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Article

Liouville Type Theorems Involving the Fractional Laplacian on the Upper Half Euclidean Space

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Abstract: In this paper, we mainly establish Liouville-type theorems for the elliptic semi-linear equations involving the fractional Laplacian on the upper half of Euclidean space. We employ a direct approach by studying an equivalent integral equation instead of using the conventional extension method. Applying the method of moving planes in integral forms, we prove the non-existence of positive solutions under very weak conditions. We also extend the results to a more general equation.

Keywords: Liouville type theorems; the fractional Laplacian; method of moving planes; non-existence

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1. Introduction

The fractional Laplacian in $\mathbb{R}^n$ is defined by

$$(-\Delta)^{\alpha/2}u(x) = C_{n,\alpha}PV\int_{\mathbb{R}^n}\frac{u(x)-u(z)}{|x-y|^{n+\alpha}}dz, \tag{1}$$

where $0<\alpha<2$ is any real number, $C_{n,\alpha} = (\int_{\mathbb{R}^n}\frac{1-\cos(2\pi\zeta_1)}{|\zeta|^{n+\alpha}})^{-1}d\zeta$ is a constant and PV stands for the Cauchy principle value. For the detailed definition about $(-\Delta)^{\alpha/2}$ and $C_{n,\alpha}$, we refer to [1]. From (1), one can see that it is a nonlocal operator.

In recent years, the fractional Laplacian has been frequently used to model diverse phenomena, for example, anomalous diffusion and quasi-geostrophic flows, turbulence and water waves, advection-diffusion, relativistic quantum mechanics of stars, molecular dynamics and other problems (see [2–13] and the references therein).

Let

$$\mathbb{R}^n_+ = \{x=(x_1,\cdots,x_n)|x_n>0\}$$

be the upper half of Euclidean space.

In this paper, we mainly establish Liouville-type theorems, the non-existence of positive solutions to the Dirichlet problem for elliptic semi-linear equations

$$\begin{cases} (-\Delta)^{\alpha/2}u(x) = x_n^\gamma u^p(x),\ u(x)>0, & x\in\mathbb{R}^n_+, \\ u(x)\equiv 0, & x\notin\mathbb{R}^n_+, \end{cases} \tag{2}$$

where $0<\alpha<2$, $\gamma\geq 0$ is any real number. And then we generalize the results to some more complicated cases.

Obviously, the operator in (1) is well defined in the Schwartz space $\mathcal{S}$ of rapidly decreasing C^∞ functions in $\mathbb{R}^n$. In this space, it can also be equivalently defined by the Fourier transform:

$$\widehat{(-\Delta)^{\alpha/2}u}(\xi) = |\xi|^\alpha\hat{u}(\xi),$$

where $\hat{u}$ is the Fourier transform of u. This definition can be extended to the distributions in the space:

$$\mathcal{L}_{\alpha/2} = \{u|\int_{\mathbb{R}^n}\frac{|u(x)|}{1+|x|^{n+\alpha}}dx<\infty\}$$

by

$$< (-\Delta)^{\alpha/2}u, \phi >= \int_{\mathbb{R}^n} u(-\Delta)^{\alpha/2}\phi dx, \quad \text{for all } \phi \in C_0^\infty(\mathbb{R}^n_+).$$

Given any $f \in L^1_{loc}(\mathbb{R}^n_+)$, we say $u \in \mathcal{L}_{\alpha/2}$ solves the problem

$$(-\Delta)^{\alpha/2}u = f(x), \qquad x \in \mathbb{R}^n_+,$$

if and only if

$$\int_{\mathbb{R}^n} u(-\Delta)^{\alpha/2}\phi dx = \int_{\mathbb{R}^n} f(x)\phi(x)dx, \quad \text{for all } \phi \in C_0^\infty(\mathbb{R}^n_+). \tag{3}$$

In this paper, we will consider the distributional solutions in the sense of (3).

To apply the method of moving planes in integral forms, we first establish the equivalence between problem (2) and the integral equation:

$$u(x) = \int_{\mathbb{R}^n_+} G_\infty(x,y) y_n^\gamma u^p(y) dy, \tag{4}$$

where

$$G_\infty(x,y) = \frac{A_{n,\alpha}}{s^{(n-\alpha)/2}}\left[1 - \frac{B_{n,\alpha}}{(s+t)^{(n-2)/2}}\int_0^{\frac{s}{t}} \frac{(s-tb)^{(n-2)/2}}{b^{\alpha/2}(1+b)}db\right], \quad x,y \in \mathbb{R}^n_+, \tag{5}$$

is the Green function in $\mathbb{R}^n_+$ with the same Dirichlet condition. Here

$$s = |x-y|^2 \quad \text{while} \quad t = 4x_n y_n.$$

We prove

Theorem 1. *Assume that u is a locally bounded positive solution of Equation (2). Then u is also a solution of integral Equation (4), and vice versa. Here we only require $\gamma > -\alpha$.*

Next, we establish the Liouville-type theorem for the integral equation.

Theorem 2. *Assume $p > \frac{n}{n-\alpha}$. If $u \in L^{\frac{n(p-1)}{\alpha}}(\mathbb{R}^n_+)$ and u is a non-negative solution of integral Equation (4), then $u(x) \equiv 0$.*

By Theorem 1, one can immediately derive the following corollary on Equation (2).

Corollary 1. *Assume $p > \frac{n}{n-\alpha}$. If $u \in L^{\frac{n(p-1)}{\alpha}}(\mathbb{R}^n_+)$ and u is a non-negative solution of Equation (2), then $u(x) \equiv 0$.*

Remark 1. *Note that here the exponent p can be any number greater than $\frac{n}{n-\alpha}$ under the global integrability condition. Hence this non-existence result also includes the supercritical case $p = \frac{n+\alpha}{n-\alpha}$.*

To prove Theorem 2, we apply the method of moving planes in integral forms. We move the plane along the x_n direction and derive that the solution must be monotone increasing in x_n. Further this is in contradiction with $u \in L^{\frac{n(p-1)}{\alpha}}(\mathbb{R}^n_+)$. For more articles concerning applications of the method of moving planes in integral forms, we refer to [14–20] and the references therein.

We next weaken the global integrability condition in Theorem 2 and exploit a Kelvin-type transform. To ensure that the half-space $\mathbb{R}^n_+$ is invariant under such a transform, we need to place the centers on the boundary $\partial\mathbb{R}^n_+$.

We consider

$$\bar{u}_{z^0}(x) = \frac{1}{|x-z^0|^{n-\alpha}} u(\frac{x-z^0}{|x-z^0|^2} + z^0),$$

which is the Kelvin type transform of $u(x)$ centered at z^0. Some new ideas are involved.

In the case of $\frac{n+\alpha}{n-\alpha} \le p \le \frac{n+2\gamma+\alpha}{n-\alpha}$, we only need to assume $u \in L_{loc}^{\frac{n(p-1)}{\alpha}}(\mathbb{R}_+^n)$, and here we consider two possibilities:

(i) There is a point $z^0 \in \partial\mathbb{R}_+^n$, such that $\bar{u}_{z^0}(x)$ is bounded near z^0. In this case, we can derive $u \in L^{\frac{n(p-1)}{\alpha}}(\mathbb{R}_+^n)$, then we can move the planes on u just as we did in the proof of Theorem 2.

(ii) For all $z^0 \in \partial\mathbb{R}_+^n$, $\bar{u}_{z^0}(x)$ is unbounded near z^0. In this case, we move the planes in $x_1, \cdots, x_{n-1}$ directions to show that, for every z^0, $\bar{u}_{z^0}(x)$ is axially symmetric about the line that is parallel to x_n-axis and passing through z^0. This implies that u depends only on x_n.

In the case of $1 < p < \frac{n+\alpha}{n-\alpha}$, we only need to suppose that u is locally bounded and only need to work on $\bar{u}_{z^0}(x)$. Then similar to the above possibility (ii), we show that for every z^0, $\bar{u}_{z^0}(x)$ is axially symmetric about the line that is parallel to x_n-axis and passing through z^0, which again implies that u depends only on x_n.

In both cases, we will be able to derive a contraction and prove the following Theorem.

Theorem 3. *Assume $1 < p \le \frac{n+2\gamma+\alpha}{n-\alpha}$ and $\gamma \ge 0$. If u is a locally bounded non-negative solution of (4), then $u(x) \equiv 0$. In particular, when $\frac{n+\alpha}{n-\alpha} \le p \le \frac{n+2\gamma+\alpha}{n-\alpha}$, we only require $u \in L_{loc}^{\frac{n(p-1)}{\alpha}}(\mathbb{R}_+^n)$.*

Corollary 2. *Assume $1 < p \le \frac{n+2\gamma+\alpha}{n-\alpha}$ and $\gamma \ge 0$. If u is a locally bounded non-negative solution of (2), then $u(x) \equiv 0$. In particular, when $\frac{n+\alpha}{n-\alpha} \le p \le \frac{n+2\gamma+\alpha}{n-\alpha}$, we only require $u \in L_{loc}^{\frac{n(p-1)}{\alpha}}(\mathbb{R}_+^n)$.*

Next, we generalize the results to the case $\gamma > -\alpha$ under weaker conditions, that is:

Theorem 4. *Assume $1 < p < \frac{n+2\gamma+\alpha}{n-\alpha}$, $\gamma > -\alpha$, and $y_n^\gamma u^{p-1}(y) \in L_{loc}^{\frac{n}{\alpha}}(\mathbb{R}_+^n)$. If u is a locally bounded non-negative solution of*

$$u(x) = \int_{\mathbb{R}_+^n} G_\infty(x,y) y_n^\gamma u^p(y) dy, \tag{6}$$

then $u(x) \equiv 0$.

Corollary 3. *Assume $1 < p < \frac{n+2\gamma+\alpha}{n-\alpha}$, $\gamma > -\alpha$, and $y_n^\gamma u^{p-1}(y) \in L_{loc}^{\frac{n}{\alpha}}(\mathbb{R}_+^n)$. If u is a locally bounded non-negative solution of*

$$\begin{cases} (-\Delta)^{\alpha/2} u(x) = x_n^\gamma u^p(x), & x \in \mathbb{R}_+^n, \\ u(x) \equiv 0, & x \notin \mathbb{R}_+^n, \end{cases} \tag{7}$$

then $u(x) \equiv 0$.

