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Existence Results for Coupled Nonlinear Sequential Fractional Differential Equations with Coupled Riemann–Stieltjes Integro-Multipoint Boundary Conditions

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Abstract: This paper is concerned with the existence of solutions for a fully coupled Riemann–Stieltjes, integro-multipoint, boundary value problem of Caputo-type sequential fractional differential equations. The given system is studied with the aid of the Leray–Schauder alternative and contraction mapping principle. A numerical example illustrating the abstract results is also presented.

Keywords: sequential fractional differential equations; Caputo fractional derivative; Riemann–Stieltjes integro-multipoint boundary conditions; existence and uniqueness; fixed point

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Citation: Alruwaily, Y.; Ahmad, B.; Ntouyas, S.K.; Alzaidi, A.S.M. Existence Results for Coupled Nonlinear Sequential Fractional Differential Equations with Coupled Riemann–Stieltjes Integro-Multipoint Boundary Conditions. *Fractal Fract.* **2022**, *6*, 123. https://doi.org/10.3390/fractalfract6020123

Academic Editor: Maria Rosaria Lancia

Received: 8 January 2022
Accepted: 16 February 2022
Published: 20 February 2022

1. Introduction

Coupled systems of fractional-order differential equations appear in the mathematical models of several real-world problems. Examples include chaos and fractional dynamics [1], bio-engineering [2], ecology [3], financial economics [4], etc. The topic of fractional differential systems, complemented by different kinds of boundary conditions, has been one a popular and important area of scientific investigation. Many researchers have contributed to the development of this subject by publishing numerous articles, Special Issues, etc. The modern methods of functional analysis areof great support in achieving existence and uniqueness results for these problems [5,6]. For some recent works on fractional or sequential fractional differential equations with nonlocal integral boundary conditions, we refer the reader to a series of papers [7–13].

In the article of [14], the authors investigated the solvability of an initial value problem involving a sequential fractional differential equation by means of fixed-point theorems in partially ordered sets. In [15], the existence and uniqueness results for a periodic boundary value problem of nonlinear sequential fractional differential equations were obtained by the method of upper and lower solutions, together with the monotone iterative technique.

Now, we briefly describe some recent works on sequential fractional-order coupled systems equipped with coupled boundary conditions. A fully coupled two-parameter system of sequential fractional integro-differential equations with nonlocal integro-multipoint boundary conditions was studied in [16]. The authors discussed the existence and uniqueness of solutions for a system of Hilfer–Hadamard sequential fractional differential equations with two-point boundary conditions in [17]. The sequential hybrid inclusion boundary value problem with three-point integro-derivative boundary conditions was investigated by using the analytic methods relying on α-ψ-contractive mappings, endpoints, and the fixed points of the product operators in [18]. The authors studied the existence and uniqueness of solutions for an initial value problem of coupled sequential fractional differential equations in [19]. The existence results for a nonlocal coupled system of sequential fractional differential equations involving ψ-Hilfer fractional derivatives were presented in [20].

The objective of the present work is to develop the existence theory for a new class of nonlinear coupled systems of sequential fractional differential equations supplemented with coupled, non-conjugate, Riemann–Stieltjes, integro-multipoint boundary conditions. In precise terms, we investigate the following system:

$$\begin{cases} ({}^cD^{q+1} + {}^cD^q)\mathcal{X}(t) = \mathfrak{f}(t, \mathcal{X}(t), \mathcal{Y}(t)), \ \ 2 < q \leq 3, \ t \in [0,1], \\ ({}^cD^{p+1} + {}^cD^p)\mathcal{Y}(t) = \mathfrak{g}(t, \mathcal{X}(t), \mathcal{Y}(t)), \ \ 2 < p \leq 3, \ t \in [0,1], \end{cases} \tag{1}$$

subject to the coupled boundary conditions:

$$\begin{cases} \mathcal{X}(0) = 0, \ \mathcal{X}'(0) = 0, \ \mathcal{X}'(1) = 0, \ \mathcal{X}(1) = k\displaystyle\int_0^\rho \mathcal{Y}(s)dA(s) + \sum_{i=1}^{n-2} \alpha_i \mathcal{Y}(\sigma_i) + k_1 \int_\nu^1 \mathcal{Y}(s)dA(s), \\ \mathcal{Y}(0) = 0, \ \mathcal{Y}'(0) = 0, \ \mathcal{Y}'(1) = 0, \ \mathcal{Y}(1) = h\displaystyle\int_0^\rho \mathcal{X}(s)dA(s) + \sum_{i=1}^{n-2} \beta_i \mathcal{X}(\sigma_i) + h_1 \int_\nu^1 \mathcal{X}(s)dA(s), \end{cases} \tag{2}$$

where ${}^cD^\xi$ denotes the Caputo fractional derivative of order $\xi \in \{q, p\}, 0 < \rho < \sigma_i < \nu < 1$, $\mathfrak{f}, \mathfrak{g} : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are given continuous functions, $k, k_1, h, h_1, \alpha_i, \ \beta_i \in \ \mathbb{R}$, i=1,2,$\cdots$, $n-2$ and A is a function of bounded variation.

Riemann–Stieltjes boundary conditions are quite general, since they include multipoint and integral boundary conditions as special cases [21]. The Riemann–Stieltjes integral is a generalization of the Riemann integral due to the Dutch astronomer T. J. Stieltjes and has potential applications in probability theory [22]. In addition, the Riemann–Stieltjes integral of the random variable with respect to its distribution function interprets the expected value of random variable [23]. Moreover, the boundary conditions (2) have useful applications in diffraction-free and self-healing optoelectronic devices. For more details, see [7].

The main emphasis in the present work is to investigate the existence criteria for the solutions to a coupled system of nonlinear sequential fractional differential equations equipped with multipoint Riemann–Stieltjes integral-type boundary conditions. Here, one can see that the coupled boundary conditions relate the value of the unknown function $\mathcal{X}(t)$ ($\mathcal{Y}(t)$) at $t = 1$ with the distributions of the unknown function $\mathcal{Y}(t)$ ($\mathcal{X}(t)$) on the segments $[0, \rho]$ and $[\nu, 1]$ in the sense of Riemann–Stieltjes integrals, together with the sum of its discrete values at $\sigma_i, i = 1, 2, \cdots, n-2$. The present study is novel in the given configuration and enriches the literature on boundary value problems of sequential fractional differential equations.

Concerning our strategy when studying the problem (1)–(2), we use the fixed-point approach, which is based on the idea of converting the given problem into a fixed-point problem, followed by the application of appropriate fixed-point theorems to show the existence of the fixed points for the operator involved in the problem at hand. We make use of the Leray–Schauder alternative to show the existence of a solution to the given problem, while the uniqueness result for the given problem is derived with the aid of the contraction mapping principle due to Banach.

The rest of this paper is organized as follows. In Section 2, we present some basic definitions of fractional calculus and prove an auxiliary lemma concerning the linear variant of the problem (1)–(2), helping to convert it into a fixed-point problem. Section 3 establishes the existence and uniqueness results for the given problem, whereas Section 4 contains an example illustrating the main results. The paper ends with a discussion in Section 5, where some special cases and possible future works are indicated.

2. Preliminary Material

First, we outline some basic concepts of fractional calculus [24].

Definition 1. *The Riemann–Liouville fractional integral of order* $\vartheta \in \mathbb{R}$ $(\vartheta > 0)$ *for a locally integrable, real-valued function* U *on* $-\infty \leq a < z < b \leq +\infty$, *denoted by* $I_a^{\vartheta}U(z)$, *is defined by*

$$I_a^{\vartheta}U(z) = \frac{1}{\Gamma(\vartheta)}\int_a^z (z-s)^{\vartheta-1}U(s)ds.$$

Here, $\Gamma(\cdot)$ *is the familiar Gamma function.*

Definition 2. *The Caputo derivative of fractional order* ϑ *for an* $(r-1)$*-times absolutely continuous function* $U : [a, \infty) \longrightarrow \mathbb{R}$ *is defined as*

$$^cD^{\vartheta}U(z) = \frac{1}{\Gamma(r-\vartheta)}\int_a^z (z-s)^{r-\vartheta-1}U^{(r)}(s)ds, \; r-1 < \vartheta < r, \; r = [\vartheta]+1,$$

where $[\vartheta]$ *denotes the integer part of the real number* ϑ.

Lemma 1. *The general solution of the fractional differential equation* $^cD^{\vartheta}\mathcal{X}(z) = 0$, $r-1 < \vartheta < r$, $z \in [a, b]$, *is*

$$\mathcal{X}(z) = \varrho_0 + \varrho_1(z-a) + \varrho_2(z-a)^2 + \cdots + \varrho_{r-1}(z-a)^{r-1},$$

where $\varrho_i \in \mathbb{R}$, $i = 0, 1, \cdots, r-1$. *Furthermore,*

$$I^{\vartheta}\, {}^cD^{\vartheta}\mathcal{X}(z) = \mathcal{X}(z) + \sum_{i=0}^{r-1}\varrho_i(z-a)^i.$$

