fractal and fractional

MDPI

Article

Fractional Order Modeling the Gemini Virus in *Capsicum annuum* with Optimal Control

Kottakkaran Sooppy Nisar [1,*], Kumararaju Logeswari [2], Veliappan Vijayaraj [2], Haci Mehmet Baskonus [3] and Chokkalingam Ravichandran [2]

[1] Department of Mathematics, College of Arts and Sciences, Prince Sattam Bin Abdulaziz University, Wadi Aldawaser 11991, Saudi Arabia
[2] Department of Mathematics, Kongunadu Arts and Science College (Autonomous), Coimbatore 641029, Tamil Nadu, India; logeswariveni@gmail.com (K.L.); vijayveliappan@gmail.com (V.V.); ravibirthday@gmail.com (C.R.)
[3] Departmant of Mathematics and Science Education, Harran University, Sanliurfa 63300, Turkey; hmbaskonus@gmail.com
* Correspondence: n.sooppy@psau.edu.sa

Abstract: In this article, a fractional model of the *Capsicum annuum* (*C. annuum*) affected by the yellow virus through whiteflies (*Bemisia tabaci*) is examined. We analyzed the model by equilibrium points, reproductive number, and local and global stability. The optimal control methods are discussed to decrease the infectious *B. tabaci* and *C. annuum* by applying the *Verticillium lecanii* (*V. lecanii*) with the Atangana–Baleanu derivative. Numerical results described the population of plants and comparison values of using *V. lecanni*. The results show that using 60% of *V. lecanni* will control the spread of the yellow virus in infected *B. tabaci* and *C. annuum* in 10 days, which helps farmers to afford the costs of cultivating chili plants.

Keywords: fractional calculus; stability analysis; Atangana–Baleanu fractional derivative; optimal control; mathematical models; yellow virus

MSC: 34A08; 26A33; 03C50; 35F21

Citation: Nisar, K.S.; Logeswari, K.; Vijayaraj, V.; Baskonus, H.M.; Ravichandran, C. Fractional Order Modeling the Gemini Virus in *Capsicum annuum* with Optimal Control. *Fractal Fract.* **2022**, *6*, 61. https://doi.org/10.3390/fractalfract6020061

Academic Editors: Riccardo Caponetto and Vassili Kolokoltsov

Received: 5 December 2021
Accepted: 17 January 2022
Published: 25 January 2022

1. Introduction

Growth of *Capsicum annuum* (chili plant) [1,2] is excessive in the mid-hill region of India. It contains vitamin C, provitamin A, and calcium, which are good for health. The spicy taste and high nutritional benefit the marketing of *C. annuum*. *Capsicum annuum* is used in the pharmaceutical industries to increase immunity, antiulcer, analgesic, antidiabetic, and antihemorrhoid agents. The extracts of *C. annuum* employ to relieve the pain of inflammation of joints, headaches, neuralgia, and burns. The framers require advantages to yield the *C. annuum* in large amounts. Due to natural obstacles like soil erosion, irrigation, and diseases spread, farmers encounter heavy losses while fertilizing *C. annuum*.

The cause of Geminivirus (yellow virus) [3–5] is one of the difficulties experienced by farmers in the cultivation of *C. annuum*. Yellow spots appear in young leaves and shoots, and the leaves turn out to be bright yellow or mixed yellow-green, which are symptoms of the yellow virus. This virus spreads by whitefly (*Bemisia tabaci*) from one host to another continuously.

Controlling these viruses using overlapping cropping methods is very difficult, since insecticide must be applied to mature plants. Controlling techniques vary depending on the conditions of the plants infected by virus variety, environment, and time. To reduce the populations of white-flies, systemic insecticides are applied to control the spread of the virus, as well as to cure the infected white-fly insect and plants, like rust fungi, etc., who have the host cyclodepsipeptide toxin. This toxin was produced via the mycelium of entomopathogenic fungi (*Verticillium lecanii*) [6–8]. However, the excess use of *V. lecanii*

generates high costs. To minimize costs of controlling the *B. tabaci* population, an optimal control method must be found.

Over the past few decades, numerous analyses, real-world problems, and numerical methods were resolved by fractional derivatives and integrals [9–16]. The applications are fluid mechanics, electrochemistry, viscoelasticity, optics, and signals processing in engineering and science. Riemann–Liouville, Caputo, Caputo–Fabrizio, and Atangana–Baleanu are some of the fractional derivatives developed by several researchers. The Atangana–Baleanu fractional derivative (AB-derivative) [17] is a new one among the Mittag–Leffler Kernel. Recently, the existence analysis [18–21], stability analysis [22,23] and models [24–27] of AB-derivative were elaborated by many authors.

Optimal control is the approach of ascertaining control and circumstances path for dynamic systems to minimize an accomplishment period. The origin of the optimal control is related to the calculus of variations. In the 1940s, the formulation of dynamic programming in the optimal control was developed by Rochard Bellman. Using the analytical method, some of the optimal control problems' solutions are difficult to find. N.H. Sweilam et al. [28] discussed the optimal control method for cancer treatment using AB-derivative. R. Amelia et al. [29] showed results to help farmers afford the costs of cultivating the red chilies by optimal control. In [30] N.H. Sweilam and S.M. AL–Mekhlafi described the fractional model of multistrain TB cure with optimal control. The optimal control problems to solve numerical procedures were investigated in [31,32].

To the best of our knowledge, the study of the *C. annuum* of the yellow virus with optimal control by applying the AB-derivative to the model is yet to come. This article was organized as follows: the basic results and definitions of the AB-derivative are discussed in Section 2. In Section 3, the formation of the *C. annuum* model with AB-derivative is explained. In Section 4, the optimality conditions demonstrate. In Sections 5 and 6, numerical results with graphs for the fractional optimal control problem have presented and conclusions.

Motivated by [22,23,28,30], this document discusses the fractional model of Geminivirus in *C. annuum* with AB-derivative via optimal control and stability analysis. The main contributions are organized as follows:

(A) The fractional model of Geminivirus in *C. annuum* with AB-derivative constructed.
(B) We obtained some stability results of this fractional model and discussed the equilibrium points and reproductive number of the model.
(C) We derived the optimal control of this fractional model and plotted the population and comparison results of each variable in the model.

2. Preliminaries

This section briefly discussed some preliminaries regarding fractional derivatives. There are few definitions for the fractional derivatives, including Riemann–Liouville, Caputo, and Caputo–Fabrizio [9,18]. Recently, a new fractional derivative with Mittag–Leffler Kernel was elaborated and implemented in a few real-world models [24–27]. We present the following definitions.

The Riemann–Liouville fractional integral (RL) is defined as follows [9,18]

$$({}_0I^{\zeta}\phi)(t) = \frac{1}{\Gamma(\zeta)}\int_0^t (t-s)^{\zeta-1}\phi(s)ds,\ \zeta > 0.$$

The Riemann–Liouville fractional order derivative (RL) is defined as follows [9,18]

$$({}^R_0D^{\zeta}\phi)(t) = \frac{d}{dt}\left(\frac{1}{\Gamma(1-\zeta)}\int_0^t (t-s)^{-\zeta}\phi(s)ds\right),\ 0 < \zeta < 1.$$

The Caputo fractional order derivative (C) is defined as follows [9,18]

$$({}_0^C D^\zeta \phi)(t) = \frac{1}{\Gamma(1-\zeta)} \int_0^t (t-s)^{-\zeta} \phi'(s) ds,\ 0 < \zeta < 1.$$

The Caputo-Fabrizio fractional order derivative in Caputo sense (CFC) is defined as follows [33]

$$({}_0^{CFC} D^\zeta \phi)(t) = \frac{M(\zeta)}{1-\zeta} \int_0^t \phi'(s) exp\left[\frac{-\zeta}{1-\zeta}(t-s)^\zeta\right] ds,\ 0 < \zeta < 1.$$

where $\phi' \in H^1(0,T)$, $M(\zeta)$ is a constant of normalization that depends on ζ, which satisfies that, $M(0) = M(1) = 1$.

The Atangana–Baleanu fractional order derivative in the Riemann–Liouville sense (ABR) is defined as follows [17]

$$({}_0^{ABR} D^\zeta \phi)(t) = \frac{B(\zeta)}{1-\zeta} \frac{d}{dt} \int_0^t \phi(s) E_\zeta\left[-\zeta \frac{(t-s)^\zeta}{1-\zeta}\right] ds,\ 0 < \zeta < 1.$$

where $\phi' \in H^1(0,T)$, $B(\zeta)$ is a normalization function, $B(0) = B(1) = 1$.

The Atangana–Baleanu fractional order derivative in the Caputo sense (ABC) is defined as follows [17]

$$({}_0^{ABC} D^\zeta \phi)(t) = \frac{B(\zeta)}{1-\zeta} \int_0^t \phi'(s) E_\zeta\left[-\zeta \frac{(t-s)^\zeta}{1-\zeta}\right] ds,\ 0 < \zeta < 1.$$

where $\phi' \in H^1(0,T)$, $B(\zeta)$ is a normalization function, $B(0) = B(1) = 1$.

The Atangana–Baleanu fractional integral of order ζ of a function $\phi(t)$ is defined as [17]

$$({}_0^{AB} I^\zeta \phi)(t) = \frac{1-\zeta}{B(\zeta)} \phi(t) + \frac{\zeta}{B(\zeta)} ({}_0 I^\zeta \phi)(t).$$

The Mittag–Leffler function of one and two parameters $E_\alpha(z), E_{\alpha,\beta}(z)$ is defined as [9]

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)} z, \alpha \in C, Re(\alpha) > 0.$$
$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} z, \alpha, \beta \in C, Re(\alpha) > 0, Re(\beta) > 0.$$

The generalized Mittag–Leffler function is defined as [9]

$$E^\gamma_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\alpha k + \beta)} \frac{z^k}{k!} z, \alpha, \beta, \gamma \in C, Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0,$$

where $\Gamma(\cdot)$ denotes the Gamma function, and note that

$$E^1_{1,1}(z) = e^z, E^1_{\alpha,1}(z) = E_\alpha(z), E^1_{\alpha,\beta}(z) = E_{\alpha,\beta}(z).$$

3. Modeling Framework of Gemini Virus

The fractional model based on the cure of yellow virus in *C. annuum* by *V. lecanii* with modified variables and parameters is presented. Here, the parameters depend on the

fractional model, which is an extension of the integer model given in [29]. The mathematical model of *C. annuum* with AB fractional derivative is represented as follows:

$$\left.\begin{aligned}
({}_0^{ABC}D_t^{\zeta})(S_v(t)) &= A - \alpha S_v - \beta_1(1-\delta_p)S_v I_{BT} - \mu_p S_v \\
({}_0^{ABC}D_t^{\zeta})(I_v(t)) &= \beta_1(1-\delta_p)S_v I_{BT} - \mu_p I_v \\
({}_0^{ABC}D_t^{\zeta})(S_g(t)) &= \alpha S_v - \beta_2(1-\delta_p)S_g I_{BT} - \mu_p S_g \\
({}_0^{ABC}D_t^{\zeta})(I_g(t)) &= \beta_2(1-\delta_p)S_g I_{BT} - \mu_p I_g \\
({}_0^{ABC}D_t^{\zeta})(S_{BT}(t)) &= BN_v - \gamma_1(1-\delta_p)I_v S_{BT} - \gamma_2(1-\delta_p)I_g S_{BT} \\
&\quad - \theta_1\delta_p S_{BT}N_p - \mu_1 S_{BT} \\
({}_0^{ABC}D_t^{\zeta})(I_{BT}(t)) &= \gamma_1(1-\delta_p)I_v S_{BT} + \gamma_2(1-\delta_p)I_g S_{BT} \\
&\quad - \theta_1\delta_p I_{BT}N_p - \mu_1 I_{BT} \\
\text{with } S_v(0) &= S_{v0}, I_v(0) = I_{v0}, \\
S_g(0) &= S_{g0}, I_g(0) = I_{g0}, \\
S_{BT}(0) &= S_{BT0}, I_{BT}(0) = I_{BT0},
\end{aligned}\right\} \quad (1)$$

where $0 < \zeta < 1$.