Furthermore, we can also generalize the results of this problem to a more complicated case. In the following equation, we substitute $f(x_n)$ for x_n^γ,

$$\begin{cases} (-\Delta)^{\alpha/2} u(x) = f(x_n) u^p(x), \ u(x) > 0, & x \in \mathbb{R}_+^n, \\ u(x) \equiv 0, & x \notin \mathbb{R}_+^n, \end{cases} \tag{8}$$

where $f(x_n)$ is a positive real-valued function in $\mathbb{R}_+^n$ and is monotone nondecreasing with respect to the variable x_n. Obviously, compared with x_n^γ, $f(x_n)$ stands for a much wider family of functions, such as $f(x_n) = x_n^2 + x_n + 1$, $f(x_n) = \ln(x_n + 1)$, $f(x_n) = e^{x_n} + 3$, and so on. Then we can establish a series of entirely similar conclusions of this equation.

Firstly, we also establish the equivalence between Equation (8) and the integral equation:

$$u(x) = \int_{\mathbb{R}^n_+} G_\infty(x,y) f(y_n) u^p(y) dy. \tag{9}$$

We prove that

Theorem 5. *Assume that u is a locally bounded positive solution of problem (8). Then u is also a solution of integral Equation (9), and vice versa.*

Next, we establish the Liouville type theorem for the integral Equation (9).

Theorem 6. *Assume $p > \frac{n}{n-\alpha}$. If $u \in L^{\frac{n(p-1)}{\alpha}}(\mathbb{R}^n_+)$ is a non-negative solution of Equation (9) , then $u(x) \equiv 0$.*

By Theorem 6, we can immediately derive that

Corollary 4. *Assume $p > \frac{n}{n-\alpha}$. If $u \in L^{\frac{n(p-1)}{\alpha}}(\mathbb{R}^n_+)$ is a non-negative solution of problem (8), then $u(x) \equiv 0$.*

Similar to the above, under much weaker conditions, we can also exploit the same type of Kelvin transform to establish the following theorem.

Theorem 7. *Assume $1 < p \leq \frac{n+\alpha}{n-\alpha}$. If u is a locally bounded non-negative solution of (9), then $u(x) \equiv 0$. In particular, when $p = \frac{n+\alpha}{n-\alpha}$, we only require $u \in L_{loc}^{\frac{2n}{n-\alpha}}(\mathbb{R}^n_+)$.*

Corollary 5. *Assume $1 < p \leq \frac{n+\alpha}{n-\alpha}$. If u is a locally bounded non-negative solution of (8), then $u(x) \equiv 0$. In particular, when $p = \frac{n+\alpha}{n-\alpha}$, we only require $u \in L_{loc}^{\frac{2n}{n-\alpha}}(\mathbb{R}^n_+)$.*

Remark 2.

(i) *In [21], the author considered a similar problem. They required that $u \in \mathcal{D}^{\alpha/2,2} \bigcap C(\mathbb{R}^n)$ and established the non-existence of positive solutions for (2) via the extension method.*

(ii) *In [22] and [23], the author considered a similar problem for a slightly different fractional operator, which is defined by the eigenvalues of the Laplacian, and showed that there exist no bounded positive solutions under the restriction that $1 \leq \alpha < 2$.*

Here, in this paper, we impose no decay conditions on u besides the natural condition $u \in L_{\alpha/2}$, and also we allow $0 < \alpha < 2$. It is well-known that these kinds of Liouville theorems play an important role in establishing a priori estimates for the solutions of a family of corresponding boundary value problems in either bounded domains or Riemannian manifolds with boundary.

The structure of the paper is the following. In Section 2, we show the equivalence between problem (2) and integral Equation (4). In Section 3, we prove non-existence of positive solutions in the half space $\mathbb{R}^n_+$ for the integral Equation (4) and thus establish Theorems 2–4. In Section 4, we point out that the non-existence of positive solutions is also true for the Equation (8), and prove Theorems 5–7 briefly.

2. Equivalence between the Two Equations on $\mathbb{R}^n_+$

In this section, we establish the equivalence between problem (2) and integral Equation (4). To prove Theorem 1, we need the following Harnack inequality for α-harmonic functions on a domain with boundary, its consequences on half-spaces, and the uniqueness of α-harmonic functions on half-spaces.

Proposition 1 ([24]). *Let $f,\ g:\ \mathbb{R}^n \to \mathbb{R}$ be two nonnegative functions such that $(-\Delta)^s f = (-\Delta)^s g = 0$ in a domain Ω. Suppose that $x_0 \in \partial\Omega$, $f(x) = g(x) = 0$ for any $x \in B_1(x_0)\backslash\Omega$, and $\partial\Omega \bigcap B_1(x_0)$ is a Lipschitz graph in the direction of x_1 with Lipschitz constant less than 1. Then there is a constant C depending only on dimension such that*

$$\sup_{x\in B_{\frac{1}{2}}(x_0)\bigcap\Omega} \frac{f(x)}{g(x)} \le C \inf_{x\in B_{\frac{1}{2}}(x_0)\bigcap\Omega} \frac{f(x)}{g(x)}.$$

Based on this Harnack inequality, we derive the uniqueness of α-harmonic functions on half-spaces.

Lemma 1 ([25]). *Assume that w is a nonnegative solution of*

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} w = 0, & x \in \mathbb{R}^n_+, \\ w \equiv 0, & x \notin \mathbb{R}^n_+. \end{cases}$$

Then there is a constants $c_0 > 0$ such that for any two points $x = (x_1, \cdots, x_n)$ and $y = (y_1, \cdots, y_n)$ in $\mathbb{R}^n_+$, we have

$$\frac{w(y)}{(y_n)^{\alpha/2}} \ge c_0 \frac{w(x)}{(x_n)^{\alpha/2}}.$$

Consequently, we have either

$$w(x) \equiv 0,\ \ x \in \mathbb{R}^n,$$

or there exists a constant $a_0 > 0$, such that

$$w(x) \ge a_0 (x_n)^{\alpha/2},\ \ \forall x \in \mathbb{R}^n_+.$$

Furthermore, we need the following maximum principle in the proof of Theorem 1.

Proposition 2 ([1]). *Let Ω be a bounded open set in $\mathbb{R}^n$, and assume that f is a lower semi-continuous function on $\overline{\Omega}$ satisfying*

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} f \ge 0, & \text{in } \Omega, \\ f \ge 0, & \text{on } \mathbb{R}^n\backslash\Omega, \end{cases}$$

then $f \ge 0$ in $\mathbb{R}^n$.

We also need another result of Silvestre, to ensure that f is lower semi-continuous.

Proposition 3 ([1]). *If $f \in \mathcal{L}_{\alpha/2}$ and $(-\Delta)^{\frac{\alpha}{2}} f \ge 0$ in an open set Ω, then f is lower semi-continuous in Ω.*

For the Green function $G_\infty(x,y)$ in (5), it has the following properties.

Proposition 4 ([25]). *If $\frac{t}{s}$ is sufficiently small, then $\forall x = (x', x_n), y = (y', y_n) \in \mathbb{R}^n_+$, one can derive that*

$$\frac{c_{n,\alpha}}{s^{(n-\alpha)/2}} \frac{t^{\alpha/2}}{s^{\alpha/2}} \le G_\infty(x,y) \le \frac{C_{n,\alpha}}{s^{(n-\alpha)/2}} \frac{t^{\alpha/2}}{s^{\alpha/2}},$$

that is

$$G_\infty(x,y) \sim \frac{t^{\alpha/2}}{s^{n/2}}, \tag{10}$$

where

$$t = 4x_n y_n, \quad s = |x-y|^2,$$

$c_{n,\alpha}$ and $C_{n,\alpha}$ stand for different positive constants and only depend on n and α.

Now, it is sufficient to prove Theorem 1.

Proof of Theorem 1. Assume that u is a positive solution of (2). We first show that

$$\int_{\mathbb{R}^n_+} G_\infty(x,y) y_n^\gamma u^p(y) dy < \infty. \tag{11}$$

Set $x_R = (0, \cdots, 0, R)$, $B_R(x_R) = \{x \,|\, |x - x_R| < R\}$. And let

$$v_R(x) = \int_{B_R(x_R)} G_R(x,y) y_n^\gamma u^p(y) dy,$$

where

$$G_R(x,y) = \frac{A_{n,\alpha}}{|x-y|^{n-\alpha}} \left[1 - \frac{B_{n,\alpha}}{\left(1 + \frac{t_R}{s_R}\right)^{\frac{n-2}{2}}} \int_0^{\frac{s_R}{t_R}} \frac{\left(1 - \frac{t_R}{s_R} b\right)^{\frac{n-2}{2}}}{b^{\alpha/2}(1+b)} db \right] \tag{12}$$

is the Green's function on $B_R(x_R)$, and

$$s_R = \frac{|x-y|^2}{R^2}, \quad t_R = \left(\frac{2x_n}{R} - \frac{|x|^2}{R^2}\right)\left(\frac{2y_n}{R} - \frac{|y|^2}{R^2}\right).$$

From the locally bounded assumption on u, one can see that for each $R > 0$, $v_R(x)$ is well-defined and continuous. Moreover

$$\begin{cases} (-\Delta)^{\alpha/2} v_R(x) = x_n^\gamma u^p(x),\ v_R(x) > 0, & x \in B_R(x_R), \\ v_R(x) \equiv 0, & x \notin B_R(x_R). \end{cases}$$

Let $w_R(x) = u(x) - v_R(x)$. Then w_R satisfies

$$\begin{cases} (-\Delta)^{\alpha/2} w_R(x) = 0, & x \in B_R(x_R), \\ w_R(x) \geq 0, & x \notin B_R(x_R). \end{cases}$$

Now from Proposition 2, we have

$$w_R(x) \geq 0, \quad \forall x \in B_R(x_R).$$

Set $w(x) = u(x) - v(x)$, where

$$v(x) = \lim_{R\to\infty} v_R(x) = \int_{\mathbb{R}^n_+} G_\infty(x,y) y_n^\gamma u^p(y) dy.$$