Lemma 2. *Let* $\psi, \phi \in (C[0,1], \mathbb{R})$ *and* $\Delta \neq 0$. *Then the unique solution of the linear system of fractional differential*

$$\begin{cases} ({}^cD^{q+1} + {}^cD^q)\mathcal{X}(t) = \psi(t), \;\; 2 < q \leq 3, \; t \in [0,1], \\ ({}^cD^{p+1} + {}^cD^p)\mathcal{Y}(t) = \phi(t), \;\; 2 < p \leq 3, \; t \in [0,1], \end{cases} \tag{3}$$

supplemented with the boundary conditions (2), can be expressed in the following formulas:

$$\mathcal{X}(t) = \int_0^t e^{-(t-s)} I_{0^+}^q \psi(s)ds + \sum_{i=1}^{4}\mathcal{Q}_i(t)\mathcal{E}_i, \; i = 1,2,3,4, \tag{4}$$

$$\mathcal{Y}(t) = \int_0^t e^{-(t-s)} I_{0^+}^p \phi(s)ds + \sum_{j=1}^{4}\mathcal{P}_j(t)\mathcal{E}_j, \; j = 1,2,3,4, \tag{5}$$

where

$$\begin{aligned}
\mathcal{E}_1 &= \int_0^1 e^{-(1-s)} I_{0^+}^q \psi(s)ds - I_{0^+}^q \psi(1), \\
\mathcal{E}_2 &= \int_0^1 e^{-(1-s)} I_{0^+}^p \phi(s)ds - I_{0^+}^p \phi(1), \\
\mathcal{E}_3 &= k\int_0^{\rho}\Big(\int_0^s e^{-(s-z)} I_{0^+}^p \phi(z)dz\Big)dA(s) + \sum_{i=1}^{n-2}\alpha_i\int_0^{\sigma_i} e^{-(\sigma_i-s)} I_{0^+}^p \phi(s)ds \\
&\quad + k_1\int_{\nu}^1\Big(\int_0^s e^{-(s-z)} I_{0^+}^p \phi(z)dz\Big)dA(s) - \int_0^1 e^{-(1-s)} I_{0^+}^q \psi(s)ds \\
\mathcal{E}_4 &= h\int_0^{\rho}\Big(\int_0^s e^{-(s-z)} I_{0^+}^q \psi(z)dz\Big)dA(s) + \sum_{i=1}^{n-2}\beta_i\int_0^{\sigma_i} e^{-(\sigma_i-s)} I_{0^+}^q \psi(s)ds
\end{aligned}$$

$$+h_1\int_\nu^1\Big(\int_0^s e^{-(s-z)}I_{0^+}^q\psi(z)dz\Big)dA(s)-\int_0^1 e^{-(1-s)}I_{0^+}^p\phi(s)ds, \tag{6}$$

$$\begin{aligned}\mathcal{Q}_i(t) &= (e^{-t}+t-1)\lambda_i+(-2e^{-t}+t^2-2t+2)\nu_i,\ i=1,2,3,4,\\ \mathcal{P}_j(t) &= (e^{-t}+t-1)\rho_j+(-2e^{-t}+t^2-2t+2)\omega_j,\ j=1,2,3,4,\end{aligned} \tag{7}$$

$$\nu_1 = \frac{e+(1-e)\lambda_1}{2},\ \nu_2=\frac{(1-e)\lambda_2}{2},\ \nu_3=\frac{(1-e)\lambda_3}{2},\ \nu_4=\frac{(1-e)\lambda_4}{2}, \tag{8}$$

$$\omega_1 = \frac{(1-e)\rho_1}{2},\ \omega_2=\frac{e+(1-e)\rho_2}{2},\ \omega_3=\frac{(1-e)\rho_3}{2},\ \omega_4=\frac{(1-e)\rho_4}{2}, \tag{9}$$

$$\lambda_1 = \frac{(2-e)\gamma_1-A_4\gamma_2 e}{2\Delta},\ \lambda_2=\frac{A_2\gamma_1 e-(2-e)\gamma_2}{2\Delta},\ \lambda_3=\frac{\gamma_1}{\Delta},\ \lambda_4=\frac{-\gamma_2}{\Delta}, \tag{10}$$

$$\rho_1 = \frac{A_4\gamma_1 e-(2-e)\gamma_3}{2\Delta},\ \rho_2=\frac{(2-e)\gamma_1-A_2\gamma_3 e}{2\Delta},\ \rho_3=\frac{-\gamma_3}{\Delta},\ \rho_4=\frac{\gamma_1}{\Delta}, \tag{11}$$

$$\Delta=\gamma_1^2-\gamma_2\gamma_3,\ \gamma_1=\frac{3-e}{2},\ \gamma_2=-A_1-A_2\frac{(1-e)}{2},\ \gamma_3=-A_3-A_4\frac{(1-e)}{2}, \tag{12}$$

$$\begin{aligned} A_1 &= k\int_0^\rho(e^{-s}+s-1)dA(s)+\sum_{i=1}^{n-2}\alpha_i(e^{-\sigma_i}+\sigma_i-1)+k_1\int_\nu^1(e^{-s}+s-1)dA(s),\\ A_2 &= k\int_0^\rho(-2e^{-s}+s^2-2s+2)dA(s)+\sum_{i=1}^{n-2}\alpha_i(-2e^{-\sigma_i}+\sigma_i^2-2\sigma_i+2)\\ &\quad +k_1\int_\nu^1(-2e^{-s}+s^2-2s+2)dA(s),\\ A_3 &= h\int_0^\rho(e^{-s}+s-1)dA(s)+\sum_{i=1}^{n-2}\beta_i(e^{-\sigma_i}+\sigma_i-1)+h_1\int_\nu^1(e^{-s}+s-1)dA(s),\\ A_4 &= h\int_0^\rho(-2e^{-s}+s^2-2s+2)dA(s)+\sum_{i=1}^{n-2}\beta_i(-2e^{-\sigma_i}+\sigma_i^2-2\sigma_i+2)\\ &\quad +h_1\int_\nu^1(-2e^{-s}+s^2-2s+2)dA(s).\end{aligned} \tag{13}$$

Proof. Rewriting the first equation in (3) as ${}^cD^q(D+1)\mathcal{X}(t)=\psi(t)$ and then applying the integral operator $I_{0^+}^q$ to it, we obtain

$$\begin{aligned}\mathcal{X}(t) &= (-e^{-t}+1)c_1+(e^{-t}+t-1)c_2+(-2e^{-t}+t^2-2t+2)c_3+e^{-t}c_4\\ &\quad +\int_0^t e^{-(t-s)}I_{0^+}^q\psi(s)ds,\end{aligned} \tag{14}$$

where $c_i\in\mathbb{R},\ i=1,2,3,4$ are unknown arbitrary constants. In a similar manner, applying the integral operator $I_{0^+}^p$ to the second equation in (3), we get

$$\begin{aligned}\mathcal{Y}(t) &= (-e^{-t}+1)b_1+(e^{-t}+t-1)b_2+(-2e^{-t}+t^2-2t+2)b_3+e^{-t}b_4\\ &\quad +\int_0^t e^{-(t-s)}I_{0^+}^p\phi(s)ds,\end{aligned} \tag{15}$$

where $b_i\in\mathbb{R},\ i=1,2,3,4$ are unknown arbitrary constants. From (14) and (15), we have

$$\begin{aligned}\mathcal{X}'(t) &= e^{-t}c_1+(-e^{-t}+1)c_2+(2e^{-t}+2t-2)c_3-e^{-t}c_4\\ &\quad -\int_0^t e^{-(t-s)}I_{0^+}^q\psi(s)ds+I_{0^+}^q\psi(t),\end{aligned} \tag{16}$$

$$\mathcal{Y}'(t) = e^{-t}b_1 + (-e^{-t}+1)b_2 + (2e^{-t}+2t-2)b_3 - e^{-t}b_4 - \int_0^t e^{-(t-s)} I_{0+}^p \phi(s)ds + I_{0+}^p \phi(t). \quad (17)$$

Using the conditions $\mathcal{X}(0) = 0, \mathcal{Y}(0) = 0, \mathcal{X}'(0) = 0, \mathcal{Y}'(0) = 0$ in Equations (14)–(17), we obtain $c_1 = c_4 = 0$ and $b_1 = b_4 = 0$. Then (14)–(17) become

$$\mathcal{X}(t) = (e^{-t}+t-1)c_2 + (-2e^{-t}+t^2-2t+2)c_3 + \int_0^t e^{-(t-s)} I_{0+}^q \psi(s)ds, \quad (18)$$

$$\mathcal{X}'(t) = (-e^{-t}+1)c_2 + (2e^{-t}+2t-2)c_3 - \int_0^t e^{-(t-s)} I_{0+}^q \psi(s)ds + I_{0+}^q \psi(t), \quad (19)$$

$$\mathcal{Y}(t) = (e^{-t}+t-1)b_2 + (-2e^{-t}+t^2-2t+2)b_3 + \int_0^t e^{-(t-s)} I_{0+}^p \phi(s)ds, \quad (20)$$

$$\mathcal{Y}'(t) = (-e^{-t}+1)b_2 + (2e^{-t}+2t-2)b_3 - \int_0^t e^{-(t-s)} I_{0+}^p \phi(s)ds + I_{0+}^p \phi(t). \quad (21)$$

Using (18)–(21) in the rest of the boundary conditions given by (2), together with notation (13), yields

$$(-e^{-1}+1)c_2 + 2e^{-1}c_3 = \mathcal{E}_1, \quad (22)$$

$$(-e^{-1}+1)b_2 + 2e^{-1}b_3 = \mathcal{E}_2, \quad (23)$$

$$e^{-1}c_2 + (-2e^{-1}+1)c_3 - A_1b_2 - A_2b_3 = \mathcal{E}_3, \quad (24)$$

$$e^{-1}b_2 + (-2e^{-1}+1)b_3 - A_3c_2 - A_4c_3 = \mathcal{E}_4, \quad (25)$$

where A_i, $i = 1,2,3,4$ are given by (13) and $\mathcal{E}_i$, $i = 1,2,3,4$ are defined by (6). Inserting the values of c_3 and b_3 from (22) and (23) into (24) and (25), we obtain

$$\gamma_1 c_2 + \gamma_2 b_2 = \frac{(2-e)}{2}\mathcal{E}_1 + \frac{A_2 e}{2}\mathcal{E}_2 + \mathcal{E}_3, \quad (26)$$

$$\gamma_3 c_2 + \gamma_1 b_2 = \frac{A_4 e}{2}\mathcal{E}_1 + \frac{(2-e)}{2}\mathcal{E}_2 + \mathcal{E}_4, \quad (27)$$

where γ_i, $i = 1,2,3$ are given by (12). Solving (26) and (27) for c_2 and b_2, we obtain

$$c_2 = \sum_{i=1}^{4} \lambda_i \mathcal{E}_i, \quad b_2 = \sum_{j=1}^{4} \rho_j \mathcal{E}_j,$$

where λ_i $(i = 1,2,3,4)$ and ρ_j $(j = 1,2,3,4)$ are given in (10) and (11), respectively. Substituting the values of c_2 and b_2 into (22) and (23) respectively, we find that

$$c_3 = \sum_{i=1}^{4} \nu_i \mathcal{E}_i, \quad b_3 = \sum_{j=1}^{4} \omega_j \mathcal{E}_j,$$

where $\nu_i, i = 1,2,3,4$, and ω_j, $j = 1,2,3,4$ are given by (8) and (9) respectively. Inserting the values of c_2, c_3, b_2 and b_3 in (18) and (20), together with the notation (7), we obtain the solution (4) and (5). One can obtain the converse of this lemma by direct computation. This completes the proof. □

