fractal and fractional

Article

Some Generalized Fractional Integral Inequalities for Convex Functions with Applications

Dafang Zhao [1,2], Muhammad Aamir Ali [3,*], Chanon Promsakon [4] and Thanin Sitthiwirattham [5,*]

1 School of Mathematics and Statistics, Hubei Normal University, Huangshi 435002, China; dafangzhao@163.com
2 Northwest Institute of Eco-Environment and Resources, Chinese Academy of Sciences, Lanzhou 730000, China
3 Jiangsu Key Laboratory for NSLSCS, School of Mathematical Sciences, Nanjing Normal University, Nanjing 210023, China
4 Department of Mathematics, Faculty of Applied Science, King Mongkut's University of Technology North Bangkok, Bangkok 10800, Thailand; chanon.p@sci.kmutnb.ac.th
5 Mathematics Department, Faculty of Science and Technology, Suan Dusit University, Bangkok 10300, Thailand
* Correspondence: mahr.muhammad.aamir@gmail.com (M.A.A.); thanin_sit@dusit.ac.th (T.S.)

Abstract: In this paper, we establish a generalized fractional integrals identity involving some parameters and differentiable functions. Then, we use the newly established identity and prove different generalized fractional integrals inequalities like midpoint inequalities, trapezoidal inequalities and Simpson's inequalities for differentiable convex functions. Finally, we give some applications of newly established inequalities in the context of quadrature formulas.

Keywords: midpoint inequalities; trapezoid inequalities; Simpson's inequalities; fractional calculus; convex functions

check for updates

Citation: Zhao, D.; Ali, M.A.; Promsakon, C.; Sitthiwirattham, T. Some Generalized Fractional Integral Inequalities for Convex Functions with Applications. *Fractal Fract.* **2022**, *6*, 94. https://doi.org/10.3390/fractalfract6020094

Academic Editor: Mark Edelman

Received: 22 January 2022
Accepted: 3 February 2022
Published: 9 February 2022

1. Introduction

Fractional calculus (that is, calculus of integrals and derivatives of any arbitrary real or complex order) has grown in popularity and relevance over the last three decades, owing to its demonstrated applications in a wide range of seemingly disparate domains of science and engineering. It does, in fact, give a number of potentially valuable tools for solving differential and integral equations, as well as a variety of other problems involving mathematical physics special functions, as well as their extensions and generalizations in one or more variables. Fluid Flow, Rheology, Dynamical Processes in Self-Similar and Porous Structures, Diffusive Transport Akin to Diffusion, Electrical Networks, Probability and Statistics, and Control Theory of Dynamical Systems are some of the current areas of application of fractional calculus.

Because of the importance of fractional calculus, researchers have utilized it to establish various fractional integral inequalities that have been shown to be quite useful in approximation theory. Inequalities such as Hermite–Hadamard, Simpson's, midpoint, Ostrowski's and trapezoidal inequalities are examples, and by using these inequalities, we can obtain the bounds of formulas used in numerical integration. In [1], Sarikaya et al. proved some Hermite–Hadamard type inequalities and trapezoidal type inequalities for the first time using the Riemann–Liouville fractional integrals. Set [2] proved a Riemann–Liouville fractional version of the Ostrowski's inequalities for differentiable functions. İşcan and Wu used harmonic convexity and proved Hermite–Hadamard type inequalities in [3]. Sarikaya and Yildrim [4] used Riemann–Liouville fractional integrals to prove some new Hermite–Hadamard type inequalities and midpoint type inequalities for differentiable convex functions. In [5], the authors used Riemann–Liouville frcational integrals and proved some Simpson's type inequalities for general convex functions. Mubeen and Habibullah [6] introduced the notions of general Riemann–Liouville fractional integrals, called *k*-fractional integrals and the authors used these integrals to prove some new Hermite–Hadamard type inequalities in [7]. Using the k-fractional integrals, Farid et al. proved Ostrowski's type inequalities for differentiable functions in [8]. In [9], Zhang et al. used k-fractional integrals and proved different integral inequalities for general convex functions. Recently, Sarikaya and Ertugral [10] defined a new class of fractional integrals, called generalized fractional, and they used these integrals to prove a general version of Hermite–Hadamard type inequalities for convex functions. In [11], Zhao et al. used generalized fractional integrals and proved some trapezoidal type inequalities for harmonic convex functions. Budak et al. [12] proved several variants of Ostrowski's and Simpson's type for differentiable convex functions via generalized fractional integrals. For more inequalities via fractional integrals, one can consult [13–24] and references therein.

Inspired by the ongoing studies, we establish some new parameterized integral inequalities for differentiable convex functions via the generalized fractional integrals. The fundamental benefit of these inequalities can be turned into midpoint type inequalities, trapezoidal type inequalities and Simpson's type inequalities via classical integrals, Riemann–Liouville fractional integrals and k-fractional integrals for convex functions without having to prove each one separately.

This paper is summarized as follows: Section 2 provides a brief overview of the fundamentals of fractional calculus as well as other related studies in this field. In Section 3, we establish an integral identity that plays a major role in establishing the main outcomes of this paper. Some new inequalities of midpoint type, trapezoidal type and Simpson's type for differentiable convex functions via different fractional operators are presented in Section 4. Some applications to quadrature formulas are discussed in Section 5 and establish several refinements of Hermite–Hadamard inequality. Section 6 concludes with some suggestions for future research.

2. Fractional Integrals and Related Inequalities

In this section, we recall some basic notations and notions of the fractional integrals. We also recall some inequalities via different fractional integrals.

Definition 1. *[25,26] Let $\Upsilon \in L_1[\eta_1, \eta_2]$. The Riemann–Liouville fractional integrals (RLFIs) $J^{\alpha}_{\eta_1+}\Upsilon$ and $J^{\alpha}_{\eta_2-}\Upsilon$ of order $\alpha > 0$ with $\eta_1 \geq 0$ are defined as follows:*

$$J^{\alpha}_{\eta_1+}\Upsilon(\varkappa) = \frac{1}{\Gamma(\eta_1)} \int_{\eta_1}^{\varkappa} (\varkappa - t)^{\alpha-1}\Upsilon(t)dt, \ \varkappa > \eta_1$$

and

$$J^{\alpha}_{\eta_2-}\Upsilon(\varkappa) = \frac{1}{\Gamma(\alpha)} \int_{\varkappa}^{\eta_2} (t - \varkappa)^{\alpha-1}\Upsilon(t)dt, \varkappa < \eta_2,$$

respectively, where Γ is the well-known Gamma function.

Definition 2. *[6] Let $\Upsilon \in L_1[\eta_1, \eta_2]$. The k-fractional integrals (KFIs) $\mathcal{J}^{\alpha,k}_{\eta_1+}\Upsilon$ and $\mathcal{J}^{\alpha,k}_{\eta_2-}\Upsilon$ of order $\alpha, k > 0$ with $\eta_1 \geq 0$ are defined as follows:*

$$\mathcal{J}^{\alpha,k}_{\eta_1+}\Upsilon(\varkappa) = \frac{1}{k\Gamma_k(\eta_1)} \int_{\eta_1}^{\varkappa} (\varkappa - t)^{\frac{\alpha}{k}-1}\Upsilon(t)dt, \ \varkappa > \eta_1$$

and

$$\mathcal{J}^{\alpha,k}_{\eta_2-}\Upsilon(\varkappa) = \frac{1}{k\Gamma_k(\alpha)} \int_{\varkappa}^{\eta_2} (t - \varkappa)^{\frac{\alpha}{k}-1}\Upsilon(t)dt, \varkappa < \eta_2,$$

respectively, where Γ_k is the well-known k-Gamma function.

Definition 3. *[10] Let $\Upsilon \in L_1[\eta_1, \eta_2]$. The generalized fractional integrals (GFIs) ${}_{\eta_1+}I_{\varphi}\Upsilon$ and ${}_{\eta_2-}I_{\varphi}\Upsilon$ with $\eta_1 \geq 0$ are defined as follows:*

$$ {}_{\eta_1+}I_{\varphi}\Upsilon(\varkappa) = \int_{\eta_1}^{\varkappa} \frac{\varphi(\varkappa - t)}{\varkappa - t}\Upsilon(t)dt,\ \varkappa > \eta_1 $$

and

$$ {}_{\eta_2-}I_{\varphi}\Upsilon(\varkappa) = \int_{\varkappa}^{\eta_2} \frac{\varphi(t - \varkappa)}{t - \varkappa}\Upsilon(t)dt,\ \varkappa < \eta_2, $$

respectively, where $\varphi : [0, \infty) \to [0, \infty)$ is a function. For more properties of the the functions φ, one can consult [10].

Remark 1. *The importance of the GFIs is that these can be turned into classical Riemann integrals, RLFIs and KFIs for $\varphi(t) = t$, $\varphi(t) = \frac{t^{\alpha}}{\Gamma(\alpha)}$ and $\varphi(t) = \frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$, respectively.*

Theorem 1. *Let $\Upsilon : I \to \mathbb{R}$ be a convex function on I with $\eta_1, \eta_2 \in I$ such that $\eta_1 < \eta_2$. If $f \in L_1[\eta_1, \eta_2]$, the the following inequality holds:*

$$ \Upsilon\left(\frac{\eta_1 + \eta_2}{2}\right) \leq \frac{1}{2\Lambda(1)}\left[{}_{\eta_1+}I_{\varphi}\Upsilon(\eta_2) +\ {}_{\eta_2-}I_{\varphi}\Upsilon(\eta_1)\right] \leq \frac{\Upsilon(\eta_1) + \Upsilon(\eta_2)}{2}, $$

where $\Lambda(1) = \int_0^1 \frac{\varphi((\eta_2 - \eta_1)t)}{t}dt$.