Then, we derive

$$\begin{cases} (-\Delta)^{\alpha/2} v(x) = x_n^\gamma u^p(x),\ v(x) \geq 0, & x \in \mathbb{R}^n_+, \\ v(x) \equiv 0, & x \notin \mathbb{R}^n_+, \end{cases}$$

and

$$\begin{cases} (-\Delta)^{\alpha/2} w(x) = 0,\ w(x) \geq 0, & x \in \mathbb{R}^n_+, \\ w(x) \equiv 0, & x \notin \mathbb{R}^n_+. \end{cases}$$

By Lemma 1, we have either

$$w(x) \equiv 0, \ \forall x \in \mathbb{R}^n,$$

or there is a constant $a_0 > 0$, such that

$$w(x) \geq a_0(x_n)^{\alpha/2}, \ \ \forall x \in \mathbb{R}^n_+.$$

If $w(x) \geq a_0(x_n)^{\alpha/2}$, we have

$$u(x) = w(x) + v(x) \geq v(x) + a_0(x_n)^{\alpha/2} \geq a_0(x_n)^{\alpha/2}.$$

And therefore,

$$u(x) \geq v(x) \geq \int_{\mathbb{R}^n_+} G_\infty(x,y) y_n^\gamma a_0^p y_n^{\alpha p/2} dy \geq C \int_{\mathbb{R}^n_+ \setminus B_R(0)} G_\infty(x,y) y_n^{\alpha p/2+\gamma} dy.$$

Denote $x = (x', x_n)$, $y = (y', y_n) \in \mathbb{R}^{n-1} \times (0, +\infty)$, $r^2 = |x' - y'|^2$ and $a^2 = |x_n - y_n|^2$. When R sufficiently large, for each fixed $x \in B_R(0)$, one can derive that $\frac{t}{s}$ is sufficiently small. Then, from (2) and (10), for each fixed $x \in B_R(0)$ and for sufficiently large R, we derive

$$\begin{aligned}
u(x) &\geq C \int_{\mathbb{R}^n_+ \setminus B_R(0)} \frac{t^{\alpha/2}}{s^{n/2}} y_n^{\alpha p/2+\gamma} dy \\
&\geq C \int_{\mathbb{R}^n_+ \setminus B_R(0)} \frac{y_n^{\frac{\alpha(p+1)}{2}+\gamma}}{|x-y|^n} dy \\
&\geq C \int_R^{+\infty} y_n^{\frac{\alpha(p+1)}{2}+\gamma} \int_R^{+\infty} \frac{r^{n-2}}{(r^2+a^2)^{n/2}} dr dy_n \\
&= C \int_R^{+\infty} y_n^{\frac{\alpha(p+1)}{2}+\gamma} \frac{1}{a} \int_{\frac{R}{a}}^{+\infty} \frac{\tau^{n-2}}{(1+\tau^2)^{n/2}} d\tau dy_n \qquad (13) \\
&\geq C \int_R^{+\infty} y_n^{\frac{\alpha(p+1)}{2}+\gamma-1} dy_n = \infty. \qquad (14)
\end{aligned}$$

One can derive (13) by letting $\tau = \frac{r}{a}$, and our assumption $\gamma > -\alpha$ verifies (14).

Obviously, (14) contradicts the locally bounded assumption on u. Therefore, we must have $w(x) \equiv 0$, that is

$$u(x) = v(x) = \int_{\mathbb{R}^n_+} G_\infty(x,y) y_n^\gamma u^p(y) dy < \infty.$$

Next, we prove that if $u(x)$ solves the integral equation, it also solves the differential equation. For any $\phi \in C_0^\infty(\mathbb{R}^n_+)$, we have

$$\begin{aligned}
< (-\Delta)^{\alpha/2} u, \phi > &= < \int_{\mathbb{R}^n_+} G_\infty(x,y) y_n^\gamma u^p(y) dy, \ (-\Delta)^{\alpha/2}\phi(x) > \\
&= \int_{\mathbb{R}^n_+} \left\{ \int_{\mathbb{R}^n_+} G_\infty(x,y) y_n^\gamma u^p(y) dy \right\} (-\Delta)^{\alpha/2}\phi(x) dx \\
&= \int_{\mathbb{R}^n_+} \left\{ \int_{\mathbb{R}^n_+} G_\infty(x,y) (-\Delta)^{\alpha/2}\phi(x) dx \right\} y_n^\gamma u^p(y) dy \\
&= \int_{\mathbb{R}^n_+} \left\{ \int_{\mathbb{R}^n_+} \delta(x-y)\phi(x) dx \right\} y_n^\gamma u^p(y) dy \\
&= \int_{\mathbb{R}^n_+} y_n^\gamma u^p(y) \phi(y) dy \\
&= < y_n^\gamma u^p(y), \ \phi(y) > .
\end{aligned}$$

This shows that $u(x)$ satisfies (2). Hence Theorem 1 is proved. □

3. Liouville Theorems for Equations (2) and (4)

In this section, we prove the non-existence of positive solutions under global and local integrability (or local boundness) assumptions respectively and thus establish Theorems 2–4.

Let λ be a positive real number and let the moving plane be

$$T_\lambda = \{x \in \mathbb{R}^n_+ \,|\, x_n = \lambda\}.$$

We denote Σ_λ the region between the plane $x_n = 0$ and the plane $x_n = \lambda$, that is

$$\Sigma_\lambda = \{x = (x_1, \cdots, x_{n-1}, x_n) \in \mathbb{R}^n_+ \,|\, 0 < x_n < \lambda\}.$$

Let

$$x^\lambda = (x_1, \cdots, x_{n-1}, 2\lambda - x_n)$$

be the reflection of the point $x = (x_1, \cdots, x_{n-1}, x_n)$ about the plane T_λ.

Set

$$\Sigma_\lambda^C = \mathbb{R}^n_+ \backslash \Sigma_\lambda,$$

which is the complement of Σ_λ, and write

$$u_\lambda(x) = u(x^\lambda),$$

and

$$w_\lambda(x) = u_\lambda(x) - u(x).$$

From [25], one can express the Green's function of the operator $(-\Delta)^{\alpha/2}$ with Dirichlet conditions on the upper Euclidean space as

$$G_\infty(x,y) = \frac{A_{n,\alpha}}{s^{(n-\alpha)/2}}\left[1 - \frac{B_{n,\alpha}}{(s+t)^{(n-2)/2}} \int_0^{\frac{s}{t}} \frac{(s-tb)^{(n-2)/2}}{b^{\alpha/2}(1+b)} db\right], \quad x, y \in \mathbb{R}^n_+,$$

where

$$s = |x-y|^2 \quad \text{while} \quad t = 4x_n y_n,$$

and have the following lemma which establishes some properties of this Green's function.

Lemma 2.

(i) *For any $x, y \in \Sigma_\lambda$, $x \neq y$, we have*

$$G_\infty(x^\lambda, y^\lambda) > \max\{G_\infty(x^\lambda, y),\ G_\infty(x, y^\lambda)\}, \tag{15}$$

and

$$G_\infty(x^\lambda, y^\lambda) - G_\infty(x,y) > \left|G_\infty(x^\lambda, y) - G_\infty(x, y^\lambda)\right|. \tag{16}$$

(ii) *or any $x \in \Sigma_\lambda$, $y \in \Sigma_\lambda^C$, it holds*

$$G_\infty(x^\lambda, y) > G_\infty(x,y). \tag{17}$$

(iii) *For any x, $y \in \mathbb{R}^n_+$, it holds*

$$\frac{\partial G_\infty}{\partial s} < 0, \quad \frac{\partial G_\infty}{\partial t} > 0, \tag{18}$$

where $s = |x-y|^2$ while $t = 4x_n y_n$.

The following lemma is a key ingredient in our integral estimate.

Lemma 3. *For any* $x \in \Sigma_\lambda$, *it holds*

$$u(x) - u_\lambda(x) \leq \int_{\Sigma_\lambda} \Big[G_\infty(x^\lambda, y^\lambda) - G_\infty(x, y^\lambda)\Big]\Big[y_n^\gamma u^p(y) - (y_n^\lambda)^\gamma u_\lambda^p(y)\Big]dy, \tag{19}$$

where $y_n^\lambda = 2\lambda - y_n$.

Proof. Let $\tilde{\Sigma}_\lambda$ be the reflection of Σ_λ about the plane T_λ. Obviously, we have

$$\begin{aligned} u(x) &= \int_{\Sigma_\lambda} G_\infty(x,y) y_n^\gamma u^p(y)dy \\ &\quad + \int_{\Sigma_\lambda} G_\infty(x, y^\lambda)(y_n^\lambda)^\gamma u_\lambda^p(y)dy + \int_{\Sigma_\lambda^C \backslash \tilde{\Sigma}_\lambda} G_\infty(x,y) y_n^\gamma u^p(y)dy, \end{aligned}$$

and

$$\begin{aligned} u(x^\lambda) &= \int_{\Sigma_\lambda} G_\infty(x^\lambda,y) y_n^\gamma u^p(y)dy \\ &\quad + \int_{\Sigma_\lambda} G_\infty(x^\lambda, y^\lambda)(y_n^\lambda)^\gamma u_\lambda^p(y)dy + \int_{\Sigma_\lambda^C \backslash \tilde{\Sigma}_\lambda} G_\infty(x^\lambda,y) y_n^\gamma u^p(y)dy. \end{aligned}$$

By Lemma 2, we have

$$\begin{aligned} u(x) - u(x^\lambda) &= \int_{\Sigma_\lambda} \Big[G_\infty(x,y) - G_\infty(x^\lambda,y)\Big] y_n^\gamma u^p(y)dy \\ &\quad + \int_{\Sigma_\lambda} \Big[G_\infty(x,y^\lambda) - G_\infty(x^\lambda,y^\lambda)\Big](y_n^\lambda)^\gamma u_\lambda^p(y)dy \\ &\quad + \int_{\Sigma_\lambda^C \backslash \tilde{\Sigma}_\lambda} \Big[G_\infty(x,y) - G_\infty(x^\lambda,y)\Big] y_n^\gamma u^p(y)dy \\ &\leq \int_{\Sigma_\lambda} \Big[G_\infty(x^\lambda,y^\lambda) - G_\infty(x,y^\lambda)\Big] y_n^\gamma u^p(y)dy \\ &\quad + \int_{\Sigma_\lambda} \Big[G_\infty(x,y^\lambda) - G_\infty(x^\lambda,y^\lambda)\Big](y_n^\lambda)^\gamma u_\lambda^p(y)dy \\ &= \int_{\Sigma_\lambda} \Big[G_\infty(x^\lambda,y^\lambda) - G_\infty(x,y^\lambda)\Big]\Big[y_n^\gamma u^p(y) - (y_n^\lambda)^\gamma u_\lambda^p(y)\Big]dy. \end{aligned}$$

This completes the proof of Lemma 3. □

We also need the following key lemma, which states an equivalent form of the Hardy-Littlewood-Sobolev inequality.

