fractal and fractional

MDPI

Article

Existence and Ulam Type Stability for Impulsive Fractional Differential Systems with Pure Delay

Chaowen Chen and Mengmeng Li *

Department of Mathematics, Guizhou University, Guiyang 550025, China
* Correspondence: mmli@gzu.edu.cn

Abstract: Through literature retrieval and classification, it can be found that for the fractional delay impulse differential system, the existence and uniqueness of the solution and UHR stability of the fractional delay impulse differential system are rarely studied by using the polynomial function of the fractional delay impulse matrix. In this paper, we firstly introduce a new concept of impulsive delayed Mittag–Leffler type solution vector function, which helps us to construct a representation of an exact solution for the linear impulsive fractional differential delay equations (IFDDEs). Secondly, by using Banach's and Schauder's fixed point theorems, we derive some sufficient conditions to guarantee the existence and uniqueness of solutions of nonlinear IFDDEs. Finally, we obtain the Ulam–Hyers stability (UHs) and Ulam–Hyers–Rassias stability (UHRs) for a class of nonlinear IFDDEs.

Keywords: fractional differential equations; impulsive delayed Mittag–Leffler type vector function; existence of solution; Ulam–Hyers stability

MSC: 34A08; 34A37; 34D20

Citation: Chen, C.; Li, M. Existence and Ulam Type Stability for Impulsive Fractional Differential Systems with Pure Delay. *Fractal Fract.* **2022**, *6*, 742. https://doi.org/10.3390/fractalfract6120742

Academic Editors: Oana Brandibur and Eva Kaslik

Received: 10 November 2022
Accepted: 13 December 2022
Published: 15 December 2022

1. Introduction

Fractional calculus and fractional differential equations (FDEs) have been widely applied in mechanics, physics, biological and the other fields of science and engineering [1–6]. In recent decades, there has been an explosion in searching for the existence, uniqueness, stability and controllability of impulsive differential equations (IDEs) as researchers in epidemic, optimal control, mechanical and engineering studies are pouring into the field of research; we refer the reader to [7–12].

Impulsive fractional differential equations (IFDEs) have attracted great interest due to their potential applications in modeling dynamical systems involving genetic phenomena and mutations. Among the numerous research results, it is worth noting that the authors in [13] introduced a formula for solutions of the Cauchy problem of IFDEs and gave a counter example to prove that the previous results were incorrect. For more research results on the recent advances of existence, uniqueness, exponential stability, uniform stability and continuous dependence of IFDEs, one can see the research papers [13–20].

Meanwhile, impulsive fractional differential delay equations and fractional differential delay equations (FDDEs) are widely used to characterize the situation of their states depending on the previous time interval subject to abrupt changes. In [21], the authors obtained finite-time stability of solution for FDDEs by using the delayed single parameter Mittag–Leffler type matrix function. In [22], the authors introduced a concept of a delayed two parameter Mittag–Leffler type matrix function and gave an explicit formula of a solution for FDDEs. In [23], an explicit solution of the conformable FDDEs was given and the UH and UHR stability were discussed.

In [24], some sufficient conditions for the finite-time stability of IFDDEs were obtained by using the generalized Bellman–Gronwall's inequality, which extended some known results. In [25], the authors proposed a class of linear fractional difference equations with discrete-time delay and impulse effects. In [26], the authors studied the controllability of an impulsive fractional differential equation with infinite state-dependent delay in an arbitrary Banach space.

Motivated by [21–23], we first study the analytic representation of a solution of linear IFDDEs:

$$\begin{cases} ({}^C\mathbb{D}^{\alpha}_{0+}z)(x) = Az(x-h) + g(x),\ x \neq x_i,\ h > 0,\ x \in J, \\ z(x_i^+) = z(x_i^-) + C_i,\ x = x_i,\ i = 1,2,\ldots,r(T,0), \\ z(x) = \varpi(x),\ \ -h \leq x \leq 0, \end{cases} \tag{1}$$

where ${}^C\mathbb{D}^{\alpha}_{0+}z(\cdot)(0 < \alpha < 1)$ is the Caputo derivative, $A \in \mathbb{R}^{n\times n}$, $g \in C(J, \mathbb{R}^n)$, $J := [0, T]$, $T = k^*h$, $k^* \in \mathbb{N} := \{0, 1, 2, \ldots\}$, $r(T, 0)$ denotes the finite number of impulsive points which belong to $(0, T)$ and $\varpi \in C^1([-h, 0], \mathbb{R}^n)$. The symbols $z(x_i^+) = \lim\limits_{\epsilon\to 0^+} z(x_i + \epsilon)$ and $z(x_i^-) = \lim\limits_{\epsilon\to 0^-} z(x_i + \epsilon)$ represent the right and left limits of $z(x)$ at $x = x_i$, respectively.

In a conference held at Wisconsin University in 1940, Ulam [27] first raised the question of the stability of functional equations. The first answer to the question of Ulam [28] was given by Hyers in 1941 in the case of Banach spaces. In recent years, many researchers have been interested in the UHs and UHR of IFDEs and FDDEs. In [29], the authors introduced the concept of piecewise continuous solutions for impulsive Cauchy problems and discussed UHs for IFDEs. In [30], the authors gave existence and uniqueness of solutions as well as UH results for IFDEs. In [31], the authors established the existence, uniqueness, UHs and UHRs of solutions for FDEs. In [32], the authors introduced four Ulam type stability concepts for non-instantaneous IFDEs with state dependent delay and obtained sufficient conditions for Ulam type stability. However, there are few studies on UHs and UHRs of IFDDE, which is complex.

Therefore, we attempt to investigate the existence, uniqueness, UHs and UHRs of the nonlinear IFDDEs in this paper:

$$\begin{cases} ({}^C\mathbb{D}^{\alpha}_{0+}z)(x) = Az(x-h) + g(x, z(x)),\ x \neq x_i,\ x \in J, \\ z(x_i^+) = z(x_i^-) + C_i,\ x = x_i,\ i = 1,2,\ldots,r(T,0), \\ z(x) = \varpi(x),\ \ -h \leq x \leq 0. \end{cases} \tag{2}$$

Compared with [21–23], the novelties of this paper are as follows:

In this paper, the explicit solution of the Caputo fractional time delay impulse differential equation is given. A different from the system studied in [23], the fractional derivative is Caputo type and adds the impulsive condition to the system in this paper. In view of this difference, the impulsive delayed Mittag–Leffler type vector function newly constructed is important to solving the problem.

Although the ideas and methods adopted in the study of existence and UHs of solutions are similar to [21–23], the considered system is different, here, we not only give the representation of solutions via the new constructed impulsive delayed Mittag–Leffler type vector function but also study the existence and uniqueness of solutions, UHs and UHRs of (2).

The structure of this paper is as follows. Firstly, we seek for the fundamental solution vector for the linear homogeneous IFDDEs and give its exact solution. Secondly, we derive the exact representation of solution of (1) by using the delayed Mittag–Leffler type matrix functions and impulsive delayed Mittag–Leffler type vector function. Furthermore, we prove the existence and uniqueness of solutions of (2). Finally, we establish the conditions for the existence of the UHs and UHRs for the nonlinear IFDDEs.

2. Preliminaries

Set $PC(J, \mathbb{R}^n) := \{z : J \to \mathbb{R}^n : z \in C((x_i, x_{i+1}], \mathbb{R}^n), i = 1, 2, \ldots, r(T, 0);$ there exist $z(x_i^+)$ and $z(x_i^-)$, $z(x_i^-) = z(x_i)\}$ with $\|z\|_{PC} := \sup\limits_{x\in J} \|z(x)\|$; $C(J, \mathbb{R}^n)$ is the space of all the continuous functions from J to $\mathbb{R}^n$ with $\|z\|_C = \max_{x\in J} \|z(x)\|$ and $C^1(J, \mathbb{R}^n) = \{z \in C(J, \mathbb{R}^n) : z' \in C(J, \mathbb{R}^n)\}$. Let $z \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$; we introduce vector norm $\|z\| = \sum_{i=1}^{n} |z_i|$ and matrix norm $\|A\| = \max_{1\le j\le n} \sum_{i=1}^{n} |a_{ij}|$. Denote $\|\varpi\|_C = \max_{x\in[-h,0]} \|\varpi(x)\|$ and $\|\varpi'\|_C = \max_{x\in[-h,0]} \|\varpi'(x)\|$.

Definition 1 (see [2]). *Let $\alpha \in (0, 1)$ and $g : [0, +\infty) \to \mathbb{R}^n$. The Caputo fractional derivative of g can be written as*

$$({}^C\mathbb{D}^{\alpha}_{0+} g)(x) = \frac{1}{\Gamma(1-\alpha)} \int_0^x (x-t)^{-\alpha} g'(t)dt, x > 0.$$

Definition 2 (see [2]). *Let $\alpha \in (0, 1)$ and $g : [0, +\infty) \to \mathbb{R}^n$. The Riemann–Liouville fractional integral of g can be written as*

$$(\mathbb{I}^{\alpha}_{0+} g)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} g(t)dt, x > 0.$$

Definition 3 (see [21]). *Delayed one-parameter Mittag–Leffler type matrix function $\mathbb{E}_h^{A\cdot^{\alpha}} : \mathbb{R} \to \mathbb{R}^{n\times n}$ is defined by*

$$\mathbb{E}_h^{Ax^{\alpha}} = \begin{cases} \Theta, \quad -\infty < x < -h, \\ E, \quad -h \le x \le 0, \\ E + A\dfrac{x^{\alpha}}{\Gamma(\alpha+1)} + A^2\dfrac{(x-h)^{2\alpha}}{\Gamma(2\alpha+1)} + \ldots + A^j\dfrac{(x-(j-1)h)^{j\alpha}}{\Gamma(j\alpha+1)}, \\ \quad (j-1)h < x \le jh, \ j \in \mathbb{N}, \end{cases} \tag{3}$$

where Θ is a zero matrix and E is an identity matrix.

