fractal and fractional

Article

On Ψ-Hilfer Fractional Integro-Differential Equations with Non-Instantaneous Impulsive Conditions

Ramasamy Arul [1], Panjayan Karthikeyan [2], Kulandhaivel Karthikeyan [3], Palanisamy Geetha [3], Ymnah Alruwaily [4], Lamya Almaghamsi [5] and El-sayed El-hady [4,6,*]

[1] Department of Mathematics, Gnanamani College of Technology, Namakkal 637018, Tamilnadu, India
[2] Department of Mathematics, Sri Vasavi College, Erode 638 316, Tamil Nadu, India
[3] Department of Mathematics, KPR Institute of Engineering and Technology, Coimbatore 641407, Tamil Nadu, India
[4] Mathematics Department, College of Science, Jouf University, P.O. Box 2014, Sakaka 72388, Saudi Arabia
[5] Department of Mathematics, College of Science, University of Jeddah, P.O. Box 80327, Jeddah 21589, Saudi Arabia
[6] Basic Science Department, Faculty of Computers and Informatics, Suez Canal University, Ismailia 41522, Egypt
* Correspondence: eaelhady@ju.edu.sa

Abstract: We establish sufficient conditions for the existence of solutions of an integral boundary value problem for a Ψ-Hilfer fractional integro-differential equations with non-instantaneous impulsive conditions. The main results are proved with a suitable fixed point theorem. An example is given to interpret the theoretical results. In this way, we generalize recent interesting results.

Keywords: fractional differential equations; Hilfer fractional integro-differential equations; fractional boundary conditions; existence and uniqueness

Citation: Arul, R.; Karthikeyan, P.; Karthikeyan, K.; Geetha, P.; Alruwaily, Y.; Almaghamsi, L.; El-hady, E.-s. On Ψ-Hilfer Fractional Integro-Differential Equations with Non-Instantaneous Impulsive Conditions. *Fractal Fract.* **2022**, *6*, 732. https://doi.org/10.3390/fractalfract6120732

Academic Editors: John R. Graef, Alina Alb Lupas and Adriana Catas

Received: 15 November 2022
Accepted: 7 December 2022
Published: 10 December 2022

1. Introduction

There has been a lot of research completed so far on fractional differential equations (FDEs) with initial and boundary conditions (BCs). The reason for this is FDEs accurately describe many real-world phenomena such as biology, physics, chemistry, signal processing, and many more (see, e.g., [1–13]). Furthermore, it should be remarked that FDEs have interesting applications in solving inverse problems, and in the modeling of heat flow in porous material (see, e.g., [14–25]).

Impulsive equations arise in fields such as engineering, biology, physics, and medicine, where objects change their state rapidly at certain points. Instantaneous impulses (InI) are known as the ones with relatively short duration of changes. On the other hand, non-instantaneous impulses (N-InI) are those in which an impulsive activity begins suddenly at some places and remains active for a set amount of time. For more details of such processes in interesting applications such as ecology and pharmacokinetics and more see, e.g., [26–30]. Hernandez and O'Regan in [31] pioneered N-InI differential equation. They reported that the InI cannot describe certain processes, such as the evolution of pharmacology. For some problems involving N-InI in psychology see [32]. For some recent works, on N-InI FDEs, see, e.g., [33–37] and references therein.

In [1], S. Asawasamrit et al. studied the Ψ-Caputo fractional derivative (FD) and N-InI BVPs. In [28], V. Gupta et al. established the FDEs with N-InI. Ψ-Hilfer FDEs with impulsive conditions was studied in [38]. In [2], M. S. Abdo et al. discussed the Ψ-Hilfer FD involving BCs. Ψ-Hilfer FD and inclusions with N-InI was established in [29].

In [26], M. I. Abbas, studied the proportional FD with respect to another function of the form

$$\begin{aligned}&{}_{\mathfrak{a}}\mathcal{D}^{\mathfrak{p},\mathfrak{q},\mathfrak{g}}\wp(\mathfrak{u}) = \mathscr{F}(\mathfrak{u},\wp(\mathfrak{u}),{}_{\mathfrak{a}}\mathscr{I}^{\mathfrak{r},\mathfrak{q},\mathfrak{g}}\wp(\mathfrak{u})), \mathfrak{u}\in(\mathfrak{s}_{\mathfrak{i}},\mathfrak{u}_{\mathfrak{i}+1}],\\ &\wp(\mathfrak{u}) = \Psi_{\mathfrak{k}}(\mathfrak{u},\wp(\mathfrak{u}_{\mathfrak{k}}^{+})),\;\; \mathfrak{u}\in(\mathfrak{u}_{\mathfrak{i}},\mathfrak{s}_{\mathfrak{i}}],\;\; \mathfrak{i}=1,\dots,\mathfrak{m},\\ &\mathscr{I}^{1-\mathfrak{p},\mathfrak{q},\mathfrak{g}}\wp(\mathfrak{a}) = \wp_0\in\mathbb{R}.\end{aligned}$$

where ${}_{\mathfrak{a}}\mathcal{D}^{\mathfrak{p},\mathfrak{q},\mathfrak{g}}$, ${}_{\mathfrak{a}}\mathscr{I}^{\mathfrak{r},\mathfrak{q},\mathfrak{g}}$ is the proportional FD and fractional integral with respect to another function and $\mathscr{F}$ is continuous.

In [11], C. Nuchpong et al. discussed the Hilfer FD with non-local BCs of the form

$$\begin{gathered}{}^{\mathfrak{H}}\mathcal{D}^{\mathfrak{p},\mathfrak{q}}\wp(\mathfrak{u}) = \mathscr{F}(\mathfrak{u},\wp(\mathfrak{u}),\mathscr{I}^{\delta}\wp(\mathfrak{u})),\;\; \mathfrak{u}\in[\mathfrak{a},\mathfrak{b}]\\ \wp(\mathfrak{a})=0,\;\; \int_{\mathfrak{a}}^{\mathfrak{b}}\wp(\mathfrak{s})\partial\mathfrak{s}+\ell = \sum_{i=1}^{m-2}\varsigma_i\wp(\vartheta_i).\end{gathered}$$

where ${}^{\mathfrak{H}}\mathcal{D}^{\mathfrak{p},\mathfrak{q}}$- Hilfer FD and $\mathscr{I}^{\delta}$- Riemann–Liouville fractional integral and the function $\mathscr{F}$ is continuous.

In [33], A. Salim et al. established the BVP for implicit fractional order generalized Hilfer-type FD with N-InI of the form

$$\begin{aligned}&\left({}^{\alpha}\mathcal{D}^{\mathfrak{p},\mathfrak{q}}_{\tau^+}\wp\right)(\mathfrak{u}) = \mathscr{F}(\mathfrak{u},\wp(\mathfrak{u}),({}^{\alpha}\mathcal{D}^{\mathfrak{p},\mathfrak{q}}\wp)(\mathfrak{u})),\;\; \mathfrak{t}\in\mathscr{J}_i,\\ &\wp(\mathfrak{u}) = \mathscr{H}_{\mathfrak{i}}(\mathfrak{u},\wp(\mathfrak{u})),\;\; \mathfrak{u}\in(\mathfrak{u}_{\mathfrak{i}},\mathfrak{s}_{\mathfrak{i}}],\;\; \mathfrak{i}=1,\dots,\mathfrak{m},\\ &\varphi_1\left({}^{\alpha}\mathscr{I}^{1-\epsilon}_{a^+}\right)(\mathfrak{a})+\varphi_2\left({}^{\alpha}\mathscr{I}^{1-\epsilon}_{\tau^+}\right)(\mathfrak{b}) = \varphi_3,\end{aligned}$$

where ${}^{\alpha}\mathcal{D}^{\mathfrak{p},\mathfrak{q}}_{\tau^+}, {}^{\alpha}\mathscr{I}^{1-\epsilon}_{a^+}$ are the generalized Hilfer-type FD and fractional integral and the function $\mathscr{F}$ is continuous.