Lemma 4 ([26,27])**.** *Assume* $0 < \alpha < n$ *and* $\Omega \in \mathbb{R}^n$. *Let* $g \in L^{\frac{np}{n+\alpha p}}(\Omega)$ *for* $\frac{n}{n-\alpha} < p < \infty$. *Define*

$$Tg(x) := \int_\Omega \frac{1}{|x-y|^{n-\alpha}} g(y)dy.$$

Then

$$\|Tg\|_{L^p(\Omega)} \leq C(n,p,\alpha)\|g\|_{L^{\frac{np}{n+\alpha p}}(\Omega)}. \tag{20}$$

From the following lemma, one can see that a nonnegative solution u of a super-harmonic function is either strictly positive or identically zero in $\mathbb{R}^n$.

Lemma 5 ([1])**.** *Let* Ω *be a bounded open set in* $\mathbb{R}^n$, *and assume that* f *is a lower semi-continuous function on* $\overline{\Omega}$ *satisfying*

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} f \geq 0, & in\ \Omega, \\ f \geq 0, & on\ \mathbb{R}^n \backslash \Omega, \end{cases}$$

then $f \geq 0$ in $\mathbb{R}^n$. Moreover, if $f(x) = 0$ for some point inside Ω, then $f \equiv 0$ in all $\mathbb{R}^n$.

By virtue of this lemma, without loss of generality, we may assume that $u > 0$ in $\mathbb{R}^n_+$ and get a contradiction.

Proof of Theorem 2. We carry out the proof in two steps. Firstly, we start from the very low end of our region $\mathbb{R}^n_+$, i.e., near $x_n = 0$. We show that for λ sufficiently small,

$$w_\lambda(x) = u_\lambda(x) - u(x) \geq 0, \ \text{ a.e. } \forall\, x \in \Sigma_\lambda. \tag{21}$$

In the second step, we will move our plane T_λ up in the positive x_n direction as long as the inequality (21) holds to show that $u(x)$ is monotone increasing in x_n and thus derive a contraction.

Step 1. Define

$$\Sigma_\lambda^- = \{x \in \Sigma_\lambda \,|\, w_\lambda(x) < 0\}.$$

We show that for λ sufficiently small, Σ_λ^- must be measure zero. In fact, for any $x \in \Sigma_\lambda^-$, by the Mean Value Theorem and Lemma 3, we have

$$\begin{aligned}
0 &< u(x) - u_\lambda(x) \\
&\leq \int_{\Sigma_\lambda} \left[G_\infty(x^\lambda, y^\lambda) - G_\infty(x, y^\lambda)\right]\left[y_n^\gamma u^p(y) - (y_n^\lambda)^\gamma u_\lambda^p(y)\right] dy \\
&\leq \int_{\Sigma_\lambda^-} \left[G_\infty(x^\lambda, y^\lambda) - G_\infty(x, y^\lambda)\right]\left[y_n^\gamma u^p(y) - (y_n^\lambda)^\gamma u_\lambda^p(y)\right] dy \\
&\leq \int_{\Sigma_\lambda^-} G_\infty(x^\lambda, y^\lambda)\left[y_n^\gamma u^p(y) - y_n^\gamma u_\lambda^p(y)\right] dy \\
&= p\int_{\Sigma_\lambda^-} G_\infty(x^\lambda, y^\lambda)\psi_\lambda^{p-1}(y) y_n^\gamma [u(y) - u_\lambda(y)] dy \\
&\leq p\int_{\Sigma_\lambda^-} G_\infty(x^\lambda, y^\lambda) u^{p-1}(y) y_n^\gamma [u(y) - u_\lambda(y)] dy,
\end{aligned} \tag{22}$$

where $\psi_\lambda(y)$ is a value between $u(y)$ and $u_\lambda(y)$. Hence on Σ_λ^-, we have

$$0 \leq u_\lambda(y) \leq \psi_\lambda(y) \leq u(y).$$

By the expression of $G_\infty(x, y)$, it is easy to see that

$$G_\infty(x, y) \leq \frac{A_{n,\alpha}}{|x - y|^{n-\alpha}}.$$

Now (22) implies

$$0 < u(x) - u_\lambda(x) \leq C\int_{\Sigma_\lambda^-} \frac{1}{|x - y|^{n-\alpha}} \left|u^{p-1}(y) y_n^\gamma\right| |u(y) - u_\lambda(y)| dy \tag{23}$$

$$\leq C\int_{\Sigma_\lambda^-} \frac{1}{|x - y|^{n-\alpha}} \left|u^{p-1}(y)\right| |u(y) - u_\lambda(y)| dy. \tag{24}$$

Notice that now λ is only a little larger than 0, so within Σ_λ^-, y_n is bounded, i.e., there exists a positive number C such that $0 < y_n \leq C$. And since $\gamma \geq 0$, we get $|y_n^\gamma| \leq C$, hence we derive (24).

We apply the Hardy-Littlewood-Sobolev inequality (20) and Hölder inequality for (24) to derive, for any $q > \frac{n}{n-\alpha}$,

$$\|w_\lambda\|_{L^q(\Sigma_\lambda^-)} \leq C\|u^{p-1} w_\lambda\|_{L^{\frac{nq}{n+\alpha q}}(\Sigma_\lambda^-)} \leq C\|u^{p-1}\|_{L^{\frac{n}{\alpha}}(\Sigma_\lambda^-)} \|w_\lambda\|_{L_q(\Sigma_\lambda^-)}. \tag{25}$$

Note here we can choose $q = \frac{n(p-1)}{\alpha}$, then by our assumption $p > \frac{n}{n-\alpha}$, we have $q > \frac{n}{n-\alpha}$ and $w_\lambda \in L^q(\mathbb{R}^n)$.

Since $u \in L^{\frac{n(p-1)}{\alpha}}(\mathbb{R}^n_+)$, we can choose sufficiently small positive λ such that

$$C\|u^{p-1}\|_{L^{\frac{n}{\alpha}}(\Sigma_\lambda^-)} = C\left\{\int_{\Sigma_\lambda^-} u^{\frac{n(p-1)}{\alpha}}(y)\right\}^{\frac{\alpha}{n}} \leq \frac{1}{2}. \tag{26}$$

By (25) and (26), we derive

$$\|w_\lambda\|_{L_q(\Sigma_\lambda^-)} = 0,$$

and hence Σ_λ^- must be measure zero. Then

$$w_\lambda(x) \geq 0, \ \text{a.e. } x \in \Sigma_\lambda. \tag{27}$$

This provides us with a starting point for moving the plane.

Step 2. Now we start from such small λ and move the plane T_λ up as long as (27) holds.

Define

$$\lambda_0 = \sup\{\lambda \,|\, w_\rho(x) \geq 0,\ \rho \leq \lambda,\ \forall\, x \in \Sigma_\rho\}.$$

We will prove

$$\lambda_0 = +\infty. \tag{28}$$

Suppose in the contrary that $\lambda_0 < +\infty$, we will show that $u(x)$ is symmetric about the plane T_{λ_0}, i.e.,

$$w_{\lambda_0} \equiv 0, \ \text{a.e. } \forall\, x \in \Sigma_{\lambda_0}. \tag{29}$$

This will contradict the strict positivity of u.

Suppose (29) does not hold, then for such a λ_0, we have $w_{\lambda_0} \geq 0$, but $w_{\lambda_0} \not\equiv 0$ a.e. on Σ_{λ_0}. We show that the plane can be moved further up. More precisely, there exists an $\epsilon > 0$ such that for all $\lambda \in [\lambda_0,\ \lambda_0 + \epsilon)$,

$$w_\lambda \geq 0, \ \text{a.e. on } \Sigma_\lambda. \tag{30}$$

To verify this, we will again resort to inequality (25). If one can prove that for ϵ sufficiently small such that for all λ in $[\lambda_0,\ \lambda_0 + \epsilon)$,

$$C\left\{\int_{\Sigma_\lambda^-} u^{\frac{n(p-1)}{\alpha}}(y)\right\}^{\frac{\alpha}{n}} \leq \frac{1}{2}, \tag{31}$$

then by (24) and (31), we derive $\|w_\lambda\|_{L^q(\Sigma_\lambda^-)} = 0$, and therefore Σ_λ^- must be measure zero. Hence for this values of $\lambda > \lambda_0$, we have (30). This contradicts the definition of λ_0. Therefore (29) must hold. Here, we also need to verify that we can get (24) from (23). Notice that $\lambda_0 < +\infty$, $\lambda \in [\lambda_0,\ \lambda_0 + \epsilon)$, and obviously we have $0 < y_n < \lambda$, then there exist a positive constants C_0 such that $|y_n^\gamma| < C_0$. And hence we can derive (24) from (23) .

We postpone the proof of (31) for a moment.

By (29), we obtain that $u(x) = 0$ on the plane $x_n = 2\lambda_0$, the symmetric image of the boundary $\partial\mathbb{R}^n_+$ with respect to the plane T_{λ_0}. This contradicts our assumption $u(x) > 0$ in $\mathbb{R}^n_+$. Therefore (28) must be valid.

Now we have proved that the positive solution of (4) is monotone increasing with respect to x_n, and this contradicts $u \in L^{\frac{n(p-1)}{\alpha}}(\mathbb{R}^n_+)$. Therefore the positive solutions of (4) do not exist.