For computational convenience, we introduce the following lemma:

Lemma 3. *For ψ, $\phi \in C([0,1],\mathbb{R})$, we have*

(i) $\left| \int_0^t e^{-(t-s)} I_{0+}^q \psi(s)ds \right| \le \frac{1}{\Gamma(q+1)}(1-e^{-1})\|\psi\|,$

$$\left|\int_0^t e^{-(t-s)} I_{0^+}^p \phi(s)ds\right| \leq \frac{1}{\Gamma(p+1)}(1-e^{-1})\|\phi\|.$$

(ii) $$\left|\int_0^1 e^{-(1-s)} I_{0^+}^q \psi(s)ds\right| \leq \frac{1}{\Gamma(q+1)}(1-e^{-1})\|\psi\|,$$

$$\left|\int_0^1 e^{-(1-s)} I_{0^+}^p \phi(s)ds\right| \leq \frac{1}{\Gamma(p+1)}(1-e^{-1})\|\phi\|.$$

(iii) $$\left|\sum_{i=1}^{n-2}\alpha_i \int_0^{\sigma_i} e^{-(\sigma_i-s)} I_{0^+}^p \phi(s)ds\right| \leq \frac{1}{\Gamma(p+1)}\sum_{i=1}^{n-2}|\alpha_i|\sigma_i^p(1-e^{-\sigma_i})\|\phi\|,$$

$$\left|\sum_{i=1}^{n-2}\beta_i \int_0^{\sigma_i} e^{-(\sigma_i-s)} I_{0^+}^q \psi(s)ds\right| \leq \frac{1}{\Gamma(q+1)}\sum_{i=1}^{n-2}|\beta_i|\sigma_i^q(1-e^{-\sigma_i})\|\psi\|.$$

(iv) $$\left|\int_0^\rho \Big(\int_0^s e^{-(s-z)} I_{0^+}^p \phi(z)dz\Big)dA(s)\right| \leq \Big[\int_0^\rho \frac{s^p}{\Gamma(p+1)}(1-e^{-s})dA(s)\Big]\|\phi\|,$$

$$\left|\int_0^\rho \Big(\int_0^s e^{-(s-z)} I_{0^+}^q \psi(z)dz\Big)dA(s)\right| \leq \Big[\int_0^\rho \frac{s^q}{\Gamma(q+1)}(1-e^{-s})dA(s)\Big]\|\psi\|.$$

(v) $$\left|\int_\nu^1 \Big(\int_0^s e^{-(s-z)} I_{0^+}^p \phi(z)dz\Big)dA(s)\right| \leq \Big[\int_\nu^1 \frac{s^p}{\Gamma(p+1)}(1-e^{-s})dA(s)\Big]\|\phi\|,$$

$$\left|\int_\nu^1 \Big(\int_0^s e^{-(s-z)} I_{0^+}^q \psi(z)dz\Big)dA(s)\right| \leq \Big[\int_\nu^1 \frac{s^q}{\Gamma(q+1)}(1-e^{-s})dA(s)\Big]\|\psi\|.$$

Proof. To prove (i), we have

$$\begin{aligned}\left|\int_0^t e^{-(t-s)} I_{0^+}^q \psi(s)ds\right| &= \left|\int_0^t e^{-(t-s)}\Big(\int_0^s \frac{(s-z)^{q-1}}{\Gamma(q)}\psi(z)dz\Big)ds\right| \\ &\leq \frac{t^q}{\Gamma(q+1)}(1-e^{-t})\|\psi\| \\ &\leq \frac{1}{\Gamma(q+1)}(1-e^{-1})\|\psi\|.\end{aligned}$$

The other cases are similar. Therefore, we omit the details. □

3. Main Results

Let $(\mathfrak{X}, \|\cdot\|)$ be a Banach space equipped with the norm $\|\mathcal{X}\| = \sup\{|\mathcal{X}(t)|, t \in [0,1]\}$, where $\mathfrak{X} = \{\mathcal{X}(t)|\mathcal{X}(t) \in (C[0,1], \mathbb{R})\}$. Then $(\mathfrak{X}\times\mathfrak{X}, \|(\cdot,\cdot)\|)$ is also a Banach space endowed with norm $\|(\mathcal{X},\mathcal{Y})\| = \|\mathcal{X}\| + \|\mathcal{Y}\|, \mathcal{X},\mathcal{Y} \in \mathfrak{X}$.

By Lemma 2, we introduce an operator $T : \mathfrak{X}\times\mathfrak{X} \to \mathfrak{X}\times\mathfrak{X}$ defined by

$$T(\mathcal{X},\mathcal{Y})(t) = \begin{pmatrix} T_1(\mathcal{X},\mathcal{Y})(t) \\ T_2(\mathcal{X},\mathcal{Y})(t)\end{pmatrix}, \tag{28}$$

where

$$\begin{aligned} T_1(\mathcal{X},\mathcal{Y})(t) &= \int_0^t e^{-(t-s)} I_{0^+}^q \mathfrak{f}(s,\mathcal{X}(s),\mathcal{Y}(s))ds \\ &\quad + \mathcal{Q}_1(t)\Big[\int_0^1 e^{-(1-s)} I_{0^+}^q \mathfrak{f}(s,\mathcal{X}(s),\mathcal{Y}(s))ds - I_{0^+}^q \mathfrak{f}(s,\mathcal{X}(s),\mathcal{Y}(s))(1)\Big] \\ &\quad + \mathcal{Q}_2(t)\Big[\int_0^1 e^{-(1-s)} I_{0^+}^p \mathfrak{g}(s,\mathcal{X}(s),\mathcal{Y}(s))ds - I_{0^+}^p \mathfrak{g}(s,\mathcal{X}(s),\mathcal{Y}(s))(1)\Big]\end{aligned}$$

$$
\begin{aligned}
&+ \mathcal{Q}_3(t)\Big[k\int_0^{\rho}\Big(\int_0^{s} e^{-(s-z)} I_{0+}^{p}\mathfrak{g}(z,\mathcal{X}(z),\mathcal{Y}(z))dz\Big)dA(s)\\
&+\sum_{i=1}^{n-2}\alpha_i\int_0^{\sigma_i} e^{-(\sigma_i-s)} I_{0+}^{p}\mathfrak{g}(s,\mathcal{X}(s),\mathcal{Y}(s))ds\\
&+k_1\int_{\nu}^{1}\Big(\int_0^{s} e^{-(s-z)} I_{0+}^{p}\mathfrak{g}(z,\mathcal{X}(z),\mathcal{Y}(z))dz\Big)dA(s)\\
&-\int_0^{1} e^{-(1-s)} I_{0+}^{q}\mathfrak{f}(s,\mathcal{X}(s),\mathcal{Y}(s))ds\Big]\\
&+ \mathcal{Q}_4(t)\Big[h\int_0^{\rho}\Big(\int_0^{s} e^{-(s-z)} I_{0+}^{q}\mathfrak{f}(z,\mathcal{X}(z),\mathcal{Y}(z))dz\Big)dA(s)\\
&+\sum_{i=1}^{n-2}\beta_i\int_0^{\sigma_i} e^{-(\sigma_i-s)} I_{0+}^{q}\mathfrak{f}(s,\mathcal{X}(s),\mathcal{Y}(s))ds\\
&+h_1\int_{\nu}^{1}\Big(\int_0^{s} e^{-(s-z)} I_{0+}^{q}\mathfrak{f}(z,\mathcal{X}(z),\mathcal{Y}(z))dz\Big)dA(s)\\
&-\int_0^{1} e^{-(1-s)} I_{0+}^{p}\mathfrak{g}(s,\mathcal{X}(s),\mathcal{Y}(s))ds\Big],
\end{aligned} \tag{29}
$$