Inspired by the aforementioned works, we studied the Ψ-Hilfer fractional integro-differential equations (Ψ-HFI-DEs) with N-InI multi-point BCs of the form (with ${}^{\mathfrak{H}}\mathcal{D}^{\mathfrak{p},\mathfrak{q};\Psi}$ is the Ψ- Hilfer FDs of order $\mathfrak{p}, 1<\mathfrak{p}<2$)

$$ {}^{\mathfrak{H}}\mathcal{D}^{\mathfrak{p},\mathfrak{q};\Psi}\wp(\mathfrak{t}) = \mathscr{F}(\mathfrak{t},\wp(\mathfrak{t}),\Psi\wp(\mathfrak{t})), \mathfrak{t}\in(\mathfrak{s}_{\mathfrak{i}},\mathfrak{t}_{\mathfrak{i}+1}], \tag{1}$$

$$\wp(\mathfrak{t}) = \mathscr{H}_{\mathfrak{i}}(\mathfrak{t},\wp(\mathfrak{t})),\;\; \mathfrak{t}\in(\mathfrak{t}_{\mathfrak{i}},\mathfrak{s}_{\mathfrak{i}}],\;\; \mathfrak{i}=1,\dots,\mathfrak{m}, \tag{2}$$

$$\wp(0)=0,\;\; \wp(\mathscr{T}) = \sum_{\mathfrak{i}=1}^{\mathfrak{m}}\nu_{\mathfrak{i}}\mathscr{I}^{\varsigma_i}\wp(\upsilon_{\mathfrak{i}}),\;\; \nu_{\mathfrak{i}}\in\mathbb{R},\;\; \upsilon_{\mathfrak{i}}\in[0,\mathscr{T}], \tag{3}$$

where $0\le\mathfrak{q}\le1$, $\nu_{\mathfrak{i}}\in\mathbb{R}$, $\upsilon_{\mathfrak{i}}\in[0,\mathscr{T}]$, $\mathscr{I}^{\varsigma_i}$-is Ψ-Riemann–Liouville fractional integral of order $\varsigma_i>0$ and $0=\mathfrak{s}_0<\mathfrak{t}_1\le\mathfrak{t}_2<\dots<\mathfrak{t}_{\mathfrak{m}}\le\mathfrak{s}_{\mathfrak{m}}\le\mathfrak{s}_{\mathfrak{m}+1}=\mathscr{T}$,- pre-fixed, $\mathscr{F}:[0,\mathscr{T}]\times\mathbb{R}\times\mathbb{R}\longrightarrow\mathbb{R}$ and $\mathscr{H}_{\mathfrak{i}}:[\mathfrak{t}_{\mathfrak{i}},\mathfrak{s}_{\mathfrak{i}}]\times\mathbb{R}\longrightarrow\mathbb{R}$ is continuous. In addition, $\Psi\wp(\mathfrak{t}) = \int_0^{\mathfrak{t}}\mathfrak{k}(\mathfrak{t},\mathfrak{s})\wp(\mathfrak{s})\partial\mathfrak{s}$ and $\mathfrak{k}\in\mathscr{C}(\mathtt{D},\mathbb{R}^+)$ with domain $\mathtt{D}=\{(\mathfrak{t},\mathfrak{s})\in\mathbb{R}^2: 0\le\mathfrak{s}\le\mathfrak{t}\le\mathscr{T}\}$.

Motivations:

1. The principal motivation for this article is to introduce a new class of N-InI Ψ-HFI-DEs with multipoint BCs by using the Ψ-Hilfer FD.
2. Krasnoselkii's and Banach's fixed point theorem (FPT) are used to investigate the existence and uniqueness of solutions of (1)–(3).
3. We extend the results in [39] by including Ψ-Hilfer FD, nonlinear integral terms, and N-InI conditions.

The organization of the article is as follows: In Section 2, some essential notations, definitions, and some useful lemmas are provided. In Section 3, we used the suitable conditions for the existence and uniqueness of the solution of (1)–(3). Section 4 focuses on an application to illustrate the results.

2. Supporting Notes

Let $\mathscr{PC}([0,\mathscr{T}],\mathbb{R}) = \{\wp : [0,\mathscr{T}] \to \mathbb{R} : \wp \in \mathscr{C}(\mathfrak{t}_{\mathfrak{k}}, \mathfrak{t}_{\mathfrak{k}+1}], \mathbb{R}\}$ be the space of continuous functions and there exists $\wp(\mathfrak{t}_{\mathfrak{k}}^{-})$ and $\wp(\mathfrak{t}_{\mathfrak{k}}^{+})$ with $\wp(\mathfrak{t}_{\mathfrak{k}}^{-}) = \wp(\mathfrak{t}_{\mathfrak{k}}^{+})$ the norm $\|\wp\|_{\mathscr{PC}} = \sup\{|\wp(\mathfrak{t})| : 0 \leq \mathfrak{t} \leq \mathscr{T}\}$. Clearly, $\mathscr{PC}([0,\mathscr{T}],\mathbb{R})$ endued with norm $\|.\|_{\mathscr{PC}}$. See [40] for the notion of the Riemann–Liouville fractional integral and derivative of order $\mathfrak{p} > 0$.

Definition 1 ([41]). *The fractional integrals and FDs for a function $\mathscr{F}$'s with regard to Ψ are defined as:*

$$\mathfrak{I}^{\mathfrak{p};\Psi}\mathscr{F}(\mathfrak{u}) = \frac{1}{\Gamma(\mathfrak{p})}\int_0^{\mathfrak{u}} \Psi'(\mathfrak{s})(\Psi(\mathfrak{u}) - \Psi(\mathfrak{s}))^{\mathfrak{p}-1}\mathscr{F}(\mathfrak{s})\,d\mathfrak{s},$$

and

$$\mathfrak{D}^{\mathfrak{p};\Psi}\mathscr{F}(\mathfrak{u}) = \frac{1}{\Gamma(\mathfrak{n}-\mathfrak{p})}\Big(\frac{1}{\Psi'(\mathfrak{u})}\frac{\partial}{\partial\mathfrak{u}}\Big)^{\mathfrak{n}}\int_0^{\mathfrak{u}} \Psi'(\mathfrak{s})(\Psi(\mathfrak{u}) - \Psi(\mathfrak{s}))^{\mathfrak{n}-\mathfrak{p}-1}\mathscr{F}(\mathfrak{s})\partial\mathfrak{s},$$

respectively.

Definition 2 ([2]). *For $\mathfrak{n} \in \mathbb{N}$, let $\mathfrak{n}-1 < \mathfrak{p} < \mathfrak{n}$ and $\mathscr{F} \in \mathscr{PC}([\mathfrak{a},\mathfrak{b}],\mathbb{R})$. The Ψ-Hilfer FD of order $\mathfrak{p}$ and type $0 \leq \mathfrak{q} \leq 1$ for a function $\mathscr{F}$ ${}^{\mathfrak{H}}\mathcal{D}^{\mathfrak{p},\mathfrak{q};\Psi}(.)$ is defined as*

$${}^{\mathfrak{H}}\mathcal{D}^{\mathfrak{p},\mathfrak{q};\Psi}\mathscr{F}(\mathfrak{t}) = \mathscr{I}^{\mathfrak{q}(\mathfrak{n}-\mathfrak{p};\Psi}\left(\frac{1}{\Psi'(\mathfrak{t})}\frac{\partial}{\partial\mathfrak{t}}\right)^{\mathfrak{n}}\mathscr{I}^{(1-\mathfrak{q})(\mathfrak{n}-\mathfrak{p};\Psi}\mathscr{F}(\mathfrak{t}).$$

Lemma 1 ([2]). *Assume $\mathfrak{p}, \iota, \delta > 0$. Then,*

$$(i)\mathscr{I}^{\mathfrak{p};\Psi}\mathscr{I}^{\iota;\Psi}\mathfrak{h}(\mathfrak{t}) = \mathscr{I}^{\mathfrak{p}+\iota;\Psi}\mathfrak{h}(\mathfrak{t}),$$
$$(ii)\mathscr{I}^{\mathfrak{p};\Psi}(\Psi(\mathfrak{t}) - \Psi(0))^{\delta-1} = \frac{\Gamma(\delta)}{\Gamma(\mathfrak{p}+\delta)}(\Psi(\mathfrak{t}) - \Psi(0))^{\mathfrak{p}+\delta-1}.$$

Note: ${}^{\mathfrak{H}}\mathcal{D}^{\mathfrak{p},\mathfrak{q};\Psi}(\Psi(\mathfrak{t}) - \Psi(0))^{\gamma-1} = 0$.

