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Fractals via Controlled Fisher Iterated Function System

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Abstract: This paper explores the generalization of the fixed-point theorem for Fisher contraction on controlled metric space. The controlled metric space and Fisher contractions are playing a very crucial role in this research. The Fisher contraction on the controlled metric space is used in this paper to generate a new type of fractal set called controlled Fisher fractals (CF-Fractals) by constructing a system named the controlled Fisher iterated function system (CF-IFS). Furthermore, the interesting results and consequences of the controlled Fisher iterated function system and controlled Fisher fractals are demonstrated. In addition, the collage theorem on controlled Fisher fractals is established as well. The newly developing IFS and fractal set in the controlled metric space can provide the novel directions in the fractal theory.

Keywords: fractals; iterated function system; controlled metric space; Fisher contraction; collage theorem



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1. Introduction

Mandelbrot developed the concept of fractals in the important book “The Fractal Geometry of Nature” to portray non-linearity in real-world objects and numerous scientific events. In the real world, the fractal geometry has shown to be a particularly efficient technique for representing complex systems with infinite intricacies [1]. Fixed-point theory is a fundamental tool in Hutchinson’s theory of iterated function systems (IFSs), and Barnsley has studied in detail the construction of deterministic fractals [2–4]. IFSs have become a valuable tool for constructing many sorts of fractals. Stochastic growth models, image processing, and random dynamical systems are only a few examples of IFS applications. The presence of a deterministic fractal or attractor of IFS in a complete metric space follows the well-known Banach contraction principle [5–7]. Fractals are often used by researchers today in the various fields of science, such as time evolution of quantum fractals, fractal-time derivative operators, Sierpinski-type fractal structures, fractionally-perturbed systems, quantum mechanics, kinetic energy, topological insulators, and other applications for physical problems [8–14].

Hutchinson’s IFS theory has massively expanded for more generalized spaces and generalized contractions, and extended to infinite IFS and multifunction systems to generate general types of fractal sets with the distinguished dimensional measures [15–31]. Hata [32] used condition functions to create IFS. Fernau [33] developed the notion of limitless IFSs. Klimek and Kosek [34], Gwozdz-Lukawska and Jachymski [35], Lesniak [36], and Mauldin and Urbanski [37] have all done outstanding work in the area for fractal theory. On a compact metric space, Secelean [38] explored countable iterated function systems. Secelean proposed the idea of creating new IFS by combining various contractions into F-contractions. The authors developed the notion of a topological IFS attractor in the reference, which generalizes the familiar IFS attractor. Every IFS attractor is also a topological IFS attractor, but the reverse is not true [39–41].

The controlled metric space (CMS) is a novel extension of b-metric space that uses a two-variable control function in the triangle inequality as a controlling component of the

system, and the fixed-point theorem and its ramifications were produced by the majority of researchers [42,43]. In this present context, the above flow of extensions directs us to instigate the notion of CMS with the Fisher contraction mapping. It also motivated us to define IFS and to discuss HB theory to evolve the new type of fractal sets in the proposed controlled metric space as a general case.

The remaining sections of the paper are organized as follows. Section 2 discusses the basic concepts of the contraction, Banach contraction principle, Hausdorff metric space, and iterated function system, which are required for this research work. Further, the fixed-point theorem for Fisher contraction on complete controlled metric space (CCMS) and other consequent results are proved in Section 3. The existence of fractals in controlled metric space using the iterated function system of Fisher contractions and other interesting results are established in Section 4. Finally, the obtained results are concluded in Section 5.

2. Preliminary Results

In the preliminary part, we recall some basic theory of an iterated function system that is required for the proposed research work.

A function \mathbb{F} on a metric space (H, ρ) is defined as a contraction if \mathbb{F} satisfies $\rho(\mathbb{F}(a), \mathbb{F}(b)) \leq r\rho(a, b), \forall a, b \in H, r \in [0, 1)$ with the contraction ratio r .

In 1922, Banach proved a classic fixed-point theorem named the Banach contraction principle, which is a strong and powerful tool in fixed-point theory [44].

