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Right Fractional Sobolev Space via Riemann–Liouville Derivatives on Time Scales and an Application to Fractional Boundary Value Problem on Time Scales

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Abstract: Using the concept of fractional derivatives of Riemann–Liouville on time scales, we first introduce right fractional Sobolev spaces and characterize them. Then, we prove the equivalence of some norms in the introduced spaces, and obtain their completeness, reflexivity, separability and some embeddings. Finally, as an application, we propose a recent method to study the existence of weak solutions of fractional boundary value problems on time scales by using variational methods and critical point theory, and, by constructing an appropriate variational setting, we obtain two existence results of the problem.

Keywords: Riemann–Liouville derivatives on time scales; fractional Sobolev's spaces on time scales; fractional boundary value problems on time scales

MSC: 34A08; 26A33; 34B15; 34N05

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1. Introduction

In the past three decades, fractional calculus and fractional differential equations have attracted extensive interest and attention in the fields of differential equations and applied mathematics, science and technology. In addition to genuine mathematical interest and curiosity, this trend is also driven by interesting scientific and technological applications that produce fractional differential equation models to better describe memory effects and nonlocal phenomena [1–5]. It is the rise of these applications that drives the field of fractional calculus and fractional differential equations in a new direction, and further research in this field is required.

As is known, discrete-time systems are as important as continuous-time systems. Therefore, it is equally important to study the solvability of boundary value problems of fractional differential equations and difference equations. At the same time, discrete-time systems are more convenient for computer processing. However, compared with the research on continuous-time systems, the research on the corresponding problems of discrete-time topics is sparse. Fortunately, the time scale theory proposed by Stefan Hilger [6] can unify the study of differential equations and difference equations. In order to study the existence and multiplicity of solutions of differential equations and difference equations by variational methods in a unified framework, Refs. [7–9] have studied some Sobolev space theories on time scales. More exactly, Agarwal et al. studied the theory of Sobolev's spaces of functions defined on a closed subinterval of an arbitrary time scale endowed with the Lebesgue Δ-measure in [7]. Zhou and Li studied Sobolev's spaces on time scales and their properties in [8]. Wang et al. introduced the theory of fractional Sobolev spaces on time scales by conformable fractional derivatives on time scales in [9]. Recently, some other classical tools or techniques, such as the coincidence degree theory, the method of upper and lower solutions with monotone iterative technique and some fixed point theorems, etc., have been used to study the existence and multiplicity of solutions of differential equations and difference equations in the literature [10–17].

However, so far, there is no right fractional Sobolev space via Riemann–Liouville derivatives on time scales. In order to fill this gap, the main purpose of this paper is to establish right fractional Sobolev spaces on time scales through Riemann–Liouville fractional derivatives, and study some of their basic properties. Then, as an application of our new theory, we study the solvability of a class of fractional boundary value problems on time scales.

The rest of this paper is organized as follows. In Section 2, we review some symbols, basic concepts and basic results of time scale calculus that will be used in this paper, and give the definitions of fractional integrals and derivatives on time scales. In Section 3, we study some basic properties of right Riemann–Liouville fractional integrals and differential operators on time scales, including the equivalence between the fractional integral on time scales defined by integrals on time scales and the fractional integral on time scales defined by the Laplace transform and the inverse Laplace transform. In Section 4, we give the definition of right fractional Sobolev spaces on time scales and study some of their important properties. In Section 5, as an application of the results of this paper, we study the solvability of a fractional boundary value problem on time scales by using the critical point theory and variational methods. In Section 6, we give a concise conclusion.

2. Preliminaries

In this section, we will recall some basic known notations, definitions and results, which are needed in what follows.

Throughout this paper, we denote by $\mathbb{T}$ a time scale. We will use the following notations: $J_{\mathbb{R}}^0 = [a,b)$, $J_{\mathbb{R}} = [a,b]$, $J^0 = J_{\mathbb{R}}^0 \cap \mathbb{T}$, $J = J_{\mathbb{R}} \cap \mathbb{T}$, $J^k = [a, \rho(b)] \cap \mathbb{T}$.

Definition 1 ([18]). *For $t \in \mathbb{T}$, we define the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$, while the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined by $\rho(t) := \sup\{s \in \mathbb{T} : s < t\}$.*

Remark 1 ([18]). *(1) In Definition 1, we put $\inf \varnothing = \sup \mathbb{T}$ (i.e., $\sigma(t) = t$ if $\mathbb{T}$ has a maximum t) and $\sup \varnothing = \inf \mathbb{T}$ (i.e., $\rho(t) = t$ if $\mathbb{T}$ has a minimum t), where $\varnothing$ denotes the empty set.*
(2) If $\sigma(t) > t$, we say that t is right-scattered, while if $\rho(t) < t$, we say that t is left-scattered. Points that are right-scattered and left-scattered at the same time are called isolated.
(3) If $t < \sup \mathbb{T}$ and $\sigma(t) = t$, we say that t is right-dense, while if $t > \inf \mathbb{T}$ and $\rho(t) = t$, we say that t is left-dense. Points that are right-dense and left-dense at the same time are called dense.
(4) The graininess function $\mu : \mathbb{T} \to [0, \infty)$ is defined by $\mu(t) := \sigma(t) - t$.
(5) The derivative makes use of the set $\mathbb{T}^k$, which is derived from the time scale $\mathbb{T}$ as follows: if $\mathbb{T}$ has a left-scattered maximum M, then $\mathbb{T}^k := \mathbb{T} \backslash \{M\}$; otherwise, $\mathbb{T}^k := \mathbb{T}$.

Definition 2 ([19]). *Assume that $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^k$. Then, we define $f^\Delta(t)$ to be the number (provided it exists) with the property that, given any $\varepsilon > 0$, there is a neighborhood U of t (i.e., $U = (t-\delta, t+\delta) \cap \mathbb{T}$ for some $\delta > 0$) such that*

$$|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)| \leq \varepsilon |\sigma(t) - s|$$

for all $s \in U$. We call $f^\Delta(t)$ the delta (or Hilger) derivative of f at t. Moreover, we say that f is delta (or Hilger) differentiable (or, in short, differentiable) on $\mathbb{T}^k$ provided that $f^\Delta(t)$ exists for all $t \in \mathbb{T}^k$. The function $f^\Delta : \mathbb{T}^k \to \mathbb{R}$ is then called the (delta) derivative of f on $\mathbb{T}^k$.

Definition 3 ([18]). *A function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$. The set of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ will be denoted by $C_{rd} = C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, \mathbb{R})$. The set of* *functions $f : \mathbb{T} \to \mathbb{R}$ that are differentiable and whose derivative is rd-continuous is denoted by $C^1_{rd} = C^1_{rd}(\mathbb{T}) = C^1_{rd}(\mathbb{T}, \mathbb{R})$.*

Definition 4 ([7]). *Let $A \subset \mathbb{T}$. A is called a Δ-null set if $\mu_\Delta(A) = 0$. We can say that a property P holds Δ-almost everywhere (Δ-a.e.) on A, or for Δ-almost all (Δ-a.a.) $t \in A$ if there is a Δ-null set $E \subset A$ such that P holds for all $t \in A \backslash E$.*

Definition 5 ([20]). *Let J denote a closed bounded interval in $\mathbb{T}$. A function $F : J \to \mathbb{R}$ is called a delta antiderivative of function $f : J^0 \to \mathbb{R}$ provided F is continuous on J, delta differentiable at J^0 and $F^\Delta(t) = f(t)$ for all $t \in J^0$. Then, we define the Δ-integral of f from a to b by $\int_a^b f(t)\Delta t := F(b) - F(a)$.*

Theorem 1 ([19]). *If $a, b \in \mathbb{T}$ and $f, g \in C_{rd}(\mathbb{T})$, then*

$$\int_{J^0} f^\sigma(t) g^\Delta(t)\Delta t = (fg)(b) - (fg)(a) - \int_{J^0} f^\Delta(t) g(t)\Delta t.$$

Proposition 1 ([21]). *Let f be an increasing continuous function on J. If F is the extension of f to the real interval $J_\mathbb{R}$ given by*

$$F(s) := \begin{cases} f(s), & if\ s \in \mathbb{T}, \\ f(t), & if\ s \in (t, \sigma(t)) \notin \mathbb{T}, \end{cases}$$

then

$$\int_a^b f(t)\Delta t \leq \int_a^b F(t)dt.$$

Theorem 2 ([22]). *$y(t,s) = h_{n-1}(t, \sigma(s))$ is the Cauchy function of $y^{\Delta^n} = 0$, where*

$$h_0(t,s) = 1, \quad h_n(t,s) = \int_s^t h_{n-1}(\tau, s)\Delta\tau, \quad n \in \mathbb{N}.$$

Theorem 3 ([22]). *For all $n \in \mathbb{N}_0$, we have*

$$\mathcal{L}_\mathbb{T}(h_n(x,0))(z) = \frac{1}{z^{n-1}}, \quad x \in \mathbb{T}_0.$$

for all $z \in \mathbb{C}\backslash\{0\}$ such that $1 + z\mu(x) \neq 0$, $x \in \mathbb{T}_0$, and

$$\lim_{x \to \infty} (h_n(x,0) e_{\ominus z}(x,0)) = 0.$$

Definition 6 ([22] (Shift (Delay) of a Function)). *For a given function $f : [t_0, \infty) \to \mathbb{C}$, the solution of the shifting problem*

$$\begin{aligned} u^{\Delta_t}(t, \sigma(s)) = -u^{\Delta_s}(t,s), \quad t, s \in \mathbb{T}, \quad t \geq t \geq s \geq t_0, \\ u(t, t_0) = f(t), \quad t \in \mathbb{T}, \quad t \geq t_0, \end{aligned}$$

is denoted by $\hat{f}$ and is called the shift or delay of f.

Definition 7 ([22] (Δ-Power Function)). *Suppose that $\alpha \in \mathbb{R}$; we define the generalized Δ-power function $h_\alpha(t, t_0)$ on $\mathbb{T}$ as follows:*

$$h_\alpha(t, t_0) = \mathcal{L}_\mathbb{T}^{-1}(\frac{1}{z^{\alpha+1}})(t), \quad t \geq t_0,$$

for all $z \in \mathbb{C} \backslash \{0\}$ such that $\mathcal{L}_{\mathbb{T}}^{-1}$ exists, $t \geq t_0$. The fractional generalized Δ-power function $h_\alpha(t,s)$ on $\mathbb{T}$, $t \geq s \geq t_0$, is defined as the shift of $h_\alpha(t,t_0)$, i.e.,

$$h_\alpha(t,s) = \widehat{h_\alpha(\cdot,t_0)}(t,s), \quad t,s \in \mathbb{T}, \quad t \geq s \geq t_0.$$

Inspired by Definition 4 in [23] and Definition 2.1 in [24], we present the right Riemann–Liouville fractional integral and derivative on time scales as follows.

Definition 8 (Fractional integral on time scales)**.** *Suppose h is an integrable function on J. Let $0 < \alpha \leq 1$. Then, the left fractional integral of order α of h is defined by*

$${}_a^{\mathbb{T}}I_t^\alpha h(t) := \int_a^t \frac{(t-\sigma(s))^{\alpha-1}}{\Gamma(\alpha)} h(s)\Delta s.$$

The right fractional integral of order α of h is defined by

$${}_t^{\mathbb{T}}I_b^\alpha h(t) := \int_t^b \frac{(s-\sigma(t))^{\alpha-1}}{\Gamma(\alpha)} h(s)\Delta s, \tag{1}$$

where Γ is the gamma function.

Definition 9 (Riemann–Liouville fractional derivative on time scales)**.** *Let $t \in \mathbb{T}$, $0 < \alpha \leq 1$, and $h : \mathbb{T} \to \mathbb{R}$. The left Riemann–Liouville fractional derivative of order α of h is defined by*

$${}_a^{\mathbb{T}}D_t^\alpha h(t) := \left({}_a^{\mathbb{T}}I_t^{1-\alpha} h(t)\right)^\Delta = \frac{1}{\Gamma(1-\alpha)} \left(\int_a^t (t-\sigma(s))^{-\alpha} h(s)\Delta s\right)^\Delta.$$

The right Riemann–Liouville fractional derivative of order α of h is defined by

$${}_t^{\mathbb{T}}D_b^\alpha h(t) := -\left({}_t^{\mathbb{T}}I_b^{1-\alpha} h(t)\right)^\Delta = \frac{-1}{\Gamma(1-\alpha)} \left(\int_t^b (s-\sigma(t))^{-\alpha} h(s)\Delta s\right)^\Delta. \tag{2}$$

Actually, ${}_t^{\mathbb{T}}D_b^\alpha h(t)$ can be rewritten as $-\Delta \circ {}_t^{\mathbb{T}}I_b^{1-\alpha} h(t)$.

Inspired by Definition 4 and Equation (21) in [23] and Theorem 2.1 in [4], we present the right Caputo fractional derivative on time scales as follows.

Definition 10 (Caputo fractional derivative on time scales)**.** *Let $t \in \mathbb{T}$, $0 < \alpha \leq 1$ and $h : \mathbb{T} \to \mathbb{R}$. The left Caputo fractional derivative of order α of h is defined by*

$${}_a^{\mathbb{T}C}D_t^\alpha h(t) := {}_a^{\mathbb{T}}I_t^{1-\alpha} h^\Delta(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t (t-\sigma(s))^{-\alpha} h^\Delta(s)\Delta s.$$

The right Caputo fractional derivative of order α of h is defined by

$${}_t^{\mathbb{T}C}D_b^\alpha h(t) := -{}_t^{\mathbb{T}}I_b^{1-\alpha} h^\Delta(t) = \frac{-1}{\Gamma(1-\alpha)} \int_t^b (s-\sigma(t))^{-\alpha} h^\Delta(s)\Delta s.$$

Definition 11 ([25])**.** *For $f : \mathbb{T} \to \mathbb{R}$, the time scale or generalized Laplace transform of f, denoted by $\mathcal{L}_{\mathbb{T}}\{f\}$ or $F(z)$, is given by*

$$\mathcal{L}_{\mathbb{T}}\{f\}(z) = F(z) := \int_0^\infty f(t) g^\sigma(t)\Delta t,$$

where $g(t) = e_{\ominus z}(t,0)$.

Theorem 4 ([25] (Inversion formula of the Laplace transform)). *Suppose that $F(z)$ is analytic in the region $Re_\mu(z) > Re_\mu(c)$ and $F(z) \to 0$ uniformly as $|z| \to \infty$ in this region. Suppose $F(z)$ has finitely many regressive poles of finite order $\{z_1, z_2, \ldots, z_n\}$ and $\widetilde{F}_{\mathbb{R}}(z)$ is the transform of the function $\tilde{f}(t)$ on $\mathbb{R}$ that corresponds to the transform $F(z) = F_{\mathbb{T}}(z)$ of $f(t)$ on $\mathbb{T}$. If*

$$\int_{c-i\infty}^{c+i\infty} |\widetilde{F}_{\mathbb{R}}(z)||dz| < \infty,$$

then

$$f(t) = \sum_{i=1}^{n} Res_{z=z_i} e_z(t,0)F(z)$$

has transform $F(z)$ for all z with $Re(z) > c$.