$$
\begin{aligned}
T_2(\mathcal{X},\mathcal{Y})(t) \;=\;& \int_0^{t} e^{-(t-s)} I_{0+}^{p}\mathfrak{g}(s,\mathcal{X}(s),\mathcal{Y}(s))ds\\
&+\mathcal{P}_1(t)\Big[\int_0^{1} e^{-(1-s)} I_{0+}^{q}\mathfrak{f}(s,\mathcal{X}(s),\mathcal{Y}(s))ds - I_{0+}^{q}\mathfrak{f}(s,\mathcal{X}(s),\mathcal{Y}(s))(1)\Big]\\
&+ \mathcal{P}_2(t)\Big[\int_0^{1} e^{-(1-s)} I_{0+}^{p}\mathfrak{g}(s,\mathcal{X}(s),\mathcal{Y}(s))ds - I_{0+}^{p}\mathfrak{g}(s,\mathcal{X}(s),\mathcal{Y}(s))(1)\Big]\\
&+ \mathcal{P}_3(t)\Big[k\int_0^{\rho}\Big(\int_0^{s} e^{-(s-z)} I_{0+}^{p}\mathfrak{g}(z,\mathcal{X}(z),\mathcal{Y}(z))dz\Big)dA(s)\\
&+\sum_{i=1}^{n-2}\alpha_i\int_0^{\sigma_i} e^{-(\sigma_i-s)} I_{0+}^{p}g(s,\mathcal{X}(s),\mathcal{Y}(s))ds\\
&+k_1\int_{\nu}^{1}\Big(\int_0^{s} e^{-(s-z)} I_{0+}^{p}\mathfrak{g}(z,\mathcal{X}(z),\mathcal{Y}(z))dz\Big)dA(s)\\
&-\int_0^{1} e^{-(1-s)} I_{0+}^{q}\mathfrak{f}(s,\mathcal{X}(s),\mathcal{Y}(s))ds\Big]\\
&+ \mathcal{P}_4(t)\Big[h\int_0^{\rho}\Big(\int_0^{s} e^{-(s-z)} I_{0+}^{q}\mathfrak{f}(z,\mathcal{X}(z),\mathcal{Y}(z))dz\Big)dA(s)\\
&+\sum_{i=1}^{n-2}\beta_i\int_0^{\sigma_i} e^{-(\sigma_i-s)} I_{0+}^{q}\mathfrak{f}(s,\mathcal{X}(s),\mathcal{Y}(s))ds\\
&+h_1\int_{\nu}^{1}\Big(\int_0^{s} e^{-(s-z)} I_{0+}^{q}\mathfrak{f}(z,\mathcal{X}(z),\mathcal{Y}(z))dz\Big)dA(s)\\
&-\int_0^{1} e^{-(1-s)} I_{0+}^{p}\mathfrak{g}(s,\mathcal{X}(s),\mathcal{Y}(s))ds\Big],
\end{aligned} \tag{30}
$$

where $\mathcal{Q}_i(t),\ i=1,2,3,4$ and $\mathcal{P}_j(t),\ j=1,2,3,4$ are given in (7).

In the forthcoming analysis, we assume that $\mathfrak{f},\mathfrak{g}:[0,1]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are continuous functions satisfying the following conditions:

($\mathcal{F}_1$) There are real constants $\eta_i,\zeta_i\geq 0, i=1,2,\ \eta_0,\zeta_0>0$ such that

$$|\mathfrak{f}(t,\mathcal{X},\mathcal{Y})|\leq \eta_0+\eta_1|\mathcal{X}|+\eta_2|\mathcal{Y}|,$$

$$|\mathfrak{g}(t,\mathcal{X},\mathcal{Y})|\leq \zeta_0+\zeta_1|\mathcal{X}|+\zeta_2|\mathcal{Y}|,$$

$\forall t\in[0,1],\mathcal{X},\mathcal{Y}\in\mathbb{R}$.

($\mathcal{F}_2$) There are positive real constants L_1 and L_2, such that

$$|\mathfrak{f}(t,\mathcal{X}_1,\mathcal{Y}_1)-\mathfrak{f}(t,\mathcal{X}_2,\mathcal{Y}_2)| \le L_1(|\mathcal{X}_1-\mathcal{X}_2|+|\mathcal{Y}_1-\mathcal{Y}_2|),$$
$$|\mathfrak{g}(t,\mathcal{X}_1,\mathcal{Y}_1)-\mathfrak{g}(t,\mathcal{X}_2,\mathcal{Y}_2)| \le L_2(|\mathcal{X}_1-\mathcal{X}_2|+|\mathcal{Y}_1-\mathcal{Y}_2|),$$

$\forall t \in [0,1], \mathcal{X}_1,\mathcal{X}_2,\mathcal{Y}_1,\mathcal{Y}_2 \in \mathbb{R}$;

In the sequel, we use the notation:

$$\Theta = \Lambda_1 L_1 + \Lambda_2 L_2, \;\; \overline{\Theta} = \overline{\Lambda}_1 L_1 + \overline{\Lambda}_2 L_2, \tag{31}$$

$$\mathcal{M} = \Lambda_1 \mathcal{N}_1 + \Lambda_2 \mathcal{N}_2, \;\; \overline{\mathcal{M}} = \overline{\Lambda}_1 \mathcal{N}_1 + \overline{\Lambda}_2 \mathcal{N}_2, \tag{32}$$

$$\begin{aligned}
\Lambda_1 &= \frac{1}{\Gamma(q+1)}\Big\{(1-e^{-1})+(2-e^{-1})\widetilde{\mathcal{Q}}_1+(1-e^{-1})\widetilde{\mathcal{Q}}_3+\widetilde{\mathcal{Q}}_4\Big[|h|\int_0^\rho s^q(1-e^{-s})dA(s) \\
&\quad + \sum_{i=1}^{n-2}|\beta_i|\sigma_i^q(1-e^{-\sigma_i})+|h_1|\int_\nu^1 s^q(1-e^{-s})dA(s)\Big]\Big\}, \\
\Lambda_2 &= \frac{1}{\Gamma(p+1)}\Big\{(2-e^{-1})\widetilde{\mathcal{Q}}_2+\widetilde{\mathcal{Q}}_3\Big[|k|\int_0^\rho s^p(1-e^{-s})dA(s) \\
&\quad + \sum_{i=1}^{n-2}|\alpha_i|\sigma_i^p(1-e^{-\sigma_i})+|k_1|\int_\nu^1 s^p(1-e^{-s})dA(s)\Big]+(1-e^{-1})\widetilde{\mathcal{Q}}_4\Big\},
\end{aligned} \tag{33}$$

$$\begin{aligned}
\overline{\Lambda}_1 &= \frac{1}{\Gamma(q+1)}\Big\{(2-e^{-1})\widetilde{\mathcal{P}}_1+(1-e^{-1})\widetilde{\mathcal{P}}_3+\widetilde{\mathcal{P}}_4\Big[|h|\int_0^\rho s^q(1-e^{-s})dA(s) \\
&\quad + \sum_{i=1}^{n-2}|\beta_i|\sigma_i^q(1-e^{-\sigma_i})+|h_1|\int_\nu^1 s^q(1-e^{-s})dA(s)\Big]\Big\}, \\
\overline{\Lambda}_2 &= \frac{1}{\Gamma(p+1)}\Big\{(1-e^{-1})+(2-e^{-1})\widetilde{\mathcal{P}}_2+\widetilde{\mathcal{P}}_3\Big[|k|\int_0^\rho s^p(1-e^{-s})dA(s) \\
&\quad + \sum_{i=1}^{n-2}|\alpha_i|\sigma_i^p(1-e^{-\sigma_i})+|k_1|\int_\nu^1 s^p(1-e^{-s})dA(s)\Big]+(1-e^{-1})\widetilde{\mathcal{P}}_4\Big\},
\end{aligned} \tag{34}$$

$$\mathcal{N}_1 = \sup_{t\in[0,1]}|\mathfrak{f}(t,0,0)|<\infty, \;\; \mathcal{N}_2 = \sup_{t\in[0,1]}|\mathfrak{g}(t,0,0,)|<\infty, \tag{35}$$

where $\widetilde{\mathcal{Q}}_i = \sup_{t\in[0,1]}|\mathcal{Q}_i(t)|$, $i=1,2,3,4$ and $\widetilde{\mathcal{P}}_j = \sup_{t\in[0,1]}|\mathcal{P}_j(t)|$, $j=1,2,3,4$,

$$\begin{aligned}
\Omega_0 &= (\Lambda_1+\overline{\Lambda}_1)\eta_0+(\Lambda_2+\overline{\Lambda}_2)\zeta_0, \\
\Omega_1 &= (\Lambda_1+\overline{\Lambda}_1)\eta_1+(\Lambda_2+\overline{\Lambda}_2)\zeta_1, \\
\Omega_2 &= (\Lambda_1+\overline{\Lambda}_1)\eta_2+(\Lambda_2+\overline{\Lambda}_2)\zeta_2,
\end{aligned} \tag{36}$$

and

$$\Omega = \max\{\Omega_1,\Omega_2\}. \tag{37}$$

The following result shows the existence of a solution for the coupled system (1)–(2) and is based on the Leray–Schauder alternative [6].

Theorem 1. *Assume that the condition ($\mathcal{F}_1$) holds and $\Omega < 1$, where Ω is given by (37). Then, the problem (1) and (2) has at least one solution on $[0,1]$.*

Proof. In the first step, it will be shown that the operator $T : \mathfrak{X} \times \mathfrak{X} \to \mathfrak{X} \times \mathfrak{X}$ is completely continuous. Note that the operator T is continuous in view of the continuity of the functions $\mathfrak{f}$ and $\mathfrak{g}$. Let $\mathcal{V} \subset \mathfrak{X} \times \mathfrak{X}$ be bounded. Then, we can find positive constants M_1 and M_2 such that $|\mathfrak{f}(t, \mathcal{X}(t), \mathcal{Y}(t))| \leq M_1$ and $|\mathfrak{g}(t, \mathcal{X}(t), \mathcal{Y}(t))| \leq M_2$, $\forall(\mathcal{X}, \mathcal{Y}) \in \mathcal{V}$. Therefore, for any $(\mathcal{X}, \mathcal{X}) \in \mathcal{V}$, we have