Theorem 1. *If $\mathbb{F} : H \rightarrow H$ is a contraction mapping on a complete metric space (H, ρ) , then \mathbb{F} has a unique fixed point $a^* \in H$. Moreover, the sequence $\{\mathbb{F}^j(a)\}_{j=1}^\infty$ converges to a^* . That is, $\lim_{j \rightarrow \infty} \mathbb{F}^j(a) = a^*$ for any element $a \in H$.*

The Theorem 1 establishes the existence and uniqueness of fixed point of particular self-mapping on a complete metric space and a constructive approach for computing the fixed point of contractions.

If (H, ρ) is a complete metric space and $\mathcal{K}_0(H)$ is the nonempty collection of all compact subsets of H , for $a \in H$ and $A, B \in \mathcal{K}_0(H)$, define

$$\rho(x, B) = \inf\{\rho(x, b) : b \in B\}$$

and

$$\rho(A, B) = \sup\{\rho(a, b) : a \in A, b \in B\}.$$

Obviously, $\rho(A, B)$ and $\rho(B, A)$ both are positive and exist. The Hausdorff metric between A and B is defined as $\mathcal{H}_\rho(A, B) = \max\{\sup_{a \in A} \rho(a, B), \sup_{b \in B} \rho(b, A)\}$. Then, the mapping \mathcal{H}_ρ is defined on $\mathcal{K}_0(H)$.

The contraction function on IFS acts on the members of the Hausdorff space, i.e., a compact subset of H . It is also known that if (H, ρ) is complete, then the Hausdorff space $(\mathcal{K}_0(H), \mathcal{H}_\rho)$ is also complete.

Definition 1 (Controlled Metric Space [43]). *Let H be a nonempty set and $\alpha : H \times H \rightarrow [1, \infty)$. A controlled metric is a function where $\rho : H \times H \rightarrow [0, \infty)$ satisfies the following conditions $\forall a, b, c \in H$.*

- (a). $\rho(a, b) = 0$ iff $a = b$,
- (b). $\rho(a, b) = \rho(b, a)$,
- (c). $\rho(a, b) \leq \alpha(a, c)\rho(a, c) + \alpha(c, b)\rho(c, b)$.

Then, the pair (H, ρ) is called the controlled metric space.

If, for all $a, b \in H, \alpha(a, b) = s \geq 1$, then (H, ρ) is a b -metric space, which leads us to conclude that every b -metric space is a controlled metric type space. Additionally, every b -metric space is a standard metric space if $s = 1$, and the converse is not always true.

Likewise, every controlled metric space is a standard metric space, and the converse is not always true.

A (hyperbolic) IFS consists of CMS (H, ρ) , along with a finite collection of contraction (continuous) mapping $\mathbb{F}_j : H \rightarrow H$ with respect to the contractivity ratio r_j , for $j = 1, 2, \dots, N_0$. The system $\{H, \mathbb{F}_j : n = 1, 2, \dots, N_0\}$ is the hyperbolic IFS which maps $\mathcal{F} : \mathcal{K}_0(H) \rightarrow \mathcal{K}_0(H)$, which is defined by

$$\mathcal{F}(B) = \cup_{j=1}^{N_0} \mathbb{F}_j(B), B \in \mathcal{K}_0(H),$$

which induces a set valued map and $\mathbb{F}_j(B) = \{\mathbb{F}_j(b) : b \in B\}$ is a contraction mapping on CMS $(\mathcal{K}_0(H), \mathcal{H}_\rho)$ with the contraction ratio r . That is,

$$\mathcal{H}_\rho(\mathcal{F}(B), \mathcal{F}(C)) \leq r \mathcal{H}_\rho(B, C).$$

Here, \mathcal{F} has a unique fixed point which is known as an attractor and denoted by $K^* \in \mathcal{K}_0(H)$.

Moreover, $K^* = \lim_{j \rightarrow \infty} \mathbb{F}^j(B)$ for any $B \in \mathcal{K}_0(H)$, where $\mathbb{F}^j = \underbrace{\mathbb{F} \circ \mathbb{F} \circ \mathbb{F} \circ \dots \circ \mathbb{F}}_{j \text{ times}}$.

Then, K^* is also known as an invariant set and $\mathcal{F}(K^*) = K^*$.

3. Fisher Fixed-Point Theorem on Controlled Metric Space

In this section, we develop a new idea of IFS by replacing the usual contraction with a more general Fisher contraction.

