve

$$\left(\frac{\mathscr{F}_*\left([\frac{a^p+m^pb^p}{2}]^{\frac{1}{p}}\right)}{\theta p}, \frac{\mathscr{F}^*\left([\frac{a^p+m^pb^p}{2}]^{\frac{1}{p}}\right)}{\theta p}\right)$$

$$\leqslant_p \left(\frac{\tau\left(\frac{1}{2^{\xi_\star}}\right)2^\theta p^{\theta-1}\Gamma(\theta)}{(b^pm^p-a^p)^\theta}\chi^{\theta,p}_{\left(\left(\frac{q}{2}(a^p-m^pb^p)+\frac{a^p}{2}+\frac{b^pm^p}{2}\right)^{\frac{1}{p}}\right)^+}\mathscr{F}_*\left(\left(\frac{q}{2}(a^p-b^pm^p)+m^pb^p\right)^{\frac{1}{p}}\right)\right.$$

$$\left.+\frac{\tau\left(1-\frac{1}{2^{\xi_\star}}\right)m^{p+\theta p}2^\theta p^{\theta-1}\Gamma(\theta)}{(b^pm^p-a^p)^\theta}\chi^{p,\theta}_{\left(\left(\frac{q}{2}(b^p-\frac{a^p}{m^p})+\frac{a^p}{2m^p}+\frac{b^p}{2}\right)^{\frac{1}{p}}\right)^-}\mathscr{F}_*\left(\left(\frac{q}{2}(b^p-\frac{a^p}{m^p})+\frac{a^p}{m^p}\right)^{\frac{1}{p}}\right),\right.$$

$$\frac{\tau\left(\frac{1}{2^{\xi_\star}}\right)2^\theta p^{\theta-1}\Gamma(\theta)}{(b^pm^p-a^p)^\theta}\chi^{\theta,p}_{\left(\left(\frac{q}{2}(a^p-m^pb^p)+\frac{a^p}{2}+\frac{b^pm^p}{2}\right)^{\frac{1}{p}}\right)^+}\mathscr{F}^*\left(\left(\frac{q}{2}(a^p-b^pm^p)+m^pb^p\right)^{\frac{1}{p}}\right)$$

$$\left.+\frac{\tau\left(1-\frac{1}{2^{\xi_\star}}\right)m^{p+\theta p}2^\theta p^{\theta-1}\Gamma(\theta)}{(b^pm^p-a^p)^\theta}\chi^{p,\theta}_{\left(\left(\frac{q}{2}(b^p-\frac{a^p}{m^p})+\frac{a^p}{2m^p}+\frac{b^p}{2}\right)^{\frac{1}{p}}\right)^-}\mathscr{F}^*\left(\left(\frac{q}{2}(b^p-\frac{a^p}{m^p})+\frac{a^p}{m^p}\right)^{\frac{1}{p}}\right)\right),$$

from which we achieve the left inequality.

Now, we obtain the right side. Using the definition of the left–right-(k,h-m)-p-convex function towards the right, we achieve

$$\tau\left(\frac{1}{2^{\xi_\star}}\right)\mathscr{F}\left(\left(\frac{q+t^p}{2}a^p+\left(1-\frac{q+t^p}{2}\right)b^pm^p\right)^{\frac{1}{p}}\right)$$

$$+\tau\left(1-\frac{1}{2^{\xi_\star}}\right)m^p\mathscr{F}\left(\left(\frac{q+t^p}{2}b^p+\left(1-\frac{q+t^p}{2}\right)\frac{a^p}{m^p}\right)^{\frac{1}{p}}\right)\leqslant_p$$

$$\left(\left(\tau\left(\frac{1}{2^{\xi_\star}}\right)\mathscr{F}(a)+\tau\left(1-\frac{1}{2^{\xi_\star}}\right)m^p\mathscr{F}(b)\right)h\left(\left(\frac{q+t^p}{2}\right)^k\right)t^{\theta p-1}\right)$$

$$+\left(\left(\tau\left(\frac{1}{2^{\xi_\star}}\right)\mathscr{F}(b)m^p+\tau\left(1-\frac{1}{2^{\xi_\star}}\right)\mathscr{F}(a)\right)h\left(1-\left(\frac{q+t^p}{2}\right)^k\right)t^{\theta p-1}\right).$$