The total population is denoted by N_p of *C. annuum* $N_p = S_v + I_v + S_g + I_g$ is taken to be constant. The total population of *B. tabaci* is denoted by $N_v = S_{BT} + I_{BT}$. Here, the total population can be divided into 6 subgroups.

- S_v denotes a set of noninfected *C. annuum* in vegetative phase liable to possible infection.
- I_v denotes a set of infected *C. annuum* in vegetative phase.
- S_g denotes a set of noninfected *C. annuum* in generative phase liable to possible infection.
- I_g denotes a set of infected *C. annuum* in generative phase.
- S_{BT} denotes a set of noninfected *B. tabaci* (white bug) liable to possible infection.
- I_{BT} denotes a set of infected *B. tabaci*.

The recruitment rate of *C. annuum* and *B. tabaci* is denoted by A and B respectively. The growth rate of *C. annuum* from vegetative to generative phase is denoted by α. β_1, and β_2 denoted the infection rate of *C. annuum* in the vegetative and generative phase respectively. γ_1, and γ_2 denoted the infection rate of *B. tabaci* in the vegetative and generative phase respectively. δ_p stands for the rate of use of *V. lecanii*. The death rate of *C. annuum* is denoted by μ_p. The natural death rate of *B. tabaci* is denoted by μ_1, and the death rate of *B. tabaci* due to curative intervention is denoted by θ_1.

4. Basic Analysis of the Model

4.1. Invariant Region

The fractional order *C. annuum* model of yellow virus (1) can be analyzed in the biological feasible region discussed as follows. The system (1) is split into two parts, namely the *C. annuum* population (N_p; with $N_p = S_v + I_v + S_g + I_g$) and the *B. tabaci* population (N_v; with $N_v = S_{BT} + I_{BT}$).

Let the feasible region $F = F_p \cup F_v \subset R_+^4 \times R_+^2$ with

$$F_p = \left\{ (S_v, I_v, S_g, I_g) \in R_+^4 : S_v + I_v + S_g + I_g \leq \frac{A}{\mu_p} \right\},$$

$$F_v = \left\{ (S_{BT}, I_{BT}) \in R_+^2 : S_{BT} + I_{BT} \leq \frac{B}{\theta_1\delta_p N_p + \mu_1} \right\}.$$

To establish the positive invariance of $F\langle$i.e., solutions in F remain in F for all $t > 0\rangle$. Adding the first four equations and the last two equations of the model (1) gives

$$ {}_0^{ABC}D_t^{\zeta} N_p(t) = A - \mu_p(S_v + I_v + S_g + I_g) = A - \mu_p N_p \tag{2} $$

$$ {}_0^{ABC}D_t^{\zeta} N_v(t) = BN_v - (\theta_1 \delta_p N_p + \mu_1)N_v = B - (\theta_1 \delta_p N_p + \mu_1)N_v \tag{3} $$

This can be used to show that the fractional order of the *C. annuum* and *B. tabaci* population in the system (1) shows that

$$ \left.\begin{aligned} {}_0^{ABC}D_t^{\zeta} N_p(t) &\leq A - \mu_p N_p, \\ {}_0^{ABC}D_t^{\zeta} N_v(t) &\leq B - (\theta_1 \delta_p N_p + \mu_1)N_v. \end{aligned}\right\} \tag{4} $$

which implies that

$$ \begin{aligned} N_p(t) &\leq At^{\zeta} E_{\zeta,\zeta+1}(-\mu_p t^{\zeta}) - N_p(0)E_{\zeta,1}(-\mu_p t^{\zeta}), \\ N_v(t) &\leq Bt^{\zeta} E_{\zeta,\zeta+1}(-(\theta_1 \delta_p N_p + \mu_1)t^{\zeta}) + N_v(0)E_{\zeta,1}(-(\theta_1 \delta_p N_p + \mu_1)t^{\zeta}). \end{aligned} $$

From above inequality, we observe that $N_p(t) \leq \frac{A}{\mu_p}$ & $N_v(t) \leq \frac{B}{(\theta_1 \delta_p N_p + \mu_1)}$. Thus, the region F is positively-invariant.

Hence, it is sufficient to consider the dynamics model of system (1) in F. The mathematical model is well-posed in the region F.

$\therefore$ Every solution of the basic model (1) with initial conditions in F remains in F for all $t > 0$. The result is summarized below.

Lemma 1. *The region $F = F_p \cup F_v \subset R_+^4 \times R_+^2$ is positively invariant for the basic model (1) with non-negative initial conditions in R_+^6.*

4.2. Disease-Free Equilibrium Point

To evaluate the equilibrium points

Let

$$ \begin{aligned} {}_0^{ABC}D_t^{\zeta} S_v(t) = 0, {}_0^{ABC}D_t^{\zeta} S_g(t) = 0, {}_0^{ABC}D_t^{\zeta} S_{BT}(t) = 0, \\ {}_0^{ABC}D_t^{\zeta} I_v(t) = 0, {}_0^{ABC}D_t^{\zeta} I_g(t) = 0, {}_0^{ABC}D_t^{\zeta} I_{BT}(t) = 0. \end{aligned} $$

Then

$$ \mathcal{E}_0(S_v^0, I_v^0, S_g^0, I_g^0, S_{BT}^0, I_{BT}^0) = \left(\frac{A}{\mu_p}, 0, 0, 0, \frac{B}{\theta_1 \delta_p N_p + \mu_1}, 0\right). $$

4.3. Reproduction Number

For the basic reproduction number for the *C. annuum* model (1), suppose that $y = (S_v, S_g, I_{BT})$ and using next generation matrix approach [34], we have

$$ \frac{dy}{dt} = \mathcal{F}(y) - \mathcal{V}(y), $$

where Jacobian of $\mathcal{F}$ and $\mathcal{V}$ at $\mathcal{E}_0$, we have

$$ \mathcal{F} = \begin{pmatrix} \alpha + \mu_p & 0 & 0 \\ -\alpha & \mu_p & 0 \\ 0 & 0 & \mu_1 \end{pmatrix} \quad \& \; \mathcal{V} = \begin{pmatrix} \frac{\beta_1(1-\delta_p)B}{\theta_1 \delta_p N_p + \mu_1} & 0 & \frac{\beta_1(1-\delta_p)A}{\mu_p} \\ 0 & \frac{\beta_2(1-\delta_p)B}{\theta_1 \delta_p N_p + \mu_1} & \frac{\beta_2(1-\delta_p)A}{\mu_p} \\ 0 & 0 & \theta_1 \delta_p N_p \end{pmatrix}. $$

The basic reproduction number ψ_0 comes from the spectral radius $\psi_0 = \rho[\mathcal{F}\mathcal{V}^{-1}]$, given by

$$\psi_0 = \frac{(\alpha+\mu_p)(\theta_1\delta_p N_p+\mu_1)}{\beta_1(1-\delta_p)B},$$

where

$$\mathcal{F}\mathcal{V}^{-1} = \begin{pmatrix} \frac{(\alpha+\mu_p)(\theta_1\delta_p N_p+\mu_1)}{\beta_1(1-\delta_p)B} & 0 & 0 \\ 0 & \frac{\mu_p(\theta_1\delta_p N_p+\mu_1)}{\beta_2(1-\delta_p)B} & 0 \\ 0 & 0 & \frac{\mu_1}{\theta_1\delta_p N_p} \end{pmatrix}.$$

Theorem 1. *There exists a unique positive endemic equilibrium point $\mathcal{E}^*$ for system (1) if $\psi_0 > 1$.*

Proof. Endemic equilibrium point is obtained from system (1), and by putting right-hand side of each equation equal to zero, we have

$$\begin{aligned} S_v^* &= \frac{A}{a} \\ I_v^* &= \frac{A\beta_1(1-\delta_p)I_{BT}}{\mu_p a} \\ S_g^* &= \frac{A\alpha}{ab} \\ I_g^* &= \frac{A\alpha\beta_2(1-\delta_p)I_{BT}}{\mu_p ab} \\ S_{BT}^* &= \frac{BN_v\mu_p ab}{\gamma_1(1-\delta_p^2 A\beta_1)I_{BT}b+\gamma_2\beta_2\alpha A(1-\delta_p)^2 I_{BT}+(\theta_1\delta_p N_p-\mu_1)\mu_p ab}, \end{aligned}$$

where $a = \alpha+\beta_1(1-\delta_p)I_{BT}-\mu_p$ and $b = \beta_2(1-\delta_p)I_{BT}+\mu_p$ and I_{BT}^* is the positive root of $\mathcal{K}(I_{BT}) = i_5 I_{BT}^5 + i_4 I_{BT}^4 + i_3 I_{BT}^3 + i_2 I_{BT}^2 + i_1 I_{BT} = 0$,
where