Now we verify inequality (31). For any small $\eta > 0$, we can choose R sufficiently large so that

$$C\left\{\int_{\mathbb{R}^n_+ \setminus B_R} u^{\frac{n(p-1)}{\alpha}}(y)\right\}^{\frac{\alpha}{n}} < \eta. \tag{32}$$

We fix this R and then show that the measure of $\Sigma_\lambda^- \bigcap B_R$ is sufficiently small for λ close to λ_0. Firstly, we have

$$w_{\lambda_0}(x) > 0 \tag{33}$$

in the interior of Σ_{λ_0}.

Actually, we can immediately derive (33) by the following fact:

$$\begin{aligned}
& u_{\lambda_0}(x) - u(x) \\
\geq\ & \int_{\Sigma_{\lambda_0}} \Big[G_\infty(x^{\lambda_0}, y^{\lambda_0}) - G_\infty(x, y^{\lambda_0})\Big]\Big[(y_n^{\lambda_0})^\gamma u^p_{\lambda_0}(y) - y_n^\gamma u^p(y)\Big]dy \\
& + \int_{\Sigma^C_{\lambda_0} \setminus \tilde{\Sigma}_{\lambda_0}} \Big[G_\infty(x^{\lambda_0}, y) - G_\infty(x, y)\Big] y_n^\gamma u^p(y) dy \\
\geq\ & \int_{\Sigma^C_{\lambda_0} \setminus \tilde{\Sigma}_{\lambda_0}} \Big[G_\infty(x^{\lambda_0}, y) - G_\infty(x, y)\Big] y_n^\gamma u^p(y) dy \qquad (34) \\
>\ & 0. \qquad (35)
\end{aligned}$$

We can easily get (35) by (34), Lemma 2 (ii) and our assumption $u > 0 \in \mathbb{R}^n_+$.

By the well-known Lusin Theorem, for any $\delta > 0$, there exists a closed subset $F_\delta \subset (\Sigma_{\lambda_0} \bigcap B_R)$ satisfies $\mu((\Sigma_{\lambda_0} \bigcap B_R) \setminus F_\delta) < \delta$ such that $w_{\lambda_0}|_{F_\delta}$ is continuous about x. Therefore, when λ is sufficiently close to λ_0, $w_\lambda|_{F_\delta}$ is continuous about λ. By (33), there exists a $\epsilon > 0$ such that for any $\lambda \in [\lambda_0, \lambda_0 + \epsilon)$ we have

$$w_\lambda(x) \geq 0, \ \forall\, x \in F_\delta.$$

And therefore, for such λ we have

$$\mu(\Sigma_\lambda^- \bigcap B_R) \leq \mu((\Sigma_{\lambda_0} \setminus F_\delta) \bigcap B_R) + \mu((\Sigma_\lambda \setminus \Sigma_{\lambda_0}) \bigcap B_R) \leq \delta + \epsilon.$$

Similar to *Step 1*, we can choose δ, ϵ sufficiently small such that

$$C\left\{\int_{\Sigma_\lambda^- \bigcap B_R} u^{\frac{n(p-1)}{\alpha}}(y)\right\}^{\frac{\alpha}{n}} < \eta. \tag{36}$$

Then, from (32) and (36), set η sufficiently small (smaller than $\frac{1}{4}$), we derive (31). Hence completes the proof of Theorem 2. $\square$

Next, we will use proper Kelvin-type transforms and obtain the non-existence of positive solutions in $\mathbb{R}^n_+$ under much weaker conditions, i.e., the solution u is only locally bounded or, in the critical case and a part of the subcritical case, only locally integrable.

Without global integrability assumption on u, we are not able to employ the method of moving planes straight forward. To circumvent this difficulty, we apply the Kelvin type transforms.

For $z^0 \in \partial\mathbb{R}^n_+$, let

$$\bar{u}_{z^0}(x) = \frac{1}{|x - z^0|^{n-\alpha}} u\left(\frac{x - z^0}{|x - z^0|^2} + z^0\right) \tag{37}$$

be the Kelvin type transforms of u centered at z^0.

Through an elementary calculation, we get

$$\begin{aligned}\bar{u}_{z^0}(x) &= \frac{1}{|x-z^0|^{n-\alpha}}\int_{\mathbb{R}^n_+} G_\infty\left(\frac{x-z^0}{|x-z^0|^2}+z^0,\ y\right)y_n^\gamma u^p(y)dy \\ &= \int_{\mathbb{R}^n_+} G_\infty(x,\ y)\frac{y_n^\gamma \bar{u}_{z^0}^p(y)}{|y-z^0|^\beta}dy,\ \ \forall\ x\in\mathbb{R}^n_+\backslash B_\epsilon(z^0),\ \epsilon>0,\end{aligned} \tag{38}$$

where $p\le\tau$, $\tau=\frac{n+2\gamma+\alpha}{n-\alpha}$, $\beta=2n+2\gamma-(n-\alpha)(p+1)\ge 0$.

Proof of Theorem 3. We consider the case $1<p<\frac{n+\alpha}{n-\alpha}$ and the case $\frac{n+\alpha}{n-\alpha}\le p\le\frac{n+2\gamma+\alpha}{n-\alpha}$ separately.

(i) We first consider the case of $\frac{n+\alpha}{n-\alpha}\le p\le\frac{n+2\gamma+\alpha}{n-\alpha}$. In this case we assume $u\in L_{loc}^{\frac{n(p-1)}{\alpha}}(\mathbb{R}^n_+)$ only.

If $u(x)$ is a solution of

$$u(x)=\int_{\mathbb{R}^n_+} G_\infty(x,\ y)y_n^\gamma u^p(y)dy, \tag{39}$$

then $\bar{u}_{z^0}(x)$ is a solution of (38). Since $u\in L_{loc}^{\frac{n(p-1)}{\alpha}}(\mathbb{R}^n_+)$, for any domain Ω that is a positive distance away from z^0, we have

$$\int_\Omega \frac{\bar{u}_{z^0}^{\frac{n(p-1)}{\alpha}}(y)}{|y-z^0|^{2n-\frac{n(p-1)(n-\alpha)}{\alpha}}}dy<\infty. \tag{40}$$

Here, we consider two possibilities.

Possibility 1. There is a $z^0=(z_1^0,\cdots,z_{n-1}^0,0)\in\partial\mathbb{R}^n_+$ such that $\bar{u}_{z^0}(x)$ is bounded near z^0. Then by (37), we obtain

$$u(y)=\frac{1}{|y-z^0|^{n-\alpha}}\bar{u}_{z^0}\left(\frac{y-z^0}{|y-z^0|^2}+z^0\right). \tag{41}$$

And we further have

$$u(y)=O\left(\frac{1}{|y|^{n-\alpha}}\right),\ \text{ as } |y|\to\infty. \tag{42}$$

Since $p\ge\frac{n+\alpha}{n-\alpha}>\frac{n}{n-\alpha}$ and $u\in L_{loc}^{\frac{n(p-1)}{\alpha}}(\mathbb{R}^n_+)$, together with (42), we have

$$\int_{\mathbb{R}^n_+} u^{\frac{n(p-1)}{\alpha}}(y)dy\le C\int_{\mathbb{R}^n_+}\frac{1}{|y|^{\frac{n(p-1)(n-\alpha)}{\alpha}}}dy<\infty. \tag{43}$$

In this situation, we still carry on the moving planes on u. By exactly the same argument as in the proof of Theorem 2, we obtain the non-existence of positive solutions for (4).

Possibility 2. For all $z^0=(z_1^0,\cdots,z_{n-1}^0,0)\in\partial\mathbb{R}^n_+$, $\bar{u}_{z^0}(x)$ are unbounded near z^0. Then for each z^0, we will carry on the moving planes on $\bar{u}_{z^0}(x)$ in $\mathbb{R}^{n-1}$ to prove that it is rotationally symmetric about the line passing through z^0 and parallel to the x_n-axis. From this, we will deduce that u is independent of the first $n-1$ variables $x_1,\cdots,x_{n-1}$. That is $u=u(x_n)$, which as we will show, contradicts the finiteness of the integral

$$\int_{\mathbb{R}^n_+} G_\infty(x,y)y_n^\gamma u^p(y)dy. \tag{44}$$

In this situation, since we only need to deal with $\bar{u}_{z^0}$, for simplicity, we denote it by $\bar{u}$. For a given real number λ, we define

$$\hat{\Sigma}_\lambda = \{x = (x_1, \cdots, x_n) \in \mathbb{R}^n_+ | \, x_1 < \lambda\}, \tag{45}$$

and in the following of this section, we let

$$x^\lambda = (2\lambda - x_1, x_2, \cdots, x_n).$$

For $x,\ y \in \hat{\Sigma}_\lambda,\ \ x \neq y$, by (18), it is easy to see that

$$G_\infty(x,\ y) = G_\infty(x^\lambda,\ y^\lambda) > G_\infty(x^\lambda,\ y) = G_\infty(x,\ y^\lambda). \tag{46}$$

By (38), obviously we have

$$\bar{u}(x) = \int_{\hat{\Sigma}_\lambda} G_\infty(x,\ y) \frac{y_n^\gamma \bar{u}^p(y)}{|y - z^0|^\beta} dy + \int_{\hat{\Sigma}_\lambda} G_\infty(x,\ y^\lambda) \frac{y_n^\gamma \bar{u}_\lambda^p(y)}{|y^\lambda - z^0|^\beta} dy,$$

and

$$\bar{u}(x^\lambda) = \int_{\hat{\Sigma}_\lambda} G_\infty(x^\lambda,\ y) \frac{y_n^\gamma \bar{u}^p(y)}{|y - z^0|^\beta} dy + \int_{\hat{\Sigma}_\lambda} G_\infty(x^\lambda,\ y^\lambda) \frac{y_n^\gamma \bar{u}_\lambda^p(y)}{|y^\lambda - z^0|^\beta} dy.$$

By (46), we get

$$\begin{aligned} & \bar{u}(x) - \bar{u}(x^\lambda) \\ = & \int_{\hat{\Sigma}_\lambda} \Big[G_\infty(x,\ y) - G_\infty(x^\lambda,\ y) \Big] \frac{y_n^\gamma \bar{u}^p(y)}{|y - z^0|^\beta} dy \\ & + \int_{\hat{\Sigma}_\lambda} \Big[G_\infty(x,\ y^\lambda) - G_\infty(x^\lambda,\ y^\lambda) \Big] \frac{y_n^\gamma \bar{u}_\lambda^p(y)}{|y^\lambda - z^0|^\beta} dy \\ = & \int_{\hat{\Sigma}_\lambda} \Big[G_\infty(x,\ y) - G_\infty(x^\lambda,\ y) \Big] y_n^\gamma \left[\frac{\bar{u}^p(y)}{|y - z^0|^\beta} - \frac{\bar{u}_\lambda^p(y)}{|y^\lambda - z^0|^\beta} \right] dy. \end{aligned} \tag{47}$$

Then, we continue in two steps. In *step 1*, we will show that for λ sufficiently negative,

$$w_\lambda(x) = \bar{u}_\lambda(x) - \bar{u}(x) \geq 0, \ \text{ a.e. } \forall\, x \in \hat{\Sigma}_\lambda. \tag{48}$$

In *step 2*, we deduce that $\hat{T}$ can be a move to the right all the way to z^0. And furthermore, we obtain $w_{z_1^0} \equiv 0, \ \ \forall\, x \in \hat{\Sigma}_{z_1^0}$.