Motivated by Definition 3.1 in [26], we present the right Riemann–Liouville fractional integral on time scales as follows.

Definition 12 (Right Riemann–Liouville fractional integral on time scales). *Let $\alpha > 0$, $\mathbb{T}$ be a time scale, and $f : \mathbb{T} \to \mathbb{R}$. The right Riemann–Liouville fractional integral of f of order α on the time scale $\mathbb{T}$, denoted by ${}_bI^\alpha_{\mathbb{T}} f$, is defined by*

$${}_bI^\alpha_{\mathbb{T}} f(t) = \mathcal{L}^{-1}_{\mathbb{T}}\left[\frac{-F(z)}{z^\alpha}\right](t).$$

Theorem 5 ([8]). *A function $f : J \to \mathbb{R}^N$ is absolutely continuous on J if and only if f is Δ-differentiable $\Delta - a.e.$ on J^0 and*

$$f(t) = f(a) + \int_{[a,t)_{\mathbb{T}}} f^\Delta(s)\Delta s, \quad \forall t \in J.$$

Theorem 6 ([27]). *A function $f : \mathbb{T} \to \mathbb{R}$ is absolutely continuous on $\mathbb{T}$ if and only if the following conditions are satisfied:*

(i) *f is Δ-differentiable $\Delta - a.e.$ on J^0 and $f^\Delta \in L^1(\mathbb{T})$.*

(ii) *The equality*

$$f(t) = f(a) + \int_{[a,t)_{\mathbb{T}}} f^\Delta(s)\Delta s$$

holds for every $t \in \mathbb{T}$.

Theorem 7 ([28]). *A function $q : J_{\mathbb{R}} \to \mathbb{R}^N$ is absolutely continuous if and only if there exist a constant $c \in \mathbb{R}^N$ and a function $\varphi \in L^1$ such that*

$$q(t) = c + (I^1_{a+}\varphi)(t), \quad t \in J_{\mathbb{R}}.$$

In this case, we have $q(a) = c$ and $q'(t) = \varphi(t)$, $t \in J_{\mathbb{R}}$ a.e.

Theorem 8 ([28] (Integral representation)). *Let $\alpha \in (0,1]$ and $q \in L^1$. Then, q has a right-sided Riemann–Liouville derivative $D^\alpha_{b-}q$ of order α if and only if there exist a constant $d \in \mathbb{R}^N$ and a function $\psi \in L^1$ such that*

$$q(t) = \frac{1}{\Gamma(\alpha)}\frac{d}{(b-t)^{1-\alpha}} + (I^\alpha_{b-}\psi)(t), \quad t \in J_R \quad a.e..$$

In this case, we have $I^{1-\alpha}_{b-}q(b) = d$ and $(D^\alpha_{b-}q)(t) = \psi(t)$, $t \in J_{\mathbb{R}}$ a.e.

Lemma 1 ([7]). *Let $f \in L^1_\Delta(J^0)$. Then, a necessary and sufficient condition for the validity of the equality*

$$\int_{J^0} (f \cdot \varphi^\Delta)(s)\Delta s = 0, \quad for\ every\ \varphi \in C^1_{0,rd}(J^k)$$

is the existence of a constant $c \in \mathbb{R}$ such that

$$f \equiv c \quad \Delta - a.e.\ on\ J^0.$$

Definition 13 ([7]). *Let $p \in \bar{\mathbb{R}}$ be such that $p \geq 1$ and $u : J \to \bar{\mathbb{R}}$. We say that u belongs to $W^{1,p}_\Delta(J)$ if and only if $u \in L^p_\Delta(J^0)$ and there exists $g : J^k \to \bar{\mathbb{R}}$ such that $g \in L^p_\Delta(J^0)$ and*

$$\int_{J^0} (u \cdot \varphi^\Delta)(s)\Delta s = -\int_{J^0} (g \cdot \varphi^\sigma)(s)\Delta s, \quad \forall \varphi \in C^1_{0,rd}(J^k),$$

where

$$C^1_{0,rd}(J^k) := \left\{ f : J \to \mathbb{R} : f \in C^1_{rd}(J^k),\ f(a) = f(b) \right\}$$

and $C^1_{rd}(J^k)$ is the set of all continuous functions on J such that they are Δ-differential on J^k and their Δ-derivatives are rd-continuous on J^k.

Theorem 9 ([7]). *Let $p \in \bar{\mathbb{R}}$ be such that $p \geq 1$. Then, the set $L^p_\Delta(J^0)$ is a Banach space together with the norm defined for every $f \in L^p_\Delta(J^0)$ as*

$$\|f\|_{L^p_\Delta} := \begin{cases} \left[\int_{J^0} |f|^p(s)\Delta s \right]^{\frac{1}{p}}, & if\ p \in \mathbb{R}, \\ \inf\{C \in \mathbb{R} : |f| \leq C\ \Delta - a.e.\ on\ J^0\}, & if\ p = +\infty. \end{cases}$$

Moreover, $L^2_\Delta(J^0)$ is a Hilbert space together with the inner product given for every $(f, g) \in L^2_\Delta(J^0) \times L^2_\Delta(J^0)$ by

$$(f, g)_{L^2_\Delta} := \int_{J^0} f(s) \cdot g(s)\Delta s.$$

Theorem 10 ([24]). *Fractional integration operators are bounded in $L^p(J_\mathbb{R})$, i.e., the following estimate*

$$\|I^\alpha_{a+}\varphi\|_{L^p(a,b)} \leq \frac{(b-a)^{Re\alpha}}{Re\alpha|\Gamma(\alpha)|}\|\varphi\|_{L^p(J_\mathbb{R})}, \quad Re\alpha > 0$$

holds.

Proposition 2 ([7]). *Suppose $p \in \bar{\mathbb{R}}$ and $p \geq 1$. Let $p' \in \bar{\mathbb{R}}$ be such that $\frac{1}{p'} + \frac{1}{p'} = 1$. Then, if $f \in L^p_\Delta(J^0)$ and $g \in L^{p'}_\Delta(J^0)$, then $f \cdot g \in L^1_\Delta(J^0)$ and*

$$\|f \cdot g\|_{L^1_\Delta} \leq \|f\|_{L^p_\Delta} \cdot \|g\|_{L^{p'}_\Delta}.$$

This expression is called Hölder's inequality and Cauchy–Schwarz's inequality whenever $p = 2$.

Theorem 11 ([19] (First Mean Value Theorem))**.** *Let f and g be bounded and integrable functions on J, and let g be nonnegative (or nonpositive) on J. Let us set*

$$m = \inf\{f(t) : t \in J^0\} \quad \text{and} \quad M = \sup\{f(t) : t \in J^0\}.$$

Then, there exists a real number Λ satisfying the inequalities $m \le \Lambda \le M$ such that

$$\int_a^b f(t)g(t)\Delta t = \Lambda \int_a^b g(t)\Delta t.$$

Corollary 1 ([19])**.** *Let f be an integrable function on J and let m and M be the infimum and supremum, respectively, of f on J^0. Then, there exists a number Λ between m and M such that*

$$\int_a^b f(t)\Delta t = \Lambda(b-a).$$

Theorem 12 ([19])**.** *Let f be a function defined on J and let $c \in \mathbb{T}$ with $a < c < b$. If f is Δ-integrable from a to c and from c to b, then f is Δ-integrable from a to b and*

$$\int_a^b f(t)\Delta t = \int_a^c f(t)\Delta t + \int_c^b f(t)\Delta t.$$

Lemma 2 ([29] (A time scale version of the Arzela–Ascoli theorem))**.** *Let X be a subset of $C(J,\mathbb{R})$ satisfying the following conditions:*

(i) *X is bounded.*
(ii) *For any given $\epsilon > 0$, there exists $\delta > 0$ such that $t_1, t_2 \in J$, $|t_1 - t_2| < \delta$ implies $|f(t_1) - f(t_2)| < \epsilon$ for all $f \in X$.*

Then, X is relatively compact.

3. Some Fundamental Properties of Right Riemann–Liouville Fractional Operators on Time Scales

Inspired by [30], we can obtain the consistency of Definitions 8 and 12 by using the above theory of the Laplace transform on time scales and the inverse Laplace transform on time scales.

Theorem 13. *Let $\alpha > 0$, $\mathbb{T}$ be a time scale, $[a,b]_{\mathbb{T}}$ be an interval of $\mathbb{T}$ and f be an integrable function on $[a,b]_{\mathbb{T}}$. Then, $\left({}_t^{\mathbb{T}}I_b^{\alpha} f\right)(t) = {}_bI_{\mathbb{T}}^{\alpha} f(t)$.*

Proof. Using the Laplace transform on $\mathbb{T}$ for (1), in view of Definition 8, Definition 3, Theorem 7, the proof of Theorem 4.14 in [22] and Definition 11, we have

$$\begin{aligned}
&\mathcal{L}_{\mathbb{T}}\left\{\left({}_t^{\mathbb{T}}I_b^{\alpha} f\right)(t)\right\}(z) \\
=&\mathcal{L}_{\mathbb{T}}\left\{\frac{1}{\Gamma(\alpha)}\int_t^b (s-\sigma(t))^{\alpha-1} f(s)\Delta s\right\}(z) \\
=&\mathcal{L}_{\mathbb{T}}\left\{\frac{-1}{\Gamma(\alpha)}\int_b^t (s-\sigma(t))^{\alpha-1} f(s)\Delta s\right\}(z) \\
=&-\mathcal{L}_{\mathbb{T}}(h_{\alpha-1}(\cdot,b) * f)(t)(z) \\
=&-\mathcal{L}_{\mathbb{T}}(h_{\alpha-1}(\cdot,b))(z)\mathcal{L}_{\mathbb{T}}(f)(t)(z) \\
=&\frac{-1}{z^{\alpha}}\mathcal{L}_{\mathbb{T}}\{f\}(z) \\
=&\frac{-F(z)}{z^{\alpha}}(t).
\end{aligned} \tag{3}$$

Taking the inverse Laplace transform on $\mathbb{T}$ for (3), with an eye to Definition 12, one arrives at

$$\left({}_t^{\mathbb{T}}I_b^{\alpha} f\right)(t) = \mathcal{L}_{\mathbb{T}}^{-1}\left[\frac{-F(z)}{z^{\alpha}}\right](t) = {}_bI_{\mathbb{T}}^{\alpha} f(t).$$

The proof is complete. □

Combining with [20,26] and Theorem 13, we can prove that Proposition 15, Proposition 16, Proposition 17, Corollary 18, Theorem 20 and Theorem 21 in [20] hold for the new definition of Riemann–Liouville fractional operators on time scales.

Proposition 3. *Let h be Δ-integrable on J and $0 < \alpha \leq 1$. Then,* ${}_t^{\mathbb{T}}D_b^{\alpha} h(t) = -\Delta \circ {}_t^{\mathbb{T}}I_b^{1-\alpha} h(t)$.

Proof. Let $h : \mathbb{T} \to \mathbb{R}$. In view of (1) and (2), we obtain

$$\begin{aligned}
{}_t^{\mathbb{T}}D_b^{\alpha} h(t) =& \frac{-1}{\Gamma(1-\alpha)}\left(\int_t^b (s-\sigma(t))^{-\alpha} h(s)\Delta s\right)^{\Delta} \\
=& -\left({}_t^{\mathbb{T}}I_b^{1-\alpha} h(t)\right)^{\Delta} \\
=& -(\Delta \circ {}_t^{\mathbb{T}}I_b^{1-\alpha})h(t).
\end{aligned}$$

The proof is complete. □

Proposition 4. *Let h be integrable on J; then, its right Riemann–Liouville Δ-fractional integral satisfies*

$${}_t^{\mathbb{T}}I_b^{\alpha} \circ {}_t^{\mathbb{T}}I_b^{\beta} = {}_t^{\mathbb{T}}I_b^{\alpha+\beta} = {}_t^{\mathbb{T}}I_b^{\beta} \circ {}_t^{\mathbb{T}}I_b^{\alpha}$$

for $\alpha > 0$ and $\beta > 0$.

Proof. Inspired by the proof of Proposition 3.4 in [26], in view of Definition 12, we have

$${}_bI_{\mathbb{T}}^{\beta}({}_bI_{\mathbb{T}}^{\alpha} f)(t) = \mathcal{L}_{\mathbb{T}}^{-1}\left[-\frac{\mathcal{L}_{\mathbb{T}}\{{}_bI_{\mathbb{T}}^{\alpha} f\}}{z^{\beta}}\right](t) = \mathcal{L}_{\mathbb{T}}^{-1}\left[\frac{F(s)}{s^{\alpha+\beta}}\right](t) = {}_bI_{\mathbb{T}}^{\alpha+\beta} f(t). \tag{4}$$

Combining with (4) and Theorem 13, one obtains that

$${}_t^{\mathbb{T}}I_b^{\alpha} \circ {}_t^{\mathbb{T}}I_b^{\beta} = {}_t^{\mathbb{T}}I_b^{\alpha+\beta}.$$

In a similar way, one arrives at

$${}_t^{\mathbb{T}}I_b^{\beta} \circ {}_t^{\mathbb{T}}I_b^{\alpha} = {}_t^{\mathbb{T}}I_b^{\alpha+\beta}.$$

Consequently, we obtain that

$${}_t^{\mathbb{T}}I_b^{\alpha} \circ {}_t^{\mathbb{T}}I_b^{\beta} = {}_t^{\mathbb{T}}I_b^{\alpha+\beta} = {}_t^{\mathbb{T}}I_b^{\beta} \circ {}_t^{\mathbb{T}}I_b^{\alpha}.$$

The proof is complete. □

Proposition 5. *If function h is integrable on J, then* ${}_t^{\mathbb{T}}D_b^{\alpha} \circ {}_t^{\mathbb{T}}I_b^{\alpha} h = h$.

Proof. Taking account of Propositions 3 and 4, one can obtain

$${}_t^{\mathbb{T}}D_b^{\alpha} \circ {}_t^{\mathbb{T}}I_b^{\alpha} h(t) = -\left({}_t^{\mathbb{T}}I_b^{1-\alpha}({}_t^{\mathbb{T}}I_b^{\alpha}(h(t))\right)^{\Delta} = -\left({}_t^{\mathbb{T}}I_b h(t)\right)^{\Delta} = h.$$

The proof is complete. □

Corollary 2. *For $0 < \alpha \leq 1$, we have*

$$ {}_t^{\mathbb{T}}D_b^{\alpha} \circ {}_t^{\mathbb{T}}D_b^{-\alpha} = Id \quad and \quad {}_t^{\mathbb{T}}I_b^{-\alpha} \circ {}_t^{\mathbb{T}}I_b^{\alpha} = Id, $$

where Id denotes the identity operator.