Definition 4 (see [21]). *The delayed two-parameter Mittag–Leffler type matrix function $\mathbb{E}_{h,\beta}^{A\cdot^{\alpha}} : \mathbb{R} \to \mathbb{R}^{n\times n}$ is defined by*

$$\mathbb{E}_{h,\beta}^{Ax^{\alpha}} = \begin{cases} \Theta, \quad -\infty < x < -h, \\ E\dfrac{(h+x)^{\alpha-1}}{\Gamma(\alpha)}, \quad -h \le x \le 0, \\ E\dfrac{(h+x)^{\alpha-1}}{\Gamma(\alpha)} + A\dfrac{x^{2\alpha-1}}{\Gamma(\alpha+\beta)} + A^2\dfrac{(x-h)^{3\alpha-1}}{\Gamma(2\alpha+\beta)} \\ + \ldots + A^j\dfrac{(x-(j-1)h)^{(j+1)\alpha-1}}{\Gamma(j\alpha+\beta)}, \\ (j-1)h < x \le jh, \ j \in \mathbb{N}. \end{cases} \tag{4}$$

Definition 5. *Let $x \in ((j-1)h, jh]$, $j = 1, 2, \ldots, k^*$; the impulsive delayed Mittag–Leffler type vector function $Z_{h,\alpha}(\cdot) : \mathbb{R} \to \mathbb{R}^n$ is defined by*

$$Z_{h,\alpha}(x) = \sum_{0<x_i<x} \mathbb{E}_h^{B(x-x_i-h)^{\alpha}} C_i. \tag{5}$$

Remark 1. *Let $x \in (jh, (j+1)h]$, $x_i = jh$, $j = 1, 2, \ldots, k^* - 1$; the impulsive delayed Mittag–Leffler type vector function $Z_{h,\alpha}(\cdot) : \mathbb{R} \to \mathbb{R}^n$ is defined by*

$$Z_{h,\alpha}(x) = \sum_{k=1}^{j} \mathbb{E}_h^{A(x-(k+1)h)^{\alpha}} C_k.$$

Lemma 1 (see [21]). *Let $x \in ((j-1)h, jh]$; $j \in \mathbb{N}$, one can obtain $({}^C\mathbb{D}^{\alpha}_{0+} \mathbb{E}_h^{A\cdot^{\alpha}})(x) = A\mathbb{E}_h^{A(x-h)^{\alpha}}$.*

Lemma 2 (see [21]). *Let* $x \in [(j-1)h, jh]$*;* $j \in \mathbb{N}$*; one can obtain* $\|\mathbb{E}_h^{Ax^\alpha}\| \le E_\alpha(\|A\|x^\alpha)$*, where* $E_\alpha(\cdot)$ *is defined by* $E_\alpha(z) = \sum\limits_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+1)}$.

Lemma 3 (see [22]). *Let* $x \in ((j-1)h, jh]$*;* $j \in \mathbb{N}$*;* $0 \le t < x$*; we have*

$$\|\mathbb{E}_{h,\alpha}^{A(x-h-t)^\alpha}\| \le \sum_{m=1}^{j} \frac{\|A\|^{m-1}}{\Gamma(m\alpha)}(x-(m-1)h)^{m\alpha-1}.$$

Lemma 4. *Let* $x \in ((j-1)h, jh]$*,* $j \in \mathbb{N}$ *,* $x_i \in \bigcup\limits_{r=2}^{j-1}(x-(j+1-r)h, x-(j-r)h]$*; we have*

$$\begin{aligned}&\int_{x_i+(j-r)h}^{x}(x-t)^{-\alpha}(t-x_i-(j-r)h)^{(j-r)\alpha-1}dt\\&=(x-x_i-(j-r)h)^{(j-r-1)\alpha}\mathbb{B}[(j-r)\alpha, 1-\alpha],\end{aligned}$$

where $\mathbb{B}[l,m] = \int_0^1 s^{l-1}(1-s)^{m-1}ds$.

Proof. By calculation, one can obtain

$$\begin{aligned}&\int_{x_i+(j-r)h}^{x}(x-t)^{-\alpha}(t-x_i-(j-r)h)^{(j-r)\alpha-1}dt\\=\;&(x-x_i-(j-r)h)^{(j-r-1)\alpha}\int_0^1 \xi^{(j-r-1)\alpha-1}(1-\xi)^{-\alpha}d\xi\\=\;&(x-x_i-(j-r)h)^{(j-r-1)\alpha}\mathbb{B}[(j-r)\alpha, 1-\alpha].\end{aligned}$$

□

Lemma 5. *Let* $x \in ((j-1)h, jh]$*,* $j \in \mathbb{N}$ *and* $x_i \in (0, x)$*; we obtain*

$$\|\mathbb{E}_h^{A(x-x_i-h)^\alpha}\| \le E_\alpha(\|A\|(x-x_i)^\alpha).$$

Proof. Let $x \in ((j-1)h, jh]$ and $x_i \in (0, x-(j-1)h]$; one can obtain

$$\begin{aligned}\|\mathbb{E}_h^{A(x-x_i-h)^\alpha}\| &\le \sum_{m=0}^{j-1}\left\|A^m\frac{(x-x_i-mh)^{m\alpha}}{\Gamma(m\alpha+1)}\right\|\\&\le \sum_{m=0}^{j-1}\|A\|^m\frac{(x-x_i)^{m\alpha}}{\Gamma(m\alpha+1)}\\&\le E_\alpha(\|A\|(x-x_i)^\alpha).\end{aligned}$$

Let $x \in ((j-1)h, jh]$ and $x_i \in (x-(j+1-r)h, x-(j-r)h]$, $r = 2, 3 \dots j-1$; one can obtain

$$\begin{aligned}\|\mathbb{E}_h^{A(x-x_i-h)^\alpha}\| &\le \sum_{m=0}^{j-r}\left\|A^m\frac{(x-x_i-mh)^{m\alpha}}{\Gamma(m\alpha+1)}\right\|\\&\le \sum_{m=0}^{j-r}\|A\|^m\frac{(x-x_i)^{m\alpha}}{\Gamma(m\alpha+1)}\\&\le E_\alpha(\|A\|(x-x_i)^\alpha).\end{aligned}$$

Let $x \in ((j-1)h, jh]$ and $x_i \in (x-h, x)$; one can obtain

$$\|\mathbb{E}_h^{A(x-x_i-h)^\alpha}\| = 0 < E_\alpha(\|A\|(x-x_i)^\alpha).$$

□

3. The General Solution of Homogeneous System

In this part, we discuss the exact solution of

$$\begin{cases} ({}^C\mathbb{D}^{\alpha}_{0^+}z)(x) = Az(x-h),\ x \neq x_i,\ x \in J, \\ z(x_i^+) = z(x_i^-) + C_i,\ x = x_i,\ i = 1,2,\dots r(T,0), \\ z(x) = \varpi(x),\ -h \leq x \leq 0, \end{cases} \tag{6}$$

by using (3)–(5).

Lemma 6. *Let $x \in ((j-1)h, jh]$; $j \in \mathbb{N}$ and $x_i \in (0,x)$ is arbitrarily fixed impulsive points; we have*

$${}^C\mathbb{D}^{\alpha}_{0^+}(\mathbb{E}_h^{A(\cdot-x_i-h)^{\alpha}}C_i)(x) = A\mathbb{E}_h^{A(x-x_i-2h)^{\alpha}}C_i.$$

Proof. Let $x_i \in (x-h, x]$; we have

$$\mathbb{E}_h^{A(x-x_i-h)^{\alpha}}C_i = EC_i. \tag{7}$$

By Definition 1 and (7), one can obtain

$$\begin{aligned} {}^C\mathbb{D}^{\alpha}_{0^+}(\mathbb{E}_h^{A(\cdot-x_i-h)^{\alpha}}C_i)(x) &= \tfrac{1}{\Gamma(1-\alpha)}\int_0^x (EC_i)'(x-t)^{-\alpha}dt \\ &= A\mathbb{E}_h^{A(x-x_i-2h)^{\alpha}}C_i. \end{aligned}$$

For any $x_i \in (x-(j+1-r)h, x-(j-r)h]$ and $r = 2,3,\dots,j-1$, we have

$$\mathbb{E}_h^{A(x-x_i-h)^{\alpha}}C_i = \sum_{m=0}^{j-r} A^m \frac{(x-x_i-mh)^{m\alpha}}{\Gamma(m\alpha+1)}C_i. \tag{8}$$

By Definition 1 and (8), one can obtain

$$\begin{aligned} &({}^C\mathbb{D}^{\alpha}_{0^+}\mathbb{E}_h^{A(\cdot-x_i-h)^{\alpha}}C_i)(x) \\ =\ & \tfrac{1}{\Gamma(1-\alpha)}\int_0^x (x-t)^{-\alpha}(\mathbb{E}_h^{A(t-x_i-h)^{\alpha}})'C_i dt \\ =\ & \tfrac{1}{\Gamma(1-\alpha)}\Bigg[\int_{x_i+h}^{x_i+2h}(x-t)^{-\alpha}A\tfrac{\alpha(t-x_i-h)^{\alpha-1}}{\Gamma(\alpha+1)}C_i dt + \\ & \dots + \int_{x_i+(j-r)h}^{x}(x-t)^{-\alpha}\sum_{m=0}^{j-r}A^m\tfrac{m\alpha(t-x_i-mh)^{m\alpha-1}}{\Gamma(m\alpha+1)}C_i dt\Bigg] \\ =\ & \tfrac{1}{\Gamma(1-\alpha)}\Bigg[\int_{x_i+h}^{x}(x-t)^{-\alpha}A\tfrac{\alpha(t-x_i-h)^{\alpha-1}}{\Gamma(\alpha+1)}C_i dt \\ & + \int_{x_i+2h}^{x}(x-t)^{-\alpha}A^2\tfrac{2\alpha(t-x_i-2h)^{2\alpha-1}}{\Gamma(2\alpha+1)}C_i dt \\ & \dots + \int_{x_i+(j-r)h}^{x}(x-t)^{-\alpha}A^{j-r}\tfrac{(j-r)\alpha(t-x_i-(j-r)h)^{(j-r)\alpha-1}}{\Gamma((j-r)\alpha+1)}C_i dt\Bigg] \\ =\ & AEC_i + A^2\tfrac{(x-x_i-2h)^{\alpha}}{\Gamma(\alpha+1)}C_i + \dots + A^{j-r}\tfrac{(x-x_i-(j-r)h)^{(j-r-1)\alpha}}{\Gamma((j-r-1)\alpha+1)}C_i \\ =\ & A\sum_{m=0}^{j-r-1}B^m\tfrac{(x-h-x_i-mh)^{m\alpha}}{\Gamma(m\alpha+1)}C_i \\ =\ & A\mathbb{E}_h^{A(x-x_i-2h)^{\alpha}}C_i. \end{aligned}$$