In 1978, Fisher developed the new type of contraction known as Fisher contraction mapping [45], defined as follows.

If there exist r and s such that $r, s \in (0, \frac{1}{2})$, and for all $a, b \in H$ and

$$\rho(\mathbb{F}(a), \mathbb{F}(b)) \leq r[\rho(a, \mathbb{F}(a)) + \rho(b, \mathbb{F}(b))] + s(\rho(a, b)),$$

then \mathbb{F} is said to be Fisher contraction mapping.

Here, r and s are contraction ratios of Fisher contraction \mathbb{F} .

Now, we can introduce the Fisher contraction on the controlled metric space as follows.

Let (H, ρ) be CCMS. If $\mathbb{F} : H \rightarrow H$ is a function with

$$\rho(\mathbb{F}(a), \mathbb{F}(b)) \leq r[\rho(a, \mathbb{F}(a)) + \rho(b, \mathbb{F}(b))] + s(\rho(a, b)) \tag{1}$$

for every $a, b \in H$ and $r, s \in (0, \frac{1}{2})$, then \mathbb{F} is called the controlled Fisher contraction. r and s are the contraction ratios.

If $r = 0$, then the Fisher contraction is reduced to a usual contraction, but the converse is not always true. Similarly, if $s = 0$, then the Fisher contraction is reduced to a Kannan contraction, but the converse is not always true.

Now, the definition of the Hausdorff metric in introduced in CMS as follows.

Definition 2. If (H, ρ) is CMS, then the Hausdorff metric on CMS is the function $\mathcal{H}_\rho : \mathcal{K}_0(H) \times \mathcal{K}_0(H) \rightarrow [0, \infty)$, defined by

$$\mathcal{H}_\rho(A, B) = \max\{sup_{a \in A} \rho(a, B), sup_{b \in B} \rho(b, A)\}.$$

Based on the definition of (hyperbolic) IFS given by Barnsley [2], let us now introduce the new IFS called the controlled Fisher IFS.

Let (H, ρ) be CCMS, and let $\mathbb{F}_j : H \rightarrow H, j = 1, 2, 3, \dots, N_0$ ($N_0 \in \mathbb{N}$) be contractions in CCMS with the corresponding contraction ratios r_j and $s_j, j = 1, 2, 3, \dots, N_0$. Then, the system

$$\{H; \mathbb{F}_j, j = 1, 2, 3, \dots, N_0\}$$

is said to be a controlled Fisher IFS (CF-IFS) of Fisher contractions with the contraction ratios $r = \max_{j=1}^{N_0} r_j$ and $s = \max_{j=1}^{N_0} s_j$. First, we present and verify some theorems that provide a relationship among \mathbb{F}^i s, $\{i = 1, 2, \dots, N_0\}$, in terms of contractivity ratios r and s ; and \mathbb{F} has a unique fixed point, if it exists.

Theorem 2. Let $\mathbb{F} : H \rightarrow H$ be a Fisher contraction with the contraction ratios r and s on CMS (H, ρ) and $a \in H$. Then, \mathbb{F} satisfies the following condition

$$\rho(\mathbb{F}^i(a), \mathbb{F}^{i+1}(a)) \leq \left(\frac{r}{1-r}\right)^i \rho(a, \mathbb{F}(a)) + (1-s)\rho(a, \mathbb{F}(a)).$$

Moreover, $\lim_{i \rightarrow \infty} \rho(\mathbb{F}^i(a), \mathbb{F}^{i+1}(a)) = 0$.

Proof. Given that \mathbb{F} is a controlled Fisher contraction mapping, then

$$\rho(\mathbb{F}^i(a), \mathbb{F}^{i+1}(a)) \leq r(\rho(\mathbb{F}^{i-1}(a), \mathbb{F}^i(a))) + s\rho(\mathbb{F}^{i-1}(a), \mathbb{F}^i(a)).$$