$$\begin{aligned} i_1 &= (\theta_1\delta_p N_p+\mu_1)^2\mu_p^4(\alpha-\mu_p)[(\alpha-\mu_p)+2\beta_1(1-\delta_p)] \\ &\quad - ABN_v(\gamma_1\beta_1\mu_p^2+\alpha\gamma_2\beta_2\mu_p)(1-\delta_p)^2[\alpha\mu_p-\mu_p^2], \\ i_2 &= (A\alpha\gamma_1\beta_1\mu_p^2(1-\delta_p)^2 - A\gamma_1\beta_1\mu_p^3(1-\delta_p)^2 + A\alpha^2\gamma_2\beta_2\mu_p^2(1-\delta_p)^2 - A\alpha\gamma_2\beta_2\mu_p^3(1-\delta_p)^2 \\ &\quad + 2\beta_2(\theta_1\delta_p N_p+\mu_1)\mu_p^3(\alpha-\mu_p)^2(1-\delta_p))(\theta_1\delta_p N_p+\mu_1) - ABN_v\gamma_1\beta_1\beta_2\mu_p(1-\delta_p)^3[\alpha\mu_p-\mu_p^2] \\ &\quad - ABN_v(\gamma_1\beta_1\mu_p^2+\alpha\gamma_2\beta_2\mu_p)(1-\delta_p)^2[\alpha\beta_2(1-\delta_p)+\mu_p\beta_1(1-\delta_p)-\mu_p\beta_2(1-\delta_p)], \\ i_3 &= (A\alpha\gamma_1\beta_1\beta_2\mu_p(1-\delta_p)^3 + A\gamma_1\beta_1^2\mu_p(1-\delta_p)^3 - A\gamma_1\beta_1\beta_2\mu_p^2(1-\delta_p)^3 + A\alpha^2\gamma_2\beta_2^2\mu_p(1-\delta_p)^3 \\ &\quad + A\alpha\gamma_2\beta_1\beta_2\mu_p^2(1-\delta_p)^3 - A\alpha\gamma_2\beta_2^2\mu_p^2(1-\delta_p)^3 \\ &\quad + (\theta_1\delta_p N_p+\mu_1)\mu_p^2\Big[\beta_2^2(\alpha-\mu_p)^2(1-\delta_p)^2+4\beta_1\beta_2\mu_p(1-\delta_p)^2(\alpha-\mu_p)+\beta_1^2\mu_p^2(1-\delta_p)^2\Big]) \\ &\quad \times(\theta_1\delta_p N_p+\mu_1) - ABN_v\gamma_1\beta_1\beta_2\mu_p(1-\delta_p)^3[\alpha\beta_2(1-\delta_p)+\mu_p\beta_1(1-\delta_p)-\mu_p\beta_2(1-\delta_p)] \\ &\quad - ABN_v\gamma_1\beta_1^2\beta_2\mu_p^2(1-\delta_p)^4 - ABN_v\alpha\gamma_2\beta_1\beta_2^2\mu_p(1-\delta_p)^4, \\ i_4 &= (A\gamma_1\beta_1^2\beta_2\mu_p(1-\delta_p)^4 + A\alpha\gamma_2\beta_1\beta_2^2\mu_p(1-\delta_p)^4 \\ &\quad + (\theta_1\delta_p N_p+\mu_1)\mu_p^2[2\beta_1(1-\delta_p)^3\beta_2^2(\alpha-\mu_p)+2\mu_p\beta_1^2\beta_2(1-\delta_p)^3])(\theta_1\delta_p N_p+\mu_1) \\ &\quad - ABN_v\gamma_1\beta_1^2\beta_2\mu_p(1-\delta_p)^5, \\ i_5 &= \beta_1^2\beta_2^2(1-\delta_p)^4(\theta_1\delta_p N_p+\mu_1)^2\mu_p^2. \end{aligned}$$

It is obvious from the values of $S_v^*, I_v^*, S_g^*, I_g^*, S_{BT}^*$ & I_{BT}^* that there exists a unique positive endemic equilibrium point $\mathcal{E}^*$, if $\psi(0) > 1$. □

Theorem 2. *The system* (1) *is locally stable at* $\mathcal{E}_0$ *for* $\psi_0 < 1$ *and unstable for* $\psi_0 > 1$.

Proof. The Jacobian of system (1) is

$$J = \begin{pmatrix} Q_1 & 0 & 0 & 0 & 0 & \frac{-\beta_1(1-\delta_p)A}{\mu_p} \\ \frac{\beta_1(1-\delta_p)B}{\theta_1\delta_p N_p+\mu_1} & -\mu_p & 0 & 0 & 0 & \frac{\beta_1(1-\delta_p)A}{\mu_p} \\ \alpha & 0 & Q_2 & 0 & 0 & \frac{-\beta_2(1-\delta_p)A}{\mu_p} \\ 0 & 0 & \frac{\beta_2(1-\delta_p)B}{\theta_1\delta_p N_p+\mu_1} & -\mu_p & 0 & \frac{\beta_2(1-\delta_p)A}{\mu_p} \\ 0 & \frac{-\gamma_1(1-\delta_p)B}{\theta_1\delta_p N_p+\mu_1} & 0 & \frac{-\gamma_2(1-\delta_p)B}{\theta_1\delta_p N_p+\mu_1} & Q_3 & 0 \\ 0 & \frac{\gamma_1(1-\delta_p)B}{\theta_1\delta_p N_p+\mu_1} & 0 & \frac{\gamma_2(1-\delta_p)B}{\theta_1\delta_p N_p+\mu_1} & Q_4 & Q_5 \end{pmatrix}$$

where

$$\begin{aligned} Q_1 &= -[\alpha + \mu_p + \beta_1(1-\delta_p)I_{BT}], \\ Q_2 &= -[\mu_p + \beta_2(1-\delta_p)I_{BT}], \\ Q_3 &= -[\gamma_1(1-\delta_p)I_v + \gamma_2(1-\delta_p)I_g + \theta_1\delta_p N_p + \mu_1], \\ Q_4 &= \gamma_1(1-\delta_p)I_v + \gamma_2(1-\delta_p)I_g, \\ Q_5 &= -[\theta_1\delta_p N_p + \mu_1]. \end{aligned}$$

Along $\mathcal{E}_0$, it implies that

$$J(\mathcal{E}_0) = \begin{pmatrix} -(\alpha+\mu_p) & 0 & 0 & 0 & 0 & 0 \\ 0 & -\mu_p & 0 & 0 & 0 & 0 \\ \alpha & 0 & -\mu_p & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mu_p & 0 & 0 \\ 0 & 0 & 0 & 0 & -(\theta_1\delta_p N_p+\mu_1) & 0 \\ 0 & 0 & 0 & 0 & 0 & -(\theta_1\delta_p N_p+\mu_1) \end{pmatrix}$$

which follows that all the eigenvalues are negative if $\psi_0 < 1$ and eigenvalues are positive for $\psi_0 > 1$. Hence, we conclude that the system (1) is locally stable under the condition $\psi_0 < 1$ and unstable for $\psi_0 > 1$. □

Theorem 3. *The system* (1) *is globally stable, if* $\psi_0 > 1$ *at* $\mathcal{E}_0$.

Proof. First, we construct the Lyapunov function $\mathcal{L}(t)$, for the system as:

$$\mathcal{L}(t) = 1 + I_{BT}(t) - \frac{lnI_{BT}(t)}{I_{BT}(0)}. \tag{5}$$

Then, differentiating the Equation (5) with respect to time, we have

$$\begin{aligned} \frac{d}{dt}(\mathcal{L}(t)) &= \frac{dI_{BT}(t)}{dt} - \frac{1}{I_{BT}(t)}\frac{dI_{BT}(t)}{dt} \\ &= \left(1 - \frac{1}{I_{BT}}\right)\frac{dI_{BT}}{dt} \\ &= \frac{dI_{BT}}{dt} - \theta_1\delta_p N_p - \mu_1. \end{aligned}$$

By manipulating along the point $\mathcal{E}_0$, we get

$$\begin{aligned}\frac{d}{dt}(\mathcal{L}(t)) &= -(\theta_1\delta_p N_p + \mu_1)\\ &\leq 0 \text{ for } \psi_0 > 1.\end{aligned}$$

Again differentiating the above equation, we have

$$\begin{aligned}\frac{d^2}{dt}(\mathcal{L}(t)) &= \frac{d^2 I_{BT}(t)}{dt} - \frac{dN_p}{dt}\\ &= \frac{d^2 I_{BT}(t)}{dt} - 4\mu_p.\end{aligned}$$

By manipulating along the point $\mathcal{E}_0$, we get

$$\begin{aligned}\frac{d^2}{dt}(\mathcal{L}(t)) &= -4\mu_p\\ &\leq 0 \text{ for } \psi_0 > 1.\end{aligned}$$

Therefore, if $\psi_0 > 1$, then $\frac{d}{dt}(\mathcal{L}(t)) < 0$, which implies that the system (1) is globally stable for $\psi_0 > 1$ at $\mathcal{E}_0$. □

Remark 1. *In the case of $\psi_0 < 1$ at $\mathcal{E}^*$, it is an interesting problem to find an effective strategy to prevent the disease.*

5. Optimal Control

The purpose of the dynamic red chili model is to minimize the population of plants infected during vegetative or generative period and insects infected by optimizing *V. lecanni* using AB-derivative [28–30].

The objective functions used are as follows:

$$J(u) = \int_0^{T_f} (A_1 I_v(t) + A_2 I_g(t) + A_3 I_{BT} + A_4 {u_1}^2(t))dt, \tag{6}$$

where u_1 is the late of giving *V. lecanni* and $A_i \geq 0$, for $i = 1, 2, \ldots, 4$ is the cost coefficient and t_f is end time in $[0, T_f]$.