Step 1. For any $\epsilon > 0$, define

$$\hat{\Sigma}_\lambda^- = \Big\{ x \in \hat{\Sigma}_\lambda \backslash B_\epsilon((z^0)^\lambda) | \, w_\lambda(x) < 0 \Big\}, \tag{49}$$

where $(z^0)^\lambda$ is the reflection of z^0 about the plane $\hat{T}_\lambda = \big\{ x \in \mathbb{R}^n_+ | \, x_1 = \lambda \big\}$.

We prove that for λ sufficiently negative, $\hat{\Sigma}_\lambda^-$ must be measure zero. In fact, by (46), (47) and Mean Value Theorem, we derive, for $x \in \hat{\Sigma}_\lambda^-$,

$$\begin{aligned}
0 &< \bar{u}(x) - \bar{u}_\lambda(x) \\
&\leq \int_{\hat{\Sigma}_\lambda^-} \Big[G_\infty(x,\, y) - G_\infty(x^\lambda,\, y)\Big] y_n^\gamma \left[\frac{\bar{u}^p(y)}{|y - z^0|^\beta} - \frac{\bar{u}_\lambda^p(y)}{|y^\lambda - z^0|^\beta}\right] dy \\
&\leq \int_{\hat{\Sigma}_\lambda^-} G_\infty(x,\, y) \frac{y_n^\gamma}{|y - z^0|^\beta} \Big[\bar{u}^p(y) - \bar{u}_\lambda^p(y)\Big] dy \\
&= p \int_{\hat{\Sigma}_\lambda^-} G_\infty(x,\, y) \frac{\psi_\lambda^{p-1}(y) y_n^\gamma}{|y - z^0|^\beta} [\bar{u}(y) - \bar{u}_\lambda(y)] dy \\
&\leq p \int_{\hat{\Sigma}_\lambda^-} G_\infty(x,\, y) \frac{y_n^\gamma \bar{u}^{p-1}(y)}{|y - z^0|^\beta} [\bar{u}(y) - \bar{u}_\lambda(y)] dy \\
&\leq C \int_{\hat{\Sigma}_\lambda^-} \frac{1}{|x - y|^{n-\alpha}} \left| \frac{y_n^\gamma \bar{u}^{p-1}(y)}{|y - z^0|^\beta} \right| |\bar{u}(y) - \bar{u}_\lambda(y)| dy. \qquad (50)
\end{aligned}$$

On the one hand, by our assumption $p \geq \frac{n+\alpha}{n-\alpha}$, we have

$$2n - \frac{n(p-1)(n-\alpha)}{\alpha} \leq 0.$$

Then by (40) we get

$$\int_\Omega \bar{u}^{\frac{n(p-1)}{\alpha}}(y) dy \leq C \int_\Omega \frac{\bar{u}^{\frac{n(p-1)}{\alpha}}(y)}{|y - z^0|^{2n - \frac{n(p-1)(n-\alpha)}{\alpha}}} dy < \infty \qquad (51)$$

for any domain Ω which is a positive distance away from z^0.

On the other hand, since $\gamma \geq 0$, we can easily obtain that y_n^γ is bounded in each bounded domain $\Omega \subset \mathbb{R}_+^n$. Therefore, by our assumption $u \in L_{loc}^{\frac{n(p-1)}{\alpha}}(\mathbb{R}_+^n)$, i.e., $u^{p-1} \in L_{loc}^{\frac{n}{\alpha}}(\mathbb{R}_+^n)$, we get

$$y_n^\gamma u^{p-1} \in L_{loc}^{\frac{n}{\alpha}}(\mathbb{R}_+^n). \qquad (52)$$

Hence, we obtain

$$\int_\Omega \left[\frac{y_n^\gamma \bar{u}^{p-1}(y)}{|y - z^0|^{2\gamma - (p-1)(n-\alpha)}}\right]^{\frac{n}{\alpha}} \frac{1}{|y - z^0|^{2n}} dy = \int_\Omega \left[\frac{y_n^\gamma \bar{u}^{p-1}(y)}{|y - z^0|^\beta}\right]^{\frac{n}{\alpha}} dy < \infty \qquad (53)$$

for any domain Ω which is a positive distance away from z^0.

From (51) and (53), we are able to apply the Hardy-Littlewood-Sobolev inequality (20) and Hölder inequality for (50) to obtain, for any $q > \frac{n}{n-\alpha}$,

$$\begin{aligned}
\|w_\lambda\|_{L^q(\hat{\Sigma}_\lambda^-)} &\leq C \left\| \frac{y_n^\gamma \bar{u}^{p-1}(y)}{|y - z^0|^\beta} w_\lambda \right\|_{L^{\frac{nq}{n+\alpha q}}(\hat{\Sigma}_\lambda^-)} \\
&\leq C \left\| \frac{y_n^\gamma \bar{u}^{p-1}(y)}{|y - z^0|^\beta} \right\|_{L^{\frac{n}{\alpha}}(\hat{\Sigma}_\lambda^-)} \|w_\lambda\|_{L^q(\hat{\Sigma}_\lambda^-)}. \qquad (54)
\end{aligned}$$

Notice that we can choose $q = \frac{n(p-1)}{\alpha}$, then our assumption $p \geq \frac{n+\alpha}{n-\alpha}$ ensures that $q \geq \frac{2n}{n-\alpha} > \frac{n}{n-\alpha}$.

By (53), we can choose N sufficiently large, such that for $\lambda \leq -N$,

$$C \left\{ \int_{\hat{\Sigma}_\lambda^-} \left[\frac{y_n^\gamma \bar{u}^{p-1}(y)}{|y - z^0|^\beta}\right]^{\frac{n}{\alpha}} dy \right\}^{\frac{\alpha}{n}} \leq \frac{1}{2}. \qquad (55)$$

Now inequality (54) and (55) imply

$$\|w_\lambda\|_{L_q(\hat{\Sigma}_\lambda^-)} = 0,$$

and hence $\hat{\Sigma}_\lambda^-$ must be measure zero. Then we get

$$w_\lambda(x) \geq 0, \ \text{ a.e. } x \in \hat{\Sigma}_\lambda. \tag{56}$$

Step 2. (Move the plane to the limiting position to derive symmetry.)

Inequality (56) provides a starting point to move the plane $\hat{T}_\lambda$. Now we start from the neighbourhood of $x_1 = -\infty$ and move the plane to the right as long as (56) holds to the limiting position.

Define

$$\lambda_0 = \sup\Big\{\lambda \leq z_1^0 | \, w_\rho(x) \geq 0, \ \rho \leq \lambda, \ \forall\, x \in \hat{\Sigma}_\rho\Big\}. \tag{57}$$

We prove that $\lambda_0 \geq z_1^0 - \epsilon$. If not, suppose that $\lambda_0 < z_1^0 - \epsilon$. We will show that $\bar{u}(x)$ is symmetric about the plane $\hat{T}_{\lambda_0}$, i.e.,

$$w_{\lambda_0}(x) \equiv 0, \ \text{ a.e. } \forall\, x \in \hat{\Sigma}_{\lambda_0} \backslash B_\epsilon((z^0)^{\lambda_0}). \tag{58}$$

Suppose (58) is not true, then for such $\lambda_0 < z_1^0 - \epsilon$, we have

$$w_{\lambda_0}(x) \geq 0, \ \text{ but } w_{\lambda_0}(x) \not\equiv 0 \text{ a.e. on } \hat{\Sigma}_{\lambda_0} \backslash B_\epsilon((z^0)^{\lambda_0}).$$

We show that the plane can be moved further to the right. More rigorously, there exists a $\zeta > 0$ such that for all $\lambda \in [\lambda_0, \ \lambda_0 + \zeta)$,

$$w_\lambda(x) \geq 0, \text{ a.e. on } \hat{\Sigma}_\lambda \backslash B_\epsilon((z^0)^\lambda).$$

This will contradict the definition of λ_0.

By inequality (54), we have

$$\|w_\lambda\|_{L^q(\hat{\Sigma}_\lambda^-)} \leq C\left\{\int_{\hat{\Sigma}_\lambda^-}\left[\frac{y_n^\gamma \bar{u}^{p-1}(y)}{|y - z^0|^\beta}\right]^{\frac{n}{\alpha}} dy\right\}^{\frac{\alpha}{n}} \|w_\lambda\|_{L^q(\hat{\Sigma}_\lambda^-)}. \tag{59}$$

Similar to the proof of (31), we can choose ζ sufficiently small so that for all $\lambda \in [\lambda_0, \ \lambda_0 + \zeta)$,

$$C\left\{\int_{\hat{\Sigma}_\lambda^-}\left[\frac{y_n^\gamma \bar{u}^{p-1}(y)}{|y - z^0|^\beta}\right]^{\frac{n}{\alpha}} dy\right\}^{\frac{\alpha}{n}} \leq \frac{1}{2}. \tag{60}$$

We postpone the proof of this inequality for a moment.