Multiplying the inequality with $t^{\theta p-1}$ and integrating the expression, then using the IVM property, we obtain the right-hand side. Now, taking the product with the constant from the left part, we achieve the original inequality

$$\mathscr{F}\left([\frac{a^p+m^pb^p}{2}]^{\frac{1}{p}}\right)\leqslant_p$$

$$\frac{\tau\left(\frac{1}{2^{\xi_\star}}\right)2^\theta p^\theta\Gamma(\theta+1)}{(b^pm^p-a^p)^\theta}\chi^{\theta,p}_{\left(\left(\frac{q}{2}(a^p-m^pb^p)+\frac{a^p}{2}+\frac{b^pm^p}{2}\right)^{\frac{1}{p}}\right)^+}\mathscr{F}\left(\left(\frac{q}{2}(a^p-b^pm^p)+m^pb^p\right)^{\frac{1}{p}}\right)$$

$$+\frac{\tau\left(1-\frac{1}{2^{\xi_\star}}\right)m^{p+\theta p}2^\theta p^\theta\Gamma(\theta+1)}{(b^pm^p-a^p)^\theta}\chi^{p,\theta}_{\left(\left(\frac{q}{2}(b^p-\frac{a^p}{m^p})+\frac{a^p}{2m^p}+\frac{b^p}{2}\right)^{\frac{1}{p}}\right)^-}\mathscr{F}\left(\left(\frac{q}{2}(b^p-\frac{a^p}{m^p})+\frac{a^p}{m^p}\right)^{\frac{1}{p}}\right)$$

$$\leqslant_p \left(\left(\tau\left(\frac{1}{2^{\xi_\star}}\right)\mathscr{F}(a)+\tau\left(1-\frac{1}{2^{\xi_\star}}\right)m^p\mathscr{F}(b)\right)\int_0^1 h\left(\left(\frac{q+t^p}{2}\right)^k\right)t^{\theta p-1}dt\right)\theta p$$

$$+\left(\left(\tau\left(\frac{1}{2^{\xi_\star}}\right)\mathscr{F}(b)m^p+\tau\left(1-\frac{1}{2^{\xi_\star}}\right)\mathscr{F}(a)\right)\int_0^1 h\left(1-\left(\frac{q+t^p}{2}\right)^k\right)t^{\theta p-1}dt\right)\theta p.$$

□

Corollary 4. *Setting $p = 3, \xi_{\star} = 3$, we achieve a new noninteger HH type inequality*

$$\mathscr{F}\left([\frac{a^3 + m^3 b^3}{2}]^{\frac{1}{3}}\right) \leqslant_p$$

$$\frac{h\left(\frac{1}{2^3}\right)2^{\theta}3^{\theta}\Gamma(\theta+1)}{(b^3m^3 - a^3)^{\theta}}\chi^{\theta,3}_{\left(\left(\frac{q}{2}(a^2 - m^3b^3) + \frac{a^3}{2} + \frac{b^3m^3}{2}\right)^{\frac{1}{3}}\right)^{+}}\mathscr{F}\left(\left(\frac{q}{2}(a^3 - b^3m^3) + m^3b^3\right)^{\frac{1}{3}}\right)$$

$$+\frac{h\left(1 - \frac{1}{2^3}\right)m^{3+3\theta}2^{\theta}3^{\theta}\Gamma(\theta+1)}{(b^3m^3 - a^3)^{\theta}}\chi^{3,\theta}_{\left(\left(\frac{q}{2}(b^3 - \frac{a^3}{m^3}) + \frac{a^3}{2m^3} + \frac{b^3}{2}\right)^{\frac{1}{3}}\right)^{-}}\mathscr{F}\left(\left(\frac{q}{2}(b^3 - \frac{a^3}{m^3}) + \frac{a^3}{m^3}\right)^{\frac{1}{3}}\right)$$