$$
\begin{aligned}
|T_1(\mathcal{X},\mathcal{Y})(t)| \leq\; & \int_0^t e^{-(t-s)} I_{0+}^q |\mathfrak{f}(s,\mathcal{X}(s),\mathcal{Y}(s))|ds \\
& + |\mathcal{Q}_1(t)| \Big[\int_0^1 e^{-(1-s)} I_{0+}^q |\mathfrak{f}(s,\mathcal{X}(s),\mathcal{Y}(s))|ds + I_{0+}^q |\mathfrak{f}(s,\mathcal{X}(s),\mathcal{Y}(s))|(1) \Big] \\
& + |\mathcal{Q}_2(t)| \Big[\int_0^1 e^{-(1-s)} I_{0+}^p |\mathfrak{g}(s,\mathcal{X}(s),\mathcal{Y}(s))|ds + I_{0+}^p |\mathfrak{g}(s,\mathcal{X}(s),\mathcal{Y}(s))|(1) \Big] \\
& + |\mathcal{Q}_3(t)| \Big[|k| \int_0^\rho \Big(\int_0^s e^{-(s-z)} I_{0+}^p |\mathfrak{g}(z,\mathcal{X}(z),\mathcal{Y}(z))|dz \Big) dA(s) \\
& + \sum_{i=1}^{n-2} |\alpha_i| \int_0^{\sigma_i} e^{-(\sigma_i - s)} I_{0+}^p |\mathfrak{g}(s,\mathcal{X}(s),\mathcal{Y}(s))|ds \\
& + |k_1| \int_\nu^1 \Big(\int_0^s e^{-(s-z)} I_{0+}^p |\mathfrak{g}(z,\mathcal{X}(z),\mathcal{Y}(z))|dz \Big) dA(s) \\
& + \int_0^1 e^{-(1-s)} I_{0+}^q |\mathfrak{f}(s,\mathcal{X}(s),\mathcal{Y}(s))|ds \Big] \\
& + |\mathcal{Q}_4(t)| \Big[|h| \int_0^\rho \Big(\int_0^s e^{-(s-z)} I_{0+}^q |\mathfrak{f}(z,\mathcal{X}(z),\mathcal{Y}(z))|dz \Big) dA(s) \\
& + \sum_{i=1}^{n-2} |\beta_i| \int_0^{\sigma_i} e^{-(\sigma_i - s)} I_{0+}^q |\mathfrak{f}(s,\mathcal{X}(s),\mathcal{Y}(s))|ds \\
& + |h_1| \int_\nu^1 \Big(\int_0^s e^{-(s-z)} I_{0+}^q |\mathfrak{f}(z,\mathcal{X}(z),\mathcal{Y}(z))|dz \Big) dA(s) \\
& + \int_0^1 e^{-(1-s)} I_{0+}^p |\mathfrak{g}(s,\mathcal{X}(s),\mathcal{Y}(s))|ds \Big] \\
\leq\; & M_1 \frac{1}{\Gamma(q+1)}(1-e^{-1}) + \widetilde{\mathcal{Q}}_1 \Big[M_1 \frac{1}{\Gamma(q+1)}(1-e^{-1}) + M_1 \frac{1}{\Gamma(q+1)} \Big] \\
& + \widetilde{\mathcal{Q}}_2 \Big[M_2 \frac{1}{\Gamma(p+1)}(1-e^{-1}) + M_2 \frac{1}{\Gamma(p+1)} \Big] \\
& + \widetilde{\mathcal{Q}}_3 \Big[|k| M_2 \int_0^\rho \frac{s^p}{\Gamma(p+1)}(1-e^{-s}) dA(s) \\
& + M_2 \sum_{i=1}^{n-2} |\alpha_i| \frac{1}{\Gamma(p+1)} \sigma_i^p (1-e^{-\sigma_i}) \\
& + M_2 |k_1| \int_\nu^1 \frac{s^p}{\Gamma(p+1)}(1-e^{-s}) dA(s) + M_1 \frac{1}{\Gamma(q+1)}(1-e^{-1}) \Big] \\
& + \widetilde{\mathcal{Q}}_4 \Big[M_1 |h| \int_0^\rho \frac{s^q}{\Gamma(q+1)}(1-e^{-s}) dA(s) + M_1 \sum_{i=1}^{n-2} |\beta_i| \sigma_i^q (1-e^{-\sigma_i}) \\
& + M_1 |h_1| \int_\nu^1 \frac{s^p}{\Gamma(p+1)}(1-e^{-s}) dA(s) + M_2 \frac{1}{\Gamma(p+1)}(1-e^{-1}) \Big] \\
\leq\; & \frac{M_1}{\Gamma(q+1)} \Big\{ (1-e^{-1}) + (2-e^{-1}) \widetilde{\mathcal{Q}}_1 + (1-e^{-1}) \widetilde{\mathcal{Q}}_3 \\
& + \widetilde{\mathcal{Q}}_4 \Big[|h| \int_0^\rho s^q (1-e^{-s}) dA(s) + \sum_{i=1}^{n-2} |\beta_i| \sigma_i^q (1-e^{-\sigma_i})
\end{aligned}
$$

$$
\begin{aligned}
& +|h_1|\int_{\nu}^{1} s^q(1-e^{-s})dA(s)\Big]\Big\} + \frac{M_2}{\Gamma(p+1)}\Big\{(2-e^{-1})\widetilde{\mathcal{Q}}_2 \\
& +\widetilde{\mathcal{Q}}_3\Big[|k|\int_0^{\rho} s^p(1-e^{-s})dA(s) + \sum_{i=1}^{n-2}|\alpha_i|\sigma_i^p(1-e^{-\sigma_i}) \\
& +|k_1|\int_{\nu}^{1} s^p(1-e^{-s})dA(s)\Big] + (1-e^{-1})\widetilde{\mathcal{Q}}_4\Big\} \\
= \; & \Lambda_1 M_1 + \Lambda_2 M_2.
\end{aligned}
$$

Thus,

$$
\|T_1(\mathcal{X},\mathcal{Y})\| \;\leq\; \Lambda_1 M_1 + \Lambda_2 M_2. \tag{38}
$$

Similarly, we have

$$
\|T_2(\mathcal{X},\mathcal{Y})\| \;\leq\; \overline{\Lambda}_1 M_1 + \overline{\Lambda}_2 M_2. \tag{39}
$$

Hence, (38) and (39) imply that the operator T uniformly bounded.

Now, we establish that the operator T is equicontinuous. For $t_1, t_2 \in [0,1]$ with $t_1 < t_2$, we obtain

$$
\begin{aligned}
& \Big|T_1(\mathcal{X},\mathcal{Y})(t_2) - T_1(\mathcal{X},\mathcal{Y})(t_1)\Big| \\
\leq \; & \Big|\int_0^{t_1}[e^{-(t_2-s)} - e^{-(t_1-s)}]I_{0+}^{q}\mathfrak{f}(s,\mathcal{X}(s),\mathcal{Y}(s))ds\Big| + \Big|\int_{t_1}^{t_2} e^{-(t_2-s)}I_{0+}^{q}\mathfrak{f}(s,\mathcal{X}(s),\mathcal{Y}(s))ds\Big| \\
& +\Big|\mathcal{Q}_1(t_2)-\mathcal{Q}_1(t_1)\Big|\Big[\Big|\int_0^1 e^{-(1-s)}I_{0+}^{q}\mathfrak{f}(s,\mathcal{X}(s),\mathcal{Y}(s))ds\Big| + \Big|I_{0+}^{q}\mathfrak{f}(s,\mathcal{X}(s),\mathcal{Y}(s))(1)\Big|\Big] \\
& +\Big|\mathcal{Q}_2(t_2)-\mathcal{Q}_2(t_1)\Big|\Big[\Big|\int_0^1 e^{-(1-s)}I_{0+}^{p}\mathfrak{g}(s,\mathcal{X}(s),\mathcal{Y}(s))ds\Big| + \Big|I_{0+}^{p}\mathfrak{g}(s,\mathcal{X}(s),\mathcal{Y}(s))(1)\Big|\Big] \\
& +\Big|\mathcal{Q}_3(t_2)-\mathcal{Q}_3(t_1)\Big|\Big[|k|\Big|\int_0^{\rho}\Big(\int_0^s e^{-(s-z)}I_{0+}^{p}\mathfrak{g}(z,\mathcal{X}(z),\mathcal{Y}(z))dz\Big)dA(s)\Big| \\
& +\sum_{i=1}^{n-2}|\alpha_i|\Big|\int_0^{\sigma_i} e^{-(\sigma_i-s)}I_{0+}^{p}\mathfrak{g}(s,\mathcal{X}(s),\mathcal{Y}(s))ds\Big| \\
& +|k_1|\Big|\int_{\nu}^1\Big(\int_0^s e^{-(s-z)}I_{0+}^{p}\mathfrak{g}(z,\mathcal{X}(z),\mathcal{Y}(z))dz\Big)dA(s)\Big| \\
& +\Big|\int_0^1 e^{-(1-s)}I_{0+}^{q}\mathfrak{f}(s,\mathcal{X}(s),\mathcal{Y}(s))ds\Big|\Big] \\
& +\Big|\mathcal{Q}_4(t_2)-\mathcal{Q}_4(t_1)\Big|\Big[|h|\Big|\int_0^{\rho}\Big(\int_0^s e^{-(s-z)}I_{0+}^{q}\mathfrak{f}(z,\mathcal{X}(z),\mathcal{Y}(z))dz\Big)dA(s)\Big| \\
& +\sum_{i=1}^{n-2}|\beta_i|\Big|\int_0^{\sigma_i} e^{-(\sigma_i-s)}I_{0+}^{q}\mathfrak{f}(s,\mathcal{X}(s),\mathcal{Y}(s))ds\Big| \\
& +|h_1|\Big|\int_{\nu}^1\Big(\int_0^s e^{-(s-z)}I_{0+}^{q}\mathfrak{f}(z,\mathcal{X}(z),\mathcal{Y}(z))dz\Big)dA(s)\Big| \\
& +\Big|\int_0^1 e^{-(1-s)}I_{0+}^{p}\mathfrak{g}(s,\mathcal{X}(s),\mathcal{Y}(s))ds\Big|\Big] \\
\leq \; & \frac{M_1}{\Gamma(q+1)}\Big[t_1^q\Big(e^{-(t_2-t_1)} - 1 - e^{-t_2} + e^{-t_1}\Big) + t_2^q\Big(1 - e^{-(t_2-t_1)}\Big)\Big]
\end{aligned}
$$