Therefore, by using u_1 *V. lecanni* in the Equation (1), it becomes

$$\left.\begin{aligned}
({}_0^{ABC}D_t^{\zeta})(S_v(t)) &= A - \alpha S_v - \beta_1(1-u_1)S_v I_{BT} - \mu_p S_v\\
({}_0^{ABC}D_t^{\zeta})(I_v(t)) &= \beta_1(1-u_1)S_v I_{BT} - \mu_p I_v\\
({}_0^{ABC}D_t^{\zeta})(S_g(t)) &= \alpha S_v - \beta_2(1-u_1)S_g I_{BT} - \mu_p S_g\\
({}_0^{ABC}D_t^{\zeta})(I_g(t)) &= \beta_2(1-u_1)S_g I_{BT} - \mu_p I_g\\
({}_0^{ABC}D_t^{\zeta})(S_{BT}(t)) &= BN_v - \gamma_1(1-u_1)I_v S_{BT} - \gamma_2(1-u_1)I_g S_{BT}\\
&\quad - \theta_1 u_1 S_{BT} N_p - \mu_1 S_{BT}\\
({}_0^{ABC}D_t^{\zeta})(I_{BT}(t)) &= \gamma_1(1-u_1)I_v S_{BT} + \gamma_2(1-u_1)I_g S_{BT}\\
&\quad - \theta_1 u_1 I_{BT} N_p - \mu_1 I_{BT}\\
\text{with } S_v(0) &= S_{v0}, I_v(0) = I_{v0},\\
S_g(0) &= S_{g0}, I_g(0) = I_{g0},\\
S_{BT}(0) &= S_{BT0}, I_{BT}(0) = I_{BT0}.
\end{aligned}\right\} \tag{7}$$

Now, to minimize the objective functional:

$$J(u) = \int_0^{T_f} \eta(S_v, I_v, S_g, I_g, S_{BT}, I_{BT}, u_1, t)dt, \tag{8}$$

subject to the constraints

$$\begin{aligned}
{}_0^{ABC}D_t^{\zeta} S_v(t) &= \xi_1,\\
{}_0^{ABC}D_t^{\zeta} I_v(t) &= \xi_2,\\
{}_0^{ABC}D_t^{\zeta} S_g(t) &= \xi_3,\\
{}_0^{ABC}D_t^{\zeta} I_g(t) &= \xi_4,\\
{}_0^{ABC}D_t^{\zeta} S_{BT}(t) &= \xi_5,\\
{}_0^{ABC}D_t^{\zeta} I_{BT}(t) &= \xi_6,
\end{aligned}$$

where, $\xi_i = \xi(S_v, I_v, S_g, I_g, S_{BT}, I_{BT}, u_1, t), i = 1, 2, \ldots, 6$, with initial conditions:

$$\begin{aligned}
S_v(0) = S_{v(0)}, S_g(0) = S_g(0), S_{BT}(0) = S_{BT}(0),\\
I_v(0) = I_{v(0)}, I_g(0) = I_g(0), I_{BT}(0) = I_{BT}(0).
\end{aligned}$$

The modified equation of (8) is [30]

$$\begin{aligned}
\tilde{J} = \int_0^{T_f} [&H_a(S_v, I_v, S_g, I_g, S_{BT}, I_{BT}, u_1, t)\\
&- \sum_{i=1}^{6} \lambda_i \xi_i(S_v, I_v, S_g, I_g, S_{BT}, I_{BT}, u_1, t)]dt,
\end{aligned} \tag{9}$$

where the Hamiltonian is:

$$\begin{aligned}
H_a(S_v, I_v, S_g, I_g, S_{BT}, I_{BT}, u_1, \lambda_i, t) = &\eta(S_v, I_v, S_g, I_g, S_{BT}, I_{BT}, u_1, t)\\
&+ \sum_{i=1}^{6} \lambda_i \xi_i(S_v, I_v, S_g, I_g, S_{BT}, I_{BT}, u_1, t).
\end{aligned} \tag{10}$$

from (9) and (10) the necessary conditions for FOCPs [35–38] are,

$$\left.\begin{aligned}
{}_0^{ABC}D_{t_f}^{\zeta} \lambda_1 &= \frac{\partial H_a}{\partial S_v},\\
{}_0^{ABC}D_{t_f}^{\zeta} \lambda_2 &= \frac{\partial H_a}{\partial I_v},\\
{}_0^{ABC}D_{t_f}^{\zeta} \lambda_3 &= \frac{\partial H_a}{\partial S_g},\\
{}_0^{ABC}D_{t_f}^{\zeta} \lambda_4 &= \frac{\partial H_a}{\partial I_g},\\
{}_0^{ABC}D_{t_f}^{\zeta} \lambda_5 &= \frac{\partial H_a}{\partial S_{BT}},\\
{}_0^{ABC}D_{t_f}^{\zeta} \lambda_6 &= \frac{\partial H_a}{\partial I_{BT}},
\end{aligned}\right\} \tag{11}$$

$$0 = \frac{\partial H}{\partial u}, \tag{12}$$

$$\left.\begin{aligned} {}_0^{ABC}D_t^{\zeta} S_v &= \frac{\partial H_a}{\partial \lambda_1}, \\ {}_0^{ABC}D_t^{\zeta} I_v &= \frac{\partial H_a}{\partial \lambda_2}, \\ {}_0^{ABC}D_t^{\zeta} S_g &= \frac{\partial H_a}{\partial \lambda_3}, \\ {}_0^{ABC}D_t^{\zeta} I_g &= \frac{\partial H_a}{\partial \lambda_4}, \\ {}_0^{ABC}D_t^{\zeta} S_{BT} &= \frac{\partial H_a}{\partial \lambda_5}, \\ {}_0^{ABC}D_t^{\zeta} I_{BT} &= \frac{\partial H_a}{\partial \lambda_6}, \end{aligned}\right\} \tag{13}$$

$$\lambda_j(T_f) = 0, \tag{14}$$

where $\lambda_j = 1, 2, 3, \ldots, 6$ are the Lagrange Multiplies.

Theorem 4. *If u_1 be the optimal controls with corresponding stats $S_v^*, I_v^*, S_g^*, I_g^*, S_{BT}^*$, and I_{BT}^*, then $\exists\ \lambda_j^*, j = 1, 2, \ldots, 6$, satisfies the following.*

(i) *Adjoint equations:*

$$\left.\begin{aligned} {}_0^{ABC}D_{t_f}^{\zeta}\lambda_1 &= \frac{\partial H_a}{\partial S_v} \\ &= \lambda_1(-\alpha - \beta_1(1-u_1)I_{BT} - \mu_p) + \lambda_2\beta_1(1-u_1)I_{BT} \\ &\quad + \lambda_3\alpha - \lambda_5\theta_1 u_1 S_{BT} - \lambda_6\theta_1 u_1 I_{BT}, \\ {}_0^{ABC}D_{t_f}^{\zeta}\lambda_2 &= \frac{\partial H_a}{\partial I_v} \\ &= A_1 - \lambda_2\mu_p - \lambda_5(\gamma_1(1-u_1)S_{BT} + \theta_1 u_1 S_{BT}), \\ {}_0^{ABC}D_{t_f}^{\zeta}\lambda_3 &= \frac{\partial H_a}{\partial S_g} \\ &= -\lambda_3\beta_2(1-u_1)I_{BT} - \lambda_3\mu_p + \lambda_4\beta_2(1-u_1)I_{BT} \\ &\quad - \lambda_5\theta_1 u_1 S_{BT} - \lambda_6\theta_1 u_1 I_{BT}, \\ {}_0^{ABC}D_{t_f}^{\zeta}\lambda_4 &= \frac{\partial H_a}{\partial I_g} \\ &= A_2 - \lambda_4\mu_p + \lambda_5[BN_v - \gamma_2(1-u_1)S_{BT} - \theta_1 u_1 S_{BT}] \\ &\quad + \lambda_6[\gamma_2(1-u_1)S_{BT} - \theta_1 u_1 I_{BT}], \\ {}_0^{ABC}D_{t_f}^{\zeta}\lambda_5 &= \frac{\partial H_a}{\partial S_{BT}} \\ &= \lambda_5[B - \gamma_1(1-u_1)I_v - \gamma_2(1-u_1)I_g - \theta_1 u_1 N_p - \mu_1 \\ &\quad + \lambda_6[\gamma_1(1-u_1)I_v + \gamma_2(1-u_1)I_g], \\ {}_0^{ABC}D_{t_f}^{\zeta}\lambda_6 &= \frac{\partial H_a}{\partial I_{BT}} \\ &= A_3 - \lambda_1\beta_1(1-u_1)S_v - \lambda_2\beta_1(1-u_1)S_v \\ &\quad - \lambda_3\beta_2(1-u_1)S_g + \lambda_4\beta_2(1-u_1)S_g - \lambda_6(\theta_1 u_1 N_p - \mu_1). \end{aligned}\right\} \tag{15}$$

(*ii*) *Transversality conditions:*

$$\lambda_j^*(T_f) = 0, j = 1, 2, \ldots, 6. \tag{16}$$

(*iii*) *Optimality conditions:*

$$H_a(S_v^*, I_v^*, S_g^*, I_g^*, S_{BT}^*, I_{BT}^*, u, \lambda_*) = \min_{0 \le u \le 1} H(S_v^*, I_v^*, S_g^*, I_g^*, S_{BT}^*, I_{BT}^*, u, \lambda^*). \tag{17}$$

Furthermore, the control functions u_1 are given by,

$$\begin{aligned} u_1^* = \max\{\min[\frac{1}{2A_4}(&\lambda_1^*\beta_1 S_v I_{BT} - \lambda_2^*\beta_1 S_v I_{BT} + \lambda_3^*\beta_2 S_g I_{BT} \\ &- \lambda_4^*\beta_2 S_g I_{BT} + \lambda_5^*\gamma_1 I_v S_{BT} + \lambda_5^*\gamma_2 I_g S_{BT} - \lambda_5^*\theta_1 S_{BT} N_p \\ &- \lambda_6^*\gamma_1 I_v S_{BT} - \lambda_6^*\gamma_2 I_g S_{BT} - \lambda_6^*\theta_1 I_{BT} N_p), 1], 0\} \end{aligned} \tag{18}$$

Proof. We can state that (7) using the conditions (11), where H_a^* is,

$$\begin{aligned} H_a^* = A_1 + A_2 + A_3 + A_4 u_1^2 &+ \lambda_{10}^{*ABC} D_t^\zeta S_v^* + \lambda_{20}^{*ABC} D_t^\zeta I_v^* \\ &+ \lambda_{30}^{*ABC} D_t^\zeta S_g^* + \lambda_{40}^{*ABC} D_t^\zeta I_g^* \\ &+ \lambda_{50}^{*ABC} D_t^\zeta S_{BT}^* + \lambda_{60}^{*ABC} D_t^\zeta I_{BT}^*. \end{aligned}$$

Moreover, $\lambda_j^*(T_f) = 0, j = 1, \ldots, 6$ holds.