Remark 2. *In Theorem 1, we have*

(i) If we set $\varphi(t) = t$, then we have the following classical Hermite–Hadamard inequality (see, [27], p. 137):

$$ \Upsilon\left(\frac{\eta_1 + \eta_2}{2}\right) \leq \frac{1}{\eta_2 - \eta_1}\int_{\eta_1}^{\eta_2} \Upsilon(\varkappa)d\varkappa \leq \frac{\Upsilon(\eta_1) + \Upsilon(\eta_2)}{2}. $$

(ii) If we set $\varphi(t) = \frac{t^{\alpha}}{\Gamma(\alpha)}$, then we have the following RLFIs Hermite–Hadamard inequality (see, [1]):

$$ \Upsilon\left(\frac{\eta_1 + \eta_2}{2}\right) \leq \frac{\Gamma(\alpha + 1)}{2(\eta_2 - \eta_1)^{\alpha}}\left[J^{\alpha}_{\eta_1+}\Upsilon(\eta_2) + J^{\alpha}_{\eta_2-}\Upsilon(\eta_1)\right] \leq \frac{\Upsilon(\eta_1) + \Upsilon(\eta_2)}{2}. $$

(iii) If we set $\varphi(t) = \frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$, then we have the following KFIs Hermite–Hadamard inequality (see, [28]):

$$ \Upsilon\left(\frac{\eta_1 + \eta_2}{2}\right) \leq \frac{\Gamma_k(\alpha + k)}{2(\eta_2 - \eta_1)^{\frac{\alpha}{k}}}\left[\mathcal{J}^{\alpha,k}_{\eta_1+}\Upsilon(\eta_2) + \mathcal{J}^{\alpha,k}_{\eta_2-}\Upsilon(\eta_1)\right] \leq \frac{\Upsilon(\eta_1) + \Upsilon(\eta_2)}{2}. $$

3. Key Equalities

In this section, we prove a generalized fractional integral identity involving some parameters and derivative of a function.

Lemma 1. *Let $\Upsilon : I \to \mathbb{R}$ be a differentiable function on I° with $\Upsilon \in L_1[\eta_1, \eta_2]$, then the following GFIs identity holds for $\lambda, \mu \in \mathbb{R}$:*

$$\begin{aligned} & \frac{\lambda\Upsilon(\eta_1) + \mu\Upsilon(\eta_2)}{2} + \frac{2-\lambda-\mu}{2}\Upsilon\left(\frac{\eta_1+\eta_2}{2}\right) \\ & -\frac{1}{2\Delta(1)}\left[{}_{\eta_1+}I_\varphi\Upsilon\left(\frac{\eta_1+\eta_2}{2}\right) + {}_{\left(\frac{\eta_1+\eta_2}{2}\right)+}I_\varphi\Upsilon(\eta_2)\right] \\ = & \frac{(\eta_2-\eta_1)}{4\Delta(1)}\left[\int_0^1(\Delta(1)-\lambda\Delta(1)-\Delta(t))\Upsilon'\left(t\eta_1+(1-t)\frac{\eta_1+\eta_2}{2}\right)dt\right. \\ & \left.+\int_0^1(\mu\Delta(1)-\Delta(t))\Upsilon'\left(t\frac{\eta_1+\eta_2}{2}+(1-t)\eta_2\right)dt\right], \end{aligned} \tag{1}$$

where

$$\Delta(\varkappa) = \int_0^\varkappa \frac{\varphi\left(\left(\frac{\eta_2-\eta_1}{2}\right)u\right)}{u}du.$$

Proof. Using integration by parts and change of variables, we have

$$\begin{aligned} H_1 = & \int_0^1(\Delta(1)-\lambda\Delta(1)-\Delta(t))\Upsilon'\left(t\eta_1+(1-t)\frac{\eta_1+\eta_2}{2}\right)dt \\ = & -\frac{2}{(\eta_2-\eta_1)}\left[(\Delta(1)-\lambda\Delta(1)-\Delta(t))\Upsilon\left(t\eta_1+(1-t)\frac{\eta_1+\eta_2}{2}\right)\Big|_0^1\right] \\ & -\frac{2}{(\eta_2-\eta_1)}\int_0^1\frac{\varphi\left(\left(\frac{\eta_2-\eta_1}{2}\right)t\right)}{t}\Upsilon\left(t\eta_1+(1-t)\frac{\eta_1+\eta_2}{2}\right)dt \\ = & \frac{2}{(\eta_2-\eta_1)}\left[\lambda\Delta(1)\Upsilon(\eta_1)+(1-\lambda)\Delta(1)\Upsilon\left(\frac{\eta_1+\eta_2}{2}\right) - {}_{\eta_1+}I_\varphi\Upsilon\left(\frac{\eta_1+\eta_2}{2}\right)\right] \end{aligned} \tag{2}$$

and similarly, we have

$$\begin{aligned} H_2 = & \int_0^1(\mu\Delta(1)-\Delta(t))\Upsilon'\left(t\frac{\eta_1+\eta_2}{2}+(1-t)\eta_2\right)dt \\ = & \frac{2}{(\eta_2-\eta_1)}\left[(1-\mu)\Delta(1)\Upsilon\left(\frac{\eta_1+\eta_2}{2}\right)+\mu\Delta(1)\Upsilon(\eta_2) - {}_{\left(\frac{\eta_1+\eta_2}{2}\right)+}I_\varphi\Upsilon(\eta_2)\right]. \end{aligned} \tag{3}$$

By adding (2), (3) and multiplying the resultant equality with the factor $\frac{(\eta_2-\eta_1)}{4\Delta(1)}$, we obtain the resultant equality (1). □

Remark 3. *The advantage of this new lemma is that it allows us to establish new lemmas for different inequalities such as midpoint, trapezoidal, and Simpson's with different choices of the parameters λ and μ. On the other hand, more new results can be established on different choices of φ such as classical, fractional and k-fractional versions of the above mentioned inequalities.*

Corollary 1. *In Lemma 1, we have*

(i) *If we set $\lambda = \mu = 1$, then we obtain the following equality:*

$$\begin{aligned} & \frac{\Upsilon(\eta_1)+\Upsilon(\eta_2)}{2} - \frac{1}{2\Delta(1)}\left[{}_{\eta_1+}I_\varphi\Upsilon\left(\frac{\eta_1+\eta_2}{2}\right) + {}_{\left(\frac{\eta_1+\eta_2}{2}\right)+}I_\varphi\Upsilon(\eta_2)\right] \\ = & \frac{(\eta_2-\eta_1)}{4\Delta(1)}\left[\int_0^1(\Delta(1)-\Delta(1)-\Delta(t))\Upsilon'\left(t\eta_1+(1-t)\frac{\eta_1+\eta_2}{2}\right)dt\right. \\ & \left.+\int_0^1(\Delta(1)-\Delta(t))\Upsilon'\left(t\frac{\eta_1+\eta_2}{2}+(1-t)\eta_2\right)dt\right]. \end{aligned}$$

This equality helps us to obtain some trapezoidal type inequalities.

(ii) *If we set $\lambda = \mu = 0$, then we obtain the following equality:*

$$\begin{aligned}
&\Upsilon\left(\frac{\eta_1+\eta_2}{2}\right) - \frac{1}{2\Delta(1)}\left[{}_{\eta_1+}I_{\varphi}\Upsilon\left(\frac{\eta_1+\eta_2}{2}\right) + {}_{\left(\frac{\eta_1+\eta_2}{2}\right)+}I_{\varphi}\Upsilon(\eta_2)\right] \\
= &\frac{(\eta_2-\eta_1)}{4\Delta(1)}\left[\int_0^1(\Delta(1)-\Delta(t))\Upsilon'\left(t\eta_1+(1-t)\frac{\eta_1+\eta_2}{2}\right)dt\right. \\
&\left. - \int_0^1 \Delta(t)\Upsilon'\left(t\frac{\eta_1+\eta_2}{2}+(1-t)\eta_2\right)dt\right].
\end{aligned}$$

This equality help us to obtain some midpoint type inequalities.

(iii) *If we set $\lambda = \mu = \frac{1}{3}$, then we obtain the following equality:*

$$\begin{aligned}
&\frac{1}{6}\left[\Upsilon(\eta_1) + 4\Upsilon\left(\frac{\eta_1+\eta_2}{2}\right) + \Upsilon(\eta_2)\right] \\
&-\frac{1}{2\Delta(1)}\left[{}_{\eta_1+}I_{\varphi}\Upsilon\left(\frac{\eta_1+\eta_2}{2}\right) + {}_{\left(\frac{\eta_1+\eta_2}{2}\right)+}I_{\varphi}\Upsilon(\eta_2)\right] \\
= &\frac{(\eta_2-\eta_1)}{12\Delta(1)}\left[\int_0^1(3\Delta(1)-\Delta(1)-3\Delta(t))\Upsilon'\left(t\eta_1+(1-t)\frac{\eta_1+\eta_2}{2}\right)dt\right. \\
&\left. + \int_0^1 (\Delta(1)-3\Delta(t))\Upsilon'\left(t\frac{\eta_1+\eta_2}{2}+(1-t)\eta_2\right)dt\right].
\end{aligned}$$

This equality helps us to obtain some Simpson's type inequalities.

Remark 4. *If we set $\varphi(t) = t$ in Lemma 1, then we obtain ([29], Lemma 2.1).*

Corollary 2. *In Lemma 1, we have*

(i) *If we set $\varphi(t) = \frac{t^{\alpha}}{\Gamma(\alpha)}$, then we obtain the following RLFIs equality:*

$$\begin{aligned}
&\frac{\lambda\Upsilon(\eta_1)+\mu\Upsilon(\eta_2)}{2} + \frac{2-\lambda-\mu}{2}\Upsilon\left(\frac{\eta_1+\eta_2}{2}\right) \\
&-\frac{\Gamma(\alpha+1)}{2^{1-\alpha}(\eta_2-\eta_1)^{\alpha}}\left[J^{\alpha}_{\eta_1+}\Upsilon\left(\frac{\eta_1+\eta_2}{2}\right) + J^{\alpha}_{\left(\frac{\eta_1+\eta_2}{2}\right)+}\Upsilon(\eta_2)\right] \\
= &\frac{(\eta_2-\eta_1)}{4}\left[\int_0^1(1-\lambda-t^{\alpha})\Upsilon'\left(t\eta_1+(1-t)\frac{\eta_1+\eta_2}{2}\right)dt\right. \\
&\left. + \int_0^1 (\mu-t^{\alpha})\Upsilon'\left(t\frac{\eta_1+\eta_2}{2}+(1-t)\eta_2\right)dt\right].
\end{aligned}$$

(ii) *If we set $\varphi(t) = \frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$, then we have the following KFIs equality:*

$$\begin{aligned}
&\frac{\lambda\Upsilon(\eta_1)+\mu\Upsilon(\eta_2)}{2} + \frac{2-\lambda-\mu}{2}\Upsilon\left(\frac{\eta_1+\eta_2}{2}\right) \\
&-\frac{\Gamma_k(\alpha+k)}{2^{\frac{k-\alpha}{k}}(\eta_2-\eta_1)^{\frac{\alpha}{k}}}\left[\mathcal{J}^{\alpha,k}_{\eta_1+}\Upsilon\left(\frac{\eta_1+\eta_2}{2}\right) + \mathcal{J}^{\alpha,k}_{\left(\frac{\eta_1+\eta_2}{2}\right)+}\Upsilon(\eta_2)\right] \\
= &\frac{(\eta_2-\eta_1)}{4}\left[\int_0^1\left(1-\lambda-t^{\frac{\alpha}{k}}\right)\Upsilon'\left(t\eta_1+(1-t)\frac{\eta_1+\eta_2}{2}\right)dt\right. \\
&\left. + \int_0^1 \left(\mu-t^{\frac{\alpha}{k}}\right)\Upsilon'\left(t\frac{\eta_1+\eta_2}{2}+(1-t)\eta_2\right)dt\right].
\end{aligned}$$

4. Main Results

In this section, we prove different types of integral inequalities for differentiable convex functions using the GFIs.