$$+\frac{M_1}{\Gamma(q+1)}\bigg\{(2-e^{-1})\Big|\mathcal{Q}_1(t_2)-\mathcal{Q}_1(t_1)\Big|+(1-e^{-1})\Big|\mathcal{Q}_3(t_2)-\mathcal{Q}_3(t_1)\Big|$$
$$+\Big|\mathcal{Q}_4(t_2)-\mathcal{Q}_4(t_1)\Big|\Big[|h|\Big|\int_0^\rho s^q(1-e^{-s})dA(s)\Big|$$
$$+\Big|\sum_{i=1}^{n-2}\beta_i\sigma_i^q(1-e^{-\sigma_i})\Big|+|h_1|\Big|\int_\nu^1 s^q(1-e^{-s})dA(s)\Big|\Big]\bigg\}$$
$$+\frac{M_2}{\Gamma(p+1)}\Big\{(2-e^{-1})\Big|\mathcal{Q}_2(t_2)-\mathcal{Q}_2(t_1)\Big|$$
$$+\Big|\mathcal{Q}_3(t_2)-\mathcal{Q}_3(t_1)\Big|\Big[|k|\Big|\int_0^\rho s^p(1-e^{-s})dA(s)\Big|$$
$$+\Big|\sum_{i=1}^{n-2}\alpha_i\sigma_i^p(1-e^{-\sigma_i})\Big|+|k_1|\Big|\int_\nu^1 s^p(1-e^{-s})dA(s)\Big|\Big]+(1-e^{-1})\Big|\mathcal{Q}_4(t_2)-\mathcal{Q}_4(t_1)\Big|\Big\},$$

and

$$\Big|T_2(\mathcal{X},\mathcal{Y})(t_2)-T_2(\mathcal{X},\mathcal{Y})(t_1)\Big|$$
$$\leq \frac{M_2}{\Gamma(p+1)}\bigg[t_1^p\Big(e^{-(t_2-t_1)}-1-e^{-t_2}+e^{-t_1}\Big)+t_2^p\Big(1-e^{-(t_2-t_1)}\Big)\bigg]$$
$$+\frac{M_1}{\Gamma(q+1)}\bigg\{(2-e^{-1})\Big|\mathcal{P}_1(t_2)-\mathcal{P}_1(t_1)\Big|+(1-e^{-1})\Big|\mathcal{P}_3(t_2)-\mathcal{P}_3(t_1)\Big|$$
$$+\Big|\mathcal{P}_4(t_2)-\mathcal{P}_4(t_1)\Big|\Big[|h|\Big|\int_0^\rho s^q(1-e^{-s})dA(s)\Big|$$
$$+\Big|\sum_{i=1}^{n-2}\beta_i\sigma_i^q(1-e^{-\sigma_i})\Big|+|h_1|\Big|\int_\nu^1 s^q(1-e^{-s})dA(s)\Big|\Big]\bigg\}$$
$$+\frac{M_2}{\Gamma(p+1)}\Big\{(2-e^{-1})\Big|\mathcal{P}_2(t_2)-\mathcal{P}_2(t_1)\Big|$$
$$+\Big|\mathcal{P}_3(t_2)-\mathcal{P}_3(t_1)\Big|\Big[|k|\Big|\int_0^\rho s^p(1-e^{-s})dA(s)\Big|$$
$$+\Big|\sum_{i=1}^{n-2}\alpha_i\sigma_i^p(1-e^{-\sigma_i})\Big|+|k_1|\Big|\int_\nu^1 s^p(1-e^{-s})dA(s)\Big|\Big]+(1-e^{-1})\Big|\mathcal{P}_4(t_2)-\mathcal{P}_4(t_1)\Big|\Big\}.$$

Clearly, $|T_1(\mathcal{X},\mathcal{Y})(t_2)-T_1(\mathcal{X},\mathcal{Y})(t_1)|\to 0$ and $|T_2(\mathcal{X},\mathcal{Y})(t_2)-T_2(\mathcal{X},\mathcal{Y})(t_1)|\to 0$ as $t_2\to t_1$ independent of $(\mathcal{X},\mathcal{Y})\in\mathcal{V}$. In consequence, the operator $T(\mathcal{X},\mathcal{Y})$ is equicontinuous. Hence, it follows, according to Arzelá-Ascoli theorem, that $T(\mathcal{X},\mathcal{Y})$ is completely continuous.

In the second step, we consider a set

$$\mathcal{U}=\{(\mathcal{X},\mathcal{Y})\in\mathfrak{X}\times\mathfrak{X}|(\mathcal{X},\mathcal{Y})=\sigma T(\mathcal{X},\mathcal{Y}),0<\sigma<1\}$$

and show that it is bounded. Let $(\mathcal{X},\mathcal{Y})\in\mathcal{U}$, then $(\mathcal{X},\mathcal{Y})=\sigma T(\mathcal{X},\mathcal{Y})$ and for any $t\in[0,1]$, we have

$$\mathcal{X}(t)=\sigma T_1(\mathcal{X},\mathcal{Y})(t),\ \mathcal{Y}(t)=\sigma T_2(\mathcal{X},\mathcal{Y})(t).$$

In consequence, we have

$$|\mathcal{X}(t)|\leq\Lambda_1(\eta_0+\eta_1|\mathcal{X}|+\eta_2|\mathcal{Y}|)+\Lambda_2(\zeta_0+\zeta_1|\mathcal{X}|+\zeta_2|\mathcal{Y}|),$$

which leads to

$$\|\mathcal{X}\|\leq\Lambda_1(\eta_0+\eta_1\|\mathcal{X}\|+\eta_2\|\mathcal{Y}\|)+\Lambda_2(\zeta_0+\zeta_1\|\mathcal{X}\|+\zeta_2\|\mathcal{Y}\|). \tag{40}$$

Likewise, one can obtain that

$$\|\mathcal{Y}\| \le \overline{\Lambda}_1(\eta_0 + \eta_1\|\mathcal{X}\| + \eta_2\|\mathcal{Y}\|) + \overline{\Lambda}_2(\zeta_0 + \zeta_1\|\mathcal{X}\| + \zeta_2\|\mathcal{Y}\|). \tag{41}$$

From (40) and (41), together with notations (36) and (37), we obtain

$$\begin{aligned}\|\mathcal{X}\| + \|\mathcal{Y}\| \quad \le \quad & \Big[(\Lambda_1 + \overline{\Lambda}_1)\eta_0 + (\Lambda_2 + \overline{\Lambda}_2)\zeta_0\Big] + \Big[(\Lambda_1 + \overline{\Lambda}_1)\eta_1 + (\Lambda_2 + \overline{\Lambda}_2)\zeta_1\Big]\|\mathcal{X}\| \\ & + \Big[(\Lambda_1 + \overline{\Lambda}_1)\eta_2 + (\Lambda_2 + \overline{\Lambda}_2)\zeta_2\Big]\|\mathcal{Y}\|,\end{aligned}$$

which implies that

$$\|(\mathcal{X}, \mathcal{Y})\| \le \Omega_0 + \max\{\Omega_1 + \Omega_2\}\|(\mathcal{X}, \mathcal{Y})\| \le \Omega_0 + \Omega\|(\mathcal{X}, \mathcal{Y})\|.$$

Thus

$$\|(\mathcal{X}, \mathcal{Y})\| \le \frac{\Omega_0}{1 - \Omega},$$

which shows that $\mathcal{U}$ is bounded. In view of the foregoing steps, we deduce that the hypothesis of the Leray–Schauder alternative [6] is satisfied; hence, its conclusion implies that the operator T has at least one fixed point. Thus, there is at least one solution to the problem (1) and (2) on $[0,1]$. □

Our next result deals with the uniqueness of solutions for the problem (1) and (2) and relies on Banach's fixed point theorem.

Theorem 2. *Let the condition* $(\mathcal{F}_2)$ *hold, and that*

$$\Theta + \overline{\Theta} < 1, \tag{42}$$

where Θ *and* $\overline{\Theta}$ *are given in (31). Then, there is a unique solution to the problem (1) and (2) on* $[0,1]$.

Proof. Let us first establish that $T\mathcal{U}_\varepsilon \subset \mathcal{U}_\varepsilon$, where the operator T is given by (28) and

$$\mathcal{U}_\varepsilon = \{(\mathcal{X}, \mathcal{Y}) \in \mathfrak{X} \times \mathfrak{X} : \|(\mathcal{X}, \mathcal{Y})\| \le \varepsilon\},$$

with $\varepsilon > \dfrac{\mathcal{M} + \overline{\mathcal{M}}}{1 - (\Theta + \overline{\Theta})}$, $\Theta, \overline{\Theta}$ and $\mathcal{M}, \overline{\mathcal{M}}$ are respectively given by (31) and (32). By the assumption $(\mathcal{F}_2)$ and (35), for $(\mathcal{X}, \mathcal{Y}) \in \mathcal{U}_\varepsilon,\ t \in [0,1]$, we have

$$\begin{aligned}|\mathfrak{f}(t, \mathcal{X}(t), \mathcal{Y}(t))| \quad & \le \quad |\mathfrak{f}(t, \mathcal{X}(t), \mathcal{Y}(t)) - \mathfrak{f}(t, 0, 0)| + |\mathfrak{f}(t, 0, 0)| \\ & \le \quad L_1(|\mathcal{X}(t)| + |\mathcal{Y}(t)|) + \mathcal{N}_1 \le L_1(\|\mathcal{X}\| + \|\mathcal{Y}\|) + \mathcal{N}_1 \le L_1\varepsilon + \mathcal{N}_1.\end{aligned}$$

Similarly, one can show that $|\mathfrak{g}(t, \mathcal{X}(t), \mathcal{Y}(t))| \le L_2\varepsilon + \mathcal{N}_2$. Taking into account (31) and (32), we obtain

$$|T_1(\mathcal{X}, \mathcal{Y})(t)| \quad \le \quad (\Lambda_1 L_1 + \Lambda_2 L_2)\varepsilon + (\Lambda_1\mathcal{N}_1 + \Lambda_2\mathcal{N}_2) = \Theta\varepsilon + \mathcal{M},$$

which yields

$$\|T_1(\mathcal{X}, \mathcal{Y})\| \quad \le \quad \Theta\varepsilon + \mathcal{M}. \tag{43}$$