$$\leqslant_p \left(\left(h\left(\frac{1}{2^3}\right)\mathscr{F}(a) + h\left(1 - \frac{1}{2^3}\right)m^3\mathscr{F}(b)\right)\int_0^1 h\left(\left(\frac{q + t^3}{2}\right)^3\right)t^{3\theta - 1}dt\right)3\theta$$

$$+\left(\left(h\left(\frac{1}{2^3}\right)\mathscr{F}(b)m^3 + h\left(1 - \frac{1}{2^3}\right)\mathscr{F}(a)\right)\int_0^1 h\left(1 - \left(\frac{q + t^3}{2}\right)^3\right)t^{3\theta - 1}dt\right)3\theta.$$

4. Conclusions

We introduced a novel type of IVM, namely the left–right-(k,h-m)-p-convex function, which generalized the previously defined p,h-convex IVM given by Khan et al. As a consequence of the generalization, many new inequalities followed. We achieved new variations of the Hermite–Hadamard inequality in combination with noninteger operators which generalized the previous HH type results. Because of the IVM environment, by letting the upper and lower bound be the same, we recovered previous results from the (k,h-m)-p-convexity to the classical convexity.

Author Contributions: Conceptualization, V.S. and S.R.; methodology, V.S., S.R., R.R. and O.A.A.A.; formal analysis, V.S., S.R. and O.A.A.A.; writing–original draft preparation, V.S. and S.R.; supervision, R.R., O.A.A.A. and S.R. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: Not applicable.

Acknowledgments: The research is supported by the Deanship of Scientific Research, Prince Sattam bin Abdulaziz University, Alkharj. The authors convey their sincere thanks to the anonymous reviewers for their valuable comments, which helped bring the manuscript to its present form.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Hardy, G.H.; Littlewood, J.E.; Pólya, G. *Inequalities*; Cambridge University Press: Cambridge, UK, 1952; 324p.
2. Siricharuanun, P.; Erden, S.; Ali, M.A.; Budak, H.; Chasreechai, S.; Sitthiwirattham, T. Some New Simpson's and Newton's Formulas Type Inequalities for Convex Functions in Quantum Calculus. *Mathematics* **2021**, *9*, 1992. [CrossRef]
3. You, X.; Ali, M.A.; Budak, H.; Reunsumrit, J.; Sitthiwirattham, T. Hermite–Hadamard–Mercer-Type Inequalities for Harmonically Convex Mappings. *Mathematics* **2021**, *9*, 2556. [CrossRef]
4. Sarikaya, M.Z.; Yildirim, H. On Hermite–Hadamard type inequalities for Riemann-Liouville fractional integrals. *Miskolc Math. Notes* **2016**, *17*, 1049–1059. [CrossRef]
5. Chen, H.; Katugampola, U.N. Hermite–Hadamard and Hermite–Hadamard-Fejr type inequalities for generalized fractional integrals. *J. Math. Anal. Appl.* **2017**, *446*, 1274–1291. [CrossRef]
6. Han, J.; Mohammed, P.O.; Zeng, H. Generalized fractional integral inequalities of Hermite–Hadamard-type for a convex function. *Open Math.* **2020**, *18*, 794–806. [CrossRef]
7. Awan, M.U.; Talib, S.; Chu, Y.M.; Noor, M.A.; Noor, K.I. Some new refinements of Hermite–Hadamard-type inequalities involving-Riemann-Liouville fractional integrals and applications. *Math. Probl. Eng.* **2020**, 2020, 3051920. [CrossRef]
8. Aljaaidi, T.A.; Pachpatte, D.B. The Minkowski's inequalities via f-Riemann-Liouville fractional integral operators. *Rendiconti del Circolo Matematico di Palermo Series 2* **2021**, *70*, 893–906. [CrossRef]