Consequently,

$$\begin{aligned} \rho(\mathbb{F}^i(a), \mathbb{F}^{i+1}(a)) &\leq \frac{r}{1-r} \rho(\mathbb{F}^{i-1}(a), \mathbb{F}^i(a)) \\ &\quad + s\rho(\mathbb{F}^{i-1}(a), \mathbb{F}^i(a)) \\ &\leq \frac{r}{1-r} \left[\frac{r}{1-r} \rho(\mathbb{F}^{i-2}(a), \mathbb{F}^{i-1}(a)) \right] \\ &\quad + s[s\rho(\mathbb{F}^{i-2}(a), \mathbb{F}^{i-1}(a))] \\ &\leq \left(\frac{r}{1-r}\right)^2 [\rho(\mathbb{F}^{i-2}(a), \mathbb{F}^{i-1}(a))] + s^2 \rho(\mathbb{F}^{i-2}(a), \mathbb{F}^{i-1}(a)) \\ &\quad \vdots \\ &\leq \left(\frac{r}{1-r}\right)^i \rho(a, \mathbb{F}(a)) + \frac{1}{1-s} \rho(a, \mathbb{F}(a)) \\ &\leq \left[\left(\frac{r}{1-r}\right)^i + (1-s)^{-1}\right] \rho(a, \mathbb{F}(a)). \end{aligned}$$

As $i \rightarrow \infty$, we have

$$\lim_{i \rightarrow \infty} \rho(\mathbb{F}^i(a), \mathbb{F}^{i+1}(a)) \leq \lim_{i \rightarrow \infty} \left[\left(\frac{r}{1-r}\right)^i + (1-s)^{-1}\right] \rho(a, \mathbb{F}(a)).$$

Since $\frac{r}{1-r} < 1$ and $(1-s) < 1$, $\lim_{i \rightarrow \infty} \rho(\mathbb{F}^i(a), \mathbb{F}^{i+1}(a)) = 0$. \square

Theorem 3. Let $\mathbb{F} : H \rightarrow H$ be a Fisher contraction mapping with the contractivity ratios r and s on CMS (H, ρ) . If \mathbb{F} has a fixed point, then it is unique.

Proof. To prove the uniqueness of the fixed point, let a^* and b^* be two fixed points of \mathbb{F} . Then, $a^* = \mathbb{F}(a^*), b^* = \mathbb{F}(b^*)$.

$$\begin{aligned} \rho(a^*, b^*) &= \rho(\mathbb{F}(a^*), \mathbb{F}(b^*)) \\ &\leq r[\rho(a^*, \mathbb{F}(a^*)) + \rho(b^*, \mathbb{F}(b^*))] + s(a^*, b^*) \\ &= r[\rho(a^*, a^*) + \rho(b^*, b^*)] + s(a^*, b^*) \\ &= 0. \end{aligned}$$

Therefore, $a^* = b^*$.

\square

Theorem 4. Let $\mathbb{F} : H \rightarrow H$ be a continuous Fisher contraction mapping on a CCMS (H, ρ) with the contractivity ratios r and s ; and \mathbb{F} has a fixed point $a^* \in H$. Then,

$$\rho(a, a^*) \leq \left\{ \left(\frac{1-r}{1-2r} \right) + \frac{1}{1-s} \right\} \rho(a, \mathbb{F}(a)).$$

Proof. For $a \in H$, we have $\lim_{i \rightarrow \infty} \mathbb{F}^i(a) = a^*$. Take the point $a \in H$ as $i \rightarrow \infty$ fixed, and the metric function $\rho(a, b)$ is continuous at the point $b \in H$. Therefore,

$$\begin{aligned} \rho(a, a^*) &= \rho(a, \lim_{i \rightarrow \infty} \mathbb{F}^i(a)) \\ &= \lim_{i \rightarrow \infty} \rho[a, \mathbb{F}^i(a)] \\ &\leq \lim_{i \rightarrow \infty} \sum_{j=1}^i \rho(\mathbb{F}^{j-1}(a), \mathbb{F}^j(a)) \\ &\leq \lim_{i \rightarrow \infty} \rho(a, \mathbb{F}(a)) \\ &\times \left\{ \left(1 + \frac{r}{1-r} + \dots + \left(\frac{r}{1-r} \right)^{i-1} \right) + (1+s + \dots + s^{i-1}) \right\} \\ &\leq \left\{ \left(1 - \frac{r}{1-r} \right)^{-1} + (1-s)^{-1} \right\} \rho(a, \mathbb{F}(a)). \end{aligned}$$