Now by (59) and (60), we have $\|w_\lambda\|_{L^q(\hat{\Sigma}_\lambda^-)} = 0$. Therefore $\hat{\Sigma}_\lambda^-$ must measure zero. Hence, for these values of $\lambda > \lambda_0$, we have

$$w_\lambda(x) \geq 0, \text{ a.e. } \forall\, x \in \hat{\Sigma}_\lambda \backslash B_\epsilon((z^0)^\lambda), \ \forall\, \epsilon > 0.$$

This contradicts the definition of λ_0. Therefore (58) must hold. That is, if $\lambda_0 < z_1^0 - \epsilon$, for any $\epsilon > 0$, then we must have

$$\bar{u}(x) = \bar{u}_{\lambda_0}(x), \text{ a.e. } \forall\, x \in \hat{\Sigma}_{\lambda_0} \backslash B_\epsilon((z^0)^{\lambda_0}).$$

Since $\bar{u}$ is singular at z^0, $\bar{u}$ must also be singular at $(z^0)^\lambda$. This is impossible because z^0 is the only singularity of $\bar{u}$. Hence we must have $\lambda_0 \geq z_1^0 - \epsilon$. Since ϵ is an arbitrary positive number, we have actually derived that

$$w_{z_1^0}(x) \geq 0, \ \text{ a.e. } \forall\, x \in \hat{\Sigma} z_1^0.$$

Entirely similarly, we can move the plane from near $x_1 = \infty$ to the left and derive that $w_{z_1^0}(x) \le 0$. Therefore we have

$$w_{z_1^0}(x) \equiv 0, \ \text{ a.e. } \forall\, x \in \hat{\Sigma} z_1^0.$$

Now we prove inequality (60). For any small $\eta > 0$, $\forall\, \epsilon > 0$, we can choose R sufficiently large so that

$$C\left\{\int_{(\mathbb{R}^n_+\setminus B_\epsilon(z^0))\setminus B_R}\left[\frac{y_n^\gamma \bar{u}^{p-1}(y)}{|y-z^0|^\beta}\right]^{\frac{n}{\alpha}} dy\right\}^{\frac{\alpha}{n}} \le \eta. \tag{61}$$

We fix this R and then show that the measure of $\hat{\Sigma}_\lambda^- \bigcap B_R$ is sufficiently small for λ close to λ_0. By (47), we have

$$w_{\lambda_0}(x) > 0 \tag{62}$$

in the interior of $\hat{\Sigma}_{\lambda_0}\setminus B_\epsilon((z^0)^{\lambda_0})$.

The rest of the proof is similar to the proof of (31). We only need to use $\hat{\Sigma}_\lambda\setminus B_\epsilon((z^0)^\lambda)$ instead of Σ_λ and $\hat{\Sigma}_{\lambda_0}\setminus B_\epsilon((z^0)^{\lambda_0})$ instead of Σ_{λ_0}.

(ii) Now we consider the case of $1 < p < \frac{n+\alpha}{n-\alpha}$. In this case, we assume u is locally bounded in $\mathbb{R}^n_+$ only, and we only need to carry the method of moving planes on $\bar{u} \equiv \bar{u}_{z^0}$ to show that it must be axially symmetric about the line passing through z^0 and parallel to x_n axis.

On the one hand, since u is locally bounded in $\mathbb{R}^n_+$, and y_n^γ, $\gamma \ge 0$ is also locally bounded in $\mathbb{R}^n_+$, similar to (52) and (53), for any domain Ω which is a positive distance away from z^0, we have

$$\int_\Omega \left[\frac{y_n^\gamma \bar{u}^{p-1}(y)}{|y-z^0|^\beta}\right]^{\frac{n}{\alpha}} dy < \infty. \tag{63}$$

On the other hand, by (38) and u is locally bounded, one can deduce

$$\bar{u}(y) = O\left(\frac{1}{|y|^{n-\alpha}}\right), \ \text{ as } |y| \to \infty. \tag{64}$$

Then, for any domain Ω which is a positive distance away from z^0, we have

$$\int_\Omega \bar{u}^q(y)dy < \infty, \tag{65}$$

as long as $q > \frac{n}{n-\alpha}$.

From (38) and (46), similar to (47), we can derive that

$$\bar{u}(x) - \bar{u}(x^\lambda) = \int_{\hat{\Sigma}_\lambda}\Big[G_\infty(x, y) - G_\infty(x^\lambda, y)\Big] y_n^\gamma \left[\frac{\bar{u}^p(y)}{|y-z^0|^\beta} - \frac{\bar{u}^p_\lambda(y)}{|y^\lambda - z^0|^\beta}\right] dy. \tag{66}$$

The proof of Theorem 3 in this case also consists of two steps.

Step 1. For any $\epsilon > 0$, define $\hat{\Sigma}_\lambda^-$ as (49). We show that for λ sufficiently negative, $\hat{\Sigma}_\lambda^-$ must be measure zero.

Similar to (50), by (46), (66) and the Mean Value Theorem, we obtain sufficiently negative values of λ and $x \in \hat{\Sigma}_\lambda^-$,

$$0 < \bar{u}(x) - \bar{u}_\lambda(x) \le C\int_{\hat{\Sigma}_\lambda^-}\frac{1}{|x-y|^{n-\alpha}}\left|\frac{y_n^\gamma \bar{u}^{p-1}(y)}{|y-z^0|^\beta}\right| |\bar{u}(y) - \bar{u}_\lambda(y)| dy. \tag{67}$$

By (63) and (65), for any $q > \frac{n}{n-\alpha}$, we can apply the Hardy-Littlewood-Sobolev inequality (20) and Hölder inequality for (67) to obtain (54).

Then, similar to the previous argument, we can also derive that $\hat{\Sigma}_\lambda^-$ must be measured zero, and hence obtain

$$w_\lambda(x) \geq 0, \text{ a.e. } \forall\, x \in \Sigma_\lambda. \tag{68}$$

Step 2. (Move the plane to the limiting position to derive symmetry.)

Inequality (68) provides a starting point to move the plane $\hat{T}_\lambda$. Now we start from the neighbourhood of $x_1 = -\infty$ and move the plane to the right as long as (68) holds to the limiting position. Define λ_0 as (57), the rest is entirely similar to the case when $\frac{n+\alpha}{n-\alpha} \leq p \leq \frac{n+2\gamma+\alpha}{n-\alpha}$. We can also conclude

$$w_{\lambda_0}(x) \equiv 0, \text{ a.e. } \forall\, x \in \hat{\Sigma}_{\lambda_0}, \ \lambda_0 = z_1^0.$$

This implies that $\bar{u}$ is symmetric about the plane $\hat{T}_{z^0}$.

Since we can choose any direction that is perpendicular to the x_n-axis as the x_1 direction, we have actually shown that the Kelvin transform of the solution $\bar{u}(x)$ is rotationally symmetric about the line parallel to x_n-axis and passing through z^0 either in *Possibility 2* of the case when $\frac{n+\alpha}{n-\alpha} \leq p \leq \frac{n+2\gamma+\alpha}{n-\alpha}$ or in the case when $1 < p < \frac{n+\alpha}{n-\alpha}$. Now for any two points X^1 and X^2, with $X^i = (x'^i,\ x_n) \in R^{n-1} \times [0,\ \infty),\ \ i = 1,2$, let z^0 be the projection of $\bar{X} = \frac{X^1+X^2}{2}$ on $\partial\mathbb{R}^n_+$. Set $Y^i = \frac{X^i - z^0}{|X^i - z^0|^2} + z^0,\ \ i = 1,2$. From the above arguments, it is easy to see $\bar{u}(Y^1) = \bar{u}(Y^1)$, hence $u(X^1) = u(X^2)$. This implies that u is independent of $x' = (x_1, \cdots, x_{n-1})$. That is $u = u(x_n)$. Then we show that this will contradict the finiteness of the integral

$$\int_{\mathbb{R}^n_+} G_\infty(x,\ y) y_n^\gamma u^p(y) dy.$$

Recall the Proposition 4 which provide the estimate of $G_\infty(x,\ y)$ while $\frac{t}{s}$ is sufficiently small, again set $x = (x', x_n), y = (y', y_n) \in \mathbb{R}^{n-1} \times (0, +\infty), r^2 = |x' - y'|^2$ and $a^2 = |x_n - y_n|^2$. If $u(x) = u(x_n)$ is a solution of

$$u(x) = \int_{\mathbb{R}^n_+} G_\infty(x,\ y) y_n^\gamma u^p(y) dy, \tag{69}$$

then for each fixed $x \in \mathbb{R}^n_+$, letting R be large enough, by Proposition 4, we have

$$\begin{aligned}
+\infty > u(x_n) &= \int_0^\infty y_n^\gamma u^p(y_n) \int_{\mathbb{R}^{n-1}} G_\infty(x,y) dy' dy_n \\
&\geq C \int_R^\infty y_n^\gamma u^p(y_n) y_n^{\alpha/2} \int_{\mathbb{R}^{n-1}\setminus B_R(0)} \frac{1}{|x-y|^n} dy' dy_n \\
&\geq C \int_R^\infty y_n^\gamma u^p(y_n) y_n^{\alpha/2} \int_R^\infty \frac{r^{n-2}}{(r^2+a^2)^{\frac{n}{2}}} dr dy_n \\
&\geq C \int_R^\infty y_n^{\gamma+\alpha/2} u^p(y_n) \frac{1}{|x_n - y_n|} \int_{R/a}^\infty \frac{\tau^{n-2}}{(\tau^2+1)^{\frac{n}{2}}} d\tau dy_n \\
&\geq C \int_R^\infty y_n^{\gamma+\alpha/2-1} u^p(y_n) dy_n.
\end{aligned} \tag{70}$$

(70) implies that there exists a sequence $\{y_n^i\} \to \infty$ as $i \to \infty$, such that

$$u^p(y_n^i)(y_n^i)^{\alpha/2+\gamma} \to 0. \tag{71}$$

Similar to (70), for any $x = (0, x_n) \in \mathbb{R}^n_+$, we derive that

$$+\infty > u(x_n) \geq C_0 \int_0^\infty y_n^\gamma u^p(y_n) y_n^{\alpha/2} \frac{1}{|x_n - y_n|} dy_n x_n^{\alpha/2}. \tag{72}$$