Lemma 2 ([2]). *Suppose $\mathfrak{n} \in \mathbb{N}$, let $\mathscr{F} \in \mathscr{L}(\mathfrak{a},\mathfrak{b}), \mathfrak{n}-1 < \mathfrak{p} \leq \mathfrak{n}, 0 \leq \mathfrak{q} \leq 1, \gamma = \mathfrak{p} + \mathfrak{n}\mathfrak{q} - \mathfrak{p}\mathfrak{q}$, $\mathscr{I}^{(\mathfrak{n}-\mathfrak{p})(1-\mathfrak{q})}\mathscr{F} \in \mathscr{AC}^{\mathfrak{k}}[\mathfrak{a},\mathfrak{b}]$. Then*

$$(\mathscr{I}^{\mathfrak{p};\Psi};\Psi^{\mathfrak{H}}\mathcal{D}^{\mathfrak{p},\mathfrak{q};\Psi}\mathscr{F})(\mathfrak{t}) = \mathscr{F}(\mathfrak{t}) - \sum_{\mathfrak{k}=1}^{\mathfrak{n}} -\frac{(\Psi(\mathfrak{t}) - \Psi(0)}{\Gamma(\gamma-\mathfrak{k}+1)}\mathscr{F}_{\Psi}^{[\mathfrak{n}-\mathfrak{k}]} lim_{\mathfrak{t}\longrightarrow\mathfrak{a}^+}(\mathscr{I}^{(\mathfrak{n}-\mathfrak{p})(1-\mathfrak{q});\Psi}\mathscr{F})(\mathfrak{t}),$$

where $\mathscr{F}_{\Psi}^{[\mathfrak{n}-\mathfrak{k}]} = \left(\frac{1}{\Psi'(\mathfrak{t})}\frac{\partial}{\partial\mathfrak{t}}\right)^{\mathfrak{n}-\mathfrak{k}}\mathscr{F}(\mathfrak{t})$.

Lemma 3. *A function $\mathscr{PC}([0,\mathscr{T}],\mathbb{R})$ given by,*
$\wp(\mathfrak{t}) =$

$$\begin{cases} \mathscr{H}_{\mathfrak{i}}(\mathfrak{s}_{\mathfrak{m}}) + \frac{1}{\Gamma(\mathfrak{p})}\int_{\mathfrak{a}}^{\mathfrak{t}} \Psi'(\mathfrak{s})(\Psi(\mathfrak{t}) - \Psi(\mathfrak{s}))^{\mathfrak{p}-1}\omega(\mathfrak{s})\partial\mathfrak{s} \\ +\frac{(\Psi(\mathfrak{t})-\Psi(0))^{\gamma-1}}{\Delta\Gamma(\mathfrak{p})}\left[\sum_{i=1}^{\mathfrak{m}} \nu_{\mathfrak{i}}\int_0^{\upsilon_i} \Psi'(\mathfrak{t})(\Psi(\upsilon_i) - \Psi(\mathfrak{s}))^{\mathfrak{p}-1}\omega(\mathfrak{s})\partial\mathfrak{s}\right], \quad \mathfrak{t} \in [0,\mathfrak{t}_1], \\ \mathscr{H}_{\mathfrak{i}}(\mathfrak{t}), \qquad \mathfrak{t} \in (\mathfrak{t}_{\mathfrak{i}}, \mathfrak{s}_{\mathfrak{i}}], \mathfrak{i} = 1,2,\ldots,\mathfrak{m}, \\ \mathscr{H}_{\mathfrak{i}}(\mathfrak{s}_{\mathfrak{i}}) + \frac{1}{\Gamma(\mathfrak{p})}\int_0^{\mathfrak{t}} \Psi'(\mathfrak{s})(\Psi(\mathfrak{t}) - \Psi(\mathfrak{s}))^{\mathfrak{p}-1}\omega(\mathfrak{s})\partial\mathfrak{s} \\ -\frac{1}{\Gamma(\mathfrak{p})}\int_0^{\mathfrak{s}_{\mathfrak{i}}} \Psi'(\mathfrak{s})(\Psi\mathfrak{s}_{\mathfrak{i}} - \Psi\mathfrak{s})^{\mathfrak{p}-1}\omega(\mathfrak{s})\partial\mathfrak{s}, \quad \mathfrak{t} \in (\mathfrak{s}_{\mathfrak{i}}, \mathfrak{t}_{\mathfrak{i}+1}], \mathfrak{i} = 1,2,\ldots,\mathfrak{m}. \end{cases} \tag{4}$$

is a solution of the system

$$
\begin{aligned}
&{}^{\mathfrak{H}}\mathcal{D}^{\mathfrak{p},\mathfrak{q};\Psi}\wp(\mathfrak{t}) = \omega(\mathfrak{t}) \;\; \mathfrak{t} \in (\mathfrak{s}_{\mathfrak{i}}, \mathfrak{t}_{\mathfrak{i}+1}] \subset [0, \mathscr{T}], 0 < \mathfrak{p} < 1,\\
&\wp(\mathfrak{t}) = \mathscr{H}_{\mathfrak{i}}(\mathfrak{t}), \;\; \mathfrak{t} \in (\mathfrak{t}_{\mathfrak{i}}, \mathfrak{s}_{\mathfrak{i}}], \quad \mathfrak{i} = 1, \ldots, \mathfrak{m},\\
&\wp(0) = 0, \;\; \wp(\mathscr{T}) = \sum_{\mathfrak{i}=1}^{\mathfrak{m}} \nu_{\mathfrak{i}} \mathscr{I}^{\varsigma_i} \wp(v_{\mathfrak{i}}).
\end{aligned} \tag{5}
$$

Proof. Assume that $\wp(\mathfrak{t})$ is satisfies for Equation (4). Integrating the first equation of (4) for $\mathfrak{t} \in [0, \mathfrak{t}_1]$, to obtain

$$\wp(\mathfrak{t}) = \wp(\mathscr{T}) + \frac{1}{\Gamma(\mathfrak{p})} \int_0^{\mathfrak{t}} \Psi'(\mathfrak{s})(\Psi(\mathfrak{t}) - \Psi(\mathfrak{s}))^{\mathfrak{p}-1} \omega(\mathfrak{s}) \mathfrak{d}\mathfrak{s}. \tag{6}$$

Now, if $\mathfrak{t} \in (\mathfrak{s}_{\mathfrak{i}}, \mathfrak{t}_{\mathfrak{i}+1}], \mathfrak{i} = 1, 2, \ldots, \mathfrak{m}$ and again integrating the first equation of (4), we have

$$\wp(\mathfrak{t}) = \wp(\mathfrak{s}_{\mathfrak{i}}) + \frac{1}{\Gamma(\mathfrak{p})} \int_{\mathfrak{s}_{\mathfrak{i}}}^{\mathfrak{t}} \Psi'(\mathfrak{s})(\Psi(\mathfrak{t}) - \Psi(\mathfrak{s}))^{\mathfrak{p}-1} \omega(\mathfrak{s}) \mathfrak{d}\mathfrak{s}. \tag{7}$$