For any fixed $x_i \in (0, x-(j-1)h]$, one can obtain

$$\mathbb{E}_h^{A(x-x_i-h)^{\alpha}}C_i = \sum_{m=0}^{j-1} A^m \frac{(x-x_i-mh)^{m\alpha}}{\Gamma(m\alpha+1)}C_i. \tag{9}$$

By Definition 1 and (9), one can obtain

$$\begin{aligned}
&({}^{C}\mathbb{D}^{\alpha}_{0+}\mathbb{E}_h^{A(\cdot-x_i-h)^{\alpha}}C_i)(x)\\
=\ &\frac{1}{\Gamma(1-\alpha)}\Bigg[\int_{x_i+h}^{x}(x-t)^{-\alpha}A\frac{\alpha(t-x_i-h)^{\alpha-1}}{\Gamma(\alpha+1)}C_idt\\
&+\int_{x_i+2h}^{x}(x-t)^{-\alpha}A^2\frac{2\alpha(t-x_i-2h)^{2\alpha-1}}{\Gamma(2\alpha+1)}C_idt\\
&\ldots+\int_{x_i+(j-1)h}^{x}(x-t)^{-\alpha}A^{j-1}\frac{j\alpha(t-x_i-(j-1)h)^{j\alpha-1}}{\Gamma(j\alpha+1)}C_idt\Bigg]\\
=\ &AEC_i+A^2\frac{(x-x_i-2h)^{\alpha}}{\Gamma(\alpha+1)}C_i+\ldots+A^{j-1}\frac{(x-x_i-(j-1)h)^{(j-2)\alpha}}{\Gamma((j-2)\alpha+1)}C_i\\
=\ &A\sum_{m=0}^{j-2}A^m\frac{(x-h-x_i-mh)^{m\alpha}}{\Gamma(m\alpha+1)}C_i\\
=\ &A\mathbb{E}_h^{A(x-x_i-2h)^{\alpha}}C_i.
\end{aligned}$$

□

Lemma 7. *Impulsive delayed Mittag–Leffler type vector function $Z_{h,\alpha}(\cdot)$ is the fundamental solution of (6).*

Proof. By Definition 1 and Lemma 6, we have

$$\begin{aligned}
{}^{C}\mathbb{D}^{\alpha}_{0+}\left(\sum_{0<x_i<x}\mathbb{E}_h^{A(\cdot-x_i-h)^{\alpha}}C_i\right)(x) &= \frac{1}{\Gamma(1-\alpha)}\int_0^x(x-t)^{-\alpha}\sum_{0<x_i<x}(\mathbb{E}_h^{A(t-x_i-h)^{\alpha}})'C_i\mathrm{d}t\\
&= \left(\sum_{0<x_i<x}{}^{C}D^{\alpha}_{0+}\mathbb{E}_h^{A(t-x_i-h)^{\alpha}}C_i\right)(x)\\
&= A\sum_{0<x_i<x}\mathbb{E}_h^{A(x-x_i-2h)^{\alpha}}C_i.
\end{aligned}$$

For any $x_i \in (0,x)$ and $i = 1,2,\ldots,r(x,0)$, we verify that $Z_{h,\alpha}(x_i^+) = Z_{h,\alpha}(x_i^-) + C_i$.

$$\begin{aligned}
Z_{h,\alpha}(x_i^+) &= \sum_{k=1}^{i}\mathbb{E}_h^{A(x_i^+-x_k-h)^{\alpha}}C_k\\
&= \sum_{k=1}^{i-1}\mathbb{E}_h^{A(x_i^--x_k-h)^{\alpha}}C_k+C_i,\ x_i^+\in(x_i,x_{i+1}],\\
Z_{h,\alpha}(x_i^-) &= \sum_{k=1}^{i-1}\mathbb{E}_h^{A(x_i^--x_k-h)^{\alpha}}C_k,\ x_i^-\in(x_{i-1},x_i],
\end{aligned}$$

which implies that $Z_{h,\alpha}(x_i^+) = Z_{h,\alpha}(x_i^-) + C_i$. □

Theorem 1. *The solution $z \in PC([-h,T],\mathbb{R}^n)$ of (6) has the following form:*

$$z(x) = \mathbb{E}_h^{Ax^{\alpha}}\varpi(-h) + \int_{-h}^{0}\mathbb{E}_h^{A(x-h-s)^{\alpha}}\varpi'(s)ds + \sum_{0<x_i<x}\mathbb{E}_h^{A(x-x_i-h)^{\alpha}}C_i.$$

Proof. The argument is similar to that in ([21] Theorem 3.2).

We find the exact solution of (6) that satisfies $z(x) = \varpi(x)$, $-h \le x \le 0$, in the form

$$z(x) = \mathbb{E}_h^{Ax^{\alpha}}c + \int_{-h}^{0}\mathbb{E}_h^{A(x-h-s)^{\alpha}}y(s)ds + Z_{h,\alpha}(x),\ \ -h \le x \le T.$$

where c is an unknown constant vector, $y(\cdot) \in C^1([0,+\infty),\mathbb{R}^n)$ is an unknown function. Note that

$$\mathbb{E}_h^{Ax^{\alpha}}c + \int_{-h}^{0}\mathbb{E}_h^{A(x-h-s)^{\alpha}}y(s)ds = \varpi(x),\ \ -h \le x \le 0.$$

Let $x = -h$; we have

$$\mathbb{E}_h^{A(-h)^{\alpha}} = E,\ \mathbb{E}_h^{A(-2h-s)^{\alpha}} = \Theta,\ \ -h < s \le 0,\ \mathbb{E}_h^{A(-2h-s)^{\alpha}} = E, s = -h.$$

Thus, we obtain $c = \varpi(-h)$ and

$$\varpi(x) = \varpi(-h) + \int_{-h}^{0} \mathbb{E}_h^{A(x-h-s)^\alpha} y(s)ds, \ -h \le x \le 0.$$

For $-h \le x \le 0$, when $-h \le s \le x$, one can obtain $-h \le x-h-s \le x$ and when $x \le s \le 0$, one can obtain $x-h \le x-h-s \le -h$. Let $-h \le x \le 0$, we obtain

$$\begin{aligned} \varpi(x) &= \varpi(-h) + \textstyle\int_{-h}^{x} \mathbb{E}_h^{A(x-h-s)^\alpha} y(s)ds \\ &= \varpi(-h) + \textstyle\int_{-h}^{x} y(s)ds. \end{aligned}$$

Thus, one can obtain $y(x) = \varpi'(x)$.

Let $x \in [0,T]$ and $x_i \in (0,x]$; we have

$$\begin{aligned} z(x_i^+) &= \mathbb{E}_h^{A(x_i^+)^\alpha}\varpi(-h) + \textstyle\int_{-h}^{0} \mathbb{E}_h^{A(x_i^+-h-s)^\alpha}\varpi'(s)ds + \sum_{k=1}^{i} \mathbb{E}_h^{A(x_i^+-x_k-h)^\alpha} C_k, \\ &= \mathbb{E}_h^{A(x_i)^\alpha}\varpi(-h) + \textstyle\int_{-h}^{0} \mathbb{E}_h^{A(x_i-h-s)^\alpha}\varpi'(s)ds + \sum_{k=1}^{i-1} \mathbb{E}_h^{A(x_i-x_k-h)^\alpha} C_k + C_i, \\ &= z(x_i) + C_i. \end{aligned}$$

□

Theorem 2. *The particular solution $\tilde{z}(x) \in PC([-h,T],\mathbb{R}^n)$ of (1) with $\tilde{z}(x) \equiv \mathbf{0} = (0,0,\dots,0)^\top$, $-h \le x \le 0$ can be written as*

$$\tilde{z}(x) = \int_0^x \mathbb{E}_{h,\alpha}^{A(x-h-t)^\alpha} g(t)dt.$$

Proof. The proof is analogous to the one in ([22] Theorem 3.1), so we omit the details. □

Combining with Theorems 1 and 2, any expression of the exact solution of (1) is obtained.