□

Theorem 5 (Controlled Fisher Fixed-Point Theorem). Let $\mathbb{F} : H \rightarrow H$ be a Fisher contraction mapping on CCMS (H, ρ) with the contractivity ratios r, s , and $\alpha : [1, \infty) \rightarrow \mathbb{R}^+$. For $a_0 \in H$, take $a_j = \mathbb{F}^j(a_0)$. Suppose that

$$\sup_{i \geq 1} \lim_{j \rightarrow \infty} \frac{\alpha(a_{j+1}, a_{j+2})}{\alpha(a_j, a_{j+2})} \alpha(a_j, a_i) < \frac{1}{r} + \frac{1}{s}.$$

In addition, for every $a \in H$, $\lim_{j \rightarrow \infty} \alpha(a_j, a)$ and $\lim_{j \rightarrow \infty} \alpha(a, a_j)$ exist. Then, \mathbb{F} has a unique fixed point.

Proof. Take $a \in H$, and given that \mathbb{F} is a Fisher contraction mapping with the contractivity ratios r and s , we have

$$\rho(\mathbb{F}^i(a), \mathbb{F}^{i+1}(a)) \leq \left\{ \left(\frac{r}{1-r} \right) + (1-s) \right\} \rho(a, \mathbb{F}(a)), \quad \forall i = 0, 1, 2, \dots$$

Then, for any fixed $a \in H$, we get

$$\rho(\mathbb{F}^j(a), \mathbb{F}^i(a)) \leq t^{i \wedge j} \rho(a, \mathbb{F}^{|j-i|}(a)) + u^{i \wedge j} \rho(a, \mathbb{F}^{|j-i|}(a)) \tag{2}$$

where $i, j \in \{0\} \cup \mathbb{N}$ and $t := \frac{r}{1-r}, u := (1-s)$. In particular, take $n = |j-i|$, for $n = 0, 1, 2, \dots$. We have

$$\begin{aligned} \rho(a, \mathbb{F}^n(a)) &\leq \rho(a, \mathbb{F}(a)) + \rho(\mathbb{F}(a), \mathbb{F}^2(a)) + \dots + \rho(\mathbb{F}^{n-1}(a), \mathbb{F}^n(a)) \\ &\leq (1+t+t^2+\dots+t^{n-1})\rho(a, \mathbb{F}(a)) \\ &\quad + (1+u+u^2+\dots+u^{n-1})\rho(a, \mathbb{F}(a)) \\ &\leq \left(\frac{1-t^k}{1-t} \right) \rho(a, \mathbb{F}(a)) + (1-u)^{-1} \rho(a, \mathbb{F}(a)). \end{aligned}$$

Then, Equation (2) becomes

$$\rho(\mathbb{F}^j(a), \mathbb{F}^i(a)) \leq \frac{t^{i \wedge j} (1-t^k)}{1-t} \rho(a, \mathbb{F}(a)) + \frac{u^{i \wedge j}}{1-u} \rho(a, \mathbb{F}(a)).$$

As a result, $\lim_{j \rightarrow \infty} \alpha(a_j, a)$ and $\lim_{j \rightarrow \infty} \alpha(a, a_j)$ exist, and we have limits. It is clear that $\{\mathbb{F}^j(a)\}_{j=0}^\infty$ is Cauchy. Since H is a complete controlled metric space, the sequence $\{\mathbb{F}^j(a)\}_{j=0}^\infty$ has a limit, say, $a^* \in H$. Hence, we have

$$\lim_{n \rightarrow \infty} \mathbb{F}^j(a) = a^*. \tag{3}$$

Now we prove that a^* is a fixed point of \mathbb{F} .

$$\begin{aligned} \rho(a^*, \mathbb{F}(a^*)) &\leq \alpha(a^*, \mathbb{F}^j(a))\rho(a^*, \mathbb{F}^j(a)) + \alpha(\mathbb{F}^j(a), \mathbb{F}(a^*))\rho(\mathbb{F}^j(a), \mathbb{F}(a^*)) \\ &\leq \alpha(a^*, \mathbb{F}^j(a))\rho(a^*, \mathbb{F}^j(a)) \\ &\quad + \alpha(\mathbb{F}^j(a), \mathbb{F}(a^*))r[\rho(\mathbb{F}^{j-1}(a), \mathbb{F}^j(a)) + \rho(a^*, \mathbb{F}(a^*))] + s\rho(a^*, a^*). \end{aligned}$$