The optimal control Equation (18) are proved by minimizing the condition (17). Substitute u_1^* in (7) we get,

$$\begin{aligned} {}_0^{ABC}D^\zeta S_v^* &= A - \alpha S_v^* - \beta_1(1-u_1^*)S_v^* I_{BT}^* - \mu_p S_v^* \\ {}_0^{ABC}D^\zeta I_v^* &= \beta_1(1-u_1^*)S_v^* I_{BT}^* - \mu_p I_v^* \\ {}_0^{ABC}D^\zeta S_g^* &= \alpha S_v^* - \beta_2(1-u_1^*)S_g^* I_{BT}^* - \mu_p S_g^* \\ {}_0^{ABC}D^\zeta I_g^* &= \beta_2(1-u_1^*)S_g^* I_{BT}^* - \mu_p I_g^* \\ {}_0^{ABC}D^\zeta S_{BT}^* &= B^* N_v^* - \gamma_1(1-u_1^*)I_v^* S_{BT}^* - \gamma_2(1-u_1^*)I_g^* S_{BT}^* - \theta_1 u_1^* S_{BT}^* N_p^* - \mu_1 S_{BT}^* \\ {}_0^{ABC}D^\zeta I_{BT}^* &= \gamma_1(1-u_1^*)I_v^* S_{BT}^* - \gamma_2(1-u_1^*)I_g^* S_{BT}^* - \theta_1 u_1^* I_{BT}^* N_p^* - \mu_1 I_{BT}^*. \end{aligned}$$

□

6. Numerical Results

Here, we examine the mathematical model of *C. annuum* with AB fractional derivative and optimal control numerically. We assume initial conditions and parameter values in Table 1 with $\zeta = 0.9$ and $N(\zeta) = 1$.

The optimal control is

$$\begin{aligned} u_1^* = \max\{\min[\frac{1}{2A}(&\lambda_1^*\beta_1 S_v I_{BT} - \lambda_2^*\beta_1 S_v I_{BT} + \lambda_3^*\beta_2 S_g I_{BT} - \lambda_4^*\beta_2 S_g I_{BT} \\ &+ \lambda_5^*\gamma_1 I_v S_{BT} + \lambda_5^*\gamma_2 I_g S_{BT} - \lambda_5^*\theta_1 S_{BT} N_p - \lambda_6^*\gamma_1 I_v S_{BT} - \lambda_6^*\gamma_2 I_g S_{BT} - \lambda_6^*\theta_1 I_{BT} N_p), 1], 0\} \end{aligned}$$

since $0 \le u_1 \le 1$. Consider $u_1 = 0.6$ and using the parametric values in (7) then,

$$
\begin{aligned}
S_{nv}(t) &= 50 + \left(0.1 + \frac{(t)^{0.9}}{\Gamma(0.9)}\right)[10 - 0.07S_{(n-1)v}(t) - 0.001 \times 0.4 \times S_{(n-1)v}(t)I_{(n-1)BT}(t) \\
&\quad - 0.03S_{(n-1)v}(t)] \\
I_v(t) &= 10 + \left(0.1 + \frac{(t)^{0.9}}{\Gamma(0.9)}\right)[0.001 \times 0.4 \times S_{(n-1)v}(t)I_{(n-1)BT}(t) - 0.03I_{(n-1)v}(t)] \\
S_g(t) &= 30 + \left(0.1 + \frac{(t)^{0.9}}{\Gamma(0.9)}\right)[0.07S_{(n-1)v}(t) - 0.001 \times 0.4 \times S_{(n-1)g}(t)I_{(n-1)BT}(t) \\
&\quad - 0.03S_{(n-1)g}(t)] \\
I_g(t) &= 10 + \left(0.1 + \frac{(t)^{0.9}}{\Gamma(0.9)}\right)[0.001 \times 0.4 \times S_{(n-1)g}(t)I_{(n-1)BT}(t) - 0.03I_{(n-1)g}(t)] \\
S_{BT}(t) &= 30 + \left(0.1 + \frac{(t)^{0.9}}{\Gamma(0.9)}\right)[10 \times 40 - 0.025 \times 0.4 \times I_{(n-1)v}(t)S_{(n-1)BT}(t) - 0.02 \\
&\quad \times 0.4 \times I_{(n-1)g}(t)S_{(n-1)BT}(t) - 0.05 \times 0.6 \times S_{(n-1)BT}(t) \times 80 - 0.07S_{(n-1)BT}(t)] \\
I_{BT}(t) &= 10 + \left(0.1 + \frac{(t)^{0.9}}{\Gamma(0.9)}\right)[0.025 \times 0.4 \times I_{(n-1)v}(t)S_{(n-1)BT}(t) + 0.02 \times 0.4 \\
&\quad \times I_{(n-1)g}(t)S_{(n-1)BT}(t) - 0.05 \times 0.6 \times I_{(n-1)BT}(t) \times 80 - 0.07I_{(n-1)BT}(t)]
\end{aligned}
$$

which gives the numerical values plotted in Figures 1–6.

In Figures 1–6, we show that the use of 60% *V. lecanni* for 5 days changed the population of susceptible and infected plants in vegetative and generative phases, as well as the variation of population of susceptible and infected white bugs.

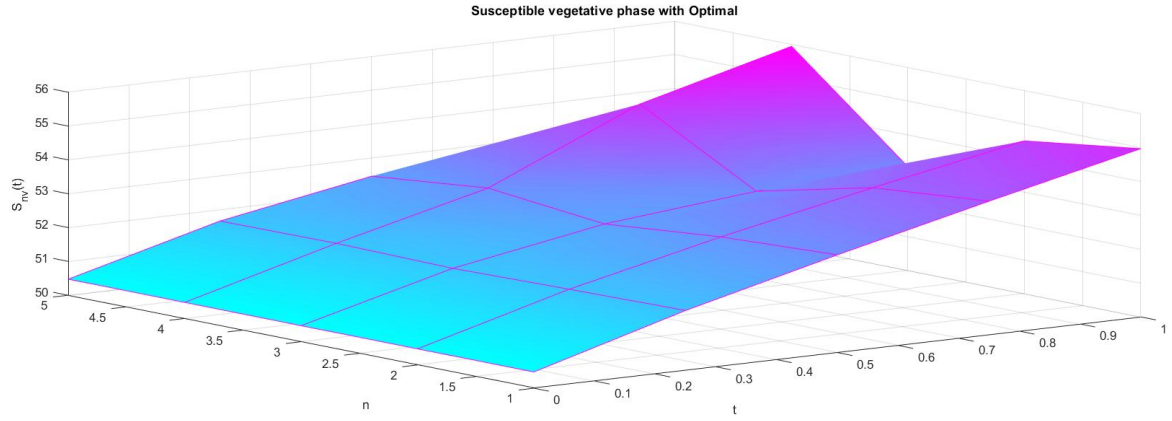

Figure 1. Optimal control of susceptible vegetative phase $S_v(t)$ of *C. annuum*.

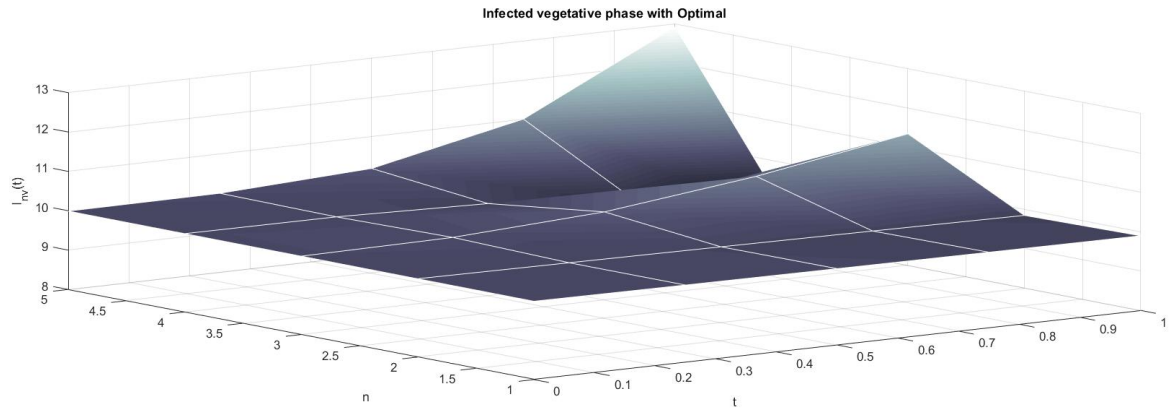

Figure 2. Optimal control of infected vegetative phase $I_v(t)$ of *C. annuum*.

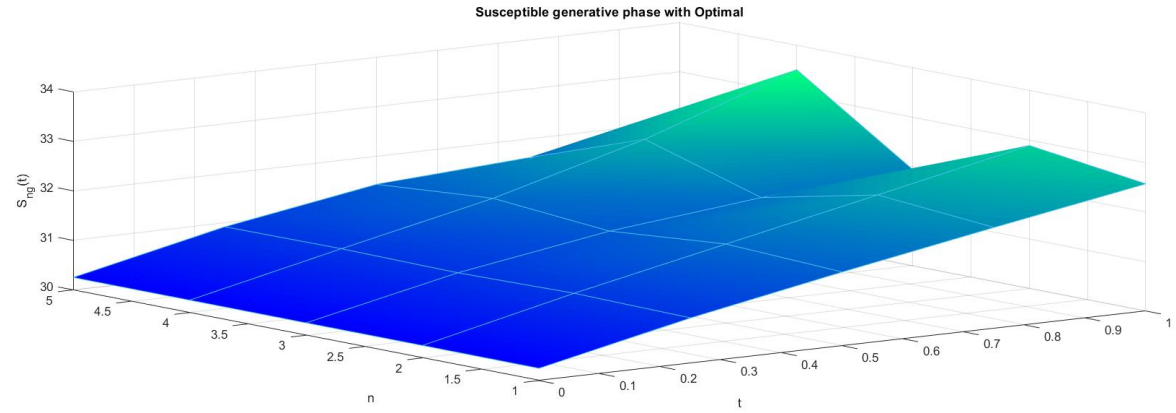

Figure 3. Optimal control of susceptible generative phase $S_g(t)$ of *C. annuum*.

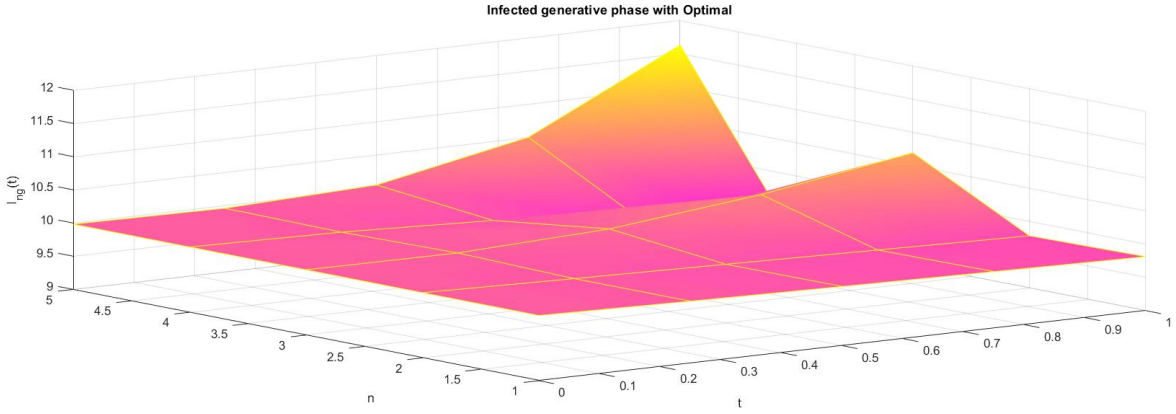

Figure 4. Optimal control of infected generative phase $I_v(g)$ of *C. annuum*.