Now, we apply impulsive condition, $\wp(\mathfrak{t}) = \mathscr{H}_{\mathfrak{i}}(\mathfrak{t}), \; \mathfrak{t} \in (\mathfrak{t}_{\mathfrak{i}}, \mathfrak{s}_{\mathfrak{i}}]$, we obtain

$$\wp(\mathfrak{s}_{\mathfrak{i}}) = \mathscr{H}_{\mathfrak{i}}(\mathfrak{s}_{\mathfrak{i}}). \tag{8}$$

Consequently, from (7) and (8), we obtain

$$\wp(\mathfrak{t}) = \mathscr{H}_{\mathfrak{i}}(\mathfrak{s}_{\mathfrak{i}}) + \frac{1}{\Gamma(\mathfrak{p})} \int_0^{\mathfrak{t}} \Psi'(\mathfrak{s})(\Psi(\mathfrak{t}) - \Psi(\mathfrak{s}))^{\mathfrak{p}-1} \omega(\mathfrak{s}) \mathfrak{d}\mathfrak{s}. \tag{9}$$

and

$$
\begin{aligned}
\wp(\mathfrak{t}) = &\mathscr{H}_{\mathfrak{i}}(\mathfrak{s}_{\mathfrak{i}}) + \frac{1}{\Gamma(\mathfrak{p})} \int_0^{\mathfrak{t}} \Psi'(\mathfrak{s})(\Psi(\mathfrak{t}) - \Psi(\mathfrak{s}))^{\mathfrak{p}-1} \omega(\mathfrak{s}) \mathfrak{d}\mathfrak{s}\\
&- \frac{1}{\Gamma(\mathfrak{p})} \int_0^{\mathfrak{s}_i} (\Psi'(\mathfrak{s})\Psi\mathfrak{s}_{\mathfrak{i}} - \Psi\mathfrak{s})^{\mathfrak{p}-1} \omega(\mathfrak{s}) \mathfrak{d}\mathfrak{s}.
\end{aligned} \tag{10}
$$

Now, we prove that $\wp$ satisfies the BCs (4). Obviously $\wp(0) = 0$.

$$\sum_{\mathfrak{i}=1}^{\mathfrak{m}} \nu_{\mathfrak{i}} \mathfrak{J}^{\varphi_i} \wp(v_{\mathfrak{i}}) \tag{11}$$

$$
\begin{aligned}
&= \sum_{i=1}^{\mathfrak{m}} \nu_{\mathfrak{i}} \mathfrak{J}^{\alpha+\varphi_i} \omega(v_{\mathfrak{i}}) + \sum_{i=1}^{\mathfrak{m}} \nu_{\mathfrak{i}} \frac{(\Psi(\mathfrak{t}) - \Psi(0))^{\mathfrak{p}-1}}{\Delta\Gamma(\gamma)} \Big[\sum_{i=1}^{\mathfrak{m}} \nu_{\mathfrak{i}} \mathfrak{J}^{\mathfrak{p}+\varphi_i;\Psi} \omega(v_{\mathfrak{i}}) - \mathfrak{J}^{\alpha;\Psi} \omega(\mathfrak{b}) \Big]\\
&= \mathfrak{J}^{\mathfrak{p};\Psi} \omega(\mathscr{T}) + \frac{(\Psi(\mathfrak{t}) - \Psi(0))^{\gamma-1}}{\Delta} \Big[\sum_{i=1}^{\mathfrak{m}} \nu_{\mathfrak{i}} \mathfrak{J}^{\mathfrak{p}+\varphi_i;\Psi} \omega(v_{\mathfrak{i}}) \Big]\\
&= \wp(\mathscr{T})
\end{aligned} \tag{12}
$$

Hence, by using the FDs, integral definitions, and Lemmas. Now it's clear that (6),(10), and (11) $\Rightarrow$ (4).
where

$$\Delta = (\Psi(\mathfrak{t}) - \Psi(0))^{\gamma-1} \sum_{i=1}^{\mathfrak{m}} \nu_{\mathfrak{i}} (\Psi(v_i) - \Psi(0))^{\gamma-1} \neq 0.$$

□

FPT plays a crucial role in many interesting results see, e.g., [12,13,42].

Theorem 4 ([43]). *(Banach FPT)*
If $\mathscr{C}$ is a closed nonempty subset of a Banach space (BSp.) $\mathbb{B}$. Let $\mathscr{N}:\mathscr{C}\to\mathscr{C}$, be a contraction mapping, then $\mathscr{N}$ has a unique FP.

Theorem 5 ([44]). *(Krasnoselkii's FPT)*
Let $\mathfrak{K}$ be a closed, convex, and nonempty subset of a BSp. $\mathfrak{X}$. Suppose $\mathscr{Q},\mathscr{R}$ are two operators satisfying:
(i) $\mathscr{Q}x+\mathscr{R}y\in\mathfrak{K}$ for any $x,y\in\mathfrak{K}$.
(ii)$\mathscr{Q}$ is completely continuous and contraction operator.
(iii) $\mathscr{R}$ is a contraction mapping. Then $\exists$ at least one FP $z_1\in\mathfrak{K}$: $z_1=\mathscr{Q}z_1+\mathscr{R}z_1$.

3. Main Results

We use this section to present our results. We employ two known FPT to investigate the existence and uniqueness of solutions of (1)–(3).

Theorem 6. *Assume the following assumption holds.*
($\mathcal{A}l_1$): $\exists$ positive constants $\mathscr{L},\mathscr{G},\mathscr{M},\mathscr{L}_{\mathfrak{h}_\mathfrak{i}}$:

$$|\mathscr{F}(\mathfrak{t},\wp_1,\omega_1)-\mathscr{F}(\mathfrak{t},\wp_2,\omega_2)|\le\mathscr{L}|\wp_1-\wp_2|+\mathscr{G}|\omega_1-\omega_2|,\ \textit{for}\ \mathfrak{t}\in[0,\mathscr{T}],$$
$$\wp_1,\wp_2,\omega_1,\omega_2\in\mathbb{R}.$$
$$|\mathfrak{k}(\mathfrak{t},\mathfrak{s},\vartheta)-\mathfrak{k}(\mathfrak{t},\mathfrak{s},\nu)|\le\mathscr{M}|\vartheta-\nu|,\ \textit{for}\ \mathfrak{t}\in[\mathfrak{t}_\mathfrak{i},\mathfrak{s}_\mathfrak{i}]\ \vartheta,\nu\in\mathbb{R}.$$
$$|\mathscr{H}_\mathfrak{i}(\mathfrak{t},\mathfrak{v}_1)-\mathscr{H}_\mathfrak{i}(\mathfrak{t},\mathfrak{v}_2)|\le\mathscr{L}_{\mathfrak{h}_\mathfrak{i}}|\mathfrak{v}_1-\mathfrak{v}_2|,\ \textit{for}\ \mathfrak{v}_1,\mathfrak{v}_2\in\mathbb{R}.$$

If

$$\begin{aligned}&\mathscr{Z}:\max\Big\{\max_{\mathfrak{i}=1,2,\dots,\mathfrak{m}}\mathscr{L}_{\mathfrak{h}_\mathfrak{i}}+\frac{(\mathscr{L}+\mathscr{G}\mathscr{M})}{\Gamma(\mathfrak{p}+1)}(\mathfrak{t}_{\mathfrak{i}+1}^{\mathfrak{p}}+\mathfrak{s}_\mathfrak{i}^{\mathfrak{p}}),\\&\mathscr{L}_{\mathfrak{h}_\mathfrak{i}}+(\mathscr{L}+\mathscr{G}\mathscr{M})\Big\{\frac{(\Psi(\mathfrak{t})-\Psi(0))^{\mathfrak{p}}}{\Gamma(\mathfrak{p}+1)}+\frac{(\Psi(\mathfrak{t})-\Psi(0))^{\gamma-1}}{|\Delta|\Gamma(\gamma)}\\&\Big[\sum_{i=1}^{\mathfrak{m}}|\nu_i|\frac{(\Psi(v_i)-\Psi(0))^{\mathfrak{p}+\varphi_i;\Psi}}{\Gamma(\mathfrak{p}+\varphi_i+1)}\Big]\Big\}\Big\}<1,\end{aligned}\tag{13}$$

then the problem (1)–(3) has a unique solution on $[0,\mathscr{T}]$.