Let $x_n = 2R$ be sufficiently large. By (72), we deduce that

$$\begin{aligned} +\infty > u(x_n) &\geq C_0 \int_0^1 y_n^\gamma u^p(y_n) y_n^{\alpha/2} \frac{1}{|x_n - y_n|} dy_n x_n^{\alpha/2} \\ &\geq \frac{C_0}{2R}(2R)^{\alpha/2} \int_0^1 y_n^\gamma u^p(y_n) y_n^{\alpha/2} dy_n \\ &\geq C_1 (2R)^{\alpha/2-1} \\ &= C_1 x_n^{\alpha/2-1}. \end{aligned} \tag{73}$$

Then by (72) and (73), for $x_n = 2R$ sufficiently large, we also obtain

$$\begin{aligned} u(x_n) &\geq C_0 \int_{R/2}^R y_n^\gamma u^p(y_n) y_n^{\alpha/2} \frac{1}{|x_n - y_n|} dy_n x_n^{\alpha/2} \\ &\geq C_0 \int_{R/2}^R y_n^\gamma C_1^p y_n^{p(\alpha/2-1)} y_n^{\alpha/2} \frac{1}{|x_n - y_n|} dy_n x_n^{\alpha/2} \\ &\geq C_0 C_1^p R^{p(\alpha/2-1)+\gamma} \frac{2}{3R} (2R)^{\alpha/2} \int_{R/2}^R y_n^{\alpha/2} dy_n \\ &:= A R^{p(\alpha/2-1)+\alpha+\gamma} \\ &:= A_1 x_n^{p(\alpha/2-1)+\alpha+\gamma}. \end{aligned} \tag{74}$$

Continuing this way m times, for $x_n = 2R$, we have

$$u(x_n) \geq A(m,p,\alpha,\gamma) x_n^{p^m(\frac{\alpha}{2}-1)+\frac{p^m-1}{p-1}(\alpha+\gamma)}. \tag{75}$$

For any fixed α and γ in their respective domain, we choose m to be an integer greater than $\frac{-\alpha^2-\alpha\gamma+\gamma+3}{\alpha+\gamma}$ and 1. That is

$$m \geq \max\left\{ \left\lceil \frac{-\alpha^2 - \alpha\gamma + \gamma + 3}{\alpha + \gamma} \right\rfloor + 1,\ 1 \right\}, \tag{76}$$

where $\lceil a \rfloor$ is the integer part of a.

We claim that for such a choice of m, it holds

$$\tau(p) := \left[p^m(\frac{\alpha}{2} - 1) + \frac{p^m - 1}{p - 1}(\alpha + \gamma) \right] p + \frac{\alpha}{2} + \gamma \geq 0. \tag{77}$$

We postpone the proof of (77) for a moment. Now by (75) and (77), we derive that

$$u^p(x_n) x_n^{\alpha/2+\gamma} \geq A(m,p,\alpha,\gamma) x_n^{\tau(p)} \geq A(m,p,\alpha,\gamma) > 0, \tag{78}$$

for all x_n sufficiently large. This contradicts (71). So there is no positive solution of (69).

Now we are left to verify (77). In fact, if we let

$$f(p) := \tau(p)(p-1) = p^{m+2}\left(\frac{\alpha}{2} - 1\right) + p^{m+1}\left(\frac{\alpha}{2} + \gamma + 1\right) - \frac{\alpha}{2}p - \frac{\alpha}{2} - \gamma, \tag{79}$$

then

$$f'(p) = p^m \left[(m+2)\left(\frac{\alpha}{2} - 1\right)p + (m+1)\left(\frac{\alpha}{2} + \gamma + 1\right) \right] - \frac{\alpha}{2}. \tag{80}$$

We show that

$$f'(p) > 0, \ \text{ for } \ 1 < p \leq \frac{n + \alpha + 2\gamma}{n - \alpha}.$$

Since $p > 1$, it suffices to show

$$(m+2)\left(\frac{\alpha}{2} - 1\right)p + (m+1)\left(\frac{\alpha}{2} + \gamma + 1\right) \geq \frac{\alpha}{2}.$$

Due to the fact $\frac{\alpha}{2} - 1 < 0$, $n \geq 3$, and $p \leq \frac{n+\alpha+2\gamma}{n-\alpha}$, we only need to verify that

$$(m+2)\Big(\frac{\alpha}{2}-1\Big)\frac{3+\alpha+2\gamma}{3-\alpha} + (m+1)\Big(\frac{\alpha}{2}+\gamma+1\Big) \geq \frac{\alpha}{2},$$

which can be derived directly from (76).

Then we complete the proof of Theorem 3. □

Proof of Theorem 4. Similar to the previous argument, without the global integrability assumptions on the solution u, we again apply the Kelvin transform above-mentioned which is centered at $z^0 \in \partial\mathbb{R}^n_+$.

The proof is similar to the case when $1 < p < \frac{n+\alpha}{n-\alpha}$ in the proof of Theorem 3. The only difference is that in this case, γ may not be non-negative, but it doesn't matter since our assumption $y_n^\gamma u^{p-1} \in L^{\frac{n}{\alpha}}_{loc}(\mathbb{R}^n_+)$ ensure that (63) still holds. Hence, we are still able to apply the Hardy-Littlewood-Sobolev inequality (20) and Hölder inequality and finally derive that $u = u(x_n)$. But this still contradicts the finiteness of the integral

$$\int_{\mathbb{R}^n_+} G_\infty(x,y) y_n^\gamma u^p(y) dy.$$

And the proof of this contradiction is entirely similar to the corresponding section in the proof of Theorem 3. Hence, we complete the proof of Theorem 4. □

4. Liouville Theorems for More Generalized Equations

In this section, we mainly prove the non-existence of positive solutions for a more generalized Equation (8) under global and local integrability (or local boundness) assumptions respectively and thus establish Theorem 6 and 7.

Similar to the above, we will first establish Theorem 5, i.e., the equivalence between Equation (8) and integral Equation (9). The proof of the equivalence is almost the same as section 2, we only need to apply the local boundness of $f(y_n)$, here we omit the details.

Because $f(y_n)$ is positive and monotone nondecreasing, so just as y_n^γ, it's also locally bounded. And since we only used the monotonicity and local boundness of y_n^γ in the proof of Theorem 3, the proof of Theorem 6 is similar to Section 3, we only need to use $f(y_n)$ instead of y_n^γ.

As for the proof of Theorem 7, we need to exploit the same type of Kelvin transform as (37), and through an elementary calculation we get

$$\bar{u}_{z^0}(x) = \int_{\mathbb{R}^n_+} G_\infty(x,\ y) f\Big(\frac{y_n}{|y-z^0|^2}\Big)\frac{\bar{u}^p_{z^0}(y)}{|y-z^0|^{\beta'}}dy,\ \ \forall\, x \in \mathbb{R}^n_+ \backslash B_\epsilon(z^0),\ \epsilon > 0, \tag{81}$$

where $p \leq \tau$, $\tau = \frac{n+\alpha}{n-\alpha}$, $\beta' = 2n - (n-\alpha)(p+1) \geq 0$.

For simplicity, we denote $\bar{u}_{z^0}$ by $\bar{u}$. From (81) and (46), similar to (47), we can also easily derive

$$\bar{u}(x) - \bar{u}(x^\lambda) = \int_{\hat{\Sigma}_\lambda}\Big[G_\infty(x,\ y) - G_\infty(x^\lambda,\ y)\Big]\left[\frac{f(\frac{y_n}{|y-z^0|^2})\bar{u}^p(y)}{|y-z^0|^{\beta'}} - \frac{f(\frac{y_n}{|y^\lambda-z^0|^2})\bar{u}^p_\lambda(y)}{|y^\lambda-z^0|^{\beta'}}\right]dy. \tag{82}$$

Define $\hat{\Sigma}^-_\lambda$ as (49). Since $f(y_n)$ is monotone nondecreasing, then for any $y \in \hat{\Sigma}_\lambda$, we have $f(\frac{y_n}{|y-z^0|^2}) \leq f(\frac{y_n}{|y^\lambda-z^0|^2})$. Hence by (46), (82) and Mean Value Theorem, similar to (50), we have, for $x \in \hat{\Sigma}^-_\lambda$,

$$0 < \bar{u}(x) - \bar{u}(x^\lambda) \leq C\int_{\hat{\Sigma}^-_\lambda}\frac{1}{|x-y|^{n-\alpha}}\left|f\Big(\frac{y_n}{|y-z^0|^2}\Big)\frac{\bar{u}^{p-1}(y)}{|y-z^0|^{\beta'}}\right||\bar{u}(y)-\bar{u}_\lambda(y)|dy. \tag{83}$$

The rest of the proof is entirely similar to Section 3. By the local boundness of $f(y_n)$ and the conditions in Theorem 7, we can also derive $u = u(x_n)$. This will still contradict the finiteness of the integral

$$\int_{\mathbb{R}^n_+} G_\infty(x,\ y) f(y_n) u^p(y) dy.$$

And the proof of this contradiction is entirely similar to Section 3, we only need to use the local boundness of $f(y_n)$, here we omit the details. Hence, we complete the proof of Theorem 7.

5. Conclusions

In this paper, we obtain Liouville-type theorems for the following Dirichlet problem involving the fractional Laplacian equation

$$\begin{cases} (-\Delta)^{\alpha/2} u(x) = x_n^\gamma u^p(x),\ u(x) > 0, & x \in \mathbb{R}^n_+, \\ u(x) \equiv 0, & x \notin \mathbb{R}^n_+, \end{cases}$$

where $\gamma > -\alpha$, $0 < \alpha < 2$. We employ a direct method by studying an equivalent integral equation

$$u(x) = \int_{\mathbb{R}^n_+} G_\infty(x,y) y_n^\gamma u^p(y) dy.$$

Applying the method of moving planes in integral forms, we prove the non-existence of positive solutions under very weak conditions. Furthermore, we also extend the results to more general $f(x_n)$ instead of x_n^γ. This type of equation is also closely related to the $\sigma-$curvature problem in conformal geometry. The Liouville-type theorem is closely related to the priori estimates of the semi-linear fractional Laplacian equation. We believe that the method and results in this paper will give some interesting ideas to treat some problems, for example, the priori estimates of solutions to sign-changing fractional Nirenberg problem.

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