Theorem 3. *The exact solution $z \in PC([-h,T],\mathbb{R}^n)$ of (1) with $z(x) = \varpi(x)$; $-h \le x \le 0$ can be written as*

$$\begin{aligned} z(x) &= \mathbb{E}_h^{Ax^\alpha}\varpi(-h) + \textstyle\int_{-h}^{0} \mathbb{E}_h^{A(x-h-s)^\alpha}\varpi'(s)ds + \sum\limits_{0<x_i<x} \mathbb{E}_h^{A(x-x_i-h)^\alpha} C_i \\ &\quad + \textstyle\int_0^x \mathbb{E}_{h,\alpha}^{A(x-h-t)^\alpha} g(t)dt. \end{aligned}$$

According to the Theorems 2 and 3, the function $z(\cdot)$ is called a solution of (2) if $z(x)$ satisfies the following form:

$$\begin{aligned} z(x) &= \mathbb{E}_h^{Ax^\alpha}\varpi(-h) + \textstyle\int_{-h}^{0} \mathbb{E}_h^{A(x-h-s)^\alpha}\varpi'(s)ds + \sum\limits_{0<x_i<x} \mathbb{E}_h^{A(x-x_i-h)^\alpha} C_i \\ &\quad + \textstyle\int_0^x \mathbb{E}_{h,\alpha}^{A(x-h-t)^\alpha} g(t,z(t))dt. \end{aligned} \tag{10}$$

4. Existence and Uniqueness of the Solution

In this part, we establish the existence and uniqueness results of (2). Let

$$\begin{aligned} \Phi(x) &= \textstyle\sum_{m=1}^{j} \frac{\|A\|^{m-1}}{\Gamma(m\alpha)}(x-(m-1)h)^{m\alpha-1}, \ x \in ((j-1)h, jh], \\ M &= \textstyle\int_{-h}^{0} \|\varpi'(s)\|ds < +\infty. \end{aligned}$$

We assume that the following conditions:

$[H_1]$ There exists an $L > 0$ such that $\|g(x,z)\| \le L\|z\| + N$ satisfies in the case of $g \in C(J \times \mathbb{R}^n, \mathbb{R}^n), x \in J, z \in \mathbb{R}^n$.

$[H_2]$ Let $\rho = LT\Phi(T) < 1$.

Theorem 4. *If* $[H_1]$ *and* $[H_2]$ *are satisfied, then* (2) *has at least one solution* $z \in PC([-h,T],\mathbb{R}^n)$.

Proof. According to Theorem 3 and (10), the operator Λ on $\mathfrak{B}_r$ can be written as

$$\begin{aligned}(\Lambda z)(x) &= \mathbb{E}_h^{Ax^\alpha}\varpi(-h) + \int_{-h}^{0} \mathbb{E}_h^{A(x-h-s)^\alpha}\varpi'(s)ds + \sum_{0<x_i<x} \mathbb{E}_h^{A(x-x_i-h)^\alpha} C_i \\ &\quad + \int_0^x \mathbb{E}_{h,\alpha}^{A(x-h-t)^\alpha} g(t,z(t))dt.\end{aligned} \tag{11}$$

where $\mathfrak{B}_r := \{z \in PC([-h,T],\mathbb{R}^n), \|z\|_{PC} \le r \text{ and } r > \frac{\kappa}{1-\rho}\}$ and $\kappa = E_\alpha(\|A\|T^\alpha)(\|\varpi(-h)\| + M) + \sum_{0<x_i<x} E_\alpha(\|A\|(T-x_i)^\alpha)\|C_i\| + \Phi(T)NT$.

Firstly, we show that $\Lambda(\mathfrak{B}_r) \subset \mathfrak{B}_r$. For any $z \in \mathfrak{B}_r$, we have

$$\begin{aligned}\|(\Lambda z)(x)\| &\le \|\mathbb{E}_h^{Ax^\alpha}\|\|\varpi(-h)\| + \int_{-h}^{0} \|\mathbb{E}_h^{A(x-h-s)^\alpha}\|\|\varpi'(s)\|ds + \sum_{0<x_i<x} \|\mathbb{E}_h^{A(x-x_i-h)^\alpha} C_i\| \\ &\quad + \int_0^x \|\mathbb{E}_{h,\alpha}^{A(x-h-t)^\alpha}\|\|g(t,z(t))\|dt \\ &\le E_\alpha(\|A\|x^\alpha)\|\varpi(-h)\| + E_\alpha(\|A\|x^\alpha)\int_{-h}^{0}\|\varpi'(s)\|ds \\ &\quad + \sum_{0<x_i<x} E_\alpha(\|A\|(x-x_i)^\alpha)\|C_i\| + \int_0^x \|\mathbb{E}_{h,\alpha}^{A(x-h-t)^\alpha}\|(L\|z\|_{PC} + N)dt \\ &\le E_\alpha(\|A\|T^\alpha)(\|\varpi(-h)\| + M) + \Phi(T)NT + LT\Phi(T)r \\ &\quad + \sum_{0<x_i<x} E_\alpha(\|A\|(T-x_i)^\alpha)\|C_i\| \\ &\le \kappa + \rho r < r,\end{aligned}$$

which implies that $\Lambda(\mathfrak{B}_r) \subset \mathfrak{B}_r$.

Secondly, we check that continuity of Λ. Let $\{z_n(\cdot)\}_{n=1}^{\infty}$ be a Cauchy sequence such that $z_n(\cdot) \to z(\cdot)(n \to \infty)$ in $\mathfrak{B}_r$, $g_n(\cdot) = g(\cdot, z_n(\cdot))$ and $g(\cdot) = g(\cdot, z(\cdot))$. For any $x \in J$, we have

$$\begin{aligned}\|(\Lambda z_n)(x) - (\Lambda z)(x)\| &\le \int_0^x \|\mathbb{E}_{h,\alpha}^{A(x-h-t)^\alpha}\|\|g(t,z_n(t)) - g(t,z(t))\|dt \\ &\le T\Phi(T)\|g_n - g\|_{PC},\end{aligned}$$

this yields that $\|(\Lambda z_n) - (\Lambda z)\|_{PC} \le T\Phi(T)\|g_n - g\|_{PC}$.

Finally, we show that Λ is equicontinuous. For any $z \in \mathfrak{B}_r$ and $0 < x \le x + \triangle x \le T$, we obtain

$$\begin{aligned}\|(\Lambda z)(x+\triangle x) - (\Lambda z)(x)\| &\le \|(\mathbb{E}_h^{A(x+\triangle x)^\alpha} - \mathbb{E}_h^{Ax^\alpha})\|\|\varpi(-h)\| \\ &\quad + \int_{-h}^{0} \|\mathbb{E}_h^{A(x+\triangle x-h-s)^\alpha} - \mathbb{E}_h^{A(x-h-s)^\alpha}\|\|\varpi'(s)\|ds \\ &\quad + \sum_{0<x_i<x} \|\mathbb{E}_h^{A(x+\triangle x-x_i-h)^\alpha} - \mathbb{E}_h^{A(x-x_i-h)^\alpha}\|\|C_i\| \\ &\quad + \int_0^x \|\mathbb{E}_{h,\alpha}^{A(x+\triangle x-h-s)^\alpha} - \mathbb{E}_{h,\alpha}^{A(x-h-s)^\alpha}\|\|g(s,z(s))\|ds \\ &\quad + \int_x^{x+\triangle x} \|\mathbb{E}_{h,\alpha}^{A(x+\triangle x-h-s)^\alpha}\|\|g(s,z(s))\|ds \\ &\le \mathfrak{I}_1 + \mathfrak{I}_2 + \mathfrak{I}_3 + \mathfrak{I}_4 + \mathfrak{I}_5,\end{aligned}$$

where

$$\begin{aligned}\mathfrak{I}_1 &= \|(\mathbb{E}_h^{A(x+\triangle x)^\alpha} - \mathbb{E}_h^{Ax^\alpha})\|\|\varpi(-h)\|, \\ \mathfrak{I}_2 &= \int_{-h}^{0} \|\mathbb{E}_h^{A(x+\triangle x-h-s)^\alpha} - \mathbb{E}_h^{A(x-h-s)^\alpha}\|\|\varpi'(s)\|ds \\ &\le \|\varpi'\|_C \int_{-h}^{0} \|\mathbb{E}_h^{A(x+\triangle x-h-s)^\alpha} - \mathbb{E}_h^{A(x-h-s)^\alpha}\|ds, \\ \mathfrak{I}_3 &= \sum_{0<x_i<x} \|\mathbb{E}_h^{A(x+\triangle x-x_i-h)^\alpha} - \mathbb{E}_h^{A(x-x_i-h)^\alpha}\|\|C_i\|, \\ \mathfrak{I}_4 &= \int_0^x \|\mathbb{E}_{h,\alpha}^{A(x+\triangle x-h-s)^\alpha} - \mathbb{E}_{h,\alpha}^{A(x-h-s)^\alpha}\|\|g(s,z(s))\|ds \\ &\le (L\|z\|_{PC} + N)\int_0^x \|\mathbb{E}_{h,\alpha}^{A(x+\triangle x-h-s)^\alpha} - \mathbb{E}_{h,\alpha}^{A(x-h-s)^\alpha}\|ds, \\ \mathfrak{I}_5 &= \int_x^{x+\triangle x} \|\mathbb{E}_{h,\alpha}^{B(x+\triangle x-h-s)^\alpha}\|\|g(s,z(s))\|ds.\end{aligned}$$

Let $-h < x \leq x + \triangle x < T$ as $\triangle x \to 0$; we have

$$\begin{array}{l} \mathbb{E}_h^{A(x+\triangle x)^\alpha} \to \mathbb{E}_h^{Ax^\alpha}, \\ \mathbb{E}_h^{A(x+\triangle x-h-s)^\alpha} \to \mathbb{E}_h^{A(x-h-s)^\alpha}, \\ \sum_{0<x_i<x} \mathbb{E}_h^{A(x+\triangle x-x_i-h)^\alpha} \to \sum_{0<x_i<x} \mathbb{E}_h^{A(x-x_i-h)^\alpha}, \\ \mathbb{E}_{h,\alpha}^{A(x+\triangle x-h-t)^\alpha} \to \mathbb{E}_{h,\alpha}^{A(x-h-t)^\alpha}, \end{array}$$

This yields that $\mathfrak{I}_1 \to 0$, $\mathfrak{I}_2 \to 0$, $\mathfrak{I}_3 \to 0$, $\mathfrak{I}_4 \to 0$ as $\triangle x \to 0$. For $\mathfrak{I}_5$, one can obtain

$$\begin{array}{rcl} \mathfrak{I}_5 & \leq & (L\|z\|_{PC} + N) \int_x^{x+\triangle x} \|\mathbb{E}_{h,\alpha}^{A(x+\triangle x-h-s)^\alpha}\| ds \\ & = & \|\mathbb{E}_{h,\alpha}^{A(x+\triangle x-h-\xi)^\alpha}\|(L\|z\|_{PC} + N)\triangle x \to 0 \text{ as } \triangle x \to 0. \end{array}$$

Therefore, one can obtain $\|(\Lambda z)(x + \triangle x) - (\Lambda z)(x)\| \to 0$ as $\triangle x \to 0$. □

$[H_3]$ There exists an $\widetilde{L} > 0$ that $\|g(x,z) - g(x,\widetilde{z})\| \leq \widetilde{L}\|z - \widetilde{z}\|$ satisfies in the case of $z, \widetilde{z} \in \mathbb{R}^n$.