In a similar manner, we obtain

$$\|T_2(\mathcal{X}, \mathcal{Y})\| \quad \le \quad \overline{\Theta}\varepsilon + \overline{\mathcal{M}}. \tag{44}$$

It then follows from (43) and (44) that

$$\|T(\mathcal{X},\mathcal{Y})\| \le (\Theta\varepsilon + \mathcal{M}) + (\overline{\Theta}\varepsilon + \overline{\mathcal{M}}) = (\Theta + \overline{\Theta})\varepsilon + (\mathcal{M} + \overline{\mathcal{M}}) \le \varepsilon.$$

Consequently, $T\mathcal{U}_\varepsilon \subset \mathcal{U}_\varepsilon$. Next, we show that the operator T is a contraction. Using conditions $(\mathcal{F}_2)$ and (31), we get

$$\begin{aligned}
& \|T_1(\mathcal{X}_1,\mathcal{Y}_1) - T_1(\mathcal{X}_2,\mathcal{Y}_2)\| \\
= & \sup_{t\in[0,1]} |T_1(\mathcal{X}_1,\mathcal{Y}_1)(t) - T_1(\mathcal{X}_2,\mathcal{Y}_2)(t)| \\
\le & \sup_{t\in[0,1]} \Big\{ \int_0^t e^{-(t-s)} \Big| I_{0+}^q \mathfrak{f}(s,\mathcal{X}_1(s),\mathcal{Y}_1(s)) - I_{0+}^q \mathfrak{f}(s,\mathcal{X}_2(s),\mathcal{Y}_2(s)) \Big| ds \\
& + |\mathcal{Q}_1(t)| \Big[\int_0^1 e^{-(1-s)} \Big| I_{0+}^q \mathfrak{f}(s,\mathcal{X}_1(s),\mathcal{Y}_1(s)) - I_{0+}^q \mathfrak{f}(s,\mathcal{X}_2(s),\mathcal{Y}_2(s)) \Big| ds \\
& + \Big| I_{0+}^q \mathfrak{f}(s,\mathcal{X}_1(s),\mathcal{Y}_1(s)) - I_{0+}^q \mathfrak{f}(s,\mathcal{X}_2(s),\mathcal{Y}_2(s)) \Big| (1) \Big] \\
& + |\mathcal{Q}_2(t)| \Big[\int_0^1 e^{-(1-s)} \Big| I_{0+}^p \mathfrak{g}(s,\mathcal{X}_1(s),\mathcal{Y}_1(s)) - I_{0+}^p \mathfrak{g}(s,\mathcal{X}_2(s),\mathcal{Y}_2(s)) \Big| ds \\
& + \Big| I_{0+}^p \mathfrak{g}(s,\mathcal{X}_1(s),\mathcal{Y}_1(s)) - I_{0+}^p \mathfrak{g}(s,\mathcal{X}_2(s),\mathcal{Y}_2(s)) \Big| (1) \Big] \\
& + |\mathcal{Q}_3(t)| \Big[|k| \int_0^\rho \Big(\int_0^s e^{-(s-z)} \Big| I_{0+}^p \mathfrak{g}(z,\mathcal{X}_1(z),\mathcal{Y}_1(z)) - I_{0+}^p \mathfrak{g}(z,\mathcal{X}_2(z),\mathcal{Y}_2(z)) \Big| dz \Big) dA(s) \\
& + \sum_{i=1}^{n-2} |\alpha_i| \int_0^{\sigma_i} e^{-(\sigma_i-s)} \Big| I_{0+}^p \mathfrak{g}(s,\mathcal{X}_1(s),\mathcal{Y}_1(s)) - I_{0+}^p \mathfrak{g}(s,\mathcal{X}_2(s),\mathcal{Y}_2(s)) \Big| ds \\
& + |k_1| \int_\nu^1 \Big(\int_0^s e^{-(s-z)} \Big| I_{0+}^p \mathfrak{g}(z,\mathcal{X}_1(z),\mathcal{Y}_1(z)) - I_{0+}^p \mathfrak{g}(z,\mathcal{X}_2(z),\mathcal{Y}_2(z)) \Big| dz \Big) dA(s) \\
& + \int_0^1 e^{-(1-s)} \Big| I_{0+}^q \mathfrak{f}(s,\mathcal{X}_1(s),\mathcal{Y}_1(s)) - I_{0+}^q \mathfrak{f}(s,\mathcal{X}_2(s),\mathcal{Y}_2(s)) \Big| ds \Big] \\
& + |\mathcal{Q}_4(t)| \Big[|h| \int_0^\rho \Big(\int_0^s e^{-(s-z)} \Big| I_{0+}^q \mathfrak{f}(z,\mathcal{X}_1(z),\mathcal{Y}_1(z)) - I_{0+}^q \mathfrak{f}(z,\mathcal{X}_2(z),\mathcal{Y}_2(z)) \Big| dz \Big) dA(s) \\
& + \sum_{i=1}^{n-2} |\beta_i| \int_0^{\sigma_i} e^{-(\sigma_i-s)} \Big| I_{0+}^q \mathfrak{f}(s,\mathcal{X}_1(s),\mathcal{Y}_1(s)) - I_{0+}^q \mathfrak{f}(s,\mathcal{X}_2(s),\mathcal{Y}_2(s)) \Big| ds \\
& + |h_1| \int_\nu^1 \Big(\int_0^s e^{-(s-z)} \Big| I_{0+}^q \mathfrak{f}(z,\mathcal{X}_1(z),\mathcal{Y}_1(z)) - I_{0+}^q \mathfrak{f}(z,\mathcal{X}_2(z),\mathcal{Y}_2(z)) \Big| dz \Big) dA(s) \\
& + \int_0^1 e^{-(1-s)} \Big| I_{0+}^p \mathfrak{g}(s,\mathcal{X}_1(s),\mathcal{Y}_1(s)) - I_{0+}^p \mathfrak{g}(s,\mathcal{X}_2(s),\mathcal{Y}_2(s)) \Big| ds \Big] \Big\} \\
\le & \Lambda_1 L_1 \Big(\|\mathcal{X}_1 - \mathcal{X}_2\| + \|\mathcal{Y}_1 - \mathcal{Y}_2\| \Big) + \Lambda_2 L_2 \Big(\|\mathcal{X}_1 - \mathcal{X}_2\| + \|\mathcal{Y}_1 - \mathcal{Y}_2\| \Big) \\
= & \Big(\Lambda_1 L_1 + \Lambda_2 L_2 \Big) \Big(\|\mathcal{X}_1 - \mathcal{X}_2\| + \|\mathcal{Y}_1 - \mathcal{Y}_2\| \Big) \\
= & \Theta \Big(\|\mathcal{X}_1 - \mathcal{X}_2\| + \|\mathcal{Y}_1 - \mathcal{Y}_2\| \Big).
\end{aligned}$$

Similarly, we can find that

$$\begin{aligned}
\|T_2(\mathcal{X}_1,\mathcal{Y}_1) - T_2(\mathcal{X}_2,\mathcal{Y}_2)\| &= \sup_{t\in[0,1]} |T_2(\mathcal{X}_1,\mathcal{Y}_1)(t) - T_2(\mathcal{X}_2,\mathcal{Y}_2)(t)| \\
&\le \Big(\overline{\Lambda}_1 L_1 + \overline{\Lambda}_2 L_2 \Big) \Big(\|\mathcal{X}_1 - \mathcal{X}_2\| + \|\mathcal{Y}_1 - \mathcal{Y}_2\| \Big) \\
&= \overline{\Theta} \Big(\|\mathcal{X}_1 - \mathcal{X}_2\| + \|\mathcal{Y}_1 - \mathcal{Y}_2\| \Big).
\end{aligned}$$

Hence we obtain

$$\|T(\mathcal{X}_1,\mathcal{Y}_1) - T(\mathcal{X}_2,\mathcal{Y}_2))\| \le (\Theta + \overline{\Theta})(\|\mathcal{X}_1 - \mathcal{X}_2\| + \|\mathcal{Y}_1 - \mathcal{Y}_2\|),$$

which, in view of the condition (42), shows that T is a contraction. Thus, the conclusion of Banach's fixed-point theorem applies and, hence, the problem (1) and (2) has a unique solution on $[0,1]$. The proof is finished. □

4. An Example

Example 1. *Consider a coupled system of fractional differential equations*

$$\begin{cases} ({}^cD^{26/7} + {}^cD^{19/7})\mathcal{X}(t) = \dfrac{135\mathcal{X}(t)}{225+t} + \dfrac{3\sin\mathcal{Y}(t)}{13+t^2} + \dfrac{3}{13\sqrt{9+t^2}}, \\ ({}^cD^{17/5} + {}^cD^{12/5})\mathcal{Y}(t) = \dfrac{\sqrt{16-t^2}}{\pi(40+t)}\sin(2\pi\mathcal{X}(t)) + \dfrac{24|\tan^{-1}\mathcal{Y}(t)|}{\pi(t^2+120)} + \dfrac{\ln 5}{2}, \quad t \in [0,1], \end{cases} \tag{45}$$

equipped with the coupled boundary conditions

$$\begin{cases} \mathcal{X}(0) = 0,\ \mathcal{X}'(0) = 0,\ \mathcal{X}'(1) = 0,\ \mathcal{X}(1) = k\displaystyle\int_0^\rho \mathcal{Y}(s)dA(s) + \sum_{i=1}^{3}\alpha_i\mathcal{Y}(\sigma_i) + k_1\int_\nu^1 \mathcal{Y}(s)dA(s), \\ \mathcal{Y}(0) = 0,\ \mathcal{Y}'(0) = 0,\ \mathcal{Y}'(1) = 0,\ \mathcal{Y}(1) = h\displaystyle\int_0^\rho \mathcal{X}(s)dA(s) + \sum_{i=1}^{3}\beta_i\mathcal{X}(\sigma_i) + h_1\int_\nu^1 \mathcal{X}(s)dA(s). \end{cases} \tag{46}$$