Proof. In view of Proposition 5, we have

$$ {}_t^{\mathbb{T}}D_b^{\alpha} \circ {}_t^{\mathbb{T}}D_b^{-\alpha} = {}_t^{\mathbb{T}}D_b^{\alpha} \circ {}_t^{\mathbb{T}}I_b^{\alpha} = Id \quad \text{and} \quad {}_t^{\mathbb{T}}I_b^{-\alpha} \circ {}_t^{\mathbb{T}}I_b^{\alpha} = {}_t^{\mathbb{T}}D_b^{\alpha} \circ {}_t^{\mathbb{T}}I_b^{\alpha} = Id. $$

The proof is complete. □

Theorem 14. *Let $f \in C(J)$ and $\alpha > 0$. Then, $f \in {}_t^{\mathbb{T}}I_b^{\alpha}(J)$ if and only if*

$$ {}_t^{\mathbb{T}}I_b^{1-\alpha} f \in C^1(J) \tag{5} $$

and

$$ \left({}_t^{\mathbb{T}}I_b^{1-\alpha} f(t) \right)\Big|_{t=b} = 0, \tag{6} $$

where ${}_t^{\mathbb{T}}I_b^{\alpha}(J)$ denotes the space of functions that can be represented by the right Riemann–Liouville Δ-integral of order α of a $C(J)$-function.

Proof. Suppose $f \in {}_t^{\mathbb{T}}I_b^{\alpha}(J)$, $f(t) = {}_t^{\mathbb{T}}I_b^{\alpha} g(t)$ for some $g \in C(J)$, and

$$ {}_t^{\mathbb{T}}I_b^{1-\alpha}(f(t)) = {}_t^{\mathbb{T}}I_b^{1-\alpha}({}_t^{\mathbb{T}}I_b^{\alpha} g(t)). $$

In view of Proposition 4, one obtains

$$ {}_t^{\mathbb{T}}I_b^{1-\alpha}(f(t)) = {}_t^{\mathbb{T}}I_b g(t) = \int_t^b g(s)\Delta s. $$

As a result, ${}_t^{\mathbb{T}}I_b^{1-\alpha} f \in C(J)$ and

$$ \left({}_t^{\mathbb{T}}I_b^{1-\alpha} f(t) \right)\Big|_{t=b} = \int_b^b g(s)\Delta s = 0. $$

Inversely, suppose that $f \in C(J)$ satisfies (5) and (6). Then, by applying Taylor's formula to function ${}_t^{\mathbb{T}}I_b^{1-\alpha} f$, we obtain

$$ {}_t^{\mathbb{T}}I_b^{1-\alpha} f(t) = \int_t^b \frac{\Delta}{\Delta s} {}_s^{\mathbb{T}}I_b^{1-\alpha} f(s)\Delta s, \quad \forall t \in J. $$

Let $\varphi(t) = \frac{\Delta}{\Delta t} {}_t^{\mathbb{T}}I_b^{1-\alpha} f(t)$. Note that $\varphi \in C(J)$ by (5). Now, by Proposition 4, one sees that

$$ {}_t^{\mathbb{T}}I_b^{1-\alpha}(f(t)) = {}_t^{\mathbb{T}}I_b^{1}\varphi(t) = {}_t^{\mathbb{T}}I_b^{1-\alpha}[{}_t^{\mathbb{T}}I_b^{\alpha}\varphi(t)] $$

and hence

$$ {}_t^{\mathbb{T}}I_b^{1-\alpha}(f(t)) - {}_t^{\mathbb{T}}I_b^{1-\alpha}[{}_t^{\mathbb{T}}I_b^{\alpha}\varphi(t)] \equiv 0. $$

Therefore, we have

$$ {}_t^{\mathbb{T}}I_b^{1-\alpha}[f(t) - {}_t^{\mathbb{T}}I_b^{\alpha}\varphi(t)] \equiv 0. $$

From the uniqueness of the solution to Abel's integral equation [31], this implies that $f - {}_{t}^{\mathbb{T}}I_{b}^{\alpha}\varphi \equiv 0$. Hence, $f = {}_{t}^{\mathbb{T}}I_{b}^{\alpha}\varphi$ and $f \in {}_{t}^{\mathbb{T}}I_{b}^{\alpha}(J)$. The proof is complete. □

Theorem 15. *Let $\alpha > 0$ and $f \in C(J)$ satisfy the condition in Theorem 14. Then,*

$$({}_{a}^{\mathbb{T}}I_{t}^{\alpha} \circ {}_{a}^{\mathbb{T}}D_{t}^{\alpha})(f) = f.$$

Proof. Combining with Theorem 14 and Proposition 5, we can see that

$${}_{t}^{\mathbb{T}}I_{b}^{\alpha} \circ {}_{t}^{\mathbb{T}}D_{b}^{\alpha}f(t) = {}_{t}^{\mathbb{T}}I_{b}^{\alpha} \circ {}_{t}^{\mathbb{T}}D_{b}^{\alpha}({}_{t}^{\mathbb{T}}I_{b}^{\alpha}\varphi(t)) = {}_{t}^{\mathbb{T}}I_{b}^{\alpha}\varphi(t) = f(t).$$

The proof is complete. □

Motivated by the proof of Equation (2.20) in [24], we present and prove the following theorem.

Theorem 16. *Let $\alpha > 0$, $p, q \geq 1$, and $\frac{1}{p} + \frac{1}{q} \leq 1 + \alpha$, where $p \neq 1$ and $q \neq 1$ in the case when $\frac{1}{p} + \frac{1}{q} = 1 + \alpha$. Moreover, let*

$${}_{a}^{\mathbb{T}}I_{t}^{\alpha}(L^{p}) := \left\{ f : f = {}_{a}^{\mathbb{T}}I_{t}^{\alpha}g,\ g \in L^{p}(J) \right\}$$

and

$${}_{t}^{\mathbb{T}}I_{b}^{\alpha}(L^{p}) := \left\{ f : f = {}_{t}^{\mathbb{T}}I_{b}^{\alpha}g,\ g \in L^{p}(J) \right\}.$$

Then, the following integration by parts formulas hold.

(a) *If $\varphi \in L^{p}(J)$ and $\psi \in L^{q}(J)$, then*

$$\int_{a}^{b} \varphi(t)\left({}_{a}^{\mathbb{T}}I_{t}^{\alpha}\psi\right)(t)\Delta t = \int_{a}^{b} \psi(t)\left({}_{t}^{\mathbb{T}}I_{b}^{\alpha}\varphi\right)(t)\Delta t.$$

(b) *If $g \in {}_{t}^{\mathbb{T}}I_{b}^{\alpha}(L^{p})$ and $f \in {}_{a}^{\mathbb{T}}I_{t}^{\alpha}(L^{q})$, then*

$$\int_{a}^{b} g(t)\left({}_{a}^{\mathbb{T}}D_{t}^{\alpha}f\right)(t)\Delta t = \int_{a}^{b} f(t)\left({}_{t}^{\mathbb{T}}D_{b}^{\alpha}g\right)(t)\Delta t.$$

(c) *For Caputo fractional derivatives, if $g \in {}_{t}^{\mathbb{T}}I_{b}^{\alpha}(L^{p})$ and $f \in {}_{a}^{\mathbb{T}}I_{t}^{\alpha}(L^{q})$, then*

$$\int_{a}^{b} g(t)\left({}_{a}^{\mathbb{T}C}D_{t}^{\alpha}f\right)(t)\Delta t = \left[{}_{t}^{\mathbb{T}}I_{b}^{1-\alpha}g(t) \cdot f(t)\right]\Big|_{t=a}^{b} + \int_{a}^{b} f(\sigma(t))\left({}_{t}^{\mathbb{T}}D_{b}^{\alpha}g\right)(t)\Delta t.$$

and

$$\int_{a}^{b} g(t)\left({}_{t}^{\mathbb{T}C}D_{b}^{\alpha}f\right)(t)\Delta t = \left[{}_{a}^{\mathbb{T}}I_{t}^{1-\alpha}g(t) \cdot f(t)\right]\Big|_{t=a}^{b} + \int_{a}^{b} f(\sigma(t))\left({}_{a}^{\mathbb{T}}D_{t}^{\alpha}g\right)(t)\Delta t.$$

Proof. (a) It follows from Definition 8 and Fubini's theorem on time scales that

$$\begin{aligned}
&\int_a^b \varphi(t)\left({}_a^{\mathbb{T}}I_t^{\alpha}\psi\right)(t)\Delta t \\
&= \int_a^b \varphi(t)\left(\int_a^t \frac{(t-\sigma(s))^{\alpha-1}}{\Gamma(\alpha)}\psi(s)\Delta s\right)\Delta t \\
&= \int_a^b \psi(s)\int_s^b \frac{(t-\sigma(s))^{\alpha-1}}{\Gamma(\alpha)}\varphi(t)\Delta t\Delta s \\
&= \int_a^b \psi(t)\int_t^b \frac{(s-\sigma(t))^{\alpha-1}}{\Gamma(\alpha)}\varphi(s)\Delta s\Delta t \\
&= \int_a^b \psi(t)\left({}_t^{\mathbb{T}}I_b^{\alpha}\varphi\right)(t)\Delta t.
\end{aligned}$$

The proof is complete.

(b) It follows from Definition 9 and Fubini's theorem on time scales that

$$\begin{aligned}
&\int_a^b g(t)\left({}_a^{\mathbb{T}}D_t^{\alpha}f\right)(t)\Delta t \\
&= \int_a^b g(t)\left(\frac{1}{\Gamma(1-\alpha)}\left(\int_a^t (t-\sigma(s))^{-\alpha}f(s)\Delta s\right)^{\Delta}\right)\Delta t \\
&= \int_a^b f(s)\left(\frac{1}{\Gamma(1-\alpha)}\left(\int_s^b (t-\sigma(s))^{-\alpha}g(t)\Delta t\right)^{\Delta}\right)\Delta s \\
&= \int_a^b f(t)\left(\frac{1}{\Gamma(1-\alpha)}\left(\int_t^b (s-\sigma(t))^{-\alpha}g(s)\Delta s\right)^{\Delta}\right)\Delta t \\
&= \int_a^b g(t)\left({}_t^{\mathbb{T}}D_b^{\alpha}f\right)(t)\Delta t.
\end{aligned}$$

The proof is complete.

(c) It follows from Definition 10, Fubini's theorem on time scales and Theorem 1 that

$$\begin{aligned}
&\int_a^b g(t)\left({}_a^{\mathbb{T}C}D_t^{\alpha}f\right)(t)\Delta t \\
&= \int_a^b g(t)\left(\frac{1}{\Gamma(1-\alpha)}\int_a^t (t-\sigma(s))^{-\alpha}f^{\Delta}(s)\Delta s\right)\Delta t \\
&= \int_a^b f^{\Delta}(s)\left(\frac{1}{\Gamma(1-\alpha)}\int_s^b (t-\sigma(s))^{-\alpha}g(t)\Delta t\right)\Delta s \\
&= \int_a^b f^{\Delta}(t)\left(\frac{1}{\Gamma(1-\alpha)}\int_t^b (s-\sigma(t))^{-\alpha}g(s)\Delta s\right)\Delta t \\
&= \left[{}_t^{\mathbb{T}}I_b^{1-\alpha}g(t)\cdot f(t)\right]\Big|_{t=a}^{b} - \int_a^b f(\sigma(t))\left(\frac{1}{\Gamma(1-\alpha)}\int_t^b (s-\sigma(t))^{-\alpha}g(s)\Delta s\right)^{\Delta} \\
&= \left[{}_t^{\mathbb{T}}I_b^{1-\alpha}g(t)\cdot f(t)\right]\Big|_{t=a}^{b} + \int_a^b f(\sigma(t))\left(\frac{-1}{\Gamma(1-\alpha)}\int_t^b (s-\sigma(t))^{-\alpha}g(s)\Delta s\right)^{\Delta} \\
&= \left[{}_t^{\mathbb{T}}I_b^{1-\alpha}g(t)\cdot f(t)\right]\Big|_{t=a}^{b} + \int_a^b f(\sigma(t))\left({}_t^{\mathbb{T}}D_b^{\alpha}g\right)(t)\Delta t.
\end{aligned}$$

The second relation is obtained in a similar way. The proof is complete.

□

4. Fractional Sobolev Spaces on Time Scales and Their Properties

In this section, inspired by the above discussion, we present and prove the following results, which are of the utmost significance for our main results. In the following, let $0 < a < b$. Suppose $a, b \in \mathbb{T}$.

Motivated by Theorems 5–8, we propose the following definition.

Definition 14. *Let $0 < \alpha \leq 1$. By $AC^{\alpha,1}_{\Delta,b^-}(J, \mathbb{R}^N)$, we denote the set of all functions $f : J \to \mathbb{R}^N$ that have the representation*

$$f(t) = \frac{1}{\Gamma(\alpha)} \frac{d}{(b-t)^{1-\alpha}} + {}_t^{\mathbb{T}}I_b^{\alpha}\psi(t), \quad t \in J \quad \Delta - a.e. \tag{7}$$

with $d \in \mathbb{R}^N$ and $\psi \in L^1_\Delta$.

Theorem 17. *Let $0 < \alpha \leq 1$ and $f \in L^1_\Delta$. Then, function f has the right Riemann–Liouville derivative ${}_t^{\mathbb{T}}D_b^{\alpha} f$ of order α on the interval J if and only if $f \in AC^{\alpha,1}_{\Delta,b^-}(J, \mathbb{R}^N)$; that is, f has the representation (7). In such a case,*

$$({}_t^{\mathbb{T}}I_b^{1-\alpha} f)(b) = d, \quad ({}_t^{\mathbb{T}}D_b^{\alpha} f)(t) = \psi(t), \quad t \in J \quad \Delta - a.e.$$

Proof. Let $f \in L^1_\Delta$ have a right Riemann–Liouville derivative ${}_t^{\mathbb{T}}D_b^{\alpha} f$. This means that ${}_t^{\mathbb{T}}I_b^{1-\alpha} f$ is (identified to) an absolutely continuous function. From the integral representation of Theorem 5, there exist a constant vector $d \in \mathbb{R}^N$ and a function $\psi \in L^1_\Delta$ such that

$$({}_t^{\mathbb{T}}I_b^{1-\alpha} f)(t) = d + ({}_t^{\mathbb{T}}I_b^{1}\psi)(t), \quad t \in J, \tag{8}$$

with $({}_t^{\mathbb{T}}I_b^{1-\alpha} f)(b) = d$ and $-\left(({}_t^{\mathbb{T}}I_b^{1-\alpha} f)(t)\right)^{\Delta} = {}_t^{\mathbb{T}}D_b^{\alpha} f(t) = \psi(t), t \in J \quad \Delta - a.e.$.

By Proposition 4 and applying ${}_t^{\mathbb{T}}I_b^{\alpha}$ to (8), we obtain

$$({}_t^{\mathbb{T}}I_b^{1} f)(t) = ({}_t^{\mathbb{T}}I_b^{\alpha} d)(t) + ({}_t^{\mathbb{T}}I_b^{1}{}_t^{\mathbb{T}}I_b^{\alpha}\psi)(t), \quad t \in J \quad \Delta - a.e.. \tag{9}$$

The result follows from the Δ-differentiability of (9).