Theorem 2. *Under the conditions of Lemma 1. If $|\Upsilon'|$ is convex function, then the following inequality holds:*

$$\begin{aligned} &\bigg|\frac{\lambda\Upsilon(\eta_1)+\mu\Upsilon(\eta_2)}{2}+\frac{2-\lambda-\mu}{2}\Upsilon\left(\frac{\eta_1+\eta_2}{2}\right) \\ &-\frac{1}{2\Delta(1)}\left[{}_{\eta_1+}I_{\varphi}\Upsilon\left(\frac{\eta_1+\eta_2}{2}\right)+{}_{\left(\frac{\eta_1+\eta_2}{2}\right)+}I_{\varphi}\Upsilon(\eta_2)\right]\bigg| \\ \leq\ &\frac{(\eta_2-\eta_1)}{4\Delta(1)}\left[(A_1(\lambda)+B_1(\mu))\left|\Upsilon'(\eta_1)\right|+(A_2(\lambda)+B_2(\mu))\left|\Upsilon'(\eta_2)\right|\right], \end{aligned}$$

where

$$\begin{aligned} A_1(\lambda)&=\int_0^1\frac{1+t}{2}|\Delta(1)-\lambda\Delta(1)-\Delta(t)|dt, \\ A_2(\lambda)&=\int_0^1\frac{1-t}{2}|\Delta(1)-\lambda\Delta(1)-\Delta(t)|dt, \\ B_1(\mu)&=\int_0^1\frac{t}{2}|\mu\Delta(1)-\Delta(t)|dt, \\ B_2(\mu)&=\int_0^1\frac{2-t}{2}|\mu\Delta(1)-\Delta(t)|dt. \end{aligned}$$

Proof. By taking modulus in (1), using properties of the modulus and convexity of $|\Upsilon'|$, we have

$$\begin{aligned} &\bigg|\frac{\lambda\Upsilon(\eta_1)+\mu\Upsilon(\eta_2)}{2}+\frac{2-\lambda-\mu}{2}\Upsilon\left(\frac{\eta_1+\eta_2}{2}\right) \\ &-\frac{1}{2\Delta(1)}\left[{}_{\eta_1+}I_{\varphi}\Upsilon\left(\frac{\eta_1+\eta_2}{2}\right)+{}_{\left(\frac{\eta_1+\eta_2}{2}\right)+}I_{\varphi}\Upsilon(\eta_2)\right]\bigg| \\ \leq\ &\frac{(\eta_2-\eta_1)}{4\Delta(1)}\left[\int_0^1|\Delta(1)-\lambda\Delta(1)-\Delta(t)|\left|\Upsilon'\left(\frac{1+t}{2}\eta_1+\frac{1-t}{2}\eta_2\right)\right|dt\right. \\ &\left.+\int_0^1|\mu\Delta(1)-\Delta(t)|\left|\Upsilon'\left(\frac{t}{2}\eta_1+\frac{2-t}{2}\eta_2\right)\right|dt\right] \\ \leq\ &\frac{(\eta_2-\eta_1)}{4\Delta(1)}\left[\int_0^1|\Delta(1)-\lambda\Delta(1)-\Delta(t)|\left(\frac{1+t}{2}|\Upsilon'(\eta_1)|+\frac{1-t}{2}|\Upsilon'(\eta_2)|\right)dt\right. \\ &\left.+\int_0^1|\mu\Delta(1)-\Delta(t)|\left(\frac{t}{2}|\Upsilon'(\eta_1)|+\frac{2-t}{2}|\Upsilon'(\eta_2)|\right)dt\right] \\ =\ &\frac{(\eta_2-\eta_1)}{4\Delta(1)}\left[(A_1(\lambda)+B_1(\mu))\left|\Upsilon'(\eta_1)\right|+(A_2(\lambda)+B_2(\mu))\left|\Upsilon'(\eta_2)\right|\right]. \end{aligned}$$

Thus, the proof is completed. □

Corollary 3. *In Theorem 2, we have*

(i) *If we set $\lambda=\mu=1$, then we obtain the following trapezoidal type inequality:*

$$\begin{aligned} &\left|\frac{\Upsilon(\eta_1)+\Upsilon(\eta_2)}{2}-\frac{1}{2\Delta(1)}\left[{}_{\eta_1+}I_{\varphi}\Upsilon\left(\frac{\eta_1+\eta_2}{2}\right)+{}_{\left(\frac{\eta_1+\eta_2}{2}\right)+}I_{\varphi}\Upsilon(\eta_2)\right]\right| \\ \leq\ &\frac{(\eta_2-\eta_1)}{4\Delta(1)}\left[(A_1(1)+B_1(1))\left|\Upsilon'(\eta_1)\right|+(A_2(1)+B_2(1))\left|\Upsilon'(\eta_2)\right|\right]. \end{aligned}$$

(ii) *If we set $\lambda = \mu = 0$, then we obtain the following midpoint type inequality:*

$$\begin{aligned}&\left|\Upsilon\left(\frac{\eta_1+\eta_2}{2}\right) - \frac{1}{2\Delta(1)}\left[{}_{\eta_1+}I_{\varphi}\Upsilon\left(\frac{\eta_1+\eta_2}{2}\right) + {}_{\left(\frac{\eta_1+\eta_2}{2}\right)+}I_{\varphi}\Upsilon(\eta_2)\right]\right| \\ \leq\ & \frac{(\eta_2-\eta_1)}{4\Delta(1)}\left[(A_1(0)+B_1(0))\left|\Upsilon'(\eta_1)\right| + (A_2(0)+B_2(0))\left|\Upsilon'(\eta_2)\right|\right].\end{aligned}$$

(iii) *If we set $\lambda = \mu = \frac{1}{3}$, then we obtain the following Simpson's type inequality:*

$$\begin{aligned}&\left|\frac{1}{6}\left[\Upsilon(\eta_1) + 4\Upsilon\left(\frac{\eta_1+\eta_2}{2}\right) + \Upsilon(\eta_2)\right]\right. \\ &\left. -\frac{1}{2\Delta(1)}\left[{}_{\eta_1+}I_{\varphi}\Upsilon\left(\frac{\eta_1+\eta_2}{2}\right) + {}_{\left(\frac{\eta_1+\eta_2}{2}\right)+}I_{\varphi}\Upsilon(\eta_2)\right]\right| \\ \leq\ & \frac{(\eta_2-\eta_1)}{4\Delta(1)}\left[\left(A_1\left(\frac{1}{3}\right) + B_1\left(\frac{1}{3}\right)\right)\left|\Upsilon'(\eta_1)\right| + \left(A_2\left(\frac{1}{3}\right) + B_2\left(\frac{1}{3}\right)\right)\left|\Upsilon'(\eta_2)\right|\right].\end{aligned}$$

Remark 5. *If we set $\varphi(t) = t$ in Theorem 2, then we obtain ([29], Theorem 3.1 for $q = 1$).*

Corollary 4. *In Theorem 2, we have*

(i) *If we set $\varphi(t) = \frac{t^{\alpha}}{\Gamma(\alpha)}$, then we obtain the following RLFIs inequality:*

$$\begin{aligned}&\left|\frac{\lambda\Upsilon(\eta_1) + \mu\Upsilon(\eta_2)}{2} + \frac{2-\lambda-\mu}{2}\Upsilon\left(\frac{\eta_1+\eta_2}{2}\right)\right. \\ &\left. -\frac{\Gamma(\alpha+1)}{2^{1-\alpha}(\eta_2-\eta_1)^{\alpha}}\left[J^{\alpha}_{\eta_1+}\Upsilon\left(\frac{\eta_1+\eta_2}{2}\right) + J^{\alpha}_{\left(\frac{\eta_1+\eta_2}{2}\right)+}\Upsilon(\eta_2)\right]\right| \\ \leq\ & \frac{(\eta_2-\eta_1)}{4}\left[(C_1(\alpha;\lambda) + C_3(\alpha;\mu))\left|\Upsilon'(\eta_1)\right| + (C_2(\alpha;\lambda) + C_4(\alpha;\mu))\left|\Upsilon'(\eta_2)\right|\right],\end{aligned}$$

where

$$\begin{aligned}&C_1(\alpha;\lambda) \\ =\ & \frac{3}{4}\lambda + (1-\lambda)^{\alpha} - \frac{1}{\alpha+1}(1-\lambda)^{\alpha^2+\alpha} - \frac{1}{2}\lambda(1-\lambda)^{2\alpha} \\ &-\frac{1}{\alpha+2}(1-\lambda)^{\alpha^2+2\alpha} + \frac{1}{2(\alpha+1)} + \frac{1}{2(\alpha+2)} \\ &+\frac{1}{2}(1-\lambda)^{2\alpha} - \lambda(1-\lambda)^{\alpha} - \frac{3}{4},\end{aligned}$$

$$\begin{aligned}&C_2(\alpha;\lambda) \\ =\ & \frac{1}{4}\lambda + (1-\lambda)^{\alpha} - \frac{1}{\alpha+1}(1-\lambda)^{\alpha^2+\alpha} + \frac{1}{2}\lambda(1-\lambda)^{2\alpha} \\ &+\frac{1}{\alpha+2}(1-\lambda)^{\alpha^2+2\alpha} + \frac{1}{2(\alpha+1)} - \frac{1}{2(\alpha+2)} \\ &-\frac{1}{2}(1-\lambda)^{2\alpha} - \lambda(1-\lambda)^{\alpha} - \frac{1}{4},\end{aligned}$$

$$\begin{aligned}C_3(\alpha;\mu) &= \frac{\mu^{2\alpha+1}}{2} - \frac{\mu^{\alpha^2+2\alpha}}{\alpha+2} + \frac{1}{2(\alpha+2)} - \frac{\mu}{4}, \\ C_4(\alpha;\mu) &= 2\left(\mu^{\alpha+1} - \frac{\mu^{\alpha^2+\alpha}}{\alpha+1}\right) + \frac{1}{\alpha+1} - \mu - C_3(\alpha;\mu).\end{aligned}$$