Proof. Expound the operator $\mathscr{N}:\mathscr{PC}([0,\mathscr{T}],\mathbb{R})\longrightarrow\mathscr{PC}([0,\mathscr{T}],\mathbb{R})$ by

$$(\mathscr{N}\wp)(\mathfrak{t})=\begin{cases}\mathscr{H}_\mathfrak{m}(\mathfrak{s}_\mathfrak{m},\wp(\mathfrak{s}_\mathfrak{m}))+\frac{1}{\Gamma(\mathfrak{p})}\int_0^{\mathfrak{t}}\Psi'(\mathfrak{s})(\Psi(\mathfrak{t})-\Psi(\mathfrak{s}))^{\mathfrak{p}-1}\mathscr{F}(\mathfrak{s},\wp(\mathfrak{s}),\mathscr{B}\wp(\mathfrak{s}))\partial\mathfrak{s}\\+\frac{(\Psi(\mathfrak{t})-\Psi(0))^{\gamma-1}}{\Delta}\Big[\sum_{i=1}^{\mathfrak{m}}\nu_\mathfrak{i}\int_0^{v_i}\Psi'(\mathfrak{t})(\Psi(v_i)-\Psi(\mathfrak{s}))^{\mathfrak{p}-1}\mathscr{F}(v_i,\wp(v_i),\mathscr{B}\wp(v_i)\Big],\mathfrak{t}\in[0,\mathfrak{t}_1],\\\mathscr{H}_\mathfrak{i}(\mathfrak{t}),\quad\mathfrak{t}\in(\mathfrak{t}_\mathfrak{i},\mathfrak{s}_\mathfrak{i}],\mathfrak{i}=1,2,\dots,\mathfrak{m},\\\mathscr{H}_\mathfrak{i}(\mathfrak{s}_\mathfrak{i})+\frac{1}{\Gamma(\mathfrak{p})}\int_0^{\mathfrak{t}}\Psi'(\mathfrak{s})(\Psi(\mathfrak{t})-\Psi(\mathfrak{s}))^{\mathfrak{p}-1}\mathscr{F}(\mathfrak{s},\wp(\mathfrak{s}),\mathscr{B}\wp(\mathfrak{s}))\partial\mathfrak{s}\\-\frac{1}{\Gamma(\mathfrak{p})}\int_0^{\mathfrak{s}_\mathfrak{i}}\Psi'(\mathfrak{s})(\Psi\mathfrak{s}_\mathfrak{i}-\Psi\mathfrak{s})^{\mathfrak{p}-1}\mathscr{F}(\mathfrak{s},\wp(\mathfrak{s}),\mathscr{B}\wp(\mathfrak{s}))\partial\mathfrak{s},\ \ \mathfrak{t}\in(\mathfrak{s}_\mathfrak{i},\mathfrak{t}_{\mathfrak{i}+1}],\mathfrak{i}=1,2,\dots,\mathfrak{m}.\end{cases}$$

Clearly $\mathscr{N}$ is well defined and $\mathscr{N}\wp\in\mathscr{PC}([0,\mathscr{T}],\mathbb{R})$. We now prove that $\mathscr{N}$ is a contraction.
Case:1. When $\wp,\overline{\wp}\in\mathscr{PC}([0,\mathscr{T}],\mathbb{R})$ and $\mathfrak{t}\in[0,\mathfrak{t}_1]$, we obtain

$$\begin{aligned}&|(\mathscr{N}\wp)(\mathfrak{t})-(\mathscr{N}\overline{\wp})(\mathfrak{t})|\\&\le\mathscr{L}_{\mathfrak{h}_\mathfrak{i}}+(\mathscr{L}+\mathscr{G}\mathscr{M})\Big\{\frac{(\Psi(\mathfrak{t})-\Psi(0))^{\mathfrak{p}}}{\Gamma(\mathfrak{p}+1)}+\frac{(\Psi(\mathfrak{t})-\Psi(0))^{\gamma-1}}{|\Delta|\Gamma(\gamma)}\\&\Big[\sum_{i=1}^{\mathfrak{m}}|\nu_i|\frac{(\Psi(v_i)-\Psi(0))^{\mathfrak{p}+\varphi_i;\Psi}}{\Gamma(\mathfrak{p}+\varphi_i+1)}\Big]\Big\}\|\wp-\overline{\wp}\|_{\mathscr{PC}}.\end{aligned}$$

Case:2. When $\mathfrak{t} \in (\mathfrak{t}_i, \mathfrak{s}_i]$, we obtain

$$\begin{aligned} |(\mathcal{N}\wp)(\mathfrak{t}) - (\mathcal{N}\overline{\wp})(\mathfrak{t})| &\leq |\mathscr{H}_i(\mathfrak{t}, \wp(\mathfrak{t})) - \mathscr{H}_i(\mathfrak{t}, \overline{\wp}(\mathfrak{t}))| \\ &\leq \mathscr{L}_{\mathfrak{h}_i} \|\wp - \overline{\wp}\|_{\mathscr{PC}}. \end{aligned}$$

Case:3. When $\mathfrak{t} \in (\mathfrak{s}_i, \mathfrak{t}_{i+1}]$, we obtain

$$\begin{aligned} &|(\mathcal{N}\wp)(\mathfrak{t}) - (\mathcal{N}\overline{\wp})(\mathfrak{t})| \\ &\leq |\mathscr{H}_i(\mathfrak{s}_i, \wp(\mathfrak{s}_i) - \mathscr{H}_i(\mathfrak{s}_i, \overline{\wp}(\mathfrak{s}_i))| + \frac{1}{\Gamma(\mathfrak{p})} \int_0^{\mathfrak{t}} (\mathfrak{t} - \mathfrak{s})^{\mathfrak{p}-1} |\mathscr{F}(\mathfrak{s}, \wp(\mathfrak{s}), \mathscr{B}\wp(\mathfrak{s})) - \mathscr{F}(\mathfrak{s}, \wp(\mathfrak{s}), \mathscr{B}\wp(\mathfrak{s}))| \partial\mathfrak{s} \\ &+ \frac{1}{\Gamma(\mathfrak{p})} \int_0^{\mathfrak{s}_i} (\mathfrak{s}_i - \mathfrak{s})^{\mathfrak{p}-1} |\mathscr{F}(\mathfrak{s}, \wp(\mathfrak{s}), \mathscr{B}\wp(\mathfrak{s})) - \mathscr{F}(\mathfrak{s}, \wp(\mathfrak{s}), \mathscr{B}\wp(\mathfrak{s}))| \partial\mathfrak{s}, \\ &\leq \left[\mathscr{L}_{\mathfrak{h}_i} + \frac{(\mathscr{L} + \mathscr{G}\mathscr{M})}{\Gamma(\mathfrak{p}+1)} (\mathfrak{t}_{i+1}^{\mathfrak{p}} + \mathfrak{s}_i^{\mathfrak{p}}) \right] \|\wp - \overline{\wp}\|_{\mathscr{PC}}. \end{aligned}$$