Conversely, now, let us assume that (7) holds true. From Proposition 4 and applying ${}_t^{\mathbb{T}}I_b^{1-\alpha}$ on (7), we obtain

$$({}_t^{\mathbb{T}}I_b^{1-\alpha} f)(t) = d + ({}_t^{\mathbb{T}}I_b^{1}\psi)(t), \quad t \in J \quad \Delta - a.e.$$

and then ${}_t^{\mathbb{T}}I_b^{1-\alpha} f$ has an absolutely continuous representation and f has a right Riemann–Liouville derivative ${}_t^{\mathbb{T}}D_b^{\alpha} f$. This completes the proof. □

Remark 2. *(i) By $AC^{\alpha,p}_{\Delta,b^-}$ $(1 \leq p < \infty)$, we denote the set of all functions $f : J \to \mathbb{R}^N$ possessing representation (7) with $d \in \mathbb{R}^N$ and $\psi \in L^p_\Delta$.*

(ii) It is easy to see that Theorem 17 implies the following (for any $1 \leq p < \infty$): f has the right Riemann–Liouville derivative ${}_t^{\mathbb{T}}D_b^{\alpha} f \in L^p_\Delta$ if and only if $f \in AC^{\alpha,p}_{\Delta,b^-}$; that is, f has the representation (7) with $\psi \in L^p_\Delta$.

Definition 15. *Let $0 < \alpha \leq 1$ and let $1 \leq p < \infty$. By the right Sobolev space of order α, we mean the set $W^{\alpha,p}_{\Delta,b^-} = W^{\alpha,p}_{\Delta,b^-}(J, \mathbb{R}^N)$ given by*

$$W^{\alpha,p}_{\Delta,b^-} := \left\{ u \in L^p_\Delta;\ \exists g \in L^p_\Delta,\ \forall \varphi \in C^\infty_{c,rd} \text{ such that } \int_a^b u(t) \cdot {}_a^{\mathbb{T}}D_t^{\alpha}\varphi(t)\Delta t = \int_a^b g(t) \cdot \varphi(t)\Delta t \right\}.$$

Remark 3. *The function g given above will be called the weak right fractional derivative of order $0 < \alpha \leq 1$ of u; let us denote it by ${}^{\mathbb{T}}u^{\alpha}_{b-}$. The uniqueness of this weak derivative follows from ([7]).*

We have the following characterization of $W^{\alpha,p}_{\Delta,b-}$.

Theorem 18. *If $0 < \alpha \leq 1$ and $1 \leq p < \infty$, then*

$$W^{\alpha,p}_{\Delta,b-} = AC^{\alpha,p}_{\Delta,b-} \cap L^p_{\Delta}.$$

Proof. On the one hand, if $u \in AC^{\alpha,p}_{\Delta,b-} \cap L^p_{\Delta}$, then, from Theorem 17, it follows that u has the derivative ${}^{\mathbb{T}}_{t}D^{\alpha}_{b}u \in L^p_{\Delta}$. Theorem 16 implies that

$$\int_a^b u(t)\, {}^{\mathbb{T}}_{a}D^{\alpha}_{t}\varphi(t)\Delta t = \int_a^b ({}^{\mathbb{T}}_{t}D^{\alpha}_{b}u)(t)\, \varphi(t)\Delta t$$

for any $\varphi \in C^{\infty}_{c,rd}$. So, $u \in W^{\alpha,p}_{\Delta,b-}$ with

$${}^{\mathbb{T}}u^{\alpha}_{b-} = g = {}^{\mathbb{T}}_{t}D^{\alpha}_{b}u \in L^p_{\Delta}.$$

On the other hand, now, let us assume that $u \in W^{\alpha,p}_{\Delta,b-}$; that is, $u \in L^p_{\Delta}$, and there exists a function $g \in L^p_{\Delta}$ such that

$$\int_a^b u(t) {}^{\mathbb{T}}_{a}D^{\alpha}_{t}\varphi(t)\Delta t = \int_a^b g(t)\varphi(t)\Delta t \tag{10}$$

for any $\varphi \in C^{\infty}_{c,rd}$.

To show that $u \in AC^{\alpha,p}_{\Delta,b-} \cap L^p_{\Delta}$, it suffices to check (Theorem 17 and definition of $AC^{\alpha,p}_{\Delta,b-}$) that u possesses the right Riemann–Liouville derivative of order α, belonging to L^p_{Δ}; that is, ${}^{\mathbb{T}}_{t}I^{1-\alpha}_{b}u$ is absolutely continuous on $[a,b]_{\mathbb{T}}$ and its delta derivative of α order (existing $\Delta - a.e.$ on J) belongs to L^p_{Δ}.

In fact, let $\varphi \in C^{\infty}_{c,rd}$, then $\varphi \in {}^{\mathbb{T}}_{a}D^{\alpha}_{t}(C_{rd})$ and ${}^{\mathbb{T}}_{a}D^{\alpha}_{t}\varphi = -({}^{\mathbb{T}}_{t}I^{1-\alpha}_{b})^{\Delta}$. From Theorem 16, it follows that

$$\begin{aligned}
\int_a^b u(t) {}^{\mathbb{T}}_{a}D^{\alpha}_{t}\varphi(t)\Delta t &= \int_a^b u(t)({}^{\mathbb{T}}_{a}I^{1-\alpha}_{t}\varphi)^{\Delta}(t)\Delta t \\
&= \int_a^b ({}^{\mathbb{T}}_{t}D^{1-\alpha}_{b}{}^{\mathbb{T}}_{t}I^{1-\alpha}_{b}u)(t)({}^{\mathbb{T}}_{a}I^{1-\alpha}_{t}\varphi)^{\Delta}(t)\Delta t \\
&= \int_a^b ({}^{\mathbb{T}}_{t}I^{1-\alpha}_{b}u)(t)(\varphi)^{\Delta}(t)\Delta t.
\end{aligned} \tag{11}$$

In view of (10) and (11), we obtain

$$\int_a^b ({}^{\mathbb{T}}_{t}I^{1-\alpha}_{b}u)(t)\varphi^{\Delta}(t)\Delta t = \int_a^b g(t)\varphi(t)\Delta t$$

for any $\varphi \in C^{\infty}_{c,rd}$. Thus, ${}^{\mathbb{T}}_{t}I^{1-\alpha}_{b}u \in W^{1,p}_{\Delta,b-}$. Consequently, ${}^{\mathbb{T}}_{t}I^{1-\alpha}_{b}u$ is absolutely continuous and its delta derivative is equal to $\Delta - a.e.$ on J to $g \in L^P_{\Delta}$. □

From the proof of Theorem 18 and the uniqueness of the weak fractional derivative, the following theorem follows.

Theorem 19. *If $0 < \alpha \leq 1$ and $1 \leq p < \infty$, then the weak left fractional derivative ${}^{\mathbb{T}}u^{\alpha}_{b-}$ of a function $u \in W^{\alpha,p}_{\Delta,b-}$ coincides with its right Riemann–Liouville fractional derivative ${}^{\mathbb{T}}_{t}D^{\alpha}_{b}u$ $\Delta - a.e.$ on J.*

Remark 4. *(1) If $0 < \alpha \leq 1$ and $(1-\alpha)p < 1$, then $AC^{\alpha,p}_{\Delta,b^-} \subset L^p_\Delta$ and, consequently,*

$$W^{\alpha,p}_{\Delta,b^-} = AC^{\alpha,p}_{\Delta,b^-} \cap L^p_\Delta = AC^{\alpha,p}_{\Delta,b^-}.$$

(2) If $0 < \alpha \leq 1$ and $(1-\alpha)p \geq 1$, then $W^{\alpha,p}_{\Delta,b^-} = AC^{\alpha,p}_{\Delta,b^-} \cap L^p_\Delta$ is the set of all functions belonging to $AC^{\alpha,p}_{\Delta,b^-}$ that satisfy the condition $({}^{\mathbb{T}}_{t}I^{1-\alpha}_{b} f)(b) = 0$.

By using the definition of $W^{\alpha,p}_{\Delta,b^-}$ with $0 < \alpha \leq 1$ and Theorem 19, one can easily prove the following result.

Theorem 20. *Let $0 < \alpha \leq 1$ and $1 \leq p < \infty$ and $u \in L^p_\Delta$. Then, $u \in W^{\alpha,p}_{\Delta,b^-}$ if and only if there exists a function $g \in L^p_\Delta$ such that*

$$\int_a^b u(t) {}^{\mathbb{T}}_{a}D^{\alpha}_{t} \varphi(t)\Delta t = \int_a^b g(t)\varphi(t)\Delta t, \quad \varphi \in C^\infty_{c,rd}.$$

In such a case, there exists the right Riemann–Liouville derivative ${}^{\mathbb{T}}_{t}D^{\alpha}_{b}u$ of u and $g = {}^{\mathbb{T}}_{t}D^{\alpha}_{b}u$.

Remark 5. *Function g will be called the weak right fractional derivative of order α of $u \in W^{\alpha,p}_{\Delta,b^-}$. Its uniqueness follows from [7]. From the above theorem, it follows that it coincides with the appropriate Riemann–Liouville derivative.*

Let us fix $0 < \alpha \leq 1$ and consider in the space $W^{\alpha,p}_{\Delta,b^-}$ a norm $\|\cdot\|_{W^{\alpha,p}_{\Delta,b^-}}$ given by

$$\|u\|^p_{W^{\alpha,p}_{\Delta,b^-}} = \|u\|^p_{L^p_\Delta} + \|{}^{\mathbb{T}}_{t}D^{\alpha}_{b}u\|^p_{L^p_\Delta}, \quad u \in W^{\alpha,p}_{\Delta,b^-}.$$

Here, $\|\cdot\|^p_{L_\Delta}$ denotes the delta norm in L^p_Δ (Theorem 9).

Lemma 3. *Let $0 < \alpha \leq 1$ and $1 \leq p < \infty$, then*

$$\|{}^{\mathbb{T}}_{t}I^{\alpha}_{b}\varphi\|^p_{L^p_\Delta} \leq K^p \|\varphi\|^p_{L^p_\Delta},$$

where $K = \frac{(b-a)^\alpha}{\Gamma(\alpha+1)}$, i.e., the fractional integration operator is bounded in L^p_Δ.

Proof. The conclusion follows from Theorem 10, Propositions 1 and 2. The proof is complete. □

Theorem 21. *If $0 < \alpha \leq 1$, then the norm $\|\cdot\|_{W^{\alpha,p}_{\Delta,b^-}}$ is equivalent to the norm $\|\cdot\|_{b,W^{\alpha,p}_{\Delta,b^-}}$ given by*

$$\|u\|^p_{b,W^{\alpha,p}_{\Delta,b^-}} = |{}^{\mathbb{T}}_{t}I^{1-\alpha}_{b}u(b)|^p + \|{}^{\mathbb{T}}_{t}D^{\alpha}_{b}u\|^p_{L^p_\Delta}, \quad u \in W^{\alpha,p}_{\Delta,b^-}.$$

Proof. (1) Assume that $(1-\alpha)p < 1$. On the one hand, for $u \in W^{\alpha,p}_{\Delta,b^-}$ given by

$$u(t) = \frac{1}{\Gamma(\alpha)} \frac{d}{(b-t)^{1-\alpha}} + {}^{\mathbb{T}}_{t}I^{\alpha}_{b}\psi(t)$$

with $d \in \mathbb{R}^N$ and $\psi \in L^p_\Delta$. Since $(b-t)^{(\alpha-1)p}$ is an increasing monotone function, by using Proposition 1, we can write that $\int_a^b (b-t)^{(\alpha-1)p}\Delta t \le \int_a^b (b-t)^{(\alpha-1)p} dt$. Moreover, taking into account Lemma 3, we have

$$\begin{aligned}\|u\|^p_{L^p_\Delta} =& \int_a^b \left| \frac{1}{\Gamma(\alpha)} \frac{d}{(b-t)^{1-\alpha}} + {}_a^{\mathbb{T}}I_t^\alpha \psi(t) \right|^p \Delta t \\ \le& 2^{p-1}\left(\frac{|d|^p}{\Gamma^p(\alpha)} \left| \int_a^b (b-t)^{(\alpha-1)p}\Delta t \right| + \| {}_t^{\mathbb{T}}I_b^\alpha \psi \|^p_{L^p_\Delta} \right) \\ \le& 2^{p-1}\left(\frac{|d|^p}{\Gamma^p(\alpha)} \left| \int_a^b (b-t)^{(\alpha-1)p} dt \right| + \| {}_t^{\mathbb{T}}I_b^\alpha \psi \|^p_{L^p_\Delta} \right) \\ \le& 2^{p-1}\left(\frac{|d|^p}{\Gamma^p(\alpha)} \frac{1}{(\alpha-1)p+1} (b-a)^{(\alpha-1)p+1} + K^p \|\psi\|^p_{L^p_\Delta} \right),\end{aligned}$$

where K is defined in Lemma 3. Noting that $d = {}_a^{\mathbb{T}}I_t^{1-\alpha}u(b)$, $\psi = -{}_a^{\mathbb{T}}D_t^\alpha u$, thus, one obtains

$$\begin{aligned}\|u\|^p_{L^p_\Delta} \le& L_{\alpha,0}(|d|^p + \|\psi\|^p_{L^p_\Delta}) \\ \le& L_{\alpha,0}\left(|{}_t^{\mathbb{T}}I_b^{1-\alpha}u(b)|^p + \|{}_t^{\mathbb{T}}D_b^\alpha u\|^p_{L^p_\Delta} \right) \\ =& L_{\alpha,0}\|u\|^p_{b,W^{\alpha,p}_{\Delta,b^-}},\end{aligned}$$

where

$$L_{\alpha,0} = 2^{p-1}\left(\frac{(b-a)^{1-(1-\alpha)p}}{\Gamma^p(\alpha)(1-(1-\alpha)p)} + K^p \right).$$

Consequently,

$$\begin{aligned}\|u\|^p_{W^{\alpha,p}_{\Delta,a^+}} =& \|u\|^p_{L^p_\Delta} + \|{}_a^{\mathbb{T}}D_t^\alpha u\|^p_{L^p_\Delta} \\ \le& L_{\alpha,1}\|u\|^p_{a,W^{\alpha,p}_{\Delta,b^-}},\end{aligned}$$

where $L_{\alpha,1} = L_{\alpha,0} + 1$.