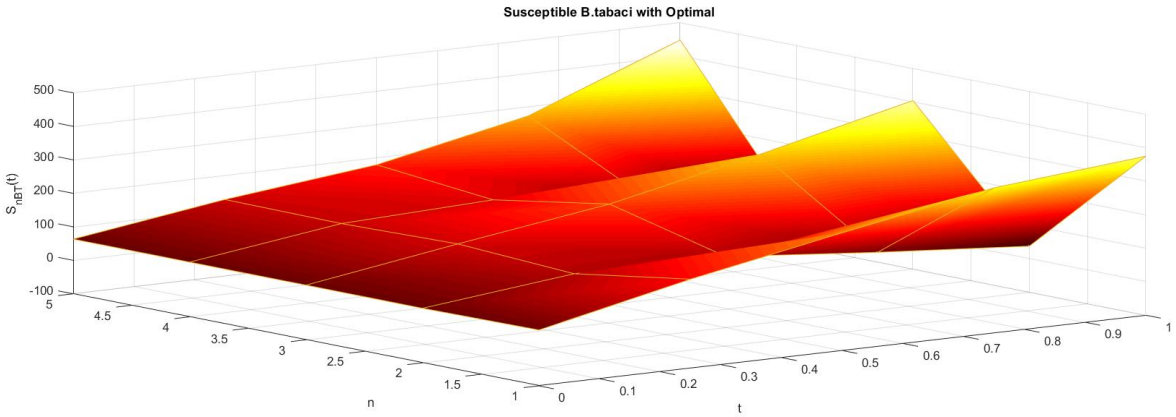

Figure 5. Optimal control of susceptible *B. tabaci* $S_{BT}(t)$ in *C. annuum*.

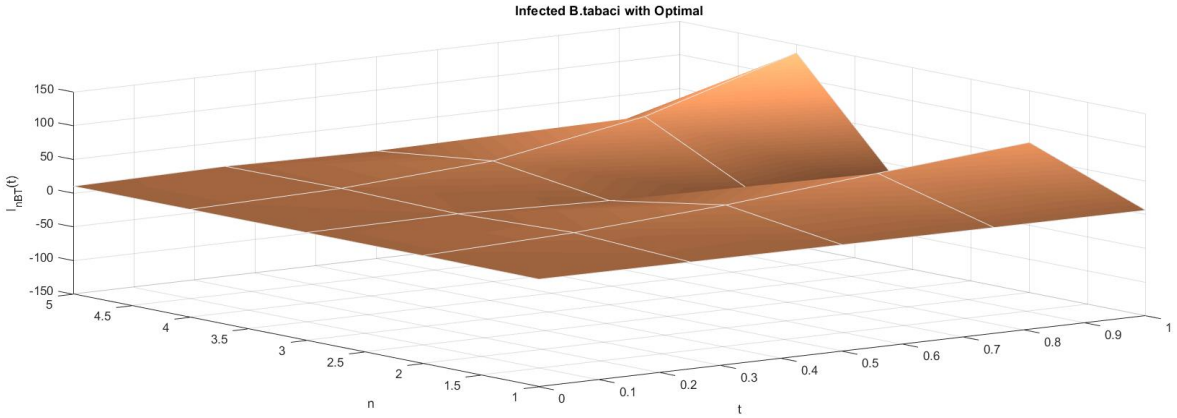

Figure 6. Optimal control of infected *B. tabaci* $I_{BT}(t)$ in *C. annuum*.

Table 1. Parametric representation with approximation values of (1) in [29].

Variable with Values	Definition
$N_p = 80$	*C. annuum* population
$N_v = 40$	*B. tabaci* population
$S_v = 50$	Vegetative phase of Susceptible *C. annuum*
$I_v = 10$	Vegetative phase of Infected *C. annuum*
$S_g = 30$	Generative phase of Susceptible *C. annuum*
$I_g = 10$	Generative phase of Infected *C. annuum*
$S_{BT} = 30$	*B. tabaci* Susceptible insect
$I_{BT} = 10$	*B. tabaci* Infected insect
$A = 10$	Recruitment of *C. annuum*
$B = 10$	Recruitment of *B. tabaci*
$\alpha = 0.07$	Rate of growth from vegetative to generative phase
$\beta_1 = 0.001$	Rate of infected *C. annuum* in the vegetaive phase
$\beta_2 = 0.001$	Rate of infected *C. annuum* in the generative phase
$\gamma_1 = 0.025$	Rate of *B. tabaci* infection in the vegetaive phase
$\gamma_2 = 0.02$	Rate of *B. tabaci* infection in the generative phase
$\delta_p = 0.2$	*V. lecanii*
$\mu_p = 0.03$	The death rate of *C. annuum*
$\mu_1 = 0.07$	Rate of natural death in *B. tabaci*
$\theta_1 = 0.05$	The death rate of *B. tabaci* due to curative intervention

When $u_1 = 0$, i.e., without *V. lecanni*, the control system reduce to the model given below:

$$
\begin{aligned}
{}_0^{ABC}D_t^{\zeta} S_v(t) &= A - \alpha S_v - \beta_1 S_v I_{BT} - \mu_p S_v \\
{}_0^{ABC}D_t^{\zeta} I_v(t) &= \beta_1 S_v I_{BT} - \mu_p I_v \\
{}_0^{ABC}D_t^{\zeta} S_g(t) &= \alpha S_v - \beta_2 S_g I_{BT} - \mu_p S_g \\
{}_0^{ABC}D_t^{\zeta} I_g(t) &= \beta_2 S_g I_{BT} - \mu_p I_g \\
{}_0^{ABC}D_t^{\zeta} S_{BT}(t) &= BN_v - \gamma_1 I_v S_{BT} - \gamma_2 I_g S_{BT} - \mu_1 S_{BT} \\
{}_0^{ABC}D_t^{\zeta} I_{BT}(t) &= \gamma_1 I_v S_{BT} + \gamma_2 I_g S_{BT} - \mu_1 I_{BT}.
\end{aligned}
$$

In Figures 7–12, we compare the numerical values of the variables $S_v(t)$, $I_v(t)$, $S_g(t)$, $I_g(t)$, $S_{BT}(t)$ and $I_{BT}(t)$ with and without *V. lecanii* u_1. The comparison of these variables of ordinary differential equations is shown in [29] of the figures are Figures 2–7. The results with *V. lecanni* in each stage of plants and insects are more effective and accurate in Atangana–Baleanu fractional derivative than ordinary differential equations stated in [29].

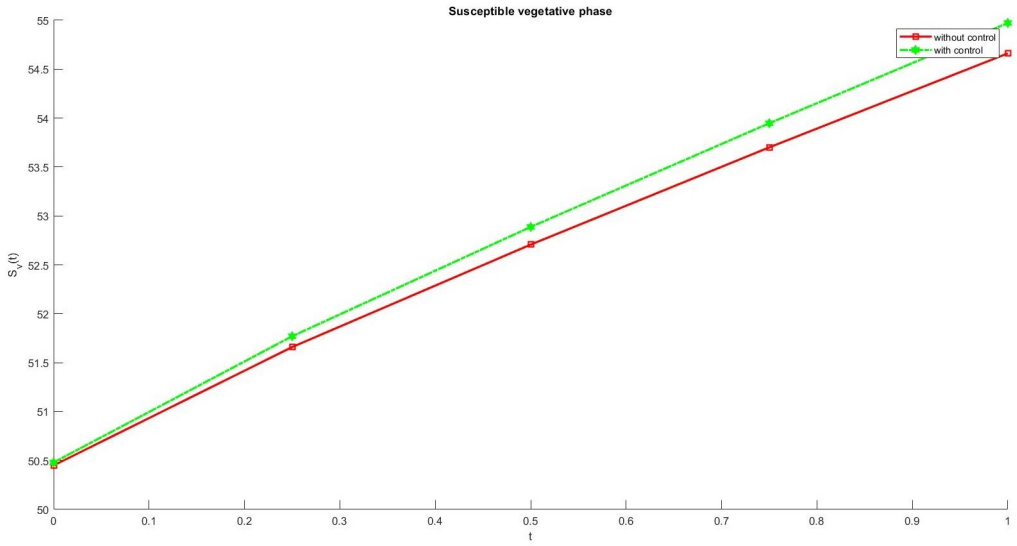

Figure 7. Comparison of susceptible vegetative phase $S_v(t)$ in *C. annuum* with and without *V. lecanni*.

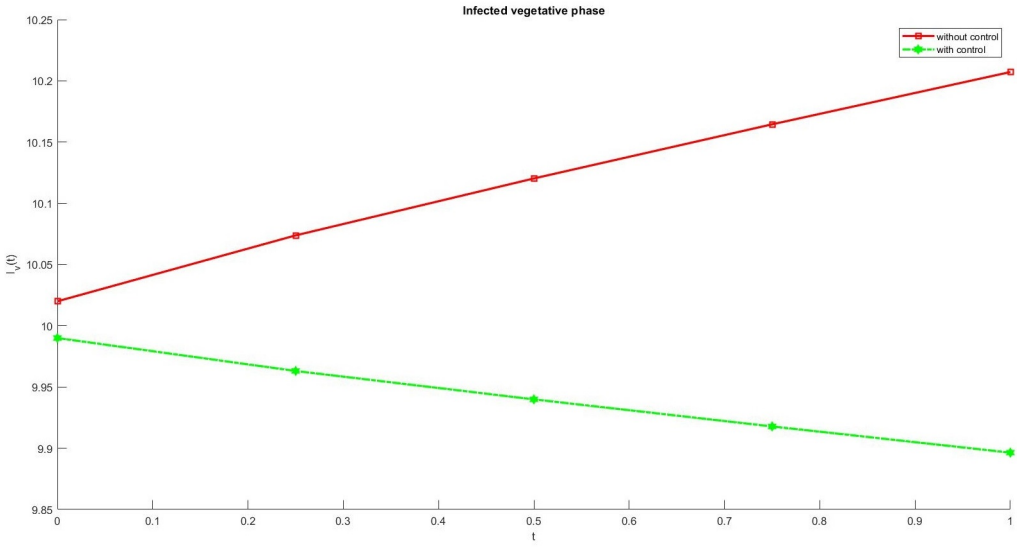

Figure 8. Comparison of infected vegetative phase $I_v(t)$ in *C. annuum* with and without *V. lecanni*.