Further, taking the limit j approaching ∞ , and considering Equation (3), Theorem 2 and $\lim_{n \rightarrow \infty} \alpha(a_j, a)$, $\lim_{j \rightarrow \infty} \alpha(a, a_j)$ exist and we have a limit. We get

$$\rho(a^*, \mathbb{F}(a^*)) \leq (1 + r)\alpha(a^*, \mathbb{F}(a^*))\rho(a^*, \mathbb{F}(a^*)).$$

Hence, $a^* = \mathbb{F}(a^*)$. By Theorem 3, a^* is a unique fixed point. \square

4. Controlled F-Iterated Function System and Fractal

The HB theorem for generating the fractals in complete controlled metric space using IFS of Fisher contractions and their consequences are proved and discussed in this section.

Theorem 6. Let $\mathbb{F} : H \rightarrow H$ be a continuous Fisher contraction mapping on CMS (H, ρ) with the contraction ratios r and s . Then, the function $\mathbb{F} : \mathcal{K}_0(H) \rightarrow \mathcal{K}_0(H)$ is defined by $\mathbb{F}(A) = \{\mathbb{F}(a) : a \in A\}$, $\forall A \in \mathcal{K}_0(H)$, a Fisher mapping on $(\mathcal{K}_0(H), \mathcal{H}_\rho)$ with the contraction ratios r and s .

Proof. Let us consider a continuous mapping \mathbb{F} . Therefore, \mathbb{F} maps $\mathcal{K}_0(H)$ into itself [2]. Let $A, B \in \mathcal{K}_0(H)$. Then,

$$\begin{aligned} \mathcal{H}_\rho(\mathbb{F}(A), \mathbb{F}(B)) &= \rho(\mathbb{F}(A), \mathbb{F}(B)) \vee \rho(\mathbb{F}(B), \mathbb{F}(A)) \\ &\leq r[\rho(A, \mathbb{F}(A)) + \rho(B, \mathbb{F}(B))] + s\rho(A, B) \\ &\quad \vee [\rho(B, \mathbb{F}(B)) + \rho(A, \mathbb{F}(A))] + s\rho(B, A) \\ &= r[\rho(A, \mathbb{F}(A)) + \rho(B, \mathbb{F}(B))] + s\rho(A, B) \\ &\leq r[\mathcal{H}_\rho(A, \mathbb{F}(A)) + \mathcal{H}_\rho(B, \mathbb{F}(B))] + s[\mathcal{H}_\rho(A, B)]. \end{aligned}$$

Therefore, $\mathcal{H}_\rho(\mathbb{F}(A), \mathbb{F}(B)) \leq r[\mathcal{H}_\rho(A, \mathbb{F}(A)) + \mathcal{H}_\rho(B, \mathbb{F}(B))] + s[\mathcal{H}_\rho(A, B)]$. \square

Theorem 7. Let (H, ρ) be CMS. Let $\{H; \mathbb{F}_j; j = 1, 2, \dots, N_0\}$ be a CF-IFS of continuous Fisher contraction mappings on $(\mathcal{K}_0(H), \mathcal{H}_\rho)$ with the corresponding contraction ratios r_j and s_j , for each j . Define $\mathcal{F} : \mathcal{K}_0(H) \rightarrow \mathcal{K}_0(H)$ by $\mathcal{F}(A) = \cup_{j=1}^{N_0} \mathbb{F}_j(A)$, for each $A \in \mathcal{K}_0(H)$. Then, \mathcal{F} is a Fisher contraction with the contraction ratios $r = \max\{r_j; j = 1, 2, \dots, N_0\}$ and $s = \max\{s_j; j = 1, 2, \dots, N_0\}$.