On the other hand, now, we will prove that there exists a constant $M_{\alpha,1}$ such that

$$\|u\|^p_{b,W^{\alpha,p}_{\Delta,b^-}} \le M_{\alpha,1}\|u\|^p_{W^{\alpha,p}_{\Delta,b^-}}, \quad u \in W^{\alpha,p}_{\Delta,b^-}.$$

Indeed, let $u \in W^{\alpha,p}_{\Delta,b^-}$ and consider coordinate functions $({}_t^{\mathbb{T}}I_b^{1-\alpha}u)^i$ of ${}_t^{\mathbb{T}}I_b^{1-\alpha}u$ with $i \in \{1,\dots,N\}$. Lemma 3, Theorem 11 and Corollary 1 imply that there exist constants

$$\Lambda_i \in \left[\inf_{t\in[a,b)_{\mathbb{T}}} ({}_t^{\mathbb{T}}I_b^{1-\alpha}u)^i(t), \sup_{t\in[a,b)_{\mathbb{T}}} ({}_t^{\mathbb{T}}I_b^{1-\alpha}u)^i(t) \right],$$

such that

$$\Lambda_i = \frac{1}{b-a}\int_a^b ({}_t^{\mathbb{T}}I_b^{1-\alpha}u)^i(s)\Delta s$$

Hence, if, for all $i = 1,2,\dots,N$, $({}_t^{\mathbb{T}}I_b^{1-\alpha}u)^i(t_0) \ne 0$, then we can take constants θ_i such that

$$\theta_i({}_t^{\mathbb{T}}I_b^{1-\alpha}u)^i(t_0) = \Lambda_i = \frac{1}{b-a}\int_a^b ({}_t^{\mathbb{T}}I_b^{1-\alpha}u)^i(s)\Delta s$$

for fixed $t_0 \in J^0$. Therefore, we have

$$({}_t^{\mathbb{T}}I_b^{1-\alpha}u)^i(t_0) = \frac{\theta_i}{b-a}\int_a^b ({}_t^{\mathbb{T}}I_b^{1-\alpha}u)^i(s)\Delta s.$$

From the absolute continuity (Theorem 6) of $({}_t^{\mathbb{T}}I_b^{1-\alpha}u)^i$, it follows that

$$({}_t^{\mathbb{T}}I_b^{1-\alpha}u)^i(t) = ({}_t^{\mathbb{T}}I_b^{1-\alpha}u)^i(t_0) + \int_{[t_0,t)_{\mathbb{T}}} \left[({}_t^{\mathbb{T}}I_b^{1-\alpha}u)^i(s)\right]^{\Delta}\Delta s$$

for any $t \in J$. Consequently, combining with Proposition 3 and Lemma 3, we see that

$$\begin{aligned}
|({}_t^{\mathbb{T}}I_b^{1-\alpha}u)^i(t)| =&\left|({}_t^{\mathbb{T}}I_b^{1-\alpha}u)^i(t_0) + \int_{[t_0,t)_{\mathbb{T}}} \left[({}_t^{\mathbb{T}}I_b^{1-\alpha}u)^i(s)\right]^{\Delta}\Delta s\right| \\
\leq&\frac{|\theta_i|}{b-a}\|{}_t^{\mathbb{T}}I_b^{1-\alpha}u\|_{L_\Delta^1} + \int_{[t_0,t)_{\mathbb{T}}} |({}_t^{\mathbb{T}}D_b^{\alpha}u)(s)|\Delta s \\
\leq&\frac{|\theta_i|}{b-a}\|{}_t^{\mathbb{T}}I_b^{1-\alpha}u\|_{L_\Delta^1} + \|{}_t^{\mathbb{T}}D_b^{\alpha}u\|_{L_\Delta^1} \\
\leq&\frac{|\theta_i|}{b-a}\frac{(b-a)^{1-\alpha}}{\Gamma(2-\alpha)}\|u\|_{L_\Delta^1} + \|{}_t^{\mathbb{T}}D_b^{\alpha}u\|_{L_\Delta^1}
\end{aligned}$$

for $t \in J$. In particular,

$$|({}_t^{\mathbb{T}}I_b^{1-\alpha}u)^i(b)| \leq \frac{|\theta_i|}{b-a}\frac{(b-a)^{1-\alpha}}{\Gamma(2-\alpha)}\|u\|_{L_\Delta^1} + \|{}_t^{\mathbb{T}}D_b^{\alpha}u\|_{L_\Delta^1}.$$

Thus,

$$\begin{aligned}
|({}_t^{\mathbb{T}}I_b^{1-\alpha}u)(b)| \leq& N\left(\frac{|\theta|(b-a)^{-\alpha}}{\Gamma(2-\alpha)} + 1\right)\left(\|u\|_{L_\Delta^1} + \|{}_t^{\mathbb{T}}D_b^{\alpha}u\|_{L_\Delta^1}\right) \\
\leq& NM_{\alpha,0}(b-a)^{\frac{p-1}{p}}\left(\|u\|_{L_\Delta^p} + \|{}_t^{\mathbb{T}}D_b^{\alpha}u\|_{L_\Delta^p}\right),
\end{aligned}$$

where $|\theta| = \max_{i\in\{1,2,\dots,N\}} |\theta_i|$ and $M_{\alpha,0} = \frac{|\theta|(b-a)^{-\alpha}}{\Gamma(2-\alpha)} + 1$. Thus,

$$|({}_t^{\mathbb{T}}I_b^{1-\alpha}u)(b)|^p \leq N^pM_{\alpha,0}^p(b-a)^{p-1}2^{p-1}\left(\|u\|_{L_\Delta^p}^p + \|{}_t^{\mathbb{T}}D_b^{\alpha}u\|_{L_\Delta^p}^p\right),$$

and, consequently,

$$\begin{aligned}
\|u\|_{b,W_{\Delta,b^-}^{\alpha,p}}^p =& |{}_t^{\mathbb{T}}I_b^{1-\alpha}u(b)|^p + \|{}_t^{\mathbb{T}}D_b^{\alpha}u\|_{L_\Delta^p}^p \\
\leq&\left(N^pM_{\alpha,0}^p(b-a)^{p-1}2^{p-1} + 1\right)\left(\|u\|_{L_\Delta^p}^p + \|{}_t^{\mathbb{T}}D_b^{\alpha}u\|_{L_\Delta^p}^p\right) \\
=& M_{\alpha,1}\|u\|_{W_{\Delta,b^-}^{\alpha,p}}^p,
\end{aligned}$$

where $M_{\alpha,1} = N^pM_{\alpha,0}^p(b-a)^{p-1}2^{p-1} + 1$.

If some of or even all of $({}_t^{\mathbb{T}}I_b^{1-\alpha}u)^i(t_0) = 0$, from the above proof process, we can see that our conclusion is still valid.

(2) When $(1-\alpha)p \geq 1$, then (Remark 4) $W^{\alpha,p}_{\Delta,b^-} = AC^{\alpha,p}_{\Delta,b^-} \cap L^p_\Delta$ is the set of all functions belonging to $AC^{\alpha,p}_{\Delta,b^-}$ that satisfy the condition $({}^{\mathbb{T}}_t I^{1-\alpha}_b u)(b) = 0$. Consequently, in the same way as in the case of $(1-\alpha)p < 1$ (putting $d = 0$), we obtain the inequality

$$\|u\|^p_{W^{\alpha,p}_{\Delta,b^-}} \leq L_{\alpha,1}\|u\|^p_{b,W^{\alpha,p}_{\Delta,b^-}}, \quad \text{with some } L_{\alpha,1} > 0.$$

The inequality

$$\|u\|^p_{b,W^{\alpha,p}_{\Delta,b^-}} \leq M_{\alpha,1}\|u\|^p_{W^{\alpha,p}_{\Delta,b^-}}, \quad \text{with some } M_{\alpha,1} > 0$$

is obvious (it is sufficient to put $M_{\alpha,1} = 1$ and use the fact that $({}^{\mathbb{T}}_a I^{1-\alpha}_t u)(b) = 0$). The proof is complete. □

We are now in a position to state and prove some basic properties of the introduced space.

Theorem 22. *The space $W^{\alpha,p}_{\Delta,b^-}$ is complete with respect to each of the norms $\|\cdot\|_{W^{\alpha,p}_{\Delta,b^-}}$ and $\|\cdot\|_{b,W^{\alpha,p}_{\Delta,b^-}}$ for any $0 < \alpha \leq 1$ and $1 \leq p < \infty$.*

Proof. In view of Theorem 21, we only need to show that $W^{\alpha,p}_{\Delta,b^-}$ with the norm $\|\cdot\|_{b,W^{\alpha,p}_{\Delta,b^-}}$ is complete. Let $\{u_k\} \subset W^{\alpha,p}_{\Delta,b^-}$ be a Cauchy sequence with respect to this norm. Thus, the sequences $\{{}^{\mathbb{T}}_t I^{1-\alpha}_b u_k(b)\}$ and $\{{}^{\mathbb{T}}_t D^{\alpha}_b u_k\}$ are Cauchy sequences in $\mathbb{R}^N$ and L^p_Δ, respectively.

Let $d \in \mathbb{R}^N$ and $\psi \in L^p_\Delta$ be the limits of the above sequences in $\mathbb{R}^N$ and L^p_Δ, respectively. Then, the function

$$u(t) = \frac{d}{\Gamma(\alpha)}(b-t)^{\alpha-1} + {}^{\mathbb{T}}_t I^{\alpha}_b \psi(t), \quad t \in J \quad \Delta - a.e.$$

belongs to $W^{\alpha,p}_{\Delta,b^-}$ and is the limit of $\{u_k\}$ in $W^{\alpha,p}_{\Delta,b^-}$ with respect to $\|\cdot\|_{b,W^{\alpha,p}_{\Delta,b^-}}$. (To assert that $u \in L^p_\Delta$, it is sufficient to consider the cases $(1-\alpha)p < 1$ and $(1-\alpha)p \geq 1$. In the second case, ${}^{\mathbb{T}}_t I^{1-\alpha}_b u_k(b) = 0$ for any $k \in \mathbb{N}$ and, consequently, $d = 0$.) The proof is complete. □

In the proofs of the next two theorems, we use the method presented in Proposition VIII.1. (b), (c) from [32].

Theorem 23. *The space $W^{\alpha,p}_{\Delta,b^-}$ is reflexive with respect to the norm $\|\cdot\|_{W^{\alpha,p}_{\Delta,b^-}}$ for any $0 < \alpha \leq 1$ and $1 < p < \infty$.*

Proof. Let us consider $W^{\alpha,p}_{\Delta,b^-}$ with the norm $\|\cdot\|_{W^{\alpha,p}_{\Delta,b^-}}$ and define a mapping

$$\lambda : W^{\alpha,p}_{\Delta,b^-} \ni u \mapsto \left(u, {}^{\mathbb{T}}_t D^{\alpha}_b u\right) \in L^p_\Delta \times L^p_\Delta.$$

It is obvious that

$$\|u\|_{W^{\alpha,p}_{\Delta,b^-}} = \|\lambda u\|_{L^p_\Delta \times L^p_\Delta},$$

where

$$\|\lambda u\|_{L^p_\Delta \times L^p_\Delta} = \left(\sum_{i=1}^{2} \|(\lambda u)_i\|^p_{L^p_\Delta}\right)^{\frac{1}{p}}, \quad \lambda u = \left(u, {}^{\mathbb{T}}_t D^{\alpha}_b u\right) \in L^p_\Delta \times L^p_\Delta,$$

which means that the operator $\lambda : u \mapsto \left(u, {}_t^{\mathbb{T}}D_b^{\alpha}u\right)$ is an isometric isomorphic mapping and the space $W_{\Delta,b^-}^{\alpha,p}$ is isometric isomorphic to the space $\Omega = \left\{\left(u, {}_t^{\mathbb{T}}D_b^{\alpha}u\right) : \forall u \in W_{\Delta,b^-}^{\alpha,p}\right\}$, which is a closed subset of $L_\Delta^p \times L_\Delta^p$ as $W_{\Delta,b^-}^{\alpha,p}$ is closed.

Since L_Δ^p is reflexive, the Cartesian product space $L_\Delta^p \times L_\Delta^p$ is also a reflexive space with respect to the norm $\|v\|_{L_\Delta^p \times L_\Delta^p} = \left(\sum_{i=1}^{2} \|v_i\|_{L_\Delta^p}^p\right)^{\frac{1}{p}}$, where $v = (v_1, v_2) \in L_\Delta^p \times L_\Delta^p$.

Thus, $W_{\Delta,b^-}^{\alpha,p}$ is reflexive with respect to the norm $\|\cdot\|_{W_{\Delta,b^-}^{\alpha,p}}$. $\square$

Theorem 24. *The space $W_{\Delta,b^-}^{\alpha,p}$ is separable with respect to the norm $\|\cdot\|_{W_{\Delta,b^-}^{\alpha,p}}$ for any $0 < \alpha \le 1$ and $1 \le p < \infty$.*

Proof. Let us consider $W_{\Delta,b^-}^{\alpha,p}$ with the norm $\|\cdot\|_{W_{\Delta,b^-}^{\alpha,p}}$ and the mapping λ defined in the proof of Theorem 23. Obviously, $\lambda(W_{\Delta,b^-}^{\alpha,p})$ is separable as a subset of separable space $L_\Delta^p \times L_\Delta^p$. Since λ is the isometry, $W_{\Delta,b^-}^{\alpha,p}$ is also separable with respect to the norm $\|\cdot\|_{W_{\Delta,b^-}^{\alpha,p}}$. $\square$

Proposition 6. *Let $0 < \alpha \le 1$ and $1 < p < \infty$. For all $u \in W_{\Delta,b^-}^{\alpha,p}$, if $1 - \alpha \ge \frac{1}{p}$ or $\alpha > \frac{1}{p}$, then*

$$\|u\|_{L_\Delta^p} \le \frac{b^\alpha}{\Gamma(\alpha+1)} \left\| {}_t^{\mathbb{T}}D_b^{\alpha}u \right\|_{L_\Delta^p}. \tag{12}$$

If $\alpha > \frac{1}{p}$ and $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\|u\|_\infty \le \frac{b^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)((\alpha-1)q+1)^{\frac{1}{q}}} \left\| {}_t^{\mathbb{T}}D_b^{\alpha}u \right\|_{L_\Delta^p}. \tag{13}$$

Proof. In view of Remark 4 and Theorem 15, in order to prove inequalities (12) and (13), we only need to prove that

$$\left\| {}_t^{\mathbb{T}}I_b^{\alpha}\left({}_t^{\mathbb{T}}D_b^{\alpha}u\right) \right\|_{L_\Delta^p} \le \frac{b^\alpha}{\Gamma(\alpha+1)} \left\| {}_t^{\mathbb{T}}D_b^{\alpha}u \right\|_{L_\Delta^p} \tag{14}$$

for $1 - \alpha \ge \frac{1}{p}$ or $\alpha > \frac{1}{p}$, and

$$\left\| {}_t^{\mathbb{T}}I_b^{\alpha}\left({}_t^{\mathbb{T}}D_b^{\alpha}u\right) \right\|_\infty \le \frac{b^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)((\alpha-1)q+1)^{\frac{1}{q}}} \left\| {}_t^{\mathbb{T}}D_b^{\alpha}u \right\|_{L_\Delta^p} \tag{15}$$

for $\alpha > \frac{1}{p}$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Firstly, we note that ${}_t^{\mathbb{T}}D_b^{\alpha}u \in L_\Delta^p([a,b]_{\mathbb{T}}, \mathbb{R}^N)$; the inequality (14) follows from Lemma 3 directly.