(ii) *If we set $\varphi(t) = \frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$, then we have the following KFIs inequality:*

$$
\begin{aligned}
&\left|\frac{\lambda \Upsilon(\eta_1)+\mu \Upsilon(\eta_2)}{2}+\frac{2-\lambda-\mu}{2}\Upsilon\left(\frac{\eta_1+\eta_2}{2}\right)\right. \\
&\left.-\frac{\Gamma_k(\alpha+k)}{2^{\frac{k-\alpha}{k}}(\eta_2-\eta_1)^{\frac{\alpha}{k}}}\left[\mathcal{J}^{\alpha,k}_{\eta_1+}\Upsilon\left(\frac{\eta_1+\eta_2}{2}\right)+\mathcal{J}^{\alpha,k}_{\left(\frac{\eta_1+\eta_2}{2}\right)+}\Upsilon(\eta_2)\right]\right| \\
\leq\; & \frac{(\eta_2-\eta_1)}{4}\left[(C_1(\alpha;\lambda;k)+C_3(\alpha;\mu;k))\left|\Upsilon'(\eta_1)\right|+(C_2(\alpha;\lambda;k)+C_4(\alpha;\mu;k))\left|\Upsilon'(\eta_2)\right|\right],
\end{aligned}
$$

where

$$
\begin{aligned}
& C_1(\alpha;\lambda;k) \\
=\; & \frac{3}{4}\lambda+(1-\lambda)^{\frac{\alpha}{k}}-\frac{1}{\frac{\alpha}{k}+1}(1-\lambda)^{\left(\frac{\alpha}{k}\right)^2+\frac{\alpha}{k}} \\
& -\frac{1}{2}\lambda(1-\lambda)^{2\frac{\alpha}{k}}-\frac{1}{\frac{\alpha}{k}+2}(1-\lambda)^{\left(\frac{\alpha}{k}\right)^2+2\frac{\alpha}{k}}+\frac{1}{2\left(\frac{\alpha}{k}+1\right)} \\
& +\frac{1}{2\left(\frac{\alpha}{k}+2\right)}+\frac{1}{2}(1-\lambda)^{2\frac{\alpha}{k}}-\lambda(1-\lambda)^{\frac{\alpha}{k}}-\frac{3}{4},
\end{aligned}
$$

$$
\begin{aligned}
& C_2(\alpha;\lambda;k) \\
=\; & \frac{1}{4}\lambda+(1-\lambda)^{\frac{\alpha}{k}}-\frac{1}{\frac{\alpha}{k}+1}(1-\lambda)^{\left(\frac{\alpha}{k}\right)^2+\frac{\alpha}{k}}+\frac{1}{2}\lambda(1-\lambda)^{2\frac{\alpha}{k}} \\
& +\frac{1}{\frac{\alpha}{k}+2}(1-\lambda)^{\left(\frac{\alpha}{k}\right)^2+2\frac{\alpha}{k}}+\frac{1}{2\left(\frac{\alpha}{k}+1\right)}-\frac{1}{2\left(\frac{\alpha}{k}+2\right)} \\
& -\frac{1}{2}(1-\lambda)^{2\frac{\alpha}{k}}-\lambda(1-\lambda)^{\frac{\alpha}{k}}-\frac{1}{4},
\end{aligned}
$$

$$
\begin{aligned}
C_3(\alpha;\mu;k) &= \frac{\mu^{2\frac{\alpha}{k}+1}}{2}-\frac{\mu^{\left(\frac{\alpha}{k}\right)^2+2\frac{\alpha}{k}}}{\frac{\alpha}{k}+2}+\frac{1}{2\left(\frac{\alpha}{k}+2\right)}-\frac{\mu}{4}, \\
C_4(\alpha;\mu;k) &= 2\left(\mu^{\frac{\alpha}{k}+1}-\frac{\mu^{\left(\frac{\alpha}{k}\right)^2+\frac{\alpha}{k}}}{\frac{\alpha}{k}+1}\right)+\frac{1}{\frac{\alpha}{k}+1}-\mu-C_3(\alpha;\mu).
\end{aligned}
$$

Theorem 3. *Under the conditions of Lemma 1. If $|\Upsilon'|^r$, $r \geq 1$ is convex function, then the following inequality holds:*

$$
\begin{aligned}
&\left|\frac{\lambda \Upsilon(\eta_1)+\mu \Upsilon(\eta_2)}{2}+\frac{2-\lambda-\mu}{2}\Upsilon\left(\frac{\eta_1+\eta_2}{2}\right)\right. \\
&\left.-\frac{1}{2\Delta(1)}\left[{}_{\eta_1+}I_\varphi \Upsilon\left(\frac{\eta_1+\eta_2}{2}\right)+{}_{\left(\frac{\eta_1+\eta_2}{2}\right)+}I_\varphi \Upsilon(\eta_2)\right]\right| \\
\leq\; & \frac{(\eta_2-\eta_1)}{4\Delta(1)}\left[A_3^{1-\frac{1}{r}}(\lambda)\left(A_1(\lambda)\left|\Upsilon'(\eta_1)\right|^r+A_2(\lambda)\left|\Upsilon'(\eta_2)\right|^r\right)^{\frac{1}{r}}\right. \\
& \left.+B_3^{1-\frac{1}{r}}(\mu)\left(B_1(\mu)\left|\Upsilon'(\eta_1)\right|^r+B_2(\mu)\left|\Upsilon'(\eta_2)\right|^r\right)^{\frac{1}{r}}\right],
\end{aligned}
$$

where

$$
\begin{aligned}
A_3(\lambda) &= \int_0^1 |\Delta(1)-\lambda\Delta(1)-\Delta(t)|dt, \\
B_3(\mu) &= \int_0^1 |\mu\Delta(1)-\Delta(t)|dt.
\end{aligned}
$$

Proof. By taking modulus in (1) and using the power mean inequality, we have

$$
\begin{aligned}
&\left|\frac{\lambda \mathrm{Y}(\eta_1)+\mu \mathrm{Y}(\eta_2)}{2}+\frac{2-\lambda-\mu}{2}\mathrm{Y}\left(\frac{\eta_1+\eta_2}{2}\right)\right.\\
&\left.-\frac{1}{2\Delta(1)}\left[{}_{\eta_1+}I_{\varphi}\mathrm{Y}\left(\frac{\eta_1+\eta_2}{2}\right)+{}_{\left(\frac{\eta_1+\eta_2}{2}\right)+}I_{\varphi}\mathrm{Y}(\eta_2)\right]\right|\\
\leq\;& \frac{(\eta_2-\eta_1)}{4\Delta(1)}\left[\int_0^1 |\Delta(1)-\lambda\Delta(1)-\Delta(t)|\left|\mathrm{Y}'\left(\frac{1+t}{2}\eta_1+\frac{1-t}{2}\eta_2\right)\right|dt\right.\\
&\left.+\int_0^1 |\mu\Delta(1)-\Delta(t)|\left|\mathrm{Y}'\left(\frac{t}{2}\eta_1+\frac{2-t}{2}\eta_2\right)\right|dt\right]\\
\leq\;& \frac{(\eta_2-\eta_1)}{4\Delta(1)}\left[\left(\int_0^1 |\Delta(1)-\lambda\Delta(1)-\Delta(t)|dt\right)^{1-\frac{1}{r}}\right.\\
&\times\left(\int_0^1 |\Delta(1)-\lambda\Delta(1)-\Delta(t)|\left|\mathrm{Y}'\left(\frac{1+t}{2}\eta_1+\frac{1-t}{2}\eta_2\right)\right|^r dt\right)^{\frac{1}{r}}\\
&+\left(\int_0^1 |\mu\Delta(1)-\Delta(t)|dt\right)^{1-\frac{1}{r}}\\
&\left.\times\left(\int_0^1 |\mu\Delta(1)-\Delta(t)|\left|\mathrm{Y}'\left(\frac{t}{2}\eta_1+\frac{2-t}{2}\eta_2\right)\right|^r dt\right)^{\frac{1}{r}}\right].
\end{aligned}
$$

By using convexity of $|\mathrm{Y}'|^r$, $r \geq 1$, we have

$$
\begin{aligned}
&\left|\frac{\lambda \mathrm{Y}(\eta_1)+\mu \mathrm{Y}(\eta_2)}{2}+\frac{2-\lambda-\mu}{2}\mathrm{Y}\left(\frac{\eta_1+\eta_2}{2}\right)\right.\\
&\left.-\frac{1}{2\Delta(1)}\left[{}_{\eta_1+}I_{\varphi}\mathrm{Y}\left(\frac{\eta_1+\eta_2}{2}\right)+{}_{\left(\frac{\eta_1+\eta_2}{2}\right)+}I_{\varphi}\mathrm{Y}(\eta_2)\right]\right|\\
\leq\;& \frac{(\eta_2-\eta_1)}{4\Delta(1)}\left[\left(\int_0^1 |\Delta(1)-\lambda\Delta(1)-\Delta(t)|dt\right)^{1-\frac{1}{r}}\right.\\
&\times\left(\int_0^1 |\Delta(1)-\lambda\Delta(1)-\Delta(t)|\left(\frac{1+t}{2}|\mathrm{Y}'(\eta_1)|^r+\frac{1-t}{2}|\mathrm{Y}'(\eta_2)|^r\right)dt\right)^{\frac{1}{r}}\\
&+\left(\int_0^1 |\mu\Delta(1)-\Delta(t)|dt\right)^{1-\frac{1}{r}}\\
&\left.\times\left(\int_0^1 |\mu\Delta(1)-\Delta(t)|\left(\frac{t}{2}|\mathrm{Y}'(\eta_1)|^r+\frac{2-t}{2}|\mathrm{Y}'(\eta_2)|^r\right)dt\right)^{\frac{1}{r}}\right]\\
=\;& \frac{(\eta_2-\eta_1)}{4\Delta(1)}\left[A_3^{1-\frac{1}{r}}(\lambda)\left(A_1(\lambda)|\mathrm{Y}'(\eta_1)|^r+A_2(\lambda)|\mathrm{Y}'(\eta_2)|^r\right)^{\frac{1}{r}}\right.\\
&\left.+B_3^{1-\frac{1}{r}}(\mu)\left(B_1(\mu)|\mathrm{Y}'(\eta_1)|^r+B_2(\mu)|\mathrm{Y}'(\eta_2)|^r\right)^{\frac{1}{r}}\right].
\end{aligned}
$$