Proof. The theorem is proved by using the mathematical induction approach and the characteristics of the metric \mathcal{H}_ρ . The assertion is plainly true for $N = 1$ and $N = 2$. Thus, we can show that

$$\begin{aligned}
 &\mathcal{H}_\rho(\mathcal{F}(A), \mathcal{F}(B)) \\
 &= \mathcal{H}_\rho(\mathbb{F}_1(A) \cup \mathbb{F}_2(A), \mathbb{F}_1(B) \cup \mathbb{F}_2(B)) \\
 &\leq \mathcal{H}_\rho(\mathbb{F}_1(A), \mathbb{F}_1(B)) \vee \mathcal{H}_\rho(\mathbb{F}_2(A), \mathbb{F}_2(B)) \\
 &\leq r_1[\mathcal{H}_\rho(A, \mathbb{F}_1(A)) + \mathcal{H}_\rho(B, \mathbb{F}_1(B))] + s_1 \mathcal{H}_\rho(A, B) \\
 &\quad \vee r_2[\mathcal{H}_\rho(A, \mathbb{F}_2(A)) + \mathcal{H}_\rho(B, \mathbb{F}_2(B))] + s_2 \mathcal{H}_\rho(B, A) \\
 &\leq (r_1 \vee r_2) \{ \mathcal{H}_\rho(A, \mathbb{F}_1(A)) \vee \mathcal{H}_\rho(A, \mathbb{F}_2(A)) \} + (s_1 \vee s_2) \mathcal{H}_\rho(A, B) \\
 &\quad + \{ \mathcal{H}_\rho(B, \mathbb{F}_1(B)) \vee \mathcal{H}_\rho(B, \mathbb{F}_2(B)) \} \\
 &= r[\mathcal{H}_\rho(A, \mathbb{F}_1(A) \cup \mathbb{F}_2(A)) + \mathcal{H}_\rho(B, \mathbb{F}_1(B) \cup \mathbb{F}_2(B))] + s \mathcal{H}_\rho(A, B).
 \end{aligned}$$

Therefore,

$$\mathcal{H}_\rho(\mathcal{F}(A), \mathcal{F}(B)) \leq r[\mathcal{H}_\rho(A, \mathcal{F}(A)) + \mathcal{H}_\rho(B, \mathcal{F}(B))] + s \mathcal{H}_\rho(A, B).$$

The principle of mathematical induction proves the theorem. \square

The following theorem for the CF-IFS can be proved based on the prior results and the concept of the CF-IFS.

Theorem 8. Let $\{H; (\mathbb{F}_0), \mathbb{F}_1, \mathbb{F}_2, \dots, \mathbb{F}_n\}$ be a CF-IFS with the condensation mapping \mathbb{F}_0 and the contraction ratios $r = \max\{r_j; j = 1, 2, \dots, N_0\}$ and $s = \max\{s_j; j = 1, 2, \dots, N_0\}$. Then, the transformation $\mathcal{F} : \mathcal{X}_0(H) \rightarrow \mathcal{X}_0(H)$, defined by

$$\mathcal{F}(A) = \cup_{j=1}^{N_0} \mathbb{F}_j(A), \forall A \in \mathcal{X}_0(H),$$

is a continuous Fisher contraction mapping on CCMS $(\mathcal{X}_0(H), \mathcal{H}_\rho)$ with the contraction ratios r and s . Additionally, \mathcal{F} has a unique fixed point $B \in \mathcal{X}_0(H)$, so it follows that

$$B = \mathcal{F}(B) = \cup_{j=1}^{N_0} \mathbb{F}_j(B),$$

given by $B = \lim_{j \rightarrow \infty} \mathcal{F}^{\circ j}(A)$, for all $A \in \mathcal{X}_0(H)$.

Proof. Since (H, ρ) is a complete CMS, the $(\mathcal{X}_0(H), \mathcal{H}_\rho)$ is also a complete CMS. Additionally, Theorem 7 makes clear that the HB operator, \mathcal{F} , is a contraction mapping on CMS. By using Theorem 5, it is concluded that \mathcal{F} has a unique fixed point. This completes our assertion. \square

Definition 3 (Controlled F-Fractals (CF-Fractals)). The fixed point $B \in \mathcal{X}_0(H)$ of HB operator \mathcal{F} for the CF-IFS described in Theorem 8 is said to be a controlled Fisher attractor or controlled Fisher fractal or Controlled F-Fractal (CF-Fractal) in CMS. Thus, $B \in \mathcal{X}_0(H)$ is said to be a fractal generated by a CF-IFS on CMS.

In this flow of extension, we can also prove the collage theorem for the CF-IFS.