We are now in a position to prove (15). For $\alpha > \frac{1}{p}$, choose q such that $\frac{1}{p} + \frac{1}{q} = 1$. For all $u \in W_{\Delta,b^-}^{\alpha,p}$, since $(s - \sigma(t))^{(\alpha-1)q}$ is an increasing monotone function, by using

Proposition 1, we find that $\int_t^b (s-\sigma(t))^{(\alpha-1)q}\Delta s \le \int_t^b (s-t)^{(\alpha-1)q}ds$. Taking into account Proposition 2, we have

$$\begin{aligned}\left|{}_t^{\mathbb{T}}I_b^\alpha({}_t^{\mathbb{T}}D_b^\alpha u(t))\right| =&\frac{1}{\Gamma(\alpha)}\left|\int_t^b (s-\sigma(t))^{\alpha-1}\,{}_t^{\mathbb{T}}D_b^\alpha u(s)\Delta s\right|\\ \le&\frac{1}{\Gamma(\alpha)}\left(\int_t^b (s-\sigma(t))^{(\alpha-1)q}\Delta s\right)^{\frac{1}{q}}\|{}_t^{\mathbb{T}}D_b^\alpha u\|_{L_\Delta^p}\\ \le&\frac{1}{\Gamma(\alpha)}\left(\int_t^b (s-t)^{(\alpha-1)q}ds\right)^{\frac{1}{q}}\|{}_t^{\mathbb{T}}D_b^\alpha u\|_{L_\Delta^p}\\ \le&\frac{b^{\frac{1}{q}+\alpha-1}}{\Gamma(\alpha)((\alpha-1)q+1)^{\frac{1}{q}}}\left\|{}_t^{\mathbb{T}}D_b^\alpha u\right\|_{L_\Delta^p}\\ =&\frac{b^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)((\alpha-1)q+1)^{\frac{1}{q}}}\left\|{}_t^{\mathbb{T}}D_b^\alpha u\right\|_{L_\Delta^p}.\end{aligned}$$

This completes the proof. □

Remark 6. *(i)* *According to (12), we can consider $W_{\Delta,b^-}^{\alpha,p}$ with respect to the norm*

$$\|u\|_{W_{\Delta,b^-}^{\alpha,p}} = \|{}_t^{\mathbb{T}}D_b^\alpha u\|_{L_\Delta^p} = \left(\int_a^b \left|{}_t^{\mathbb{T}}D_b^\alpha u(t)\right|^p \Delta t\right)^{\frac{1}{p}}, \tag{16}$$

in the following analysis.

(ii) *It follows from (12) and (13) that $W_{\Delta,b^-}^{\alpha,p}$ is continuously immersed into $C(J,\mathbb{R}^N)$ with the natural norm $\|\cdot\|_\infty$.*

Proposition 7. *Let $0<\alpha\le 1$ and $1<p<\infty$. Assume that $\alpha>\frac{1}{p}$ and the sequence $\{u_k\}$ converges weakly to u in $W_{\Delta,b^-}^{\alpha,p}$. Then, $u_k\to u$ in $C(J,\mathbb{R}^N)$, i.e., $\|u-u_k\|_\infty=0$, as $k\to\infty$.*

Proof. If $\alpha>\frac{1}{p}$, then by (13) and (16), the injection of $W_{\Delta,b^-}^{\alpha,p}$ into $C(J,\mathbb{R}^N)$, with its natural norm $\|\cdot\|_\infty$, is continuous, i.e., $u_k\to u$ in $W_{\Delta,b^-}^{\alpha,p}$, then $u_k\to u$ in $C(J,\mathbb{R}^N)$.

Since $u_k\rightharpoonup u$ in $W_{\Delta,b^-}^{\alpha,p}$, it follows that $u_k\rightharpoonup u$ in $C(J,\mathbb{R}^N)$. In fact, for any $h\in\left(C(J,\mathbb{R}^N)\right)^*$, if $u_k\to u$ in $W_{\Delta,b^-}^{\alpha,p}$, then $u_k\to u$ in $C(J,\mathbb{R}^N)$, and thus $h(u_k)\to h(u)$. Therefore, $h\in\left(W_{\Delta,b^-}^{\alpha,p}\right)^*$, which means that $\left(C(J,\mathbb{R}^N)\right)^*\subset\left(W_{\Delta,b^-}^{\alpha,p}\right)^*$. Hence, if $u_k\rightharpoonup u$ in $W_{\Delta,b^-}^{\alpha,p}$, then for any $h\in\left(C(J,\mathbb{R}^N)\right)^*$, we have $h\in\left(W_{\Delta,b^-}^{\alpha,p}\right)^*$, and thus $h(u_k)\to h(u)$, i.e., $u_k\rightharpoonup u$ in $C(J,\mathbb{R}^N)$.

By the Banach–Steinhaus theorem, $\{u_k\}$ is bounded in $W_{\Delta,b^-}^{\alpha,p}$ and, hence, in $C(J,\mathbb{R}^N)$. We are now in a position to prove that the sequence $\{u_k\}$ is equicontinuous. Let $\frac{1}{p}+\frac{1}{q}=1$

and $t_1, t_2 \in [a,b]_{\mathbb{T}}$, $t_1 \leq t_2$, $\forall f \in L^p_\Delta(J, \mathbb{R}^N)$, by using Proposition 2, Proposition 1 and Theorem 12, and noting $\alpha > \frac{1}{p}$, we have

$$\begin{aligned}
&\left| {}_{t_1}^{\mathbb{T}}I_b^\alpha f(t_1) - {}_{t_2}^{\mathbb{T}}I_b^\alpha f(t_2) \right| \\
=&\frac{1}{\Gamma(\alpha)} \left| \int_{t_1}^b (s-\sigma(t_1))^{\alpha-1} f(s)\Delta s - \int_{t_2}^b (s-\sigma(t_2))^{\alpha-1} f(s)\Delta s \right| \\
\leq&\frac{1}{\Gamma(\alpha)} \left| \int_{t_1}^b (s-\sigma(t_1))^{\alpha-1} f(s)\Delta s - \int_{t_1}^b (s-\sigma(t_2))^{\alpha-1} f(s)\Delta s \right| \\
&+ \frac{1}{\Gamma(\alpha)} \left| \int_{t_1}^{t_2} (s-\sigma(t_2))^{\alpha-1} f(s)\Delta s \right| \\
\leq&\frac{1}{\Gamma(\alpha)} \left| \int_{t_1}^b \left((s-\sigma(t_1))^{\alpha-1} - (s-\sigma(t_2))^{\alpha-1} \right) \right| |f(s)|\Delta s \\
&+ \frac{1}{\Gamma(\alpha)} \left| \int_{t_1}^{t_2} (s-\sigma(t_2))^{\alpha-1} \right| |f(s)|\Delta s \\
\leq&\frac{1}{\Gamma(\alpha)} \left| \left(\int_{t_1}^b \left((s-\sigma(t_1))^{\alpha-1} - (s-\sigma(t_2))^{\alpha-1} \right)^q \Delta s \right)^{\frac{1}{q}} \right| \|f\|_{L^p_\Delta} \\
&+ \frac{1}{\Gamma(\alpha)} \left| \left(\int_{t_1}^{t_2} (s-\sigma(t_2))^{(\alpha-1)q} \Delta s \right)^{\frac{1}{q}} \right| \|f\|_{L^p_\Delta} \\
\leq&\frac{1}{\Gamma(\alpha)} \left| \left(\int_{t_1}^b \left((s-\sigma(t_1))^{(\alpha-1)q} - (s-\sigma(t_2))^{(\alpha-1)q} \right) \Delta s \right)^{\frac{1}{q}} \right| \|f\|_{L^p_\Delta} \\
&+ \frac{1}{\Gamma(\alpha)} \left| \left(\int_{t_1}^{t_2} (s-\sigma(t_2))^{(\alpha-1)q} \Delta s \right)^{\frac{1}{q}} \right| \|f\|_{L^p_\Delta} \qquad (17) \\
\leq&\frac{1}{\Gamma(\alpha)} \left| \left(\int_{t_1}^b \left((s-t_1)^{(\alpha-1)q} - (s-t_2)^{(\alpha-1)q} \right) ds \right)^{\frac{1}{q}} \right| \|f\|_{L^p_\Delta} \\
&+ \frac{1}{\Gamma(\alpha)} \left| \left(\int_{t_1}^{t_2} (s-t_2)^{(\alpha-1)q} ds \right)^{\frac{1}{q}} \right| \|f\|_{L^p_\Delta} \\
=&\frac{\|f\|_{L^p_\Delta}}{\Gamma(\alpha)(1+(\alpha-1)q)^{\frac{1}{q}}} \left| \left((b-t_1)^{(\alpha-1)q+1} - (b-t_2)^{(\alpha-1)q+1} + (t_2-t_1)^{(\alpha-1)q+1} \right)^{\frac{1}{q}} \right| \\
&+ \frac{\|f\|_{L^p_\Delta}}{\Gamma(\alpha)(1+(\alpha-1)q)^{\frac{1}{q}}} \left((t_2-t_1)^{(\alpha-1)q+1} \right)^{\frac{1}{q}} \\
\leq&\frac{\|f\|_{L^p_\Delta}}{\Gamma(\alpha)(1+(\alpha-1)q)^{\frac{1}{q}}} \left((b-t_2)^{(\alpha-1)q+1} - (b-t_1)^{(\alpha-1)q+1} + (t_2-t_1)^{(\alpha-1)q+1} \right)^{\frac{1}{q}}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\|f\|_{L^p_\Delta}}{\Gamma(\alpha)(1+(\alpha-1)q)^{\frac{1}{q}}}\left((t_2-t_1)^{(\alpha-1)q+1}\right)^{\frac{1}{q}}\\
\leq&\frac{\|f\|_{L^p_\Delta}}{\Gamma(\alpha)(1+(\alpha-1)q)^{\frac{1}{q}}}\left((t_2-t_1)^{(\alpha-1)q+1}\right)^{\frac{1}{q}}\\
&+\frac{\|f\|_{L^p_\Delta}}{\Gamma(\alpha)(1+(\alpha-1)q)^{\frac{1}{q}}}\left((t_2-t_1)^{(\alpha-1)q+1}\right)^{\frac{1}{q}}\\
=&\frac{2\|f\|_{L^p_\Delta}}{\Gamma(\alpha)(1+(\alpha-1)q)^{\frac{1}{q}}}(t_2-t_1)^{\alpha-\frac{1}{p}}.
\end{aligned}
$$

Therefore, the sequence $\{u_k\}$ is equicontinuous since, for $t_1, t_2 \in [a,b]_{\mathbb{T}}$, $t_1 \leq t_2$, by applying (17) and in view of (16), we have

$$
\begin{aligned}
|u_k(t_1)-u_k(t_2)| =&\left|{}_{t_1}^{\mathbb{T}}I_b^{\alpha}({}_{t_1}^{\mathbb{T}}D_b^{\alpha}u_k(t_1)) - {}_{t_2}^{\mathbb{T}}I_b^{\alpha}({}_{t_2}^{\mathbb{T}}D_b^{\alpha}u_k(t_2))\right|\\
\leq&\frac{2(t_2-t_1)^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)(1+(\alpha-1)q)^{\frac{1}{q}}}\|{}_{t}^{\mathbb{T}}D_b^{\alpha}u_k\|_{L^p_\Delta}\\
=&\frac{2(t_2-t_1)^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)(1+(\alpha-1)q)^{\frac{1}{q}}}\|{}_{t}^{\mathbb{T}}D_b^{\alpha}u_k\|_{L^p_\Delta}\\
\leq&\frac{2(t_2-t_1)^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)((\alpha-1)q+1)^{\frac{1}{q}}}\left\|{}_{t}^{\mathbb{T}}D_b^{\alpha}u\right\|_{L^p_\Delta}\\
=&\frac{2(t_2-t_1)^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)((\alpha-1)q+1)^{\frac{1}{q}}}\|u_k\|_{W^{\alpha,p}_{\Delta,b^-}}\\
\leq&C(t_2-t_1)^{\alpha-\frac{1}{p}},
\end{aligned}
$$

where $\frac{1}{p}+\frac{1}{q}=1$ and $C \in \mathbb{R}^+$ is a constant. By the Ascoli–Arzela theorem on time scales (Lemma 2), $\{u_k\}$ is relatively compact in $C(J,\mathbb{R}^N)$. By the uniqueness of the weak limit in $C(J,\mathbb{R}^N)$, every uniformly convergent subsequence of $\{u_k\}$ converges uniformly on J to u. □

Remark 7. *It follows from Proposition 7 that $W^{\alpha,p}_{\Delta,b^-}$ is compactly immersed into $C(J,\mathbb{R}^N)$ with the natural norm $\|\cdot\|_\infty$.*

Theorem 25. *Let $1<p<\infty$, $\frac{1}{p}<\alpha\leq 1$, $\frac{1}{p}+\frac{1}{q}=1$, $L: J\times\mathbb{R}^N\times\mathbb{R}^N\to\mathbb{R}$, $(t,x,y)\mapsto L(t,x,y)$ satisfies*

(i) For each $(x,y)\in\mathbb{R}^N\times\mathbb{R}^N$, $L(t,x,y)$ is Δ-measurable in t;
(ii) For Δ-almost every $t\in J$, $L(t,x,y)$ is continuously differentiable in (x,y).

If there exists $m_1\in C(\mathbb{R}^+,\mathbb{R}^+)$, $m_2\in L^1_\Delta(J,\mathbb{R}^+)$ and $m_3\in L^q_\Delta(J,\mathbb{R}^+)$, $1<q<\infty$, such that, for Δ-a.e. $t\in J$ and every $(x,y)\in\mathbb{R}^N\times\mathbb{R}^N$, one has

$$
\begin{aligned}
|L(t,x,y)| &\leq m_1(|x|)(m_2(t)+|y|^p),\\
|D_xL(t,x,y)| &\leq m_1(|x|)(m_2(t)+|y|^p),\\
|D_yL(t,x,y)| &\leq m_1(|x|)(m_3(t)+|y|^{p-1}).
\end{aligned}
$$

Then, the functional Φ defined by

$$
\Phi(u)=\int_a^b L(t,u(t),{}_{t}^{\mathbb{T}}D_b^{\alpha}u(t))\Delta t
$$

is continuously differentiable on $W^{\alpha,p}_{\Delta,b^-}$*, and for all* $u, v \in W^{\alpha,p}_{\Delta,b^-}$*, we have*

$$\begin{aligned}\langle \Phi'(u), v\rangle = \int_a^b \Big[&\Big(D_x L(t, u(t), {}^{\mathbb{T}}_{t}D^{\alpha}_{b}u(t), v(t)\Big) \\ &+ \Big(D_y L(t, u(t), {}^{\mathbb{T}}_{t}D^{\alpha}_{b}u(t), {}^{\mathbb{T}}_{t}D^{\alpha}_{b}v(t)\Big)\Big]\Delta t. \end{aligned} \tag{18}$$

Proof. It suffices to prove that φ has, at every point u, a directional derivative $\Phi'(u) \in (W^{\alpha,p}_{\Delta,b^-})^*$ given by (18) and that the mapping

$$\Phi' : W^{\alpha,p}_{\Delta,b^-} \ni u \mapsto \Phi'(u) \in (W^{\alpha,p}_{\Delta,b^-})^*$$

is continuous. The rest of the proof is similar to the proof of [33] P_{10} Theorem 1.4. We omit it here. The proof is complete. □

5. An Application

In this section, we present a recent approach via variational methods and critical point theory to obtain the existence of weak solutions for the following fractional boundary value problem (FBVP for short) on time scales

$$\begin{cases} {}^{\mathbb{T}}_{a}D^{\alpha}_{t}\big({}^{\mathbb{T}}_{t}D^{\alpha}_{b}u(t)\big) = \nabla G(\sigma(t), u^{\sigma}(t)), & \Delta - a.e.\ t \in J, \\ u(a) = u(b) = 0, \end{cases} \tag{19}$$

where ${}^{\mathbb{T}}_{a}D^{\alpha}_{t}$ and ${}^{\mathbb{T}}_{t}D^{\alpha}_{b}$ are the left and right Riemann–Liouville fractional derivative operators of order $\alpha \in (0, 1]$ defined on $\mathbb{T}$, respectively, and function $G : J \times \mathbb{R}^N \to \mathbb{R}$ satisfies the following assumption:

(A_1) $G(t, x)$ is Δ-measurable in t for each $x \in \mathbb{R}^N$, continuously differentiable in x for Δ-a.e. $t \in J$ and there are $p \in C(\mathbb{R}^+, \mathbb{R}^+)$, $q \in L^1_{\Delta}(J, \mathbb{R}^+)$ such that

$$|G(t, x)| \le p(|x|)q(t), \quad |\nabla G(t, x)| \le p(|x|)q(t),$$

for all $x \in \mathbb{R}^N$ and Δ-a.e. $t \in J$, and $\nabla G(t, x)$ is the gradient of G at x.