Thus, the proof is completed. □

Corollary 5. *In Theorem 3, we have*

(i) *If we set* $\lambda = \mu = 1$, *then we obtain the following trapezoidal type inequality:*

$$\begin{aligned}
&\left|\frac{\Upsilon(\eta_1)+\Upsilon(\eta_2)}{2} - \frac{1}{2\Delta(1)}\left[{}_{\eta_1+}I_{\varphi}\Upsilon\left(\frac{\eta_1+\eta_2}{2}\right) + {}_{\left(\frac{\eta_1+\eta_2}{2}\right)+}I_{\varphi}\Upsilon(\eta_2)\right]\right| \\
\leq\ & \frac{(\eta_2-\eta_1)}{4\Delta(1)}\left[A_3^{1-\frac{1}{r}}(1)\left(A_1(1)\left|\Upsilon'(\eta_1)\right|^r + A_2(1)\left|\Upsilon'(\eta_2)\right|^r\right)^{\frac{1}{r}}\right. \\
&\left. +B_3^{1-\frac{1}{r}}(1)\left(B_1(1)\left|\Upsilon'(\eta_1)\right|^r + B_2(1)\left|\Upsilon'(\eta_2)\right|^r\right)^{\frac{1}{r}}\right].
\end{aligned}$$

(ii) *If we set* $\lambda = \mu = 0$, *then we obtain the following midpoint type inequality:*

$$\begin{aligned}
&\left|\Upsilon\left(\frac{\eta_1+\eta_2}{2}\right) - \frac{1}{2\Delta(1)}\left[{}_{\eta_1+}I_{\varphi}\Upsilon\left(\frac{\eta_1+\eta_2}{2}\right) + {}_{\left(\frac{\eta_1+\eta_2}{2}\right)+}I_{\varphi}\Upsilon(\eta_2)\right]\right| \\
\leq\ & \frac{(\eta_2-\eta_1)}{4\Delta(1)}\left[A_3^{1-\frac{1}{r}}(0)\left(A_1(0)\left|\Upsilon'(\eta_1)\right|^r + A_2(0)\left|\Upsilon'(\eta_2)\right|^r\right)^{\frac{1}{r}}\right. \\
&\left. +B_3^{1-\frac{1}{r}}(0)\left(B_1(0)\left|\Upsilon'(\eta_1)\right|^r + B_2(0)\left|\Upsilon'(\eta_2)\right|^r\right)^{\frac{1}{r}}\right].
\end{aligned}$$

(iii) *If we set* $\lambda = \mu = \frac{1}{3}$, *then we obtain the following Simpson's type inequality:*

$$\begin{aligned}
&\left|\frac{1}{6}\left[\Upsilon(\eta_1) + 4\Upsilon\left(\frac{\eta_1+\eta_2}{2}\right) + \Upsilon(\eta_2)\right]\right. \\
&\left. - \frac{1}{2\Delta(1)}\left[{}_{\eta_1+}I_{\varphi}\Upsilon\left(\frac{\eta_1+\eta_2}{2}\right) + {}_{\left(\frac{\eta_1+\eta_2}{2}\right)+}I_{\varphi}\Upsilon(\eta_2)\right]\right| \\
\leq\ & \frac{(\eta_2-\eta_1)}{4\Delta(1)}\left[A_3^{1-\frac{1}{r}}\left(\frac{1}{3}\right)\left(A_1\left(\frac{1}{3}\right)\left|\Upsilon'(\eta_1)\right|^r + A_2\left(\frac{1}{3}\right)\left|\Upsilon'(\eta_2)\right|^r\right)^{\frac{1}{r}}\right. \\
&\left. +B_3^{1-\frac{1}{r}}\left(\frac{1}{3}\right)\left(B_1\left(\frac{1}{3}\right)\left|\Upsilon'(\eta_1)\right|^r + B_2\left(\frac{1}{3}\right)\left|\Upsilon'(\eta_2)\right|^r\right)^{\frac{1}{r}}\right].
\end{aligned}$$

Remark 6. *If we set* $\varphi(t) = t$ *in Theorem 2, then we obtain ([29], Theorem 3.1).*

Corollary 6. *In Theorem 3, we have*

(i) *If we set* $\varphi(t) = \frac{t^{\alpha}}{\Gamma(\alpha)}$, *then we obtain the following RLFIs inequality:*

$$\begin{aligned}
&\left|\frac{\lambda\Upsilon(\eta_1)+\mu\Upsilon(\eta_2)}{2} + \frac{2-\lambda-\mu}{2}\Upsilon\left(\frac{\eta_1+\eta_2}{2}\right)\right. \\
&\left. - \frac{\Gamma(\alpha+1)}{2^{1-\alpha}(\eta_2-\eta_1)^{\alpha}}\left[J^{\alpha}_{\eta_1+}\Upsilon\left(\frac{\eta_1+\eta_2}{2}\right) + J^{\alpha}_{\left(\frac{\eta_1+\eta_2}{2}\right)+}\Upsilon(\eta_2)\right]\right| \\
\leq\ & \frac{(\eta_2-\eta_1)}{4}\left[C_5^{1-\frac{1}{r}}(\alpha;\lambda)\left(C_1(\alpha;\lambda)\left|\Upsilon'(\eta_1)\right|^r + C_2(\alpha;\lambda)\left|\Upsilon'(\eta_2)\right|^r\right)^{\frac{1}{r}}\right. \\
&\left. +C_6^{1-\frac{1}{r}}(\alpha;\mu)\left(C_3(\alpha;\mu)\left|\Upsilon'(\eta_1)\right|^r + C_4(\alpha;\mu)\left|\Upsilon'(\eta_2)\right|^r\right)^{\frac{1}{r}}\right],
\end{aligned}$$

where

$$\begin{aligned}
C_5(\alpha;\lambda) &= 2\left((1-\lambda)^{\alpha+1} - \frac{(1-\lambda)^{\alpha^2+\alpha}}{\alpha+1}\right) + \frac{1}{\alpha+1} + \lambda - 1 \\
C_6(\alpha;\mu) &= 2\left(\mu^{\alpha+1} - \frac{\mu^{\alpha^2+\alpha}}{\alpha+1}\right) + \frac{1}{\alpha+1} - \mu.
\end{aligned}$$

(ii) *If we set* $\varphi(t) = \frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$*, then we have the following KFIs inequality:*

$$\begin{aligned} &\left| \frac{\lambda \Upsilon(\eta_1) + \mu \Upsilon(\eta_2)}{2} + \frac{2-\lambda-\mu}{2} \Upsilon\left(\frac{\eta_1+\eta_2}{2}\right) \right. \\ &\left. - \frac{\Gamma_k(\alpha+k)}{2^{\frac{k-\alpha}{k}} (\eta_2-\eta_1)^{\frac{\alpha}{k}}} \left[\mathcal{J}^{\alpha,k}_{\eta_1+} \Upsilon\left(\frac{\eta_1+\eta_2}{2}\right) + \mathcal{J}^{\alpha,k}_{\left(\frac{\eta_1+\eta_2}{2}\right)+} \Upsilon(\eta_2) \right] \right| \\ \leq\ & \frac{(\eta_2-\eta_1)}{4} \left[C_5^{1-\frac{1}{r}}(\alpha;\lambda;k) \left(C_1(\alpha;\lambda;k) |\Upsilon'(\eta_1)|^r + C_2(\alpha;\lambda;k) |\Upsilon'(\eta_2)|^r \right)^{\frac{1}{r}} \right. \\ &\left. + C_6^{1-\frac{1}{r}}(\alpha;\mu) \left(C_3(\alpha;\lambda;k) |\Upsilon'(\eta_1)|^r + C_4(\alpha;\lambda;k) |\Upsilon'(\eta_2)|^r \right)^{\frac{1}{r}} \right], \end{aligned}$$

where

$$\begin{aligned} C_5(\alpha;\lambda;k) &= 2\left((1-\lambda)^{\frac{\alpha}{k}+1} - \frac{(1-\lambda)^{\left(\frac{\alpha}{k}\right)^2+\frac{\alpha}{k}}}{\frac{\alpha}{k}+1} \right) + \frac{1}{\frac{\alpha}{k}+1} + \lambda - 1, \\ C_6(\alpha;\mu;k) &= 2\left(\mu^{\frac{\alpha}{k}+1} - \frac{\mu^{\left(\frac{\alpha}{k}\right)^2+\frac{\alpha}{k}}}{\frac{\alpha}{k}+1} \right) + \frac{1}{\frac{\alpha}{k}+1} - \mu. \end{aligned}$$

Theorem 4. *Under the conditions of Lemma 1. If* $|\Upsilon'|^r$*,* $r > 1$ *is convex function, then the following inequality holds:*

$$\begin{aligned} &\left| \frac{\lambda \Upsilon(\eta_1) + \mu \Upsilon(\eta_2)}{2} + \frac{2-\lambda-\mu}{2} \Upsilon\left(\frac{\eta_1+\eta_2}{2}\right) \right. \\ &\left. - \frac{1}{2\Delta(1)} \left[{}_{\eta_1+}I_\varphi \Upsilon\left(\frac{\eta_1+\eta_2}{2}\right) + {}_{\left(\frac{\eta_1+\eta_2}{2}\right)+}I_\varphi \Upsilon(\eta_2) \right] \right| \\ \leq\ & \frac{(\eta_2-\eta_1)}{4\Delta(1)} \left[A_4(\lambda;s) \left(\frac{3|\Upsilon'(\eta_1)|^r + |\Upsilon'(\eta_2)|^r}{4} \right)^{\frac{1}{r}} \right. \\ &\left. + B_4(\mu;s) \left(\frac{|\Upsilon'(\eta_1)|^r + 3|\Upsilon'(\eta_2)|^r}{4} \right)^{\frac{1}{r}} \right], \end{aligned}$$

where

$$\begin{aligned} A_4(\lambda;s) &= \left(\int_0^1 |\Delta(1) - \lambda\Delta(1) - \Delta(t)|^s dt \right)^{\frac{1}{s}}, \\ B_4(\mu;s) &= \left(\int_0^1 |\mu\Delta(1) - \Delta(t)|^s dt \right)^{\frac{1}{s}} \end{aligned}$$

and $s^{-1} + r^{-1} = 1$.