Theorem 9 (Collage Theorem). Let (H, ρ) be CCMS. Given that $Z \in \mathcal{X}_0(H)$ and $\epsilon \geq 0$, suppose $\{H; \mathbb{F}_j, j = 0, 1, 2, \dots, N_0\}$ is a CF-IFS, with the condensation mapping \mathbb{F}_0 and Fisher contraction ratios r and s such that

$$\mathcal{H}_\rho\left(Z, \cup_{j=0, j=1}^{N_0} \mathbb{F}_j(Z)\right) \leq \epsilon.$$

Then,

$$\mathcal{H}_\rho(Z, B) \leq \left[\left(\frac{1-r}{1-2r} \right) + \left(\frac{1}{1-s} \right) \right] \epsilon,$$

where B is the attractor or controlled F-Fractal of a CF-IFS.

Proof. Let us assume that $Z \in \mathcal{X}_0(H)$ and $\epsilon \geq 0$. Choose the CF-IFS $\{H; \mathbb{F}_j, j = 0, 1, 2, \dots, N_0\}$, where \mathbb{F}_0 is the condensation mapping with the Fisher contraction ratios r and s , so that

$$\mathcal{H}_\rho\left(Z, \bigcup_{j=0, j=1}^{N_0} \mathbb{F}_j(Z)\right) \leq \epsilon.$$

By Theorem 4, for $a \in H$, we have $\lim_{i \rightarrow \infty} \mathbb{F}^i(a) = a^*$. Consider a point $a \in H$ as fixed and the metric function $\rho(a, b)$ as continuous at a point $b \in H$. Therefore,

$$\begin{aligned} \rho(a, a^*) &= \rho\left(a, \lim_{i \rightarrow \infty} \mathbb{F}^i(a)\right) \\ &\leq \left\{ \left(1 - \frac{r}{1-r}\right)^{-1} + (1-s)^{-1} \right\} \rho(a, \mathbb{F}(a)). \end{aligned}$$

This implies

$$\mathcal{H}_\rho(Z, B) \leq \left[\left(\frac{1-r}{1-2r}\right) + \left(\frac{1}{1-s}\right) \right] \epsilon.$$

□

In the proposed theory, if we choose all the contractivity factors $r_j = 0$ ($j = 1, 2, \dots, N_0$), then the controlled Fisher IFS becomes a standard IFS; and if all the contractivity factors $s_j = 0$ ($j = 1, 2, \dots, N_0$), then the controlled Fisher IFS becomes a Kannan IFS; the converse of both cases are not always true. Hence, the method of generating controlled Fisher fractal is a generalized case of the method of generating the usual metric fractal through the classical IFS [2,4,5] and Kannan IFS (K-IFS) [41].

To date, the research on the generation of fractals has not been discussed in controlled metric space by using Fisher contractions. The importance of this research was to generate fractal sets in controlled metric space through the an iterated function system of Fisher contractions. It was demonstrated with the idea of constructing a new type of fractals in a controlled metric space through interesting theorems and results. It is believed that this research will lead to a new path for developing the controlled multifractals and their consequences based on Fisher contractions.

5. Conclusions

In this paper, a generalization of the fixed-point theorem for the Fisher contraction on a controlled metric space was explored. The Fisher contraction over controlled metric spaces was utilized in this study to develop a new type of iterated function system, the CF-IFS. Essentially, an iterated function system of Fisher contractions has been constructed in a controlled metric space to generate controlled Fisher fractals. The subsequent results proved interesting on the controlled Fisher-iterated function system and controlled Fisher fractals. It was observed that the controlled fractals are a general form of the classical fractals and Kannan-type fractals. The proposed controlled Fisher fractals and their implications will provide a prominent idea for analyzing the multi-level fractal objects in controlled metric spaces.

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Notations

H	Nonempty Set or Nonempty Space
ρ	Metric
(H, ρ)	Metric Space
\mathbb{F}	Contraction Mapping
$\mathcal{K}_0(H)$	Collection of all Nonempty Compact Subsets of H
\mathcal{H}_ρ	Hausdorff Metric
$(\mathcal{K}_0(H), \mathcal{H}_\rho)$	Hausdorff Metric Space
r, s	Contractivity Factors
\mathcal{F}	Hutchinson-Barnsley Operator
K^*	Invariant Set or Attractor
\vee	Maximum
\wedge	Minimum

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