By constructing a variational structure on $W^{\alpha,2}_{\Delta,b^-}$, we can reduce the problem of finding weak solutions of (26) to one of seeking the critical points of a corresponding functional.

In particular, when $\mathbb{T} = \mathbb{R}$, FBVP (26) reduces to the standard fractional boundary value problem of the following form

$$\begin{cases} {}_{a}D^{\alpha}_{t}\big({}_{t}D^{\alpha}_{b}u(t)\big) = \nabla G(t, u(t)), & a.e.\ t \in J_{\mathbb{R}}, \\ u(a) = u(b) = 0. \end{cases}$$

When $\alpha = 1$, FBVP (26) reduces to the second-order Hamiltonian system on time scale $\mathbb{T}$

$$\begin{cases} u^{\Delta^2}(t) = \nabla G(\sigma(t), u^{\sigma}(t)), & \Delta - a.e.\ t \in J^{\kappa^2}, \\ u(a) - u(b) = 0, \quad u^{\Delta}(a) - u^{\Delta}(b) = 0. \end{cases}$$

Although many excellent results have been obtained based on the existence of solutions for fractional boundary value problems [34–40] and the second-order Hamiltonian systems on time scale $\mathbb{T}$ [41–45], it seems that no similar results have been obtained in the literature for FBVP (26) on time scales. The present section seeks to show that the critical point theory is an effective approach to deal with the existence of solutions for FBVP Theorem 26 on time scales.

By Theorem 22, the space $W^{\alpha,2}_{\Delta,b^-}$ with the inner product

$$\langle u,v\rangle = \langle u,v\rangle_{W^{\alpha,2}_{\Delta,b^-}} = \int_a^b (u(t),v(t))\Delta t + \int_a^b ({}_t^{\mathbb{T}}D_b^{\alpha}u(t), {}_t^{\mathbb{T}}D_b^{\alpha}v(t))\Delta t$$

and the induced norm

$$\|u\| = \|u\|_{W^{\alpha,2}_{\Delta,b^-}} = \left(\int_a^b |u(t)|^2\Delta t + \int_a^b |{}_t^{\mathbb{T}}D_b^{\alpha}u(t)|^2\Delta t\right)^{\frac{1}{2}}$$

is a Hilbert space.

Consider the functional $\Phi : W^{\alpha,2}_{\Delta,b^-} \to \mathbb{R}$ defined by

$$\Phi(u) = \frac{1}{2}\int_a^b |{}_t^{\mathbb{T}}D_b^{\alpha}u(t)|^2\Delta t - \int_a^b G(\sigma(t),u^{\sigma}(t))\Delta t, \quad \forall u \in W^{\alpha,2}_{\Delta,b^-}. \tag{20}$$

From now on, H, which we defined in (20), will be considered as a functional on $W^{\alpha,2}_{\Delta,b^-}$ with $\frac{1}{2} < \alpha \le 1$. We have the following facts.

Theorem 26. *The functional Φ is continuously differentiable on $W^{\alpha,2}_{\Delta,b^-}$ and*

$$\langle \Phi'(u),v\rangle = \int_a^b \Big[({}_t^{\mathbb{T}}D_b^{\alpha}u(t),\, {}_t^{\mathbb{T}}D_b^{\alpha}v(t)) - (\nabla G(\sigma(t),u^{\sigma}(t)),v^{\sigma}(t))\Big]\Delta t$$

for all $v \in W^{\alpha,2}_{\Delta,b^-}$.

Proof. Let $L(t,x,y) = \frac{1}{2}|y|^2 - G(t,x)$ for all $x,y \in \mathbb{R}^N$ and $t \in J$. Then, by condition $(\mathbf{A_1})$, $L(t,x,y)$ meets all the requirements of Theorem 25. Therefore, by Theorem 25, it follows that the functional φ is continuously differentiable on $W^{\alpha,p}_{\Delta,b^-}$ and

$$\langle \Phi'(u),v\rangle = \int_a^b \Big[({}_t^{\mathbb{T}}D_b^{\alpha}u(t),\, {}_t^{\mathbb{T}}D_b^{\alpha}v(t)) - (\nabla G(\sigma(t),u^{\sigma}(t)),v^{\sigma}(t))\Big]\Delta t$$

for all $v \in W^{\alpha,2}_{\Delta,b^-}$. The proof is complete. □

Definition 16. *A function $u : \Phi \to \mathbb{R}^N$ is called a solution of FBVP (26) if*

(i) ${}_a^{\mathbb{T}}D_t^{\alpha-1}({}_t^{\mathbb{T}}D_b^{\alpha}u(t))$ and ${}_t^{\mathbb{T}}D_b^{\alpha-1}u(t)$ are differentiable for Δ-a.e. $t \in J^0$ and

(ii) u satisfies FBVP (26).

For a solution $u \in W^{\alpha,2}_{\Delta,b^-}$ of FBVP (26) such that $\nabla G(\cdot,u(\cdot)) \in L^1_{\Delta}(J,\mathbb{R}^N)$, multiplying FBVP (26) by $v \in C^{\infty}_{0,rd}(J,\mathbb{R}^N)$ yields

$$\begin{aligned}
&\int_a^b \Big[{}_a^{\mathbb{T}}D_t^{\alpha}({}_t^{\mathbb{T}}D_b^{\alpha}u(t),\, v(t)) - \nabla G(\sigma(t),u^{\sigma}(t))\Big]\Delta t\\
=&\int_a^b \Big[({}_t^{\mathbb{T}}D_b^{\alpha}u(t),\, {}_t^{\mathbb{T}}D_b^{\alpha}v(t))\Delta t - \nabla G(\sigma(t),u^{\sigma}(t))\Big]\Delta t\\
=&0,
\end{aligned} \tag{21}$$

after applying (b) of Theorem 16 and Definition 24. Hence, we can give the definition of a weak solution for FBVP (26) as follows.

Definition 17. *By a weak solution for FBVP (26), we mean that a function $u \in W^{\alpha,2}_{\Delta,b^-}$ such that $\nabla G(\cdot,u(\cdot)) \in L^1_{\Delta}(J,\mathbb{R}^N)$ and satisfies (21) for all $v \in C^{\infty}_{0,rd}(J,\mathbb{R}^N)$.*

By our above remarks, any solution $u \in W^{\alpha,2}_{\Delta,b^-}$ of FBVP (26) is a weak solution provided that $\nabla G(\cdot, u(\cdot)) \in L^1_\Delta(J, \mathbb{R}^N)$. Our task is now to establish a variational structure on $W^{\alpha,2}_{\Delta,b^-}$ with $\alpha \in \left(\frac{1}{2}, 1\right]$, which enables us to reduce the existence of weak solutions of FBVP (26) to the one of finding critical points of the corresponding functional.

Theorem 27. *If $\frac{1}{2} < \alpha \leq 1$, $u \in W^{\alpha,2}_{\Delta,b^-}$ is a critical point of Φ in $W^{\alpha,2}_{\Delta,b^-}$, i.e., $\Phi'(u) = 0$, then u is a weak solution of system* (26) *with $\frac{1}{2} < \alpha \leq 1$.*

Proof. Because of $\varphi'(u) = 0$, it follows from Theorem 26 that

$$\int_a^b \left[({}^{\mathbb{T}}_t D^\alpha_b u(t), {}^{\mathbb{T}}_t D^\alpha_b v(t)) - \nabla G(\sigma(t), u^\sigma(t)) \right] \Delta t = 0$$

for all $v \in W^{\alpha,2}_{\Delta,b^-}$, and hence for all $v \in C_0^\infty(J, \mathbb{R}^N)$. Therefore, according to Definition 17, u is a weak solution of FBVP (26) and the proof is complete. □

According to Theorem 27, we see that in order to find weak solutions of FBVP (26), it suffices to obtain the critical points of the functional φ given by (20). We need to use some critical point theorems. For the reader's convenience, we present some necessary definitions and theorems and skip the proofs.

Let X be a real Banach space and $C^1(X, \mathbb{R}^N)$ denote the set of functionals that are Fréchet differentiable and their Fréchet derivatives are continuous on X.

Definition 18 ([46])**.** *Let $\psi \in C^1(X, \mathbb{R}^N)$. If any sequence $\{u_k\} \subset H$ for which $\{\psi(u_k)\}$ is bounded and $\psi'(u_k) \to 0$ as $k \to \infty$ possesses a convergent subsequence, then we say ψ satisfies the Palais–Smale condition (denoted as P.S. condition for short).*

Theorem 28 ([33])**.** *Let X be a real reflexive Banach space. If the functional $\psi : X \to \mathbb{R}^N$ is weakly lower semi-continuous and coercive, i.e., $\lim_{\|z\| \to \infty} \psi(z) = +\infty$, then there exists $z_0 \in X$ such that $\psi(z_0) = \inf_{z \in X} \psi(z)$. Moreover, if ψ is also Fréchet differentiable on X, then $\psi'(z_0) = 0$.*

Theorem 29 ([46] (Mountain pass theorem))**.** *Let X be a real Banach space and $\psi \in C^1(X, \mathbb{R}^N)$, satisfying the P.S. condition. Assume that*

(i) $\psi(0) = 0$,
(ii) *there exist $\rho > 0$ and $\sigma > 0$ such that $\psi(z) \geq \sigma$ for all $z \in X$ with $\|z\| = \rho$,*
(iii) *there exists z_1 in X with $\|z_1\| \geq \rho$ such that $\psi(z_1) < \sigma$.*

Then, ψ possesses a critical value $c \geq \sigma$. Moreover, c can be characterized as

$$c = \inf_{\omega \in \overline{\Omega}} \max_{z \in \omega([0,1])} \psi(z),$$

where $\overline{\Omega} = \{\omega \in C([0,1], X) : \omega(0) = 0, \omega(1) = z_1\}$.

First, we can solve the existence of weak solutions for FBVP (26) by using Theorem 28. Suppose that the assumption $(\mathbf{A_1})$ is satisfied. Looking at (20), the corresponding functional Φ on $W^{\alpha,2}_{\Delta,b^-}$ given by

$$\Phi(u) = \frac{1}{2} \int_a^b |{}^{\mathbb{T}}_t D^\alpha_b u(t)|^2 \Delta t - \int_a^b G(\sigma(t), u^\sigma(t)) \Delta t,$$

is continuously differentiable according to Theorem 26 and is also weakly lower semi-continuous functional on $W^{\alpha,2}_{\Delta,b^-}$ as the sum of a convex continuous function and a weakly continuous function.

Actually, in view of Proposition 7, if $u_k \rightharpoonup u$ in $W^{\alpha,2}_{\Delta,b^-}$, then $u_k \to u$ in $C(J,\mathbb{R}^N)$. As a result, $G(\sigma(t),u_k^\sigma(t)) \to G(\sigma(t),u^\sigma(t))$ Δ-a.e. $t\in[a,b]_{\mathbb{T}}$. Using the Lebesgue-dominated convergence theorem on time scales, we obtain $\int_a^b G(\sigma(t),u_k^\sigma(t))\Delta t \to \int_a^b G(\sigma(t),u^\sigma(t))\Delta t$, which implies that the functional $u \to \int_a^b G(\sigma(t),u^\sigma(t))\Delta t$ is weakly continuous on $W^{\alpha,2}_{\Delta,b^-}$. Furthermore, because the fractional derivative operator on $\mathbb{T}$ is a linear operator, the functional $u \to \int_a^b |{}_a^{\mathbb{T}}D_t^\alpha u(t)|^2\Delta t$ is convex and continuous on $W^{\alpha,2}_{\Delta,b^-}$.

If φ is coercive, using Theorem 28, Φ has a minimum so that FBVP (26) is solvable. It remains to find conditions under which Φ is coercive on $W^{\alpha,2}_{\Delta,b^-}$, i.e., $\lim_{\|z\|\to\infty}\varphi(z)=+\infty$, for $u\in W^{\alpha,2}_{\Delta,b^-}$. We shall know that it suffices to require that $G(t,x)$ is bounded by a function for Δ-a.e. $t\in J$ and all $x\in\mathbb{R}^N$.

Theorem 30. *Let $\frac{1}{2}<\alpha\le 1$, and suppose that G satisfies ($\mathbf{A_1}$). If*

$$|G(t,x)| \le \bar{e}|x|^2 + \bar{f}(t)|x|^{2-\gamma} + \bar{h}(t), \quad \Delta - a.e.\ t\in J,\ x\in\mathbb{R}^N, \tag{22}$$

where $\bar{e}\in\left[0,\frac{\Gamma^2(\alpha+1)}{2b^{2\alpha}}\right)$, $\gamma\in(0,2)$, $\bar{f}\in L_\Delta^{\frac{2}{\gamma}}([a,b]_{\mathbb{T}},\mathbb{R})$ and $\bar{h}\in L^1_\Delta(J,\mathbb{R})$, then FBVP (26) has at least one weak solution that minimizes φ on $W^{\alpha,2}_{\Delta,b^-}$.