Therefore $\mathcal{N}$ is a contraction as in the above inequality

$$\mathscr{Z} = \left[\mathscr{L}_{\mathfrak{h}_i} + \frac{(\mathscr{L} + \mathscr{G}\mathscr{M})}{\Gamma(\mathfrak{p}+1)} (\mathfrak{t}_{i+1}^{\mathfrak{p}} + \mathfrak{s}_i^{\mathfrak{p}}) \right] < 1.$$

Thus, the (1)–(3) problem has a unique solution $\forall\ \wp \in \mathscr{PC}([0, \mathscr{T}], \mathbb{R})$. □

Theorem 7. *Let condition* $(\mathtt{Al}_1)$ *be satisfied and the following assumption holds:*
$(\mathtt{Al}_2)$*: $\exists$ a constant* $\mathscr{L}_{g_i} > 0$*:*

$$|\mathscr{F}(t, W_1, \omega_1)| \leq \mathscr{L}_{g_i}(1 + |W_1| + |\omega_1|), \ \ \mathfrak{t} \in [\mathfrak{s}_i, \mathfrak{t}_{i+1}], \forall\ W_1, \omega_1 \in \mathbb{R}.$$

$(\mathtt{Al}_3)$*: $\exists$ a function* $\kappa_i(\mathfrak{t})$*,* $i = 1, 2, \ldots, m$*:*

$$|\mathscr{H}_i(t, W_1, \omega_1)| \leq \kappa_i(\mathfrak{t}), \ \ \mathfrak{t} \in [\mathfrak{t}_i, \mathfrak{s}_i], \forall\ \ W_1, \omega_1 \in \mathbb{R}.$$

Assume that $\mathscr{M}_i : \sup_{\mathfrak{t} \in [\mathfrak{t}_i, \mathfrak{s}_i]} \kappa_i(\mathfrak{t}) < \infty$*, and* $\mathscr{K} := \max \mathscr{L}_{h_i} < 1$*, for all* $i = 1, 2, .., m$*. Then the (1)–(3) problem has at least one solution on* $[0, \mathscr{T}]$*.*

Proof. Suppose that $\mathscr{B}_{\mathfrak{p},\mathfrak{r}} = \{\wp \in \mathscr{PC}([0, \mathscr{T}], \mathbb{R}) : \|\wp\|_{\mathscr{PC}} \leq \mathfrak{r}\}$. Let $\mathscr{Q}$ and $\mathscr{R}$ be two operators on $\mathscr{B}_{\mathfrak{p},\mathfrak{r}}$ defined as follows:

$$\mathscr{Q}\wp(\mathfrak{t}) = \begin{cases} \mathscr{H}_{\mathfrak{m}}(\mathfrak{s}_{\mathfrak{m}}, \wp(\mathfrak{s}_{\mathfrak{m}})), & \mathfrak{t} \in [0, \mathfrak{t}_1], \\ \mathscr{H}_i(\mathfrak{t}, \wp(\mathfrak{t})), & \mathfrak{t} \in (\mathfrak{t}_i, \mathfrak{s}_i], i = 1, 2, \ldots, \mathfrak{m}, \\ \mathscr{H}_i(\mathfrak{s}_i, \wp(\mathfrak{s}_i)), & \mathfrak{t} \in (\mathfrak{s}_i, \mathfrak{t}_{i+1}], i = 1, 2, \ldots, \mathfrak{m}. \end{cases}$$

and

$$\mathscr{R}\wp(\mathfrak{t}) = \begin{cases} \frac{1}{\Gamma(\mathfrak{p})} \int_{\mathfrak{a}}^{\mathfrak{t}} \Psi'(\mathfrak{s})(\Psi(\mathfrak{t}) - \Psi(\mathfrak{s}))^{\mathfrak{p}-1} \mathscr{F}(\mathfrak{s}, \wp(\mathfrak{s}), \mathscr{B}\wp(\mathfrak{s})) \partial\mathfrak{s} \\ + \frac{(\Psi(\mathfrak{t}) - \Psi(0))^{\gamma-1}}{\Delta} \left[\sum_{i=1}^{\mathfrak{m}} \nu_i \int_0^{v_i} \Psi'(\mathfrak{t})(\Psi(v_i) - \Psi(\mathfrak{s}))^{\mathfrak{p}-1} \mathscr{F}(v_i, \wp(v_i), \mathscr{B}\wp(v_i)) \right], \mathfrak{t} \in [0, \mathfrak{t}_1], \\ 0, \mathfrak{t} \in (\mathfrak{t}_i, \mathfrak{s}_i], i = 1, 2, \ldots, \mathfrak{m}, \\ \frac{1}{\Gamma(\mathfrak{p})} \int_0^{\mathfrak{t}} \Psi'(\mathfrak{s})(\Psi(\mathfrak{t}) - \Psi(\mathfrak{s}))^{\mathfrak{p}-1} \mathscr{F}(\mathfrak{s}, \wp(\mathfrak{s}), \mathscr{B}\wp(\mathfrak{s})) \partial\mathfrak{s} \\ - \frac{1}{\Gamma(\mathfrak{p})} \int_0^{\mathfrak{s}_i} \Psi'(\mathfrak{s})(\Psi\mathfrak{s}_{(i)} - \Psi(\mathfrak{s}))^{\mathfrak{p}-1} \mathscr{F}(\mathfrak{s}, \wp(\mathfrak{s}), \mathscr{B}\wp(\mathfrak{s})) \partial\mathfrak{s}, \ \ \mathfrak{t} \in (\mathfrak{s}_i, \mathfrak{t}_{i+1}], i = 1, 2, \ldots, \mathfrak{m}. \end{cases}$$

step.1 For $\wp \in \mathscr{B}_{\mathfrak{p},\mathfrak{r}}$ then $\mathscr{Q}\wp + \mathscr{R}\wp \in \mathscr{B}_{\mathfrak{p},\mathfrak{r}}$.
case.1 For $\mathfrak{t} \in [0, \mathfrak{t}_1]$,

$$
\begin{aligned}
|\mathscr{Q}\wp+\mathscr{R}\overline{\wp}| \leq{}& |\mathscr{H}_{\mathfrak{m}}(\mathfrak{s}_{\mathfrak{m}},\wp(\mathfrak{s}_{\mathfrak{m}}))|+\frac{1}{\Gamma(\mathfrak{p})}\int_0^{\mathfrak{t}}(\mathfrak{t}-\mathfrak{s})^{\mathfrak{p}-1}|\mathscr{F}(\mathfrak{s},\wp(\mathfrak{s}),\mathscr{B}\wp(\mathfrak{s}))|\mathfrak{d}\mathfrak{s}\\
&+\frac{(\Psi(\mathfrak{t})-\Psi(0))^{\gamma-1}}{\Delta}\Big[\sum_{i=1}^{\mathfrak{m}}\nu_{\mathfrak{i}}\int_0^{v_i}\Psi'(\mathfrak{t})(\Psi(v_i)-\Psi(\mathfrak{s}))^{\mathfrak{p}-1}\mathscr{F}(v_i,\wp(v_i),\mathscr{B}\wp(v_i)\mathfrak{d}v_i\Big],\\
\leq{}& \Big[\mathscr{L}_{\mathfrak{h}_{\mathfrak{i}}}+(\mathscr{L}+\mathscr{G}\mathscr{M})\Big\{\frac{(\Psi(\mathfrak{t})-\Psi(0))^{\mathfrak{p}}}{\Gamma(\mathfrak{p}+1)}+\frac{(\Psi(\mathfrak{t})-\Psi(0))^{\gamma-1}}{|\Delta|\Gamma(\gamma)}\\
&\Big[\sum_{i=1}^{\mathfrak{m}}|\nu_i|\frac{(\Psi(v_i)-\Psi(0))^{\mathfrak{p}+\varphi_i;\Psi}}{\Gamma(\mathfrak{p}+\varphi_i+1)}\Big]\Big\}\Big](1+\mathfrak{r})\leq\mathfrak{r}.
\end{aligned}
$$