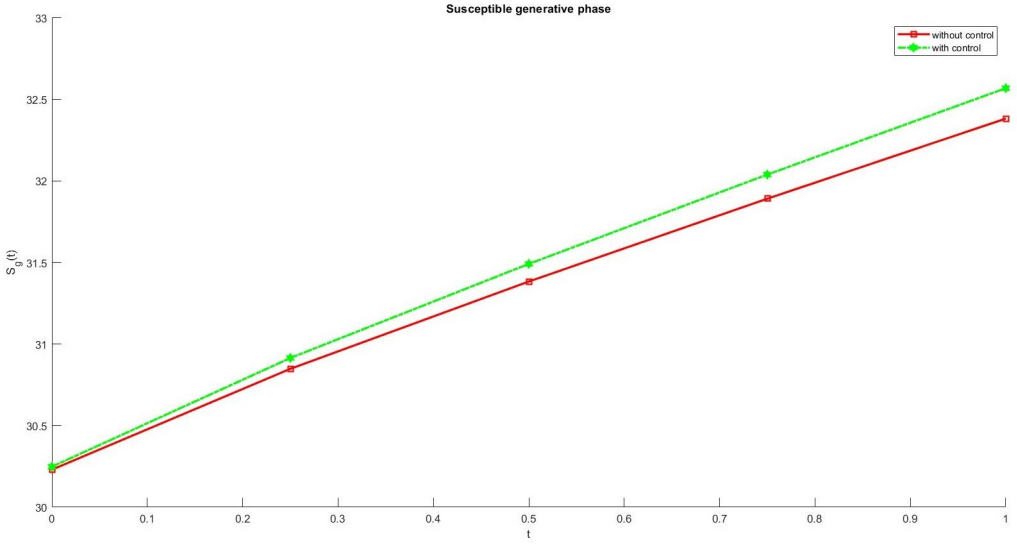

Figure 9. Comparison of susceptible generative phase $S_g(t)$ in *C. annuum* with and without *V. lecanni*.

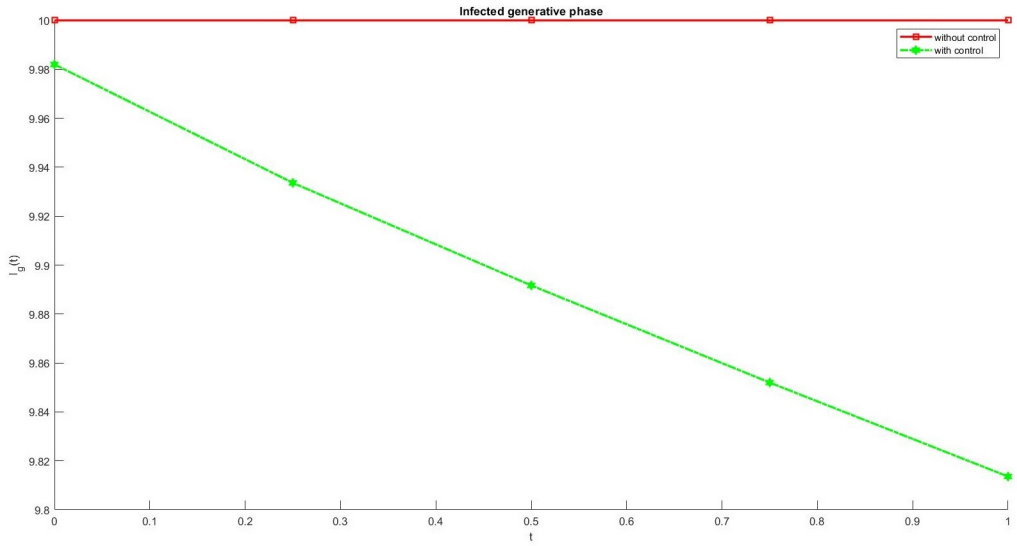

Figure 10. Comparison of infected generative phase $I_v(g)$ in *C. annuum* with and without *V. lecanni*.

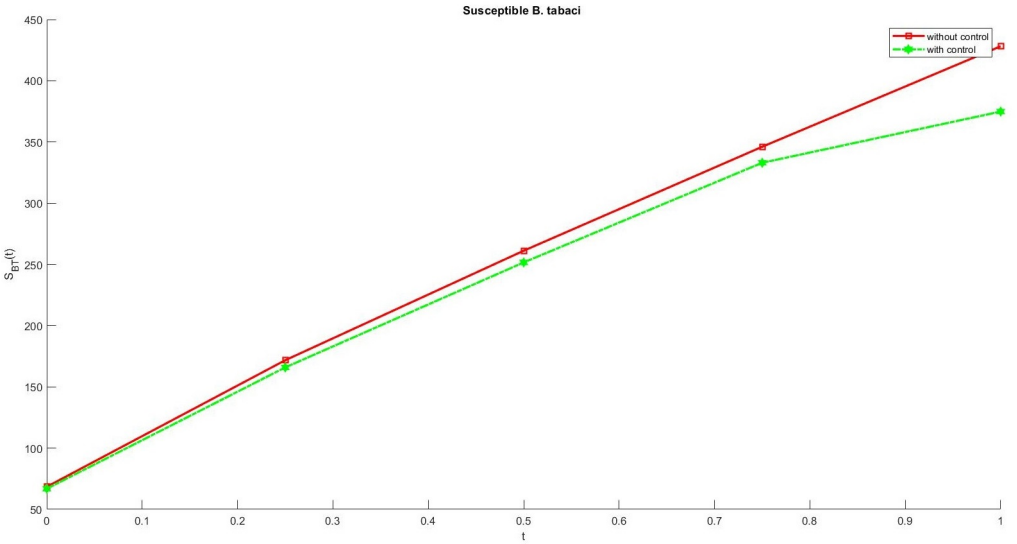

Figure 11. Comparison of susceptible *B. tabaci* $S_{BT}(t)$ in *C. annuum* with and without *V. lecanni*.

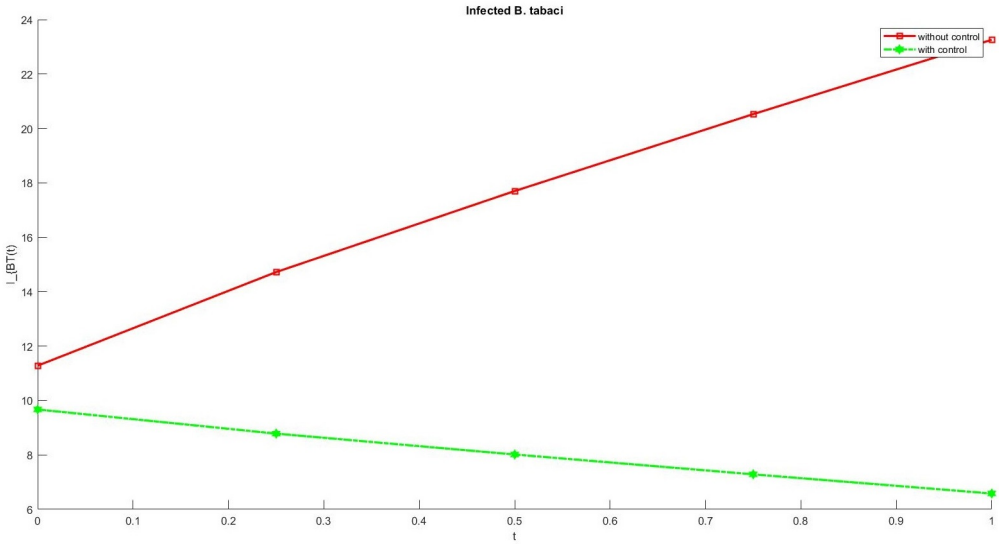

Figure 12. Comparison of infected *B. tabaci* $I_{BT}(t)$ in *C. annuum* with and without *V. lecanni*.

Figures 7 and 9 show that the susceptible *C. annuum* in vegetative and generative phases, which increased the population by 1% with *V. lecanni* compared to that of without *V. lecanni* because the infected *B. tabaci* cannot transmit the virus through chili plants. Figures 8 and 10 show the comparison of infected *C. annuum* in vegetative and generative phases, which decrease the population by 1% with *V. lecanni* compared to that of without *V. lecanni*.

In Figure 11, the comparison of susceptible *B. tabaci* population decreases with *V. lecanni* by 50% compared to that of without *V. lecanni* because the infected *B. tabaci* cannot infect the healthy one with an antidote. In Figure 11, the comparison of infected *B. tabaci* population decreases with *V. lecanni* by 4% compared to that of without *V. lecanni* because the infected ones were either cured or dead due to curative intervention.

In Figure 13, the measure of implementing 60% of *V. lecanni* per day will reduce 1% of infected *C. annuum* and 1% of infected *B. tabaci*. By continuing this process, the 60% of *V. lecanni* control the spread of the yellow virus within 10 days, which helps the farmers to afford the costs of cultivating the *C. annuum*.

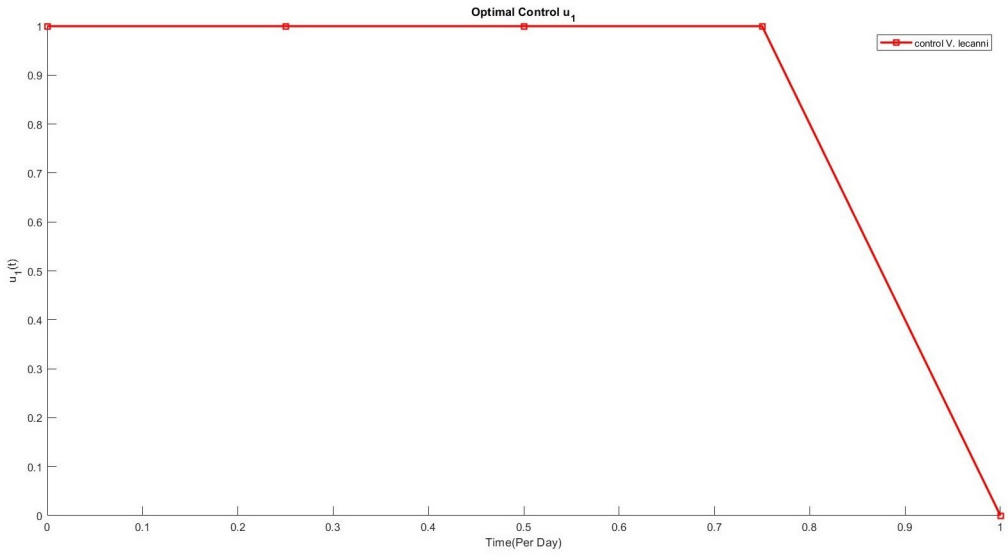

Figure 13. Control of *V. lecanni*.