$[H_4]$ Let $\widetilde{\rho} = \widetilde{L}T\Phi(T) < 1$.

Theorem 5. *If* $[H_3]$ *and* $[H_4]$ *are satisfied, then* (2) *has a unique solution* $z \in PC(J, \mathbb{R}^n)$.

Proof. It is easy to prove that $\Lambda : \mathfrak{B}_{\widetilde{r}} \to \mathfrak{B}_{\widetilde{r}}$ defined in (11) is uniformly bounded by using the Theorem 4. Now, we check that Λ is a Banach operator. For any $z, \widetilde{z} \in \mathfrak{B}_{\widetilde{r}}$, where $\mathfrak{B}_{\widetilde{r}} := \{z \in PC([-h,T], \mathbb{R}^n), \|z\|_{PC} \leq \widetilde{r}$ with $\widetilde{r} > \frac{\widetilde{\kappa}}{1-\widetilde{\rho}}\}$, $\widetilde{\kappa} = (\|\varpi(-h)\| + M)E_\alpha(\|A\|T^\alpha) + \sum_{0<x_i<x} E_\alpha(\|A\|(T-x_i)^\alpha)\|C_i\| + \Phi(T)T\|\widetilde{g}\|$ and $\|\widetilde{g}\| = \sup_{x\in J}\|g(x,\mathbf{0})\|$. For any $x \in [-h,T]$, one can obtain

$$\begin{array}{rcl} \|(\Lambda z)(x) - (\Lambda \widetilde{z})(x)\| & \leq & \int_0^x \|\mathbb{E}_{h,\alpha}^{A(x-h-t)^\alpha}\| \|g(t,z(t)) - g(t,\widetilde{z}(t))\| dt \\ & \leq & \widetilde{L} \int_0^x \|\mathbb{E}_{h,\alpha}^{A(x-h-t)^\alpha}\| \|z(t) - \widetilde{z}(t)\| dt \\ & \leq & \widetilde{L}\|z - \widetilde{z}\|_{PC} \int_0^x \|\mathbb{E}_{h,\alpha}^{A(x-h-t)^\alpha}\| dt \\ & \leq & \widetilde{L}T\Phi(T)\|z - \widetilde{z}\|_{PC}, \end{array}$$

which implies that $\|\Lambda z - \Lambda \widetilde{z}\|_{PC} \leq \widetilde{\rho}\|z - \widetilde{z}\|_{PC}$. □

5. Ulam Type Stability Results of (2)

In this part, we establish the Ulam type stability results of nonlinear IFDDEs. Let $\varepsilon, \phi > 0$, $\widetilde{J} := [-h,0] \cup J$ and $\psi \in C(J, \mathbb{R}^+ := (0,+\infty))$. Consider (2) and the following inequalities:

$$\begin{cases} \|{}^C\mathbb{D}_{0+}^\alpha \Psi(x) - A\Psi(x-h) - g(x,\Psi(x))\| \leq \varepsilon, \ x \in J, \\ \|\Psi(x_i^+) - \Psi(x_i^-) - C_i\| \leq \phi, \ i = 1,2,\ldots,r(T,0), \\ \Psi(x) = \varpi(x), \ x \in [-h,0], \end{cases} \quad (12)$$

and

$$\begin{cases} \|{}^C\mathbb{D}_{0+}^\alpha \Psi(x) - A\Psi(x-h) - g(x,\Psi(x))\| \leq \varepsilon\psi(x), \ x \in J \\ \|\Psi(x_i^+) - \Psi(x_i^-) - C_i\| \leq \varepsilon\phi, \ i = 1,2,\ldots,r(T,0), \\ \Psi(x) = \varpi(x), \ x \in [-h,0]. \end{cases} \quad (13)$$

Definition 6. *System* (2) *is said to be Ulam–Hyers stable if there exists* $K > 0$ *such that for every* $\varepsilon > 0$ *and for any solution* $\Psi \in PC(\widetilde{J}, \mathbb{R}^n)$ *satisfying* (12)*, there exists a solution* $z \in PC(\widetilde{J}, \mathbb{R}^n)$ *of* (2) *such that*

$$\|\Psi(x) - z(x)\| \leq K(\varepsilon + \phi), \ x \in \widetilde{J}.$$

Remark 2. *If $\Psi \in PC(\widetilde{J}, \mathbb{R}^n)$ is a solution of inequality (12), then there exist $D_i \in \mathbb{R}^n$ and $\mathcal{Z} \in C(J, \mathbb{R}^n)$ such that*

(a) $\|\mathcal{Z}(x)\| \leq \varepsilon, \|D_i\| \leq \phi,\ x \in J,\ i = 1, 2, \ldots, r(T, 0)$.
(b) $({}^{C}\mathbb{D}^{\alpha}_{0+}\Psi)(x) = A\Psi(x-h) + g(x, \Psi(x)) + \mathcal{Z}(x),\ x \in J$.
(c) $\Psi(x_i^+) = \Psi(x_i^-) + C_i + D_i, i = 1, 2, \ldots, r(T, 0)$.
(d) $\Psi(x) = \varpi(x),\ x \in [-h, 0]$.

Definition 7. *System (2) is said to be Ulam–Hyers–Rassias stable with respect to $\psi(\cdot)$ and ϕ if there exists $\widetilde{K} > 0$ such that for every $\varepsilon > 0$ and for every solution $\Psi \in PC(\widetilde{J}, \mathbb{R}^n)$ of inequality (13), there exists a solution $z \in PC(\widetilde{J}, \mathbb{R}^n)$ of (2) such that*

$$\|\Psi(x) - z(x)\| \leq \varepsilon\widetilde{K}(\psi(x) + \phi),\ x \in \widetilde{J}.$$

Remark 3. *The function $\Psi \in PC(\widetilde{J}, \mathbb{R}^n)$ is said to be a solution of inequality (13) if there exist $E_i \in \mathbb{R}^n$ and $\mathcal{Z} \in PC(J, \mathbb{R}^n)$ such that*

(a) $\|\mathcal{Z}(x)\| \leq \varepsilon\psi(x), \|E_i\| \leq \varepsilon\phi,\ x \in J,\ i = 1, 2, \ldots, r(T, 0)$.
(b) $({}^{C}\mathbb{D}^{\alpha}_{0+}\Psi)(x) = A\Psi(x-h) + g(x, \Psi(x)) + \mathcal{Z}(x),\ x \in J$.
(c) $\Psi(x_i^+) = \Psi(x_i^-) + C_i + E_i,\ i = 1, 2, \ldots, r(T, 0)$.
(d) $\Psi(x) = \varpi(x),\ x \in [-h, 0]$.

Lemma 8. *If $\Psi \in PC(\widetilde{J}, \mathbb{R}^n)$ is a solution of inequality (12), then Ψ satisfies the following integral inequality*

$$\begin{aligned} &\left\| \Psi(x) - \mathbb{E}_h^{Ax^\alpha}\varpi(-h) - \int_{-h}^{0} \mathbb{E}_h^{A(x-h-s)^\alpha}\varpi'(s)ds - \sum_{0<x_i<x} \mathbb{E}_h^{A(x-x_i-h)^\alpha}C_i \right. \\ &\quad \left. - \int_0^x \mathbb{E}_{h,\alpha}^{A(x-h-t)^\alpha} g(t, \Psi(t))dt \right\| \\ \leq\ & \phi \sum_{0<x_i<x} E_\alpha(\|A\|(T-x_i)^\alpha) + T\Phi(T)\varepsilon. \end{aligned}$$

Proof. By Remark 2, one can obtain

$$\begin{cases} ({}^{C}\mathbb{D}^{\alpha}_{0+}\Psi)(x) = A\Psi(x-h) + g(x, \Psi(x)) + \mathcal{Z}(x), x \in J, \\ \Psi(x_i^+) = \Psi(x_i^-) + C_i + D_i, i = 1, 2, \ldots, r(T, 0), \\ \Psi(x) = \varpi(x),\ x \in [-h, 0], \end{cases}$$

and

$$\begin{aligned} \Psi(x) \ =\ & \mathbb{E}_h^{Ax^\alpha}\varpi(-h) + \int_{-h}^{0} \mathbb{E}_h^{A(x-h-s)^\alpha}\varpi'(s)ds + \sum_{0<x_i<x} \mathbb{E}_h^{A(x-x_i-h)^\alpha}(C_i + D_i) \\ & + \int_0^x \mathbb{E}_{h,\alpha}^{A(x-h-t)^\alpha}(g(t, \Psi(t)) + \mathcal{Z}(t))dt. \end{aligned}$$

Let $x \in J$; we obtain

$$\begin{aligned} &\left\| \Psi(x) - \mathbb{E}_h^{Ax^\alpha}\varpi(-h) - \int_{-h}^{0} \mathbb{E}_h^{A(x-h-s)^\alpha}\varpi'(s)ds - \sum_{0<x_i<x} \mathbb{E}_h^{A(x-x_i-h)^\alpha}C_i \right. \\ &\quad \left. - \int_0^x \mathbb{E}_{h,\alpha}^{A(x-h-t)^\alpha} g(t, \Psi(t))dt \right\| \\ \leq\ & \sum_{0<x_i<x} \|\mathbb{E}_h^{A(x-x_i-h)^\alpha}D_i\| + \int_0^x \|\mathbb{E}_{h,\alpha}^{A(x-h-t)^\alpha}\mathcal{Z}(t)\|dt \\ \leq\ & \sum_{0<x_i<x} E_\alpha(\|A\|(x-x_i)^\alpha)\|D_i\| + \Phi(T)\int_0^x \varepsilon\, dt \\ \leq\ & \phi \sum_{0<x_i<x} E_\alpha(\|A\|(T-x_i)^\alpha) + \varepsilon\Phi(T)T. \end{aligned}$$