case:2 $\forall\ \mathfrak{t}\in(\mathfrak{t}_{\mathfrak{i}},\mathfrak{s}_{\mathfrak{i}}]$,

$$|\mathscr{Q}\wp+\mathscr{R}\overline{\wp}|\leq|\mathscr{H}_{\mathfrak{i}}(\mathfrak{t},\mathscr{W}_1(\mathfrak{t}))|\leq\mathscr{M}_{\mathfrak{i}}.$$

case:3. $\forall\ \mathfrak{t}\in(\mathfrak{s}_{\mathfrak{i}},\mathfrak{t}_{\mathfrak{i}+1}]$,

$$
\begin{aligned}
|\mathscr{Q}\wp+\mathscr{R}\overline{\wp}(\mathfrak{t})|\leq{}&|\mathscr{H}_{\mathfrak{i}}(\mathfrak{s}_{\mathfrak{i}},\wp(\mathfrak{s}_{\mathfrak{i}}))|+\frac{1}{\Gamma(\mathfrak{p})}\int_0^{\mathfrak{t}}(\mathfrak{t}-\mathfrak{s})^{\mathfrak{p}-1}|\mathscr{F}(\mathfrak{s},\wp(\mathfrak{s}),\mathscr{B}\wp(\mathfrak{s}))|\mathfrak{d}\mathfrak{s}\\
&+\frac{1}{\Gamma(\mathfrak{p})}\int_0^{\mathfrak{s}_{\mathfrak{i}}}(\mathfrak{s}_{\mathfrak{i}}-\mathfrak{s})^{\mathfrak{p}-1}|\mathscr{F}(\mathfrak{s},\wp(\mathfrak{s}),\mathscr{B}\wp(\mathfrak{s}))|\mathfrak{d}\mathfrak{s},\\
&\leq\mathscr{M}_{\mathfrak{i}}+\left[\frac{\mathscr{L}_{\mathfrak{g}_{\mathfrak{i}}}(\mathfrak{s}_{\mathfrak{i}}^{\mathfrak{p}}+\mathfrak{t}_{\mathfrak{i}+1}^{\mathfrak{p}})}{\Gamma(\mathfrak{p}+1)}\right](1+\mathfrak{r})\leq\mathfrak{r}.
\end{aligned}
$$

Thus,

$$\mathscr{Q}\wp+\mathscr{R}\wp\in\mathscr{B}_{\mathfrak{p},\mathfrak{r}}.$$

step:2 $\mathscr{Q}$ is contraction on $\mathscr{B}_{\mathfrak{p},\mathfrak{r}}$.
case:1.$\wp_1,\wp_2\in\mathscr{B}_{\mathfrak{p},\mathfrak{r}}$ then $\mathfrak{t}\in[0,\mathfrak{t}_1]$,

$$|\mathscr{Q}\wp_1(\mathfrak{t})-\mathscr{Q}\wp_2(\mathfrak{t})|\leq\mathscr{L}_{\mathfrak{g}_{\mathfrak{m}}}|\wp_1(\mathfrak{s}_{\mathfrak{m}})-\wp_2(\mathfrak{s}_{\mathfrak{m}})|\leq\mathscr{L}_{\mathfrak{g}_{\mathfrak{m}}}\|\wp_1-\wp_2\|_{\mathscr{P}\mathscr{C}}.$$

case:2. $\forall\ \mathfrak{t}\in(\mathfrak{t}_{\mathfrak{i}},\mathfrak{s}_{\mathfrak{i}}],\mathfrak{i}=1,2,\ldots,\mathfrak{m}$,

$$|\mathscr{Q}\wp_1(\mathfrak{t})-\mathscr{Q}\wp_2(\mathfrak{t})|\leq\mathscr{L}_{\mathfrak{g}_{\mathfrak{i}}}\|\wp_1-\wp_2\|_{\mathscr{P}\mathscr{C}}.$$

case:3. For $\mathfrak{t}\in(\mathfrak{s}_{\mathfrak{i}},\mathfrak{t}_{\mathfrak{i}+1}]$,

$$|\mathscr{Q}\wp_1(\mathfrak{t})-\mathscr{Q}\wp_2(\mathfrak{t})|\leq\mathscr{L}_{\mathfrak{g}_{\mathfrak{i}}}\|\wp_1-\wp_2\|_{\mathscr{P}\mathscr{C}}.$$

We can deduce the following from the above inequalities:

$$|\mathscr{Q}\wp_1(\mathfrak{t})-\mathscr{Q}\wp_2(\mathfrak{t})|\leq\mathscr{K}\|\wp_1-\wp_2\|_{\mathscr{P}\mathscr{C}}.$$

Hence, $\mathscr{Q}$ is a contraction.
step:3. We prove that $\mathscr{R}$ is continuous.
Assume $\wp_{\mathfrak{n}}$ be a $\ni\wp_{\mathfrak{n}}\to\overline{\wp}$ sequence in $\mathscr{P}\mathscr{C}([0,\mathscr{T}],\mathbb{R})$.
case:1. $\forall\ \mathfrak{t}\in[0,\mathfrak{t}_1]$,

$$
\begin{aligned}
|\mathscr{Q}\wp_{\mathfrak{n}}(\mathfrak{t})-\mathscr{Q}\wp(\mathfrak{t})|\leq{}&\left[\frac{(\Psi(\mathfrak{t})-\Psi(0))^{\mathfrak{p}}}{\Gamma(\mathfrak{p}+1)}+\frac{(\Psi(\mathfrak{t})-\Psi(0))^{\gamma-1}}{|\Delta|\Gamma(\gamma)}\quad\Big[\sum_{i=1}^{\mathfrak{m}}|\nu_i|\frac{(\Psi(v_i)-\Psi(0))^{\mathfrak{p}+\varphi_i;\Psi}}{\Gamma(\mathfrak{p}+\varphi_i+1)}\Big]\right]\\
&\|\mathscr{F}(.,\wp_{\mathfrak{n}}(.),.,)-\mathscr{F}(.,\wp(.),.,)\|_{\mathscr{P}\mathscr{C}}.
\end{aligned}
$$

case:2. $\forall\, \mathfrak{t} \in (\mathfrak{t}_i, \mathfrak{s}_i]$, we obtain

$$|\mathscr{Q}\wp_n(\mathfrak{t}) - \mathscr{Q}\wp(\mathfrak{t})| = 0.$$

case:3. $\forall\, \mathfrak{t} \in (\mathfrak{s}_i, \mathfrak{t}_{i+1}], i = 1, 2, \ldots, \mathfrak{m}$,

$$|\mathscr{Q}\wp_n(\mathfrak{t}) - \mathscr{Q}\wp(\mathfrak{t})| \leq \frac{(\mathfrak{t}_{i+1} - \mathfrak{s}_i)}{\Gamma(\mathfrak{p}+1)} \|\mathscr{F}(., \wp_n(.), .,) - \mathscr{F}(., \wp(.), .,)\|_{\mathscr{PC}}.$$

Thus, we conclude from the above cases that $\|\mathscr{Q}\wp_n(\mathfrak{t}) - \mathscr{Q}\wp(\mathfrak{t})\|_{\mathscr{PC}} \longrightarrow 0$ as $\mathfrak{n} \to \infty$.
step:4. We prove that $\mathscr{Q}$ is compact.

First $\mathscr{Q}$ is uniformly bounded on $\mathscr{B}_{\mathfrak{p},\mathfrak{r}}$.