Proof. Taking absolute value of (1) and using the Hölder's inequality, we have

$$\begin{aligned}
&\left|\frac{\lambda\Upsilon(\eta_1)+\mu\Upsilon(\eta_2)}{2}+\frac{2-\lambda-\mu}{2}\Upsilon\left(\frac{\eta_1+\eta_2}{2}\right)\right.\\
&\left.-\frac{1}{2\Delta(1)}\left[{}_{\eta_1+}I_{\varphi}\Upsilon\left(\frac{\eta_1+\eta_2}{2}\right)+{}_{\left(\frac{\eta_1+\eta_2}{2}\right)+}I_{\varphi}\Upsilon(\eta_2)\right]\right|\\
\leq\ &\frac{(\eta_2-\eta_1)}{4\Delta(1)}\left[\left(\int_0^1|\Delta(1)-\lambda\Delta(1)-\Delta(t)|^s dt\right)^{\frac{1}{s}}\left(\int_0^1\left|\Upsilon'\left(\frac{1+t}{2}\eta_1+\frac{1-t}{2}\eta_2\right)\right|^r dt\right)^{\frac{1}{r}}\right.\\
&\left.+\left(\int_0^1|\mu\Delta(1)-\Delta(t)|^s dt\right)^{\frac{1}{s}}\left(\int_0^1\left|\Upsilon'\left(\frac{t}{2}\eta_1+\frac{2-t}{2}\eta_2\right)\right|^r dt\right)^{\frac{1}{r}}\right].
\end{aligned}$$

From convexity of $|\Upsilon'|^r, r>1$, we have

$$\begin{aligned}
&\left|\frac{\lambda\Upsilon(\eta_1)+\mu\Upsilon(\eta_2)}{2}+\frac{2-\lambda-\mu}{2}\Upsilon\left(\frac{\eta_1+\eta_2}{2}\right)\right.\\
&\left.-\frac{1}{2\Delta(1)}\left[{}_{\eta_1+}I_{\varphi}\Upsilon\left(\frac{\eta_1+\eta_2}{2}\right)+{}_{\left(\frac{\eta_1+\eta_2}{2}\right)+}I_{\varphi}\Upsilon(\eta_2)\right]\right|\\
\leq\ &\frac{(\eta_2-\eta_1)}{4\Delta(1)}\left[\left(\int_0^1|\Delta(1)-\lambda\Delta(1)-\Delta(t)|^s dt\right)^{\frac{1}{s}}\right.\\
&\times\left(\int_0^1\left(\frac{1+t}{2}|\Upsilon'(\eta_1)|^r+\frac{1-t}{2}|\Upsilon'(\eta_2)|^r\right)dt\right)^{\frac{1}{r}}\\
&\left.+\left(\int_0^1|\mu\Delta(1)-\Delta(t)|^s dt\right)^{\frac{1}{s}}\left(\int_0^1\left(\frac{t}{2}|\Upsilon'(\eta_1)|^r+\frac{2-t}{2}|\Upsilon'(\eta_2)|^r\right)dt\right)^{\frac{1}{r}}\right]\\
=\ &\frac{(\eta_2-\eta_1)}{4\Delta(1)}\left[A_4(\lambda;s)\left(\frac{3|\Upsilon'(\eta_1)|^r+|\Upsilon'(\eta_2)|^r}{4}\right)^{\frac{1}{r}}\right.\\
&\left.+B_4(\mu;s)\left(\frac{|\Upsilon'(\eta_1)|^r+3|\Upsilon'(\eta_2)|^r}{4}\right)^{\frac{1}{r}}\right].
\end{aligned}$$

Thus, the proof is ended. □

Corollary 7. *In Theorem 4, we have*

(i) *If we set $\lambda=\mu=1$, then we obtain the following trapezoidal type inequality:*

$$\begin{aligned}
&\left|\frac{\Upsilon(\eta_1)+\Upsilon(\eta_2)}{2}-\frac{1}{2\Delta(1)}\left[{}_{\eta_1+}I_{\varphi}\Upsilon\left(\frac{\eta_1+\eta_2}{2}\right)+{}_{\left(\frac{\eta_1+\eta_2}{2}\right)+}I_{\varphi}\Upsilon(\eta_2)\right]\right|\\
\leq\ &\frac{(\eta_2-\eta_1)}{4\Delta(1)}\left[A_4(1;s)\left(\frac{3|\Upsilon'(\eta_1)|^r+|\Upsilon'(\eta_2)|^r}{4}\right)^{\frac{1}{r}}\right.\\
&\left.+B_4(1;s)\left(\frac{|\Upsilon'(\eta_1)|^r+3|\Upsilon'(\eta_2)|^r}{4}\right)^{\frac{1}{r}}\right].
\end{aligned}$$

(ii) *If we set $\lambda = \mu = 0$, then we obtain the following midpoint type inequality:*

$$\begin{aligned}&\left|\Upsilon\left(\frac{\eta_1+\eta_2}{2}\right) - \frac{1}{2\Delta(1)}\left[{}_{\eta_1+}I_\varphi \Upsilon\left(\frac{\eta_1+\eta_2}{2}\right) + {}_{\left(\frac{\eta_1+\eta_2}{2}\right)+}I_\varphi \Upsilon(\eta_2)\right]\right| \\ \le\ & \frac{(\eta_2-\eta_1)}{4\Delta(1)}\left[A_4(0;s)\left(\frac{3|\Upsilon'(\eta_1)|^r + |\Upsilon'(\eta_2)|^r}{4}\right)^{\frac{1}{r}}\right. \\ &\left. + B_4(0;s)\left(\frac{|\Upsilon'(\eta_1)|^r + 3|\Upsilon'(\eta_2)|^r}{4}\right)^{\frac{1}{r}}\right].\end{aligned}$$

(iii) *If we set $\lambda = \mu = \frac{1}{3}$, then we obtain the following Simpson's type inequality:*

$$\begin{aligned}&\left|\frac{1}{6}\left[\Upsilon(\eta_1) + 4\Upsilon\left(\frac{\eta_1+\eta_2}{2}\right) + \Upsilon(\eta_2)\right]\right. \\ &\left. - \frac{1}{2\Delta(1)}\left[{}_{\eta_1+}I_\varphi \Upsilon\left(\frac{\eta_1+\eta_2}{2}\right) + {}_{\left(\frac{\eta_1+\eta_2}{2}\right)+}I_\varphi \Upsilon(\eta_2)\right]\right| \\ \le\ & \frac{(\eta_2-\eta_1)}{4\Delta(1)}\left[A_4\left(\frac{1}{3};s\right)\left(\frac{3|\Upsilon'(\eta_1)|^r + |\Upsilon'(\eta_2)|^r}{4}\right)^{\frac{1}{r}}\right. \\ &\left. + B_4\left(\frac{1}{3};s\right)\left(\frac{|\Upsilon'(\eta_1)|^r + 3|\Upsilon'(\eta_2)|^r}{4}\right)^{\frac{1}{r}}\right].\end{aligned}$$

Remark 7. *If we set $\varphi(t) = t$ in Theorem 4, then we obtain the following inequality*

$$\begin{aligned}&\left|\frac{\lambda\Upsilon(\eta_1) + \mu\Upsilon(\eta_2)}{2} + \frac{2-\lambda-\mu}{2}\Upsilon\left(\frac{\eta_1+\eta_2}{2}\right) - \frac{1}{\eta_2-\eta_1}\int_{\eta_1}^{\eta_2}\Upsilon(\varkappa)d\varkappa\right| \\ \le\ & \frac{(\eta_2-\eta_1)}{4}\left[\left(\frac{(1-\lambda)^{s+1} + \lambda^{s+1}}{s+1}\right)^{\frac{1}{s}}\left(\frac{3|\Upsilon'(\eta_1)|^r + |\Upsilon'(\eta_2)|^r}{4}\right)^{\frac{1}{r}}\right. \\ &\left. + \left(\frac{(1-\mu)^{s+1} + \mu^{s+1}}{s+1}\right)^{\frac{1}{s}}\left(\frac{|\Upsilon'(\eta_1)|^r + 3|\Upsilon'(\eta_2)|^r}{4}\right)^{\frac{1}{r}}\right].\end{aligned}$$

Corollary 8. *In Theorem 4, we have*

(i) *If we set $\varphi(t) = \frac{t^\alpha}{\Gamma(\alpha)}$, then we obtain the following RLFIs inequality:*

$$\begin{aligned}&\left|\frac{\lambda\Upsilon(\eta_1) + \mu\Upsilon(\eta_2)}{2} + \frac{2-\lambda-\mu}{2}\Upsilon\left(\frac{\eta_1+\eta_2}{2}\right)\right. \\ &\left. - \frac{\Gamma(\alpha+1)}{2^{1-\alpha}(\eta_2-\eta_1)^\alpha}\left[J^\alpha_{\eta_1+}\Upsilon\left(\frac{\eta_1+\eta_2}{2}\right) + J^\alpha_{\left(\frac{\eta_1+\eta_2}{2}\right)+}\Upsilon(\eta_2)\right]\right| \\ \le\ & \frac{(\eta_2-\eta_1)}{4}\left[A_4^\alpha(\lambda;s)\left(\frac{3|\Upsilon'(\eta_1)|^r + |\Upsilon'(\eta_2)|^r}{4}\right)^{\frac{1}{r}}\right. \\ &\left. + B_4^\alpha(\lambda;s)\left(\frac{|\Upsilon'(\eta_1)|^r + 3|\Upsilon'(\eta_2)|^r}{4}\right)^{\frac{1}{r}}\right],\end{aligned}$$

where

$$A_4^{\alpha}(\lambda;s) = \left(\int_0^1 |1-\lambda-t^{\alpha}|^s dt\right)^{\frac{1}{s}},$$
$$B_4^{\alpha}(\lambda;s) = \left(\int_0^1 |\mu-t^{\alpha}|^s\right)^{\frac{1}{s}}.$$

(ii) *If we set* $\varphi(t)=\frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$, *then we have the following KFIs inequality:*

$$\begin{aligned}
&\left|\frac{\lambda\Upsilon(\eta_1)+\mu\Upsilon(\eta_2)}{2}+\frac{2-\lambda-\mu}{2}\Upsilon\left(\frac{\eta_1+\eta_2}{2}\right)\right.\\
&\left.-\frac{\Gamma_k(\alpha+k)}{2^{\frac{k-\alpha}{k}}(\eta_2-\eta_1)^{\frac{\alpha}{k}}}\left[\mathcal{J}^{\alpha,k}_{\eta_1+}\Upsilon\left(\frac{\eta_1+\eta_2}{2}\right)+\mathcal{J}^{\alpha,k}_{\left(\frac{\eta_1+\eta_2}{2}\right)+}\Upsilon(\eta_2)\right]\right|\\
\leq\ &\frac{(\eta_2-\eta_1)}{4}\left[A_4^{\frac{\alpha}{k}}(\lambda;s)\left(\frac{3|\Upsilon'(\eta_1)|^r+|\Upsilon'(\eta_2)|^r}{4}\right)^{\frac{1}{r}}\right.\\
&\left.+B_4^{\frac{\alpha}{k}}(\lambda;s)\left(\frac{|\Upsilon'(\eta_1)|^r+3|\Upsilon'(\eta_2)|^r}{4}\right)^{\frac{1}{r}}\right],
\end{aligned}$$

where

$$A_4^{\frac{\alpha}{k}}(\lambda;s) = \left(\int_0^1 \left|1-\lambda-t^{\frac{\alpha}{k}}\right|^s dt\right)^{\frac{1}{s}},$$
$$B_4^{\frac{\alpha}{k}}(\lambda;s) = \left(\int_0^1 \left|\mu-t^{\frac{\alpha}{k}}\right|^s\right)^{\frac{1}{s}}.$$

5. Applications to Quadrature Formulas

In this section, we discuss some special cases of newly established inequalities and obtain midpoint type inequalities, trapezoidal type inequalities and Simpson's type inequalities.