7. Conclusions

In this study, we described the *C. annuum* model of the yellow virus in two discrete aspects. First, we examined the *C. annuum* model and applied the optimal control. Second, we analyzed the *C. annuum* model using the Atangana–Baleanu derivative. The threshold quantity is less than one when the presented model is locally stable. Furthermore, the model is globally stable when $\psi_0 > 1$. With the help of V. lecanii (an entomopathogenic fungus), $u_1(t)$ optimal control reduced the population of infected *B. tabaci* and *C. annuum*. The numerical results of optimal conditions of the *C. annuum* model with AB-derivative are described detailly by successive approximation method. The infected population of *C. annuum* increases and decreases according to V. lecanii use and vice-versa for susceptible. The results show that using 60% of *V. lecanni* controls the spread of the yellow virus in infected *B. tabaci* and *C. annuum* over 10 days, which helps farmers to afford the costs of cultivation.

Author Contributions: Conceptualization, K.S.N., K.L. and C.R.; methodology, K.S.N.; software, V.V., H.M.B.; validation, K.S.N., C.R.; formal analysis, K.L.; investigation, K.S.N., C.R.; resources, K.L.; data curation, K.L.; writing—original draft preparation, K.S.N., K.L. and C.R.; writing—review and editing, V.V., H.M.B.; visualization, H.M.B.; supervision, C.R.; project administration, K.S.N.; funding acquisition, K.S.N. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: Authors sincerely thank the reviewers and editors for providing constructive comments for improvement of the manuscript. The second author, third author and fifth author wishes to thank the M. Aruchami and C. A. Vasuki, Kongunadu Arts and Science College (Autonomous), Coimbatore, Tamil Nadu, India, for their constant encouragement and support for this research work.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Mariyono, J. Impacts seed technology improvement on economic aspects of chilli production in central java-Indonesia. *J. Ekon. Pembang.* **2016**, *17*, 1–14. [CrossRef]
2. Khan, F.A.; Mahmood, T.; Ali, M.; Saeed, A.; Maalik, A. Pharmacological importance of an ethnobotanical plant: *Capsicum annuum* L. *Nat. Prod. Res.* **2014**, *28*, 1267–1274. [CrossRef] [PubMed]
3. Subban, R.; Sundaram, K. Effect of Antiviral Formulations on Chilli Leaf Curl Virus (CLCV) Disease of Chilli Plant (*Capsicum annuum* L.). *J. Pharm. Res.* **2012**, *5*, 5363–5366.
4. Solahudin, M.; Pramudya, B.; Manaf, R. Gemini Virus Attack Analysis in Field of Chili (*Capsicum annuum* L.) Using Aerial Photography and Bayesian Segmentation Method. *Procedia Environ. Sci.* **2015**, *24*, 254–257. [CrossRef]
5. Ganefianti, D.W.; Hidayat, H.S.; Syukur, M. Susceptible Phase of Chili Pepper Due to Yellow Leaf Curl Begomovirus Infection. *Int. J. Adv. Sci. Eng. Inf. Technol.* **2017**, *7*, 594–601. [CrossRef]
6. Bouhous, M.; Larous, L. Efficiency of the entomopathogenic fungus Verticillium lecanii in the biological control of Trialeurodes vaporariorum, (Homoptera: Aleyrodidae), a greenhouse culture pest. *Afr. J. Microbiol. Res.* **2012**, *6*, 2435–2442.
7. Alavo, T.B.C. The insect pathogenic fungus Verticillium lecanii (Zimm.) Viegas and its use for pests control: A review. *J. Exp. Biol. Agric. Sci.* **2015**, *3*, 337–345.
8. Sopp, P.I.; Gillespie, A.T.; Palmer, A. Application of Verticillium lecanii for the control of Aphis gossypii by a low-volume electrostatic rotary atomiser and a high-volume hydraulic sprayer. *Entomophaga* **1989**, *34*, 417–428. [CrossRef]
9. Podlubny, I. *Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications*; Academic Press: San Diego, CA, USA, 1999.
10. Ross, B. A brief history and exposition of the fundamental theory of fractional calculus. *Fract. Calc. Appl.* **2006**, *457*, 1–36.
11. Caputo, M.; Fabrizio, M. On the notion of fractional derivative and applications to the hysteresis phenomena. *Meccanica* **2017**, *52*, 3043–3052. [CrossRef]
12. Atangana, A.; Hammouch, Z. Fractional calculus with power law: The cradle of our ancestors. *Eur. Phys. J. Plus* **2019**, *134*, 429. [CrossRef]

13. Arqub, O.A.; Al-Smadi, M. Numerical solutions of Riesz fractional diffusion and advection-dispersion equations in porous media using iterative reproducing kernel algorithm. *J. Porous Media* **2020**, *23*, 783–804. [CrossRef]
14. Arqub, O.A.; Shawagfeh, N. Application of reproducing kernel algorithm for solving Dirichlet time-fractional diffusion-Gordon types equations in porous media. *J. Porous Media* **2019**, *22*, 411–434. [CrossRef]
15. Arqub, O.A. Application of residual power series method for the solution of time-fractional Schrodinger equations in one-dimensional space. *Fundam. Informaticae* **2019**, *166*, 87–110. [CrossRef]
16. Djennadi, S.; Shawagfeh, N.; Arqub, O.A. A fractional Tikhonov regularization method for an inverse backward and source problems in the time-space fractional diffusion equations. *Chaos Solitons Fractals* **2021**, *150*, 111127. [CrossRef]
17. Atangana, A.; Baleanu, D. New fractional derivatives with non-local and non-singular kernel: Theory and application to heat transfer model. *Therm. Sci.* **2016**, *20*, 763–769. [CrossRef]
18. Jarad, F.; Abdeljawad, T.; Hammouch, Z. On a class of ordinary differential equations in the frame of Atangana–Baleanu derivative. *Chaos Solitons Fractals* **2018**, *117*, 16–20. [CrossRef]
19. Ravichandran, C.; Logeswari, K.; Jarad, F. New results on existence in the framework of Atangana–Baleanu derivative for fractional integro-differential equations. *Chaos Solitons Fractals* **2019**, *125*, 194–200. [CrossRef]
20. Logeswari, K.; Ravichandran, C. A new exploration on existence of fractional neutral integro-differential equations in the concept of Atangana–Baleanu derivative. *Physica A* **2020**, *544*, 123454. [CrossRef]
21. Ravichandran, C.; Logeswari, K.; Panda, S.K.; Nisar, K.S. On new approach of fractional derivative by Mittag–Leffler kernel to neutral integro-differential systems with impulsive conditions. *Chaos Solitons Fractals* **2020**, *139*, 110012. [CrossRef]
22. Bonyah, E.; Gómez-Aguilar, J.F.; Adu, A. Stability analysis and optimal control of a fractional human African trypanosomiasis model. *Chaos Solitons Fractals* **2018**, *117*, 150–160. [CrossRef]
23. Alzahrani, E.O.; Khan, M.A. Modeling the dynamics of Hepatitis E with optimal control. *Chaos Solitons Fractals* **2018**, *116*, 287–301. [CrossRef]
24. Logeswari, K.; Ravichandran, C.; Nisar, K.S. Mathematical model for spreading of COVID-19 virus with the Mittag–Leffler kernel. *Numer. Methods Partial. Differ. Equ.* **2020**, 1–16. [CrossRef] [PubMed]
25. Tassaddiq, A.; Khan, I.; Nisar, K.S. Heat transfer analysis in sodium alginate based nanofluid using MoS2 nanoparticles: Atangana–Baleanu fractional model. *Chaos Solitons Fractals* **2020**, *130*, 109445. [CrossRef]
26. Korpinar, Z.; Inc, M.; Bayram, M. Theory and application for the system of fractional Burger equations with Mittag leffler kernel. *Appl. Math. Comput.* **2020**, *367*, 124781. [CrossRef]
27. Panda, S.K.; Ravichandran, C.; Hazarika, B. Results on system of Atangana–Baleanu fractional order Willis aneurysm and nonlinear singularly perturbed boundary value problems. *Chaos Solitons Fractals* **2021**, *142*, 110390. [CrossRef]
28. Sweilam, N.H.; Al-Mekhlafi, S.M.; Assiri, T.; Atangana, A. Optimal control for cancer treatment mathematical model using Atangana–Baleanu-Caputo fractional derivative. *Adv. Differ. Equ.* **2020**, *2020*, 334. [CrossRef]
29. Amelia, R.; Anggriani, N.; Supriatna, A.K. Optimal Control Model of Verticillium lecanii Application in the Spread of Yellow Red Chili Virus. *WSEAS Trans. Math.* **2019**, *18*, 351–358.
30. Sweilam, N.H.; L-Mekhlafi, S.M.A. On the optimal control for fractional multi-strain TB model. *Optim. Control Appl. Methods* **2016**, *37*, 1355–1374. [CrossRef]
31. Trigeassou, J.-C.; Maamri, N. Optimal State Control of Fractional Order Differential Systems: The Infinite State Approach. *Fractal Fract.* **2021**, *5*, 29. [CrossRef]
32. Wang, F.; Li, X.; Zhou, Z. Spectral Galerkin Approximation of Space Fractional Optimal Control Problem with Integral State Constraint. *Fractal Fract.* **2021**, *5*, 102. [CrossRef]
33. Caputo, M.; Fabrizio, M. A new Definition of Fractional Derivative without Singular Kernel. *Prog. Fract. Differ. Appl.* **2015**, *1*, 73–85.
34. Brauer, F.; den Driessche, P.V.; Wu, J. *Mathematical Epidemiology*; Springer: Berlin/Heidelberg, Germany, 2008.
35. Agrawal, O.P. On a general formulation for the numerical solution of optimal control problems. *Int. J. Control* **1989**, *50*, 627–638. [CrossRef]
36. Agrawal, O.P. Formulation of Euler-Lagrange equations for fractional variational problems. *J. Math. Anal. Appl.* **2002**, *272*, 368–379. [CrossRef]
37. Agrawal, O.P. A formulation and numerical scheme for fractional optimal control problems. *IFAC Proc. Vol.* **2006**, *39*, 68–72. [CrossRef]
38. Agrawal, O.P.; Defterli, O.; Baleanu, D. Fractional optimal control problems with several state and control variables. *J. Vib. Control* **2010**, *16*, 1967–1976. [CrossRef]