□

Lemma 9. *If $\Psi \in PC(\widetilde{J}, \mathbb{R}^n)$ is a solution of inequality (13), then Ψ satisfies the following integral inequality*

$$\begin{aligned} &\Bigg\| \Psi(x) - \mathbb{E}_h^{Ax^\alpha}\varpi(-h) - \int_{-h}^{0} \mathbb{E}_h^{A(x-h-s)^\alpha}\varpi'(s)\mathrm{d}s - \sum_{0<x_i<x} \mathbb{E}_h^{A(x-x_i-h)^\alpha} C_i \\ &\quad - \int_0^x \mathbb{E}_{h,\alpha}^{A(x-h-t)^\alpha} g(t,\Psi(t))dt \Bigg\| \\ \leq\ & \varepsilon\phi \sum_{0<x_i<x} E_\alpha(\|A\|(T-x_i)^\alpha) + \varepsilon\Phi(T)\int_0^x \psi(t)dt. \end{aligned}$$

Proof. By Remark 3, one can obtain

$$\begin{cases} ({}^C\mathbb{D}_{0^+}^\alpha \Psi)(x) = A\Psi(x-h) + g(x,\Psi(x)) + \mathcal{Z}(x), x \in J, \\ \Psi(x_i^+) = \Psi(x_i^-) + C_i + E_i, i = 1,2,\ldots,r(T,0), \\ \Psi(x) = \varpi(x),\ x \in [-h,0], \end{cases}$$

and

$$\begin{aligned} \Psi(x) \ =\ & \mathbb{E}_h^{Ax^\alpha}\varpi(-h) + \int_{-h}^{0} \mathbb{E}_h^{A(x-h-s)^\alpha}\varpi'(s)ds + \sum_{0<x_i<x} \mathbb{E}_h^{A(x-x_i-h)^\alpha}(C_i + E_i) \\ & + \int_0^x \mathbb{E}_{h,\alpha}^{A(x-h-t)^\alpha}(g(t,\Psi(t)) + \mathcal{Z}(t))dt. \end{aligned}$$

Let $x \in J$; we obtain

$$\begin{aligned} &\Bigg\| \Psi(x) - \mathbb{E}_h^{Ax^\alpha}\varpi(-h) - \int_{-h}^{0} \mathbb{E}_h^{A(x-h-s)^\alpha}\varpi'(s)\mathrm{d}s - \sum_{0<x_i<x} \mathbb{E}_h^{A(x-x_i-h)^\alpha} C_i \\ &\quad - \int_0^x \mathbb{E}_{h,\alpha}^{A(x-h-t)^\alpha} g(t,\Psi(t))dt \Bigg\| \\ \leq\ & \sum_{0<x_i<x} \|\mathbb{E}_h^{A(x-x_i-h)^\alpha} E_i\| + \int_0^x \|\mathbb{E}_{h,\alpha}^{A(x-h-t)^\alpha}\mathcal{Z}(t)\|dt \\ \leq\ & \sum_{0<x_i<x} E_\alpha(\|A\|(x-x_i)^\alpha)\|E_i\| + \Phi(x)\int_0^x \|\mathcal{Z}(t)\|dt \\ \leq\ & \varepsilon\phi \sum_{0<x_i<x} E_\alpha(\|A\|(T-x_i)^\alpha) + \varepsilon\Phi(T)\int_0^x \psi(t)dt. \end{aligned}$$

□

Theorem 6. *Suppose that $[H_3]$ and $[H_4]$ are satisfied. Then (2) is UH on $\widetilde{J}$.*

Proof. Let $z \in PC(\widetilde{J}, \mathbb{R}^n)$; we have

$$\begin{aligned} z(x) \ =\ & \mathbb{E}_h^{Ax^\alpha}\varpi(-h) + \int_{-h}^{0} \mathbb{E}_h^{A(x-h-s)^\alpha}\varpi'(s)ds + \sum_{0<x_i<x} \mathbb{E}_h^{A(x-x_i-h)^\alpha} C_i \\ & + \int_0^x \mathbb{E}_{h,\alpha}^{A(x-h-t)^\alpha} g(t,z(t))dt. \end{aligned}$$

Let $x \in [-h,0]$; we have

$$\|\Psi(x) - z(x)\| = \|\varpi(x) - \varpi(x)\| = 0 < K(\varepsilon + \phi).$$

According to Lemma 8, for any $x \in J$, one can obtain

$$\begin{aligned} & \|\Psi(x) - z(x)\| \\ \leq\ & \Bigg\| \Psi(x) - \mathbb{E}_h^{Ax^\alpha}\varpi(-h) - \int_{-h}^{0} \mathbb{E}_h^{A(x-h-s)^\alpha}\varpi'(s)ds - \sum_{0<x_i<x} \mathbb{E}_h^{A(x-x_i-h)^\alpha} C_i \\ & - \int_0^x \mathbb{E}_{h,\alpha}^{A(x-h-t)^\alpha} g(t,\Psi(t))dt \Bigg\| + \int_0^x \|\mathbb{E}_{h,\alpha}^{A(x-h-t)^\alpha}\| \|g(t,\Psi(t)) - g(t,z(t))\|dt \\ \leq\ & \phi \sum_{0<x_i<T} E_\alpha(\|A\|(T-x_i)^\alpha) + \varepsilon T\Phi(T) + \widetilde{L}T\Phi(T)\|\Psi - z\|_{PC}, \end{aligned}$$

which implies that

$$\begin{aligned}\|\Psi(x)-z(x)\| &\leq \frac{\phi \sum\limits_{0<x_i<T} E_\alpha(\|A\|(T-x_i)^\alpha)}{1-\widetilde{L}T\Phi(T)} + \frac{\varepsilon T\Phi(T)}{1-\widetilde{L}T\Phi(T)} \\ &\leq K(\varepsilon+\phi),\end{aligned}$$

where

$$K = \frac{1}{1-\widetilde{L}T\Phi(T)} \max\left\{ \sum_{0<x_i<T} E_\alpha(\|A\|(T-x_i)^\alpha), T\Phi(T) \right\}.$$

□

$[H_5]$ For any $t \in J$, there is a monotone function $\psi(\cdot) \in C(J, \mathbb{R}^+)$ such that

$$\int_0^x \psi(t)dt \leq \widetilde{M}\psi(x).$$

Theorem 7. *Suppose that $[H_3]$–$[H_5]$ are satisfied. Then (2) is UHR on $\widetilde{J}$.*

Proof. Let $x \in [-h, 0]$; one can obtain

$$\|\Psi(x)-z(x)\| = \|\varpi(x)-\varpi(x)\| = 0 < \varepsilon\widetilde{K}(\phi+\psi(x)).$$

According to Lemma 9, for any $x \in J$, one can obtain

$$\begin{aligned} &\|\Psi(x)-z(x)\| \\ \leq\ & \left\| \Psi(x) - \mathbb{E}_h^{Ax^\alpha}\varpi(-h) - \int_{-h}^0 \mathbb{E}_h^{A(x-h-s)^\alpha}\varpi'(s)ds - \sum_{0<x_i<x} \mathbb{E}_h^{A(x-x_i-h)^\alpha} C_i \right. \\ & \left. - \int_0^x \mathbb{E}_{h,\alpha}^{A(x-h-t)^\alpha} g(t,\Psi(t))dt \right\| + \int_0^x \|\mathbb{E}_{h,\alpha}^{A(x-h-t)^\alpha}\| \|g(t,\Psi(t)) - g(t,z(t))\| dt \\ \leq\ & \varepsilon\phi \sum_{0<x_i<T} E_\alpha(\|A\|(T-x_i)^\alpha) + \varepsilon\widetilde{M}\Phi(T)\psi(x) + \widetilde{L}T\Phi(T)\|\Psi-z\|_{PC}, \end{aligned}$$

which implies that

$$\begin{aligned}\|\Psi(x)-z(x)\| &\leq \frac{\varepsilon\phi \sum\limits_{0<x_i<T} E_\alpha(\|A\|(T-x_i)^\alpha)}{1-\widetilde{L}T\Phi(T)} + \frac{\varepsilon\widetilde{M}\Phi(T)\psi(x)}{1-\widetilde{L}T\Phi(T)} \\ &\leq \varepsilon\widetilde{K}(\phi+\psi(x)),\end{aligned}$$

where

$$\widetilde{K} = \frac{1}{1-\widetilde{L}T\Phi(T)} \max\left\{ \sum_{0<x_i<T} E_\alpha(\|A\|(T-x_i)^\alpha), \widetilde{M}\Phi(T) \right\}.$$

□

6. Examples

In this part, we illustrate the obtained results with a couple of examples.