Proof. Taking account of the arguments above, our task reduces to testifying that Φ is coercive on $W^{\alpha,2}_{\Delta,b^-}$. For $u\in W^{\alpha,2}_{\Delta,b^-}$, together with (22), (12) and the Hölder inequality on time scales, we obtain that

$$
\begin{aligned}
&\Phi(u)\\
=&\frac{1}{2}\int_a^b |{}_t^{\mathbb{T}}D_b^\alpha u(t)|^2\Delta t - \int_a^b G(\sigma(t),u^\sigma(t))\Delta t\\
\ge&\frac{1}{2}\int_a^b |{}_t^{\mathbb{T}}D_b^\alpha u(t)|^2\Delta t - \bar{e}\int_a^b |u(t)|^2\Delta t - \int_a^b \bar{f}(t)|u(t)|^{2-\gamma}\Delta t - \int_a^b \bar{h}(t)\Delta t\\
\ge&\frac{1}{2}\|u\|^2 - \bar{e}\|u\|^2_{L^2_\Delta} - \left(\int_a^b |\bar{f}(t)|^{\frac{2}{\gamma}}\Delta t\right)^{\frac{\gamma}{2}}\left(\int_a^b |u(t)|^2\Delta t\right)^{1-\frac{\gamma}{2}}\Delta t - \int_a^b \bar{h}(t)\Delta t\\
=&\frac{1}{2}\|u\|^2 - \bar{e}\|u\|^2_{L^2_\Delta} - \left(\int_a^b |\bar{f}(t)|^{\frac{2}{\gamma}}\Delta t\right)^{\frac{\gamma}{2}}\|u\|^{2-\gamma}_{L^2_\Delta} - \int_a^b \bar{h}(t)\Delta t\\
\ge&\frac{1}{2}\|u\|^2 - \frac{\bar{e}b^{2\alpha}}{\Gamma^2(\alpha+1)}\|u\|^2 - \left(\int_a^b |\bar{f}(t)|^{\frac{2}{\gamma}}\Delta t\right)^{\frac{\gamma}{2}}\left(\frac{b^\alpha}{\Gamma(\alpha+1)}\right)^{2-\gamma}\|u\|^{2-\gamma} - \int_a^b \bar{h}(t)\Delta t\\
=&\left(\frac{1}{2} - \frac{\bar{e}b^{2\alpha}}{\Gamma^2(\alpha+1)}\right)\|u\|^2 - \left(\int_a^b |\bar{f}(t)|^{\frac{2}{\gamma}}\Delta t\right)^{\frac{\gamma}{2}}\left(\frac{b^\alpha}{\Gamma(\alpha+1)}\right)^{2-\gamma}\|u\|^{2-\gamma} - \int_a^b \bar{h}(t)\Delta t.
\end{aligned}
$$

Noting that $\bar{e}\in\left[0,\frac{\Gamma^2(\alpha+1)}{2b^{2\alpha}}\right)$ and $\gamma\in(0,2)$, we obtain $\Phi(u)=+\infty$ as $\|u\|\to\infty$, and so Φ is coercive, which completes the proof. □

Let $e_0 = \min_{\lambda\in[\frac{1}{2},1]}\left\{\frac{\Gamma^2(\lambda+1)}{2b^{2\lambda}}\right\}$. As a result, we can obtain the following result by Theorem 29.

Corollary 3. *For $\frac{1}{2}<\alpha\le 1$, if F satisfies the condition ($\mathbf{A_1}$) and (22) with $\bar{e}\in[0,e_0)$, then FBVP (26) has at least one weak solution that minimizes Φ on $W^{\alpha,2}_{\Delta,b^-}$.*

It is time for us to apply Theorem 29 (Mountain pass theorem) to find a nonzero critical point of functional Φ on $W^{\alpha,2}_{\Delta,b^-}$.

Theorem 31. *Let $\frac{1}{2}<\alpha\le 1$, and suppose that G satisfies ($\mathbf{A_1}$). If*

(A_2) *$G \in C(J \times \mathbb{R}^N, \mathbb{R})$, and there are $\mu > 2$ and $M > 0$ such that $0 < \mu G(t,x) \leq (\nabla G(t,x), x)$ for all $x \in \mathbb{R}^N$ with $|x| \geq M$ and $t \in J$,*

(A_3) $\limsup\limits_{|x| \to 0} \frac{G(t,x)}{|x|^2} < \frac{\Gamma^2(\alpha+1)}{2b^{2\alpha}}$ *uniformly for $t \in J$ and $x \in \mathbb{R}^N$*

are satisfied, then FBVP (26) has at least one nonzero weak solution on $W^{\alpha,2}_{\Delta,b^-}$.

Proof. We will demonstrate that Φ satisfies all the conditions of Theorem 29.

First, we will verify that Φ satisfies the P.S. condition. Because $G(t,x) - \frac{1}{\mu}(\nabla G(t,x), x)$ is continuous for $t \in J$ and $|x| \leq M$, there is $c \in \mathbb{R}^+$ such that

$$G(t,x) \leq \frac{1}{\mu}(\nabla G(t,x), x) + c, \quad t \in J, \ |x| \leq M.$$

In view of condition $(\mathbf{A_2})$, one has

$$G(t,x) \leq \frac{1}{\mu}(\nabla G(t,x), x) + c, \quad t \in J, \ x \in \mathbb{R}^N. \tag{23}$$

Let $\{u_k\} \subset W^{\alpha,2}_{\Delta,b^-}$, $|\Phi(u_k)| \leq K$, $k = 1, 2, \cdots$, $\Phi'(u_k) \to 0$. Notice that

$$\begin{aligned} \langle \Phi'(u_k), u_k \rangle = & \int_a^b \Big[({}_t^{\mathbb{T}}D_b^{\alpha} u_k(t), {}_t^{\mathbb{T}}D_b^{\alpha} u_k(t)) - (\nabla G(\sigma(t), u_k^{\sigma}(t)), u_k^{\sigma}(t)) \Big] \Delta t \\ = & \|u_k\|^2 - \int_a^b \nabla G(\sigma(t), u_k^{\sigma}(t)), u_k^{\sigma}(t)) \Delta t. \end{aligned} \tag{24}$$

Combining with (23) and (24), one arrives at

$$\begin{aligned} K \geq & \Phi(u_k) \\ = & \frac{1}{2} \int_a^b |{}_t^{\mathbb{T}}D_b^{\alpha} u_k(t)|^2 \Delta t - \int_a^b G(\sigma(t), u_k^{\sigma}(t)) \Delta t \\ \geq & \frac{1}{2} \|u_k\|^2 - \mu \int_a^b (\nabla G(\sigma(t), u_k^{\sigma}(t)), u_k^{\sigma}(t)) \Delta t - cb \\ = & \left(\frac{1}{2} - \frac{1}{\mu}\right) \|u_k\|^2 + \frac{1}{\mu} \langle \varphi'(u_k), u_k \rangle - cb \\ \geq & \left(\frac{1}{2} - \frac{1}{\mu}\right) \|u_k\|^2 - \frac{1}{\mu} \|\varphi'(u_k)\| \|u_k\| - cb. \end{aligned}$$

It follows from $\Phi'(u_k) \to 0$ that there is $N_0 \in \mathbb{N}$ such that

$$K \geq \left(\frac{1}{2} - \frac{1}{\mu}\right) \|u_k\|^2 - \|u_k\| - cb, \quad k > N_0,$$

which means that $\{u_k\} \subset W^{\alpha,2}_{\Delta,b^-}$ is bounded. In view of $W^{\alpha,2}_{\Delta,b^-}$ being a reflexive space, going to a subsequence if necessary, we may suppose that $u_k \rightharpoonup u$ weakly in $W^{\alpha,2}_{\Delta,b^-}$; therefore, one obtains

$$\begin{aligned} & \langle \Phi'(u_k) - \Phi'(u), u_k - u \rangle \\ = & \langle \Phi'(u_k), u_k - u \rangle - \langle \Phi'(u), u_k - u \rangle \\ \leq & \|\Phi'(u_k)\| \|u_k - u\| - \langle \Phi'(u), u_k - u \rangle \to 0, \end{aligned} \tag{25}$$

as $k \to \infty$. Furthermore, in view of (13) and Proposition 7, one can find that u_k is bounded in $C(J, \mathbb{R}^N)$ and $\|u_k - u\|_\infty = 0$ as $k \to \infty$. As a result, one has

$$\int_a^b \nabla G(\sigma(t), u_k^{\sigma}(t)) \Delta t \to \int_a^b \nabla G(\sigma(t), u^{\sigma}(t)) \Delta t, \quad k \to \infty. \tag{26}$$

Noting that

$$\begin{aligned}
&\langle \Phi'(u_k) - \Phi'(u), u_k - u\rangle \\
&= \int_a^b ({}_t^{\mathbb{T}}D_b^{\alpha} u_k(t) - {}_t^{\mathbb{T}} D_b^{\alpha} u(t))^2 \Delta t - \int_a^b (\nabla G(\sigma(t), u_k^{\sigma}(t)) - \nabla G(\sigma(t), u^{\sigma}(t))) \\
&\times (u_k^{\sigma}(t) - u^{\sigma}(t))\Delta t \\
&\geq \|u_k - u\|^2 - \left| \int_a^b (\nabla G(\sigma(t), u_k^{\sigma}(t)) - \nabla G(\sigma(t), u^{\sigma}(t)))\Delta t \right| \|u_k - u\|_{\infty}.
\end{aligned}$$

Together with (25) and (26), it is not difficult for us to prove that $\|u_k - u\|^2 \to 0$ as $k \to \infty$, and so that $u_k \to u$ in $W^{\alpha,2}_{\Delta,b^-}$. Hence, we obtain the desired convergence property.

By condition $(\mathbf{A_3})$, there are $\epsilon \in (0,1)$ and $\delta > 0$ such that $G(t,x) \leq (1-\epsilon)\left(\frac{\Gamma^2(\alpha+1)}{2b^{2\alpha}}\right)|x|^2$ for all $t \in J$ and $x \in \mathbb{R}^N$ with $|x| \leq \delta$.

Let $\rho = \frac{\Gamma(\alpha)(2(\alpha-1)+1)^{\frac{1}{2}}}{b^{\alpha-\frac{1}{2}}}\delta$ and $\sigma = \frac{\epsilon\rho^2}{2} > 0$. Then, in light of (13), one sees that

$$\|u\|_{\infty} \leq \frac{b^{\alpha-\frac{1}{2}}}{\Gamma(\alpha)(2(\alpha-1)+1)^{\frac{1}{2}}}\|u\| = \delta$$

for all $u \in W^{\alpha,2}_{\Delta,b^-}$ with $\|u\| = \rho$. Hence, combining with (12), one obtains

$$\begin{aligned}
\Phi(u) &= \frac{1}{2}\int_a^b |{}_t^{\mathbb{T}}D_b^{\alpha} u(t)|^2 \Delta t - \int_a^b G(\sigma(t), u^{\sigma}(t))\Delta t \\
&= \frac{1}{2}\|u\|^2 - \int_a^b G(\sigma(t), u^{\sigma}(t))\Delta t \\
&\geq \frac{1}{2}\|u\|^2 - (1-\epsilon)\frac{\Gamma^2(\alpha+1)}{2b^{2\alpha}}\int_a^b |u(t)|^2 \Delta t \\
&\geq \frac{1}{2}\|u\|^2 - \frac{1}{2}(1-\epsilon)\|u\|^2 \\
&= \frac{1}{2}\epsilon\|u\|^2 \\
&= \sigma
\end{aligned}$$

for all $u \in W^{\alpha,2}_{\Delta,b^-}$ with $\|u\| = \rho$. This implies that **(ii)** in Theorem 29 is satisfied.

It follows from the definition of Φ and condition $(\mathbf{A_3})$ that $\Phi(0) = 0$, and so it suffices to prove that Φ satisfies **(iii)** in Theorem 29.

For $s \in \mathbb{R}$, $|x| \geq M$ and $t \in J$, let

$$F(s) = G(t, sx), \qquad H(s) = F'(s) - \frac{\mu}{s}F(s) \tag{27}$$

In view of $(\mathbf{A_2})$, when $s \geq \frac{M}{|x|}$, one obtains

$$H(s) = \frac{\nabla G(t, sx)sx - \mu G(t, sx)}{s} \geq 0$$

In addition, taking the expression of $F(\cdot)$ and $H(\cdot)$ in (27) into account, we can easily obtain that $F(s)$ satisfies

$$F'(s) = H(s) + \frac{\mu}{s}F(s)$$

Therefore, when $s \geq \frac{M}{|x|}$, we have

$$G(t, sx) = s^{\mu}\left[G(t, x) + \int_1^s \tau^{-\mu} H(\tau) d\tau\right].$$

Thus, for $|x| \geq M$ and $t \in J$, together with $(\mathbf{A_1})$, one obtains

$$\left(\frac{M}{|x|}\right)^{\mu} G(t, x) \leq G\left(t, x\frac{M}{|x|}\right) \leq \max_{|x| \leq M} p(|x|)q(t),$$

which implies that

$$G(t, x) \leq \frac{|x|^{\mu}}{M^{\mu}} \max_{|x| \leq M} p(|x|)q(t).$$

Thus, one obtains

$$G(t, x) \geq \frac{|x|^{\mu}}{M^{\mu}} \min_{|x| \leq M} p(|x|)q(t), \tag{28}$$

For any $u \in W^{\alpha,2}_{\Delta,b^-}$ with $u \neq 0$, $\kappa > 0$ and noting that $\mu > 2$, one has

$$\begin{aligned}
\Phi(\kappa u) =& \frac{1}{2}\int_a^b |{}^{\mathbb{T}}_t D^{\alpha}_b \kappa u(t)|^2 \Delta t - \int_a^b G(\sigma(t), \kappa u^{\sigma}(t))\Delta t \\
\leq& \frac{1}{2}\|\kappa u\|^2 - \int_a^b \frac{|\kappa u|^{\mu}}{M^{\mu}} \min_{|\kappa u| \leq M} p(|\kappa u|)q(t)\Delta t \\
\leq& \frac{1}{2}\kappa^2\|u\|^2 - \frac{\kappa^{\mu}}{M^{\mu}} \min_{|\kappa u| \leq M} p(|\kappa u|)\|u\|^{\mu}_{L^1_{\Delta}} \inf_{t \in [a,b]_{\mathbb{T}}} q(t) \int_a^b |u|^{\mu}\Delta t \\
=& \frac{1}{2}\kappa^2\|u\|^2 - \frac{\kappa^{\mu}}{M^{\mu}} \min_{|\kappa u| \leq M} p(|\kappa u|) \inf_{t \in [a,b)_{\mathbb{T}}} q(t)\|u\|^{\mu}_{L^1_{\Delta}} \\
\to& -\infty
\end{aligned}$$

as $k \to \infty$. Then, there is a sufficiently large κ_0 such that $\Phi(\kappa_0 u) \leq 0$. As a result, **(iii)** of Theorem 29 holds.

Lastly, note that $\Phi(0) = 0$, while, for our critical point u, $\Phi(u) \geq \sigma > 0$. Therefore, u is a nontrivial weak solution of the FBVP (26), and this completes the proof. □

Corollary 4. *For all $\frac{1}{2} < \alpha \leq 1$, assume that G satisfies conditions $(\mathbf{A_1})$ and $(\mathbf{A_2})$. If*
(A_4) $G(t, x) = o(|x|^2)$, as $|x| \to 0$ uniformly for $t \in J$ and $x \in \mathbb{R}^N$
is satisfied, then the FBVP (26) has at least one nonzero weak solution on $W^{\alpha,2}_{\Delta,b^-}$.

6. Conclusions

In this paper, we have proven the equivalence between the fractional integrals and fractional derivatives on time scales defined by integral and the fractional integrals and fractional derivatives on time scales defined by the Laplace transform and the inverse Laplace transform. We give the definition of right fractional Sobolev spaces on time scales and study some of their important properties. As an application of the results of this paper, we have shown the solvability of a fractional boundary value problem on time scales by using the critical point theory. The methods and results of this paper provide a basic workspace for using variational methods and critical point theory to study the solvability of dynamic equations on time scales.

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