9. Mohammed, P.O.; Aydi, H.; Kashuri, A.; Hamed, Y.S.; Abualnaja, K.M. Midpoint inequalities in fractional calculus defined using positive weighted symmetry function kernels. *Symmetry* **2021**, *13*, 550. [CrossRef]
10. Mohammed, P.O.; Abdeljawad, T.; Jarad, F.; Chu, Y.M. Existence and uniqueness of uncertain fractional backward difference equations of Riemann-Liouville type. *Math. Probl. Eng.* **2020**, *2020*, 6598682. [CrossRef]
11. Mitrinović, D.S. *Analytic Inequalities*; Springer: Berlin/Heidelberg, Germany, 1970.
12. Pečarić, J.; Proschan, F.; Tong, Y. *Convex Functions, Partial Orderings, and Statistical Applications*; Academic Press, Inc.: Cambridge, MA, USA, 1992.
13. Hadamard, J. Étude sur les propriétés des fonctions entières en particulier d'une fonction considéré par Riemann. *J. Math. Pures Appl.* **1893**, *58*, 171–215.
14. Cristescu, G. Hadamard type inequalities for convolution of h-convex functions. *Ann. Tiberiu Popoviciu Semin. Funct. Equ. Approx. Convexity* **2010**, *8*, 3–11.
15. Dragomir, S.S. An inequality improving the first Hermite-Hadamard inequality for convex functions defined on linear spaces and applications for semi-inner products. *J. Inequal. Pure Appl. Math.* **2002**, *3*, 31.
16. Dragomir, S.S.; Fitzpatrick, S. The Hadamard inequalities for sconvex functions in the second sense. *Demonstr. Math.* **1999**, *32*, 687–696.
17. El Farissi, A. Simple proof and refeinment of Hermite-Hadamard inequality. *J. Math. Ineq.* **2010**, *4*, 365–369. [CrossRef]
18. Kikianty, E.; Dragomir, S.S. Hermite-Hadamard's inequality and the p-HH-norm on the Cartesian product of two copies of a normed space. *Math. Inequal. Appl.* **2010**. [CrossRef]
19. Mitrinović, D.S.; Lacković, I.B. Hermite and convexity. *Aequ. Math.* **1985**, *28*, 229–232. [CrossRef]
20. Hermann, R. *Fractional Calculus an Introduction for Physicists*; World Scientific Publishing Co. Pte. Ltd.: Singapore, 2011; p. 596224.
21. Oldham, K.B.; Spanier, J. *The Fractional Calculus Theory and Applications of Differentation and Integration to Arbitrary Order*; Academic Press, Inc.: London, UK, 1974.
22. Stojiljković, V.; Ramaswamy, R.; Alshammari, F.; Ashour, O.A.; Alghazwani, M.L.H.; Radenović, S. Hermite–Hadamard Type Inequalities Involving (k-p) Fractional Operator for Various Types of Convex Functions. *Fractal Fract.* **2022**, *6*, 376. [CrossRef]
23. Stojiljković, V.; Ramaswamy, R.; Ashour Abdelnaby, O.A.; Radenović, S. Riemann-Liouville Fractional Inclusions for Convex Functions Using Interval Valued Setting. *Mathematics* **2022**, *10*, 3491. [CrossRef]
24. Afzal, W.; Abbas, M.; Macías-Díaz, J.E.; Treanţă, S. Some H-Godunova–Levin Function Inequalities Using Center Radius (Cr) Order Relation. *Fractal Fract.* **2022**, *6*, 518. [CrossRef]
25. Afzal, W.; Alb Lupaş, A.; Shabbir, K. Hermite–Hadamard and Jensen-Type Inequalities for Harmonical (h1, h2)-Godunova–Levin Interval-Valued Functions. *Mathematics* **2022**, *10*, 2970. [CrossRef]
26. Afzal, W.; Shabbir, K.; Treanţă, S.; Nonlaopon, K. Jensen and Hermite-Hadamard type inclusions for harmonical h-Godunova-Levin functions. *AIMS Math.* **2023**, *8*, 3303–3321. [CrossRef]