Example 1. *Let $\alpha = 0.3$, $h = 0.4$, $k^* = 5$, $r(T,0) = 4$, $T = 2$, $x_i = 0.4i$ and $i = 1,2,3,4$. Consider*

$$\begin{cases} ({}^C\mathbb{D}_{0^+}^\alpha z)(x) = Az(x-0.4) + g(x),\ x \in [0,2], \\ z(x_i^+) = z(x_i^-) + C_i,\ x = x_i,\ i = 1,2,3,4, \\ \varpi(x) = (2x^2+1, x^2+2)^\top,\ -0.4 \leq x \leq 0, \end{cases} \tag{14}$$

where

$$z(x) = \begin{pmatrix} z_1(x) \\ z_2(x) \end{pmatrix},\ A = \begin{pmatrix} 0.2 & 0.8 \\ 0.3 & 0.5 \end{pmatrix},\ C_i = \begin{pmatrix} \frac{i}{2} \\ \frac{i}{4} \end{pmatrix}, g(x) = \begin{pmatrix} x \\ x^2 \end{pmatrix}. \tag{15}$$

By Theorem 1, we have

$$
\begin{aligned}
z(x) \;=\;& \mathbb{E}_{0.4}^{Ax^{0.3}}\varpi(-0.4) + \int_{-0.4}^{0} \mathbb{E}_{0.4}^{A(x-0.4-s)^{0.3}}\varpi'(s)ds + \sum_{0<x_i<x} \mathbb{E}_{0.4}^{A(x-0.4(i+1))^{0.3}} C_i \\
&+ \int_0^x \mathbb{E}_{0.4,0.3}^{A(x-0.4-t)^{0.3}} g(t)dt,
\end{aligned}
$$

where

$$
\mathbb{E}_{0.4}^{Ax^{0.3}} = \begin{cases}
E + A\dfrac{x^{0.3}}{\Gamma(1.3)},\ x \in [0, 0.4], \\
E + A\dfrac{x^{0.3}}{\Gamma(1.3)} + A^2\dfrac{(x-0.4)^{0.6}}{\Gamma(1.6)},\ x \in (0.4, 0.8], \\
E + A\dfrac{x^{0.3}}{\Gamma(1.3)} + A^2\dfrac{(x-0.4)^{0.6}}{\Gamma(1.6)} + A^3\dfrac{(x-0.8)^{0.9}}{\Gamma(1.9)},\ x \in (0.8, 1.2], \\
E + A\dfrac{x^{0.3}}{\Gamma(1.3)} + A^2\dfrac{(x-0.4)^{0.6}}{\Gamma(1.6)} + A^3\dfrac{(x-0.8)^{0.9}}{\Gamma(1.9)} + A^4\dfrac{(x-1.2)^{1.2}}{\Gamma(2.2)},\ x \in (1.2, 1.6], \\
E + A\dfrac{x^{0.3}}{\Gamma(1.3)} + A^2\dfrac{(x-0.4)^{0.6}}{\Gamma(1.6)} + A^3\dfrac{(x-0.8)^{0.9}}{\Gamma(1.9)} + A^4\dfrac{(x-1.2)^{1.2}}{\Gamma(2.2)} \\
\quad + A^5\dfrac{(x-1.6)^{1.5}}{\Gamma(2.5)},\ x \in (1.6, 2],
\end{cases}
$$

and

$$
\sum_{0<x_i<x} \mathbb{E}_{0.4}^{A(x-0.4(i+1))^{0.3}} C_i = \begin{cases}
\begin{pmatrix} 0 \\ 0 \end{pmatrix},\ x \in [0, 0.4], \\
\begin{pmatrix} \frac{1}{2} \\ \frac{1}{4} \end{pmatrix},\ x \in (0.4, 0.8], \\
\left(E + A\dfrac{(x-0.8)^{0.3}}{\Gamma(1.3)}\right)\begin{pmatrix} \frac{1}{2} \\ \frac{1}{4} \end{pmatrix} + \begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix},\ x \in (0.8, 1.2], \\
\left(E + A\dfrac{(x-0.8)^{0.3}}{\Gamma(1.3)} + A^2\dfrac{(x-1.2)^{0.6}}{\Gamma(1.6)}\right)\begin{pmatrix} \frac{1}{2} \\ \frac{1}{4} \end{pmatrix} \\
\quad + \left(E + A\dfrac{(x-1.2)^{0.3}}{\Gamma(1.3)}\right)\begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix} + \begin{pmatrix} \frac{3}{2} \\ \frac{3}{4} \end{pmatrix},\ x \in (1.2, 1.6], \\
\left(E + A\dfrac{(x-0.8)^{0.3}}{\Gamma(1.3)} + A^2\dfrac{(x-1.2)^{0.6}}{\Gamma(1.6)} + A^3\dfrac{(x-1.6)^{0.9}}{\Gamma(1.9)}\right)\begin{pmatrix} \frac{1}{2} \\ \frac{1}{4} \end{pmatrix} \\
\quad + \left(E + A\dfrac{(x-1.2)^{0.3}}{\Gamma(1.3)} + A^2\dfrac{(x-1.6)^{0.6}}{\Gamma(1.6)}\right)\begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix} \\
\quad + \left(E + A\dfrac{(x-1.6)^{0.3}}{\Gamma(1.3)}\right)\begin{pmatrix} \frac{3}{2} \\ \frac{3}{4} \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix},\ x \in (1.6, 2].
\end{cases}
$$

Example 2. *Let* $\alpha = 0.3$, $h = 0.4, k^* = 3$, $T = 1.2$, $r(T, 0) = 3$, $x_1 = 0.1$, $x_2 = 0.6$ *and* $x_3 = 1.1$. *Consider*

$$
\begin{cases}
({}^{C}\mathbb{D}_{0+}^{\alpha} z)(x) = Az(x-0.4) + g(x, z(x)),\ x \in [0, 1.2], \\
z(x_i^+) = z(x_i^-) + C_i,\ x = x_i,\ i = 1, 2, 3, \\
\varpi(x) = (2x^2+1, x^2+2)^\top,\ -0.4 \le x \le 0,
\end{cases} \tag{16}
$$

where A *and* C_i *are defined in* (15) *and* $g(x, z(x)) = \left(\frac{x}{12}z_1(x),\ \frac{x}{12}z_2(x)\right)^\top$.

Let $x \in [0, 1.2]$ and $z, \widetilde{z} \in \mathbb{R}^2$; one can obtain

$$\begin{aligned} \|g(x, z(x)) - g(x, \widetilde{z}(x))\| &\leq \tfrac{x}{12}(|z_1(x) - \widetilde{z}_1(x)| + |z_2(x) - \widetilde{z}_2(x)|) \\ &\leq \tfrac{1}{10}\|z - \widetilde{z}\|_{PC}. \end{aligned}$$

By calculation, one has $\|A\| = 1.3$, $\widetilde{L} = \frac{1}{10}$, $L = \frac{1}{10}$, $\Phi(1.2) = 2.9819$, $M = 1.92$. Hence, $[H_1]$, $[H_2]$, $[H_3]$ and $[H_4]$ are satisfied. By Theorems 4 and 5, the solution $z \in PC([-0.4, 1.2], \mathbb{R}^2)$ of (16) can be given by

$$\begin{aligned} z(x) &= \mathbb{E}_{0.4}^{Ax^{0.3}}\varpi(-0.4) + \int_{-0.4}^{0} \mathbb{E}_{0.4}^{A(x-0.4-s)^{0.3}}\varpi'(s)ds + \sum_{0<x_i<x} \mathbb{E}_{0.4}^{A(x-x_i-0.4)^{0.3}}C_i \\ &\quad + \int_0^x \mathbb{E}_{0.4,0.3}^{A(x-0.4-t)^{0.3}} g(t, z(t))dt, \end{aligned}$$

where

$$\sum_{0<x_i<x} \mathbb{E}_{0.4}^{A(x-x_i-0.4)^{0.3}}C_i = \begin{cases} \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \ 0 < x \leq 0.1, \\ \begin{pmatrix} \frac{1}{2} \\ \frac{1}{4} \end{pmatrix}, \ 0.1 < x \leq 0.5, \\ \left(E + A\frac{(x-0.5)^{0.3}}{\Gamma(1.3)}\right)\begin{pmatrix} \frac{1}{2} \\ \frac{1}{4} \end{pmatrix}, \ 0.5 < x \leq 0.6, \\ \left(E + A\frac{(x-0.5)^{0.3}}{\Gamma(1.3)}\right)\begin{pmatrix} \frac{1}{2} \\ \frac{1}{4} \end{pmatrix} + \begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix}, \ 0.6 < x \leq 1, \\ \left(E + A\frac{(x-0.5)^{0.3}}{\Gamma(1.3)} + A^2\frac{(x-0.9)^{0.6}}{\Gamma(1.6)}\right)\begin{pmatrix} \frac{1}{2} \\ \frac{1}{4} \end{pmatrix} \\ + \left(E + A\frac{(x-1)^{0.3}}{\Gamma(1.3)}\right)\begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix}, \ 1 < x \leq 1.1, \\ \left(E + A\frac{(x-0.5)^{0.3}}{\Gamma(1.3)} + A^2\frac{(x-0.9)^{0.6}}{\Gamma(1.6)}\right)\begin{pmatrix} \frac{1}{2} \\ \frac{1}{4} \end{pmatrix} \\ + \left(E + A\frac{(x-1)^{0.3}}{\Gamma(1.3)}\right)\begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix} + \begin{pmatrix} \frac{3}{2} \\ \frac{3}{4} \end{pmatrix}, 1.1 < x \leq 1.2, \end{cases}$$

and

$$\mathbb{E}_{0.4,0.3}^{A(x-0.4-t)^{0.3}} = \begin{cases} E\frac{(x-t)^{-0.7}}{\Gamma(0.3)}, x \in [0, 0.4], t \in [0, x], \\ E\frac{(x-t)^{-0.7}}{\Gamma(0.3)} + A\frac{(x-0.4-t)^{-0.4}}{\Gamma(0.6)}, x \in [0.4, 0.8], t \in [0, x-0.4], \\ E\frac{(x-t)^{-0.7}}{\Gamma(0.3)}, x \in [0.4, 0.8], t \in [x-0.4, x], \\ E\frac{(x-t)^{-0.7}}{\Gamma(0.3)} + A\frac{(x-0.4-t)^{-0.4}}{\Gamma(0.6)} + A^2\frac{(x-0.8-t)^{-0.1}}{\Gamma(0.9)}, \\ x \in [0.8, 1.2], t \in [0, x-0.8], \\ E\frac{(x-t)^{-0.7}}{\Gamma(0.3)} + A\frac{(x-0.4-t)^{-0.4}}{\Gamma(0.6)}, x \in [0.8, 1.2], t \in [x-0.8, x-0.4], \\ E\frac{(x-t)^{-0.7}}{\Gamma(0.3)}, x \in [0.8, 1.2], t \in [x-0.4, x]. \end{cases}$$

If $\Psi \in PC([-0.4, 1.2], \mathbb{R}^2)$ is solution of (12), then there exist $\mathcal{Z}(x) = (\frac{\varepsilon}{2}e^{-x}, \frac{\varepsilon}{2}e^{-x})^\top \in C([0, 1.2], \mathbb{R}^2)$ and $D_i = (\frac{i}{100}, \frac{i}{100})^\top$ such that $\|\mathcal{Z}(x)\| \leq \varepsilon$ and $\|D_i\| \leq \phi = 0.1$. Choose

$$K = \frac{1}{1 - \widetilde{L}T\Phi(T)} \max\left\{ \sum_{0<x_i<T} E_\alpha(\|A\|(T - x_i)^\alpha), T\Phi(T) \right\} \approx 96.21.$$

According to Theorem 6, we obtain

$$\|\Psi(x) - z(x)\| \leq K(\varepsilon + \phi).$$

Then (2) is UH on $[-0.4, 1.2]$.