Here $q = 19/7$, $p = 12/5$, $k = 3/16, k_1 = 2/175, h = 5/88, h_1 = 3/104$, $A(s) = 1 + \frac{s^{r+1}}{r+1}$, $r \in \mathbb{N}$, $\rho = 2/7, \nu = 6/7, \sigma_1 = 3/7, \sigma_2 = 4/7, \sigma_3 = 5/7, \alpha_1 = 1/10, \alpha_2 = 1/414$, $\alpha_3 = 3/313, \beta_1 = 1/3, \beta_2 = 1/41, \beta_3 = 7/121$. *Clearly*

$$\begin{aligned} |\mathfrak{f}(t,\mathcal{X}(t),\mathcal{Y}(t))| &\le \frac{1}{13} + \frac{3}{5}\|\mathcal{X}\| + \frac{3}{13}\|\mathcal{Y}\|, \\ |\mathfrak{g}(t,\mathcal{X}(t),\mathcal{Y}(t))| &\le \frac{\ln 5}{2} + \frac{1}{5}\|\mathcal{X}\| + \frac{1}{10}\|\mathcal{Y}\|, \end{aligned}$$

and hence $\eta_0 = 1/13, \eta_1 = 3/5, \eta_2 = 3/13$, $\zeta_0 = (\ln 5)/2, \zeta_1 = 1/5, \zeta_2 = 1/10$. *Using (36) and (37) with the given data and* $r = 2$, *we find that* $\Omega_1 \simeq 0.331501$, $\Omega_2 \simeq 0.138843$ *and* $\Omega = \max\{\Omega_1, \Omega_2\} \simeq 0.331501 < 1$. *Therefore, by Theorem 1, the problem (45) and (46) has at least one solution on* $[0,1]$.

To explain Theorem 2, we consider the following system of sequential fractional differential equations supplemented with the boundary conditions (46):

$$\begin{cases} ({}^cD^{26/7} + {}^cD^{19/7})\mathcal{X}(t) = \dfrac{3e^{-t}}{\sqrt{(t^4+25)}}\dfrac{|\mathcal{X}(t)|}{(1+|\mathcal{X}(t)|)} + \dfrac{18}{(t^2+30)}\sin(\mathcal{Y}(t)) + \dfrac{9}{2\sqrt{5+t}}, \\ ({}^cD^{17/5} + {}^cD^{12/5})\mathcal{Y}(t) = \dfrac{1}{(t+10)}\tan^{-1}\mathcal{X}(t) + \dfrac{e^{-t}}{10}\dfrac{|\mathcal{Y}(t)|^3}{(1+|\mathcal{Y}(t)|^3)} + \dfrac{\cos(t+1)}{(9+t)}, \end{cases} \tag{47}$$

$t \in [0,1]$. *It is easy to check whether* $|\mathfrak{f}(t,\mathcal{X}_1,\mathcal{Y}_1) - \mathfrak{f}(t,\mathcal{X}_2,\mathcal{Y}_2)| \le L_1(\|\mathcal{X}_1 - \mathcal{X}_2\| + \|\mathcal{Y}_1 - \mathcal{Y}_2\|)$ *with* $L_1 = 3/5$ *and* $|\mathfrak{g}(t,\mathcal{X}_1,\mathcal{Y}_1) - \mathfrak{g}(t,\mathcal{X}_2,\mathcal{Y}_2)| \le L_2(\|\mathcal{X}_1 - \mathcal{X}_2\| + \|\mathcal{Y}_1 - \mathcal{Y}_2\|)$ *with* $L_2 = 1/10$. *Additionally,* $\Theta + \overline{\Theta} \simeq 0.282351 < 1$. *Therefore, the hypothesis of Theorem 2 is satisfied. Hence, by the conclusion of Theorem 2, there is a unique solution to the system (47) equipped with the boundary conditions (46) on* $[0,1]$.

5. Discussion

We have presented the criteria ensuring the existence and uniqueness of solutions for a coupled system of higher-order sequential Caputo fractional differential equations complemented with Riemann–Stieltjes integro-multipoint boundary conditions on the interval $[0,1]$. A characteristic of the method employed in the present study is its generality, as it can be applied to a variety of boundary value problems. As a special case, our results become associated with multipoint boundary conditions:

$$\begin{cases} \mathcal{X}(0)=0,\ \mathcal{X}'(0)=0,\ \mathcal{X}'(1)=0,\ \mathcal{X}(1)=\sum_{i=1}^{n-2}\alpha_i\mathcal{Y}(\sigma_i), \\ \mathcal{Y}(0)=0,\ \mathcal{Y}'(0)=0,\ \mathcal{Y}'(1)=0,\ \mathcal{Y}(1)=\sum_{i=1}^{n-2}\beta_i\mathcal{X}(\sigma_i), \end{cases} \tag{48}$$

if we take $k=k_1=h=h_1=0$ in (2). In this case, the corresponding operators take the form:

$$\begin{aligned} \widehat{T}_1(\mathcal{X},\mathcal{Y})(t) &= \int_0^t e^{-(t-s)} I_{0+}^q \mathfrak{f}(s,\mathcal{X}(s),\mathcal{Y}(s))ds \\ &+\mathcal{Q}_1(t)\Big[\int_0^1 e^{-(1-s)} I_{0+}^q \mathfrak{f}(s,\mathcal{X}(s),\mathcal{Y}(s))ds - I_{0+}^q \mathfrak{f}(s,\mathcal{X}(s),\mathcal{Y}(s))(1)\Big] \\ &+\mathcal{Q}_2(t)\Big[\int_0^1 e^{-(1-s)} I_{0+}^p \mathfrak{g}(s,\mathcal{X}(s),\mathcal{Y}(s))ds - I_{0+}^p \mathfrak{g}(s,\mathcal{X}(s),\mathcal{Y}(s))(1)\Big] \\ &+\mathcal{Q}_3(t)\Big[\sum_{i=1}^{n-2}\alpha_i\int_0^{\sigma_i} e^{-(\sigma_i-s)} I_{0+}^p \mathfrak{g}(s,\mathcal{X}(s),\mathcal{Y}(s))ds \\ &-\int_0^1 e^{-(1-s)} I_{0+}^q \mathfrak{f}(s,\mathcal{X}(s),\mathcal{Y}(s))ds\Big] \\ &+\mathcal{Q}_4(t)\Big[\sum_{i=1}^{n-2}\beta_i\int_0^{\sigma_i} e^{-(\sigma_i-s)} I_{0+}^q \mathfrak{f}(s,\mathcal{X}(s),\mathcal{Y}(s))ds \\ &-\int_0^1 e^{-(1-s)} I_{0+}^p \mathfrak{g}(s,\mathcal{X}(s),\mathcal{Y}(s))ds\Big], \end{aligned}$$

$$\begin{aligned} \widehat{T}_2(\mathcal{X},\mathcal{Y})(t) &= \int_0^t e^{-(t-s)} I_{0+}^p \mathfrak{g}(s,\mathcal{X}(s),\mathcal{Y}(s))ds \\ &+\mathcal{P}_1(t)\Big[\int_0^1 e^{-(1-s)} I_{0+}^q \mathfrak{f}(s,\mathcal{X}(s),\mathcal{Y}(s))ds - I_{0+}^q \mathfrak{f}(s,\mathcal{X}(s),\mathcal{Y}(s))(1)\Big] \\ &+\mathcal{P}_2(t)\Big[\int_0^1 e^{-(1-s)} I_{0+}^p \mathfrak{g}(s,\mathcal{X}(s),\mathcal{Y}(s))ds - I_{0+}^p \mathfrak{g}(s,\mathcal{X}(s),\mathcal{Y}(s))(1)\Big] \\ &+\mathcal{P}_3(t)\Big[\sum_{i=1}^{n-2}\alpha_i\int_0^{\sigma_i} e^{-(\sigma_i-s)} I_{0+}^p \mathfrak{g}(s,\mathcal{X}(s),\mathcal{Y}(s))ds \\ &-\int_0^1 e^{-(1-s)} I_{0+}^q \mathfrak{f}(s,\mathcal{X}(s),\mathcal{Y}(s))ds\Big] \\ &+\mathcal{P}_4(t)\Big[\sum_{i=1}^{n-2}\beta_i\int_0^{\sigma_i} e^{-(\sigma_i-s)} I_{0+}^q \mathfrak{f}(s,\mathcal{X}(s),\mathcal{Y}(s))ds \\ &-\int_0^1 e^{-(1-s)} I_{0+}^p \mathfrak{g}(s,\mathcal{X}(s),\mathcal{Y}(s))ds\Big]. \end{aligned}$$

In future, we plan to develop the existence theory for the multivalued analogue of the problem (1) and (2). Moreover, the boundary value problem considered in this paper can be studied for other kinds of derivatives, such as Hadamard, Caputo–Hadamard, Hilfer, Hilfer–Hadamard, etc.

Author Contributions: Conceptualization, Y.A., B.A. and S.K.N.; methodology, Y.A., B.A., S.K.N. and A.S.M.A.; validation, Y.A., B.A., S.K.N. and A.S.M.A.; formal analysis, Y.A., B.A., S.K.N. and A.S.M.A.; writing—original draft preparation, Y.A., B.A., S.K.N. and A.S.M.A.; funding acquisition, A.S.M.A. All authors have read and agreed to the published version of the manuscript.

Funding: Taif University Researchers Supporting Project number (TURSP-2020/303), Taif University, Taif, Saudi Arabia.

Acknowledgments: Taif University Researchers Supporting Project number (TURSP-2020/303), Taif University, Taif, Saudi Arabia. The authors thank the reviewers for their constructive remarks on their work.

Conflicts of Interest: The authors declare no conflict of interest.

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