Remark 8. *In Corollary 4 part (i), we have*

(i) *If we set* $\lambda=\mu=1$, *then we obtain the following trapezoidal type inequality:*

$$\begin{aligned}
&\left|\frac{\Upsilon(\eta_1)+\Upsilon(\eta_2)}{2}-\frac{\Gamma(\alpha+1)}{2^{1-\alpha}(\eta_2-\eta_1)^{\alpha}}\left[J^{\alpha}_{\eta_1+}\Upsilon\left(\frac{\eta_1+\eta_2}{2}\right)+J^{\alpha}_{\left(\frac{\eta_1+\eta_2}{2}\right)+}\Upsilon(\eta_2)\right]\right|\\
\leq\ &\frac{(\eta_2-\eta_1)}{4}\left[(C_1(\alpha;1)+C_3(\alpha;1))\left|\Upsilon'(\eta_1)\right|+(C_2(\alpha;1)+C_4(\alpha;1))\left|\Upsilon'(\eta_2)\right|\right].
\end{aligned}$$

(ii) *If we set* $\lambda=\mu=0$, *then we obtain the following midpoint type inequality:*

$$\begin{aligned}
&\left|\Upsilon\left(\frac{\eta_1+\eta_2}{2}\right)-\frac{\Gamma(\alpha+1)}{2^{1-\alpha}(\eta_2-\eta_1)^{\alpha}}\left[J^{\alpha}_{\eta_1+}\Upsilon\left(\frac{\eta_1+\eta_2}{2}\right)+J^{\alpha}_{\left(\frac{\eta_1+\eta_2}{2}\right)+}\Upsilon(\eta_2)\right]\right|\\
\leq\ &\frac{(\eta_2-\eta_1)}{4}\left[(C_1(\alpha;0)+C_3(\alpha;0))\left|\Upsilon'(\eta_1)\right|+(C_2(\alpha;0)+C_4(\alpha;0))\left|\Upsilon'(\eta_2)\right|\right].
\end{aligned}$$

(iii) *If we* $\lambda=\mu=\frac{1}{3}$, *then we obtain the following Simpson's type inequality:*

$$\begin{aligned}
&\left|\frac{1}{6}\left[\Upsilon(\eta_1)+4\Upsilon\left(\frac{\eta_1+\eta_2}{2}\right)+\Upsilon(\eta_2)\right]\right.\\
&\left.-\frac{\Gamma(\alpha+1)}{2^{1-\alpha}(\eta_2-\eta_1)^{\alpha}}\left[J^{\alpha}_{\eta_1+}\Upsilon\left(\frac{\eta_1+\eta_2}{2}\right)+\ J^{\alpha}_{\left(\frac{\eta_1+\eta_2}{2}\right)+}\Upsilon(\eta_2)\right]\right|\\
\leq\ &\frac{(\eta_2-\eta_1)}{4}\left[\left(C_1\left(\alpha;\frac{1}{3}\right)+C_3\left(\alpha;\frac{1}{3}\right)\right)\left|\Upsilon'(\eta_1)\right|+\left(C_2\left(\alpha;\frac{1}{3}\right)+C_4\left(\alpha;\frac{1}{3}\right)\right)\left|\Upsilon'(\eta_2)\right|\right].
\end{aligned}$$

Remark 9. *In Corollary 4 part (ii), we have*

(i) *If we set $\lambda=\mu=1$, then we obtain the following trapezoidal type inequality:*

$$\begin{aligned}
&\left|\frac{\Upsilon(\eta_1)+\Upsilon(\eta_2)}{2}-\frac{\Gamma_k(\alpha+k)}{2^{\frac{k-\alpha}{k}}(\eta_2-\eta_1)^{\frac{\alpha}{k}}}\left[\mathcal{J}^{\alpha,k}_{\eta_1+}\Upsilon\left(\frac{\eta_1+\eta_2}{2}\right)+\ \mathcal{J}^{\alpha,k}_{\left(\frac{\eta_1+\eta_2}{2}\right)+}\Upsilon(\eta_2)\right]\right|\\
\leq\ &\frac{(\eta_2-\eta_1)}{4}\left[(C_1(\alpha;1;k)+C_3(\alpha;1;k))\left|\Upsilon'(\eta_1)\right|+(C_2(\alpha;1;k)+C_4(\alpha;1;k))\left|\Upsilon'(\eta_2)\right|\right].
\end{aligned}$$

(ii) *If we set $\lambda=\mu=0$, then we obtain the following midpoint type inequality:*

$$\begin{aligned}
&\left|\Upsilon\left(\frac{\eta_1+\eta_2}{2}\right)-\frac{\Gamma_k(\alpha+k)}{2^{\frac{k-\alpha}{k}}(\eta_2-\eta_1)^{\frac{\alpha}{k}}}\left[\mathcal{J}^{\alpha,k}_{\eta_1+}\Upsilon\left(\frac{\eta_1+\eta_2}{2}\right)+\ \mathcal{J}^{\alpha,k}_{\left(\frac{\eta_1+\eta_2}{2}\right)+}\Upsilon(\eta_2)\right]\right|\\
\leq\ &\frac{(\eta_2-\eta_1)}{4}\left[(C_1(\alpha;0;k)+C_3(\alpha;0;k))\left|\Upsilon'(\eta_1)\right|+(C_2(\alpha;0;k)+C_4(\alpha;0;k))\left|\Upsilon'(\eta_2)\right|\right].
\end{aligned}$$

(iii) *If we $\lambda=\mu=\frac{1}{3}$, then we obtain the following Simpson's type inequality:*

$$\begin{aligned}
&\left|\frac{1}{6}\left[\Upsilon(\eta_1)+4\Upsilon\left(\frac{\eta_1+\eta_2}{2}\right)+\Upsilon(\eta_2)\right]\right.\\
&\left.-\frac{\Gamma_k(\alpha+k)}{2^{\frac{k-\alpha}{k}}(\eta_2-\eta_1)^{\frac{\alpha}{k}}}\left[\mathcal{J}^{\alpha,k}_{\eta_1+}\Upsilon\left(\frac{\eta_1+\eta_2}{2}\right)+\ \mathcal{J}^{\alpha,k}_{\left(\frac{\eta_1+\eta_2}{2}\right)+}\Upsilon(\eta_2)\right]\right|\\
\leq\ &\frac{(\eta_2-\eta_1)}{4}\left[\left(C_1\left(\alpha;\frac{1}{3};k\right)+C_3\left(\alpha;\frac{1}{3};k\right)\right)\left|\Upsilon'(\eta_1)\right|+\left(C_2\left(\alpha;\frac{1}{3};k\right)+C_4\left(\alpha;\frac{1}{3};k\right)\right)\left|\Upsilon'(\eta_2)\right|\right].
\end{aligned}$$

Remark 10. *In Corollary 6 part (i), we have*

(i) *If we set $\lambda=\mu=1$, then we obtain the following trapezoidal type inequality:*

$$\begin{aligned}
&\left|\frac{\Upsilon(\eta_1)+\Upsilon(\eta_2)}{2}-\frac{\Gamma(\alpha+1)}{2^{1-\alpha}(\eta_2-\eta_1)^{\alpha}}\left[J^{\alpha}_{\eta_1+}\Upsilon\left(\frac{\eta_1+\eta_2}{2}\right)+\ J^{\alpha}_{\left(\frac{\eta_1+\eta_2}{2}\right)+}\Upsilon(\eta_2)\right]\right|\\
\leq\ &\frac{(\eta_2-\eta_1)}{4}\left[C_5^{1-\frac{1}{r}}(\alpha;1)\left(C_1(\alpha;1)\left|\Upsilon'(\eta_1)\right|^r+C_2(\alpha;1)\left|\Upsilon'(\eta_2)\right|^r\right)^{\frac{1}{r}}\right.\\
&\left.+C_6^{1-\frac{1}{r}}(\alpha;1)\left(C_3(\alpha;1)\left|\Upsilon'(\eta_1)\right|^r+C_4(\alpha;1)\left|\Upsilon'(\eta_2)\right|^r\right)^{\frac{1}{r}}\right].
\end{aligned}$$

(ii) *If we set $\lambda = \mu = 0$, then we obtain the following midpoint type inequality:*

$$\begin{aligned}&\left|\Upsilon\left(\frac{\eta_1+\eta_2}{2}\right)-\frac{\Gamma(\alpha+1)}{2^{1-\alpha}(\eta_2-\eta_1)^{\alpha}}\left[J^{\alpha}_{\eta_1+}\Upsilon\left(\frac{\eta_1+\eta_2}{2}\right)+\ J^{\alpha}_{\left(\frac{\eta_1+\eta_2}{2}\right)+}\Upsilon(\eta_2)\right]\right|\\ \leq\ &\frac{(\eta_2-\eta_1)}{4}\left[C_5^{1-\frac{1}{r}}(\alpha;0)\left(C_1(\alpha;0)\left|\Upsilon'(\eta_1)\right|^r+C_2(\alpha;0)\left|\Upsilon'(\eta_2)\right|^r\right)^{\frac{1}{r}}\right.\\ &\left.+C_6^{1-\frac{1}{r}}(\alpha;0)\left(C_3(\alpha;0)\left|\Upsilon'(\eta_1)\right|^r+C_4(\alpha;0)\left|\Upsilon'(\eta_2)\right|^r\right)^{\frac{1}{r}}\right].\end{aligned}$$