If $\Psi \in PC([-0.4, 1.2], \mathbb{R}^2)$ is a solution of (12), then there exist $\mathcal{Z}(x) = (\frac{\varepsilon}{2}e^{x}, \frac{\varepsilon}{3}e^{x}) \in C([0, 1.2], \mathbb{R}^2)$ and $E_i = (\frac{i}{100}, \frac{i}{100})^\top$ such that $\|\mathcal{Z}(x)\| \leq \varepsilon e^x := \varepsilon\psi(x)$, $\|E_i\| \leq \phi = 0.1$. Moreover,

$$\int_0^x \psi(t)dt = \int_0^x e^t dt < e^x,\ x \in [0, 1.2].$$

Choose $\widetilde{M} = 1$ and $\widetilde{K} = \frac{1}{1 - \widetilde{L}T\Phi(T)} \max\left\{ \sum_{0<x_i<T} E_\alpha(\|A\|(T - x_i)^\alpha), \widetilde{M}\Phi(T) \right\} \approx 96.21$. According to Theorem 7, we obtain

$$\|\Psi(x) - z(x)\| \leq \varepsilon\widetilde{K}(\phi + \psi(x)),$$

then (2) is UHR on $[-0.4, 1.2]$.

7. Conclusions

In this paper, a new concept of impulsive delayed Mittag–Leffler type vector function was described, which helps us to construct a representation of an exact solution for Caputo fractional time delay impulse differential systems. By using the fixed point technique, fractional calculus, the delayed Mittag–Leffler type matrix functions and the impulsive delayed Mittag–Leffler type vector function, the existence and Ulam type stability of the considered systems were investigated. Moreover, we provided two examples to illustrate the applicability of the results.

Author Contributions: The contributions of all authors (C.C. and M.L.) are equal. All the main results were developed together. All authors have read and agreed to the published version of the manuscript.

Funding: This work is partially supported by the National Natural Science Foundation of China (12201148), Guizhou Provincial Science and Technology Projects (No.QKHJC-ZK[2022]YB069) and Natural Science Special Project of Guizhou University (No.GZUTGHZ(2021)08).

Data Availability Statement: No applicable.

Acknowledgments: The authors thank the help from the editor too.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Hilfer, R. *Application of Fractional Calculus in Physics*; World Scientific: Singapore, 2001.
2. Kilbas, A.; Srivastava, H.; Trujillo, J. *Theory and Applications of Fractional Differential Equations*; Elsevier: Amsterdam, The Netherlands, 2006.
3. Sabatier, J.; Agrawal, O.; Machado, J. *Advances in Fractional Calculus*; Springer: Dordrecht, The Netherlands, 2007.
4. Wang, J.; Fečkan, M.; Zhou, Y. Center stable manifold for planar fractional damped equations. *Appl. Math. Comput.* **2017**, *296*, 257–269. [CrossRef]
5. Singh, B.K.; Agrawal, S. Study of time fractional proportional delayed multi-pantograph system and integro-differential equations. *Math. Meth. Appl. Sci.* **2022**, *45*, 8305–8328. [CrossRef]
6. Nieto, J.J. Fractional Euler numbers and generalized proportional fractional logistic differential equation. *Fract. Calc. Appl. Anal.* **2022**, *25*, 876–886. [CrossRef] [PubMed]

7. Kaviya, R.; Priyanka, M.; Muthukumar, P. Mean-square exponential stability of impulsive conformable fractional stochastic differential system with application on epidemic model. *Chaos Solitons Fractals* **2022**, *160*, 112070. [CrossRef]
8. Ahmed, N.U.; Wang, S. Optimal control of nonlinear hybrid systems driven by signed measures with variable intensities and supports. *SIAM J. Control Optim.* **2021**, *59*, 4268–4294. [CrossRef]
9. Sisodiya, O.S.; Misra, O.P.; Dhar, J. Modeling effects of impulsive control strategies on the spread of mosquito borne disease: Role of latent period. *J. Appl. Math. Comput.* **2021**, *68*, 2589–2615. [CrossRef]
10. Yang, D.; Wang, J.; O'Regan, D. On the orbital Hausdorff dependence of differential equations with non-instantaneous impulses. *Comptes Rendus Math.* **2018**, *356*, 150–171. [CrossRef]
11. Cao, W.; Pan, T. Dynamics of an impulsive stochastic SIR epidemic model with saturated incidence rate. *J. Appl. Anal. Comput.* **2020**, *10*, 1396–1415. [CrossRef]
12. Wang, Y.; Tian, Y.; Li, X. Global exponential synchronization of interval neural networks with mixed delays via delayed impulsive control. *Neurocomputing* **2021**, *420*, 290–298. [CrossRef]
13. Fečkan, M.; Zhou, Y.; Wang, J. On the concept and existence of solution for impulsive fractional differential equations. *Commun. Nonlinear Sci. Numer. Simul.* **2012**, *17*, 3050–3060. [CrossRef]
14. Ali, A.; Rabiei, F.; Shah, K. On Ulam's type stability for a class of impulsive fractional differential equations with nonlinear integral boundary conditions. *J. Nonlinear Sci. Appl.* **2017**, *10*, 4760–4775. [CrossRef]
15. Church, K.E.M. Uniqueness of solutions and linearized stability for impulsive differential equations with state-dependent delay. *J. Differ. Equ.* **2022**, *338*, 415–440. [CrossRef]
16. Wang, J.; Fečkan, M.; Zhou, Y. On the stability of first order impulsive evolution equations. *Opusc. Math.* **2014**, *34*, 639–657. [CrossRef]
17. Wang, J.; Zhang, Y. A class of nonlinear differential equations with fractional integrable impulses. *Commun. Nonlinear Sci. Numer. Simul.* **2014**, *19*, 3001–3010. [CrossRef]
18. Li, S.; Shu, L.; Shu, X.; Xu, F. Existence and Hyers-Ulam stability of random impulsive stochastic functional differential equations with finite delays. *Stochastics* **2019**, *91*, 857–872. [CrossRef]
19. You, Z.; Wang, J.; O'Regan, D.; Zhou, Y. Relative controllability of delay differential systems with impulses and linear parts defined by permutable matrices. *Math. Meth. Appl. Sci.* **2019**, *42*, 954–968. [CrossRef]
20. Wang, X.; Wang, J.; Fečkan, M. Controllability of conformable differential systems. *Nonlinear Anal. Model. Control* **2020**, *25*, 658–674. [CrossRef]
21. Li, M.; Wang, J. Finite time stability of fractional delay differential equations. *Appl. Math. Lett.* **2017**, *64*, 170–176. [CrossRef]
22. Li, M.; Wang, J. Exploring delayed Mittag–Leffler type matrix functions to study finite time stability of fractional delay differential equations. *Appl. Math. Comput.* **2018**, *324*, 254–265. [CrossRef]
23. Li, M.; Wang, J. Existence results and Ulam type stability for a conformable fractional oscillating system with pure delay. *Chaos Solitons Fract.* **2022**, *161*, 112317. [CrossRef]
24. Wang, Q.; Lu, D.; Fang, Y. Stability analysis of impulsive fractional differential systems with delay. *Appl. Math. Lett.* **2015**, *40*, 1–6. [CrossRef]
25. Wu, G.; Baleanu, D.; Huang, L. Novel Mittag–Leffler stability of linear fractional delay difference equations with impulse. *Appl. Math. Lett.* **2018**, *82*, 71–78. [CrossRef]
26. Aimene, D.; Seba, D.; Laoubi, K. Controllability of impulsive fractional functional evolution equations with infinite state-dependent delay in Banach spaces. *Math. Meth. Appl. Sci.* **2021**, *44*, 7979–7994. [CrossRef]
27. Ulam, S.M. *A Collection of Mathematical Problem*; Hassell Street Press: New York, NY, USA, 1968.
28. Hyers, D.H. On the stability of the linear functional equation. *Proc. Natl. Acad. Sci. USA* **1941**, *27*, 222–224. [CrossRef] [PubMed]
29. Wang, J.; Zhou, Y.; Fečkan, M. Nonlinear impulsive problems for fractional differential equations and Ulam stability. *Comput. Math. Appl.* **2012**, *64*, 3389–3405. [CrossRef]
30. Wang, J.; Shah, K.; Ali, A. Existence and Hyers-Ulam stability of fractional nonlinear impulsive switched coupled evolution equations. *Math. Meth. Appl. Sci.* **2018**, *41*, 2392–2402. [CrossRef]
31. Ahmad, M.; Zada, A.; Alzabut, J. Hyers-Ulam stability of a coupled system of fractional differential equations of Hilfer-Hadamard type. *Demonstr. Math.* **2019**, *52*, 283–295. [CrossRef]
32. Agarwal, R.; Hristova, S.; O'Regan, D. Ulam type stability for non-instantaneous impulsive Caputo fractional differential equations with finite state dependent delay. *Georgian Math. J.* **2021**, *28*, 499–517. [CrossRef]