27. Khan, M.B.; Noor, M.A.; Noor, K.I.; Ab Ghani, A.T.; Abdullah, L. Extended Perturbed Mixed Variational-Like Inequalities for Fuzzy Mappings. *J. Math.* **2021**, *2021*, 6652930. [CrossRef]
28. Khan, M.B.; Noor, M.A.; Noor, K.I.; Chu, Y.-M. New Hermite-Hadamard Type Inequalities for (h1, h2)-Convex Fuzzy-IntervalValued Functions. *Adv. Differ. Equ.* **2021**, *2021*, 6–20. [CrossRef]
29. Khan, M.B.; Mohammed, P.O.; Noor, M.A.; Hamed, Y.S. New Hermite–Hadamard Inequalities in Fuzzy-Interval Fractional Calculus and Related Inequalities. *Symmetry* **2021**, *13*, 673. [CrossRef]
30. Yang, X.J. *General Fractional Derivatives Theory, Methods and Applications*; Taylor and Francis Group: London, UK, 2019.
31. Budak, H.; Tunç, T.; Sarikaya, M.Z. Fractional Hermite–Hadamard type inequalities for interval-valued functions. *Proc. Am. Math. Soc.* **2019**, *148*, 705–718. [CrossRef]
32. Moore, R.E. *Interval Analysis*; Prentice Hall: Englewood Cliffs, NJ, USA, 1966.
33. Jensen, J.L.W.V. Sur les fonctions convexes et les inégalités entre les valeurs moyennes. *Acta Math.* **1906**, *30*, 175–193. [CrossRef]
34. Khan, M.B.; Cătas, A.; Saeed, T. Generalized Fractional Integral Inequalities for p-Convex Fuzzy Interval-Valued Mappings. *Fractal Fract.* **2022**, *6*, 324. [CrossRef]
35. Katugampola, U.N. A new approach to generalized fractional derivatives. *Bull. Math. Anal. Appl.* **2014**, *6*, 1–15.
36. Abramowitz, M.; Stegun, I.A. *Handbook of Mathematical Functions: With Formulas, Graphs, and Mathematical Tables*; MR1225604; Dover Publications: New York, NY, USA, 1992.
37. Fang, Z.; Shi, R. On the (p,h)-convex function and some integral inequalities. *J. Inequalities Appl.* **2014**, *2014*, 45. [CrossRef]
38. Khan, M.B.; Mohammed, P.O.; Noor, M.A.; Baleanu, D.; Guirao, J.L.G. Some New Fractional Estimates of Inequalities for LR-p-Convex Interval-Valued Functions by Means of Pseudo Order Relation. *Axioms* **2021**, *10*, 175. 10.3390/axioms10030175. [CrossRef]
39. Zhang, D.; Guo, C.; Chen, D.; Wang, G. Jensen's inequalities for set-valued and fuzzy set-valued functions. *Fuzzy Sets Syst.* **2020**, *404*, 178–204. [CrossRef]
40. Costa, T.M. Jensen's inequality type integral for fuzzy-interval-valued functions. *Fuzzy Sets Syst.* **2017**, *327*, 31–47. [CrossRef]
41. Jia, W.; Yussouf, M.; Farid, G.; Khan, K.A. Hadamard and Fejér–Hadamard inequalities for (α, h–m)– p–convex functions via Riemann–Liouville fractional integrals. *Math. Probl. Eng.* **2021**, *2021*, 9945114. [CrossRef]
42. Khan, M.B.; Noor, M.A.; Noor, K.I. Some Novel Inequalities for LR-(p,h)-Convex Interval Valued Functions by Means of Pseudo Order Relation. *Math. Methods Appl. Sci.* **2021**, *45*, 1310–1340. [CrossRef]

43. Özdemir, M.E.; Yıldız, Ç.; Akdemir, A.O.; Set, E. On some inequalities for s-convex functions and applications. *J. Inequalities Appl.* **2013**, *2013*, 333. [CrossRef]
44. Varošanec, S. On h-convexity. *J. Math. Anal. Appl.* **2007**, *326*, 303–311. [CrossRef]
45. Noor, M.A.; Awan, M.U.; Noor, K.I.; Postolache, M. Some Integral Inequalities for p-Convex Functions. *Filomat* **2016**, *30*, 2435–2444. [CrossRef]