Since $\|\mathscr{Q}\wp\| \leq \frac{\mathscr{L}_{\mathfrak{g}_i}(\mathscr{T})}{\Gamma(1+\mathfrak{p})} < \mathfrak{r}$,

First $\mathscr{Q}$ is uniformly bounded on $\mathscr{B}_{\mathfrak{p},\mathfrak{r}}$. We prove that $\mathscr{Q}$ maps a bounded set to a $\mathscr{B}_{\mathfrak{p},\mathfrak{r}}$ equicontinuous set.
case:1. For interval $\mathfrak{t} \in [0, \mathfrak{t}_1], 0 \leq \mathscr{E}_1 \leq \mathscr{E}_2 \leq \mathfrak{t}_1, \wp \in \mathscr{B}_{\mathfrak{r}}$, we obtain

$$|\mathscr{Q}\mathscr{E}_2 - \mathscr{Q}\mathscr{E}_1| \leq \frac{\mathscr{L}_{\mathfrak{g}_i}(1+\mathfrak{r})}{\Gamma(\mathfrak{p}+1)}(\mathscr{E}_2 - \mathscr{E}_1).$$

case:2. $\forall\, \mathfrak{t} \in (\mathfrak{t}_i, \mathfrak{s}_i], \mathfrak{t}_i < \mathscr{E}_1 < \mathscr{E}_2 \leq \mathfrak{s}_i, \wp \in \mathscr{B}_{\mathfrak{p},\mathfrak{r}}$, we obtain

$$|\mathscr{Q}\mathscr{E}_2 - \mathscr{Q}\mathscr{E}_1| = 0.$$

case:3. $\forall\, \mathfrak{t} \in (\mathfrak{s}_i, \mathfrak{t}_{i+1}], \mathfrak{s}_i < \mathscr{E}_1 < \mathscr{E}_2 \leq \mathfrak{t}_{i+1}, \wp \in \mathscr{B}_{\mathfrak{p},\mathfrak{r}}$, we establish

$$|\mathscr{Q}\mathscr{E}_2 - \mathscr{Q}\mathscr{E}_1| \leq \frac{\mathscr{L}_{\mathfrak{g}_i}(1+\mathfrak{r})}{\Gamma(\mathfrak{p}+1)}(\mathscr{E}_2 - \mathscr{E}_1).$$

From the above cases, we obtain $|\mathscr{Q}\mathscr{E}_2 - \mathscr{Q}\mathscr{E}_1| \longrightarrow 0$ as $\mathscr{E}_2 \longrightarrow \mathscr{E}_1$ and $\mathscr{Q}$ is equicontinuous. As a result, $\mathscr{Q}(\mathscr{B}_{\mathfrak{p},\mathfrak{r}})$ is relatively compact, and $\mathscr{Q}$ is compact using the Ascoli–Arzela theorem (see, e.g., [45]). Hence the (1)–(3) problems have at least one fixed point on $[0, \mathscr{T}]$. □

4. Example

Consider the Ψ-Hlifer fractional BVP,

$$D^{\mathfrak{p},\mathfrak{q};\Psi}\wp(\mathfrak{u}) = \frac{\mathfrak{e}^{-\mathfrak{u}}|\mathfrak{w}|}{9 + \mathfrak{e}^{\mathfrak{u}}(1 + |\wp|)} + \frac{1}{3}\int_0^{\mathfrak{u}} \mathfrak{e}^{-(\mathfrak{s}-\mathfrak{u})}\wp(\mathfrak{s})\mathfrak{d}\mathfrak{s}, \;\; \mathfrak{u} \in (0, 1], \tag{14}$$

$$\wp(\mathfrak{u}) = \frac{|\wp(\mathfrak{u})|}{2(1 + |\wp(\mathfrak{u})|)}, \; \mathfrak{u} \in (\frac{1}{2}, 1], \tag{15}$$

$$\wp(0) = 0, \;\; \wp(1) = \frac{1}{2}\mathscr{I}^{\frac{2}{3}}\wp\left(\frac{2}{7}\right) + \frac{2}{3}\mathscr{I}^{\frac{4}{5}}\wp\left(\frac{5}{9}\right) + \frac{2}{5}\mathscr{I}^{\frac{3}{4}}\wp\left(\frac{1}{7}\right), \tag{16}$$

and $\mathscr{L} = \mathscr{G} = \frac{1}{10}$, $\mathscr{M} = \frac{1}{3}$, $\mathfrak{p} = \frac{5}{7}$ $\gamma = \frac{2}{5}$, $\mathscr{L}_{\mathfrak{h}_1} = \frac{1}{3}$, $\nu_1 = \frac{1}{2}$, $\nu_2 = \frac{2}{3}$, $\nu_3 = \frac{2}{5}$, $v_1 = \frac{2}{7}$, $v_2 = \frac{5}{9}$, $v_3 = \frac{1}{7}$, $\varphi_1 = \frac{2}{3}$, $\varphi_2 = \frac{4}{5}$, $\varphi_3 = \frac{3}{4}$. We shall check that condition (12) of Theorem 6 for $p \in (1,2)$. By using theorem 6, we determine that (with $m = 3$)

$$\mathscr{L}_{\mathfrak{h}_i} + \frac{(\mathscr{L} + \mathscr{G}\mathscr{M})}{\Gamma(\mathfrak{p}+1)}(\mathfrak{u}_{i+1}^{\mathfrak{p}} + \mathfrak{s}_i^{\mathfrak{p}}) \approx 0.41 < 1.$$

and

$$\mathscr{L}_{\mathfrak{h}_i} + (\mathscr{L} + \mathscr{G}\mathscr{M})\Big\{ \frac{(\Psi(\mathfrak{u}) - \Psi(0))^{\mathfrak{p}}}{\Gamma(\mathfrak{p}+1)} + \frac{(\Psi(\mathfrak{u}) - \Psi(0))^{\gamma-1}}{|\Delta|\Gamma(\gamma)} \Big[\sum_{i=1}^{m} |\nu_i| \frac{(\Psi(v_i) - \Psi(0))^{\mathfrak{p}+\varphi_i;\Psi}}{\Gamma(\mathfrak{p}+\varphi_i+1)} \Big] \Big\} \approx 0.485 < 1.$$

Hence, in view of Theorem 6 the problem (13)–(15) has a unique solution $[0, \mathscr{T}]$. This example illustrates the obtained results.

5. Conclusions

In this paper, we discussed a new class of nonlinear $\Psi-$HFI-DE with NInI Conditions. Existence and uniqueness results are established. Banach's FPT is employed to show the uniqueness results, while Krasnoselskii's FPT is used to analyze the existence results. At the end, an example is presented to demonstrate the consistency of the findings. Potential future work could be to develop a numerical algorithm for the R-L IBVPs with different types of FDs. Moreover, we plan to investigate our results based on other FDs such as, e.g., Katugampola derivative, Abu-Shady–Kaabar FD, and conformable derivative.

Author Contributions: Conceptualization, R.A., P.K., K.K., P.G., Y.A., L.A., and E.-s.E.-h.; methodology, R.A., P.K., K.K., and P.G.; software, R.A., P.K., K.K., Y.A., L.A., and E.-s.E.-h.; validation, R.A., P.K., K.K., Y.A., L.A., and E.-s.E.-h.; formal analysis, R.A., P.K., K.K., and P.G.; investigation, R.A., P.K., K.K., Y.A., L.A., and E.-s.E.-h.; data curation, R.A., P.K., K.K., Y.A., L.A., and E.-s.E.-h.; writing—original draft preparation, R.A., P.K., K.K., and P.G.; writing—review and editing, R.A., P.K., K.K., Y.A., L.A., and E.-s.E.-h.; visualization, R.A., P.K., K.K., P.G., Y.A., L.A., and E.-s.E.-h.; supervision, K.K.; project administration, R.A., P.K., K.K., Y.A., L.A., and E.-s.E.-h. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not Applicable.

Informed Consent Statement: Not Applicable.

Data Availability Statement: Not Applicable.

Conflicts of Interest: The authors declare no conflict of interest.

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