(iii) *If we $\lambda = \mu = \frac{1}{3}$, then we obtain the following Simpson's type inequality:*

$$\begin{aligned}&\left|\frac{1}{6}\left[\Upsilon(\eta_1)+4\Upsilon\left(\frac{\eta_1+\eta_2}{2}\right)+\Upsilon(\eta_2)\right]\right.\\ &\left.-\frac{\Gamma(\alpha+1)}{2^{1-\alpha}(\eta_2-\eta_1)^{\alpha}}\left[J^{\alpha}_{\eta_1+}\Upsilon\left(\frac{\eta_1+\eta_2}{2}\right)+\ J^{\alpha}_{\left(\frac{\eta_1+\eta_2}{2}\right)+}\Upsilon(\eta_2)\right]\right|\\ \leq\ &\frac{(\eta_2-\eta_1)}{4}\left[C_5^{1-\frac{1}{r}}\left(\alpha;\frac{1}{3}\right)\left(C_1\left(\alpha;\frac{1}{3}\right)\left|\Upsilon'(\eta_1)\right|^r+C_2\left(\alpha;\frac{1}{3}\right)\left|\Upsilon'(\eta_2)\right|^r\right)^{\frac{1}{r}}\right.\\ &\left.+C_6^{1-\frac{1}{r}}\left(\alpha;\frac{1}{3}\right)\left(C_3\left(\alpha;\frac{1}{3}\right)\left|\Upsilon'(\eta_1)\right|^r+C_4\left(\alpha;\frac{1}{3}\right)\left|\Upsilon'(\eta_2)\right|^r\right)^{\frac{1}{r}}\right].\end{aligned}$$

Remark 11. *In Corollary 6 part (ii), we have*

(i) *If we set $\lambda = \mu = 1$, then we obtain the following trapezoidal type inequality:*

$$\begin{aligned}&\left|\frac{\Upsilon(\eta_1)+\Upsilon(\eta_2)}{2}-\frac{\Gamma_k(\alpha+k)}{2^{\frac{k-\alpha}{k}}(\eta_2-\eta_1)^{\frac{\alpha}{k}}}\left[\mathcal{J}^{\alpha,k}_{\eta_1+}\Upsilon\left(\frac{\eta_1+\eta_2}{2}\right)+\ \mathcal{J}^{\alpha,k}_{\left(\frac{\eta_1+\eta_2}{2}\right)+}\Upsilon(\eta_2)\right]\right|\\ \leq\ &\frac{(\eta_2-\eta_1)}{4}\left[C_5^{1-\frac{1}{r}}(\alpha;1;k)\left(C_1(\alpha;1;k)\left|\Upsilon'(\eta_1)\right|^r+C_2(\alpha;1;k)\left|\Upsilon'(\eta_2)\right|^r\right)^{\frac{1}{r}}\right.\\ &\left.+C_6^{1-\frac{1}{r}}(\alpha;1)\left(C_3(\alpha;1;k)\left|\Upsilon'(\eta_1)\right|^r+C_4(\alpha;1;k)\left|\Upsilon'(\eta_2)\right|^r\right)^{\frac{1}{r}}\right].\end{aligned}$$

(ii) *If we set $\lambda = \mu = 0$, then we obtain the following midpoint type inequality:*

$$\begin{aligned}&\left|\Upsilon\left(\frac{\eta_1+\eta_2}{2}\right)-\frac{\Gamma_k(\alpha+k)}{2^{\frac{k-\alpha}{k}}(\eta_2-\eta_1)^{\frac{\alpha}{k}}}\left[\mathcal{J}^{\alpha,k}_{\eta_1+}\Upsilon\left(\frac{\eta_1+\eta_2}{2}\right)+\ \mathcal{J}^{\alpha,k}_{\left(\frac{\eta_1+\eta_2}{2}\right)+}\Upsilon(\eta_2)\right]\right|\\ \leq\ &\frac{(\eta_2-\eta_1)}{4}\left[C_5^{1-\frac{1}{r}}(\alpha;0;k)\left(C_1(\alpha;0;k)\left|\Upsilon'(\eta_1)\right|^r+C_2(\alpha;0;k)\left|\Upsilon'(\eta_2)\right|^r\right)^{\frac{1}{r}}\right.\\ &\left.+C_6^{1-\frac{1}{r}}(\alpha;0)\left(C_3(\alpha;0;k)\left|\Upsilon'(\eta_1)\right|^r+C_4(\alpha;0;k)\left|\Upsilon'(\eta_2)\right|^r\right)^{\frac{1}{r}}\right].\end{aligned}$$

(iii) *If we* $\lambda = \mu = \frac{1}{3}$, *then we obtain the following Simpson's type inequality:*

$$\begin{aligned}
&\left|\frac{1}{6}\left[\Upsilon(\eta_1) + 4\Upsilon\left(\frac{\eta_1+\eta_2}{2}\right) + \Upsilon(\eta_2)\right]\right. \\
&\left.-\frac{\Gamma_k(\alpha+k)}{2^{\frac{k-\alpha}{k}}(\eta_2-\eta_1)^{\frac{\alpha}{k}}}\left[\mathcal{J}^{\alpha,k}_{\eta_1+}\Upsilon\left(\frac{\eta_1+\eta_2}{2}\right) + \mathcal{J}^{\alpha,k}_{\left(\frac{\eta_1+\eta_2}{2}\right)+}\Upsilon(\eta_2)\right]\right| \\
\leq\; & \frac{(\eta_2-\eta_1)}{4}\left[C_5^{1-\frac{1}{r}}\left(\alpha;\frac{1}{3};k\right)\left(C_1\left(\alpha;\frac{1}{3};k\right)\left|\Upsilon'(\eta_1)\right|^r + C_2\left(\alpha;\frac{1}{3};k\right)\left|\Upsilon'(\eta_2)\right|^r\right)^{\frac{1}{r}}\right. \\
&\left.+C_6^{1-\frac{1}{r}}\left(\alpha;\frac{1}{3}\right)\left(C_3\left(\alpha;\frac{1}{3};k\right)\left|\Upsilon'(\eta_1)\right|^r + C_4\left(\alpha;\frac{1}{3};k\right)\left|\Upsilon'(\eta_2)\right|^r\right)^{\frac{1}{r}}\right].
\end{aligned}$$

Remark 12. *In Corollary 8 part (i), we have*

(i) *If we set* $\lambda = \mu = 1$, *then we obtain the following trapezoidal type inequality:*

$$\begin{aligned}
&\left|\frac{\Upsilon(\eta_1)+\Upsilon(\eta_2)}{2} - \frac{\Gamma(\alpha+1)}{2^{1-\alpha}(\eta_2-\eta_1)^{\alpha}}\left[J^{\alpha}_{\eta_1+}\Upsilon\left(\frac{\eta_1+\eta_2}{2}\right) + J^{\alpha}_{\left(\frac{\eta_1+\eta_2}{2}\right)+}\Upsilon(\eta_2)\right]\right| \\
\leq\; & \frac{(\eta_2-\eta_1)}{4}\left[A_4^{\alpha}(1;s)\left(\frac{3|\Upsilon'(\eta_1)|^r + |\Upsilon'(\eta_2)|^r}{4}\right)^{\frac{1}{r}}\right. \\
&\left.+B_4^{\alpha}(1;s)\left(\frac{|\Upsilon'(\eta_1)|^r + 3|\Upsilon'(\eta_2)|^r}{4}\right)^{\frac{1}{r}}\right].
\end{aligned}$$

(ii) *If we set* $\lambda = \mu = 0$, *then we obtain the following midpoint type inequality:*

$$\begin{aligned}
&\left|\Upsilon\left(\frac{\eta_1+\eta_2}{2}\right) - \frac{\Gamma(\alpha+1)}{2^{1-\alpha}(\eta_2-\eta_1)^{\alpha}}\left[J^{\alpha}_{\eta_1+}\Upsilon\left(\frac{\eta_1+\eta_2}{2}\right) + J^{\alpha}_{\left(\frac{\eta_1+\eta_2}{2}\right)+}\Upsilon(\eta_2)\right]\right| \\
\leq\; & \frac{(\eta_2-\eta_1)}{4}\left[A_4^{\alpha}(0;s)\left(\frac{3|\Upsilon'(\eta_1)|^r + |\Upsilon'(\eta_2)|^r}{4}\right)^{\frac{1}{r}}\right. \\
&\left.+B_4^{\alpha}(0;s)\left(\frac{|\Upsilon'(\eta_1)|^r + 3|\Upsilon'(\eta_2)|^r}{4}\right)^{\frac{1}{r}}\right].
\end{aligned}$$

(iii) *If we* $\lambda = \mu = \frac{1}{3}$, *then we obtain the following Simpson's type inequality:*

$$\begin{aligned}
&\left|\frac{1}{6}\left[\Upsilon(\eta_1) + 4\Upsilon\left(\frac{\eta_1+\eta_2}{2}\right) + \Upsilon(\eta_2)\right]\right. \\
&\left.-\frac{\Gamma(\alpha+1)}{2^{1-\alpha}(\eta_2-\eta_1)^{\alpha}}\left[J^{\alpha}_{\eta_1+}\Upsilon\left(\frac{\eta_1+\eta_2}{2}\right) + J^{\alpha}_{\left(\frac{\eta_1+\eta_2}{2}\right)+}\Upsilon(\eta_2)\right]\right| \\
\leq\; & \frac{(\eta_2-\eta_1)}{4}\left[A_4^{\alpha}\left(\frac{1}{3};s\right)\left(\frac{3|\Upsilon'(\eta_1)|^r + |\Upsilon'(\eta_2)|^r}{4}\right)^{\frac{1}{r}}\right. \\
&\left.+B_4^{\alpha}\left(\frac{1}{3};s\right)\left(\frac{|\Upsilon'(\eta_1)|^r + 3|\Upsilon'(\eta_2)|^r}{4}\right)^{\frac{1}{r}}\right].
\end{aligned}$$

Remark 13. *In Corollary 8 part (ii), we have*

(i) *If we set* $\lambda = \mu = 1$, *then we obtain the following trapezoidal type inequality:*

$$\left|\frac{\Upsilon(\eta_1)+\Upsilon(\eta_2)}{2} - \frac{\Gamma_k(\alpha+k)}{2^{\frac{k-\alpha}{k}}(\eta_2-\eta_1)^{\frac{\alpha}{k}}}\left[\mathcal{J}^{\alpha,k}_{\eta_1+}\Upsilon\left(\frac{\eta_1+\eta_2}{2}\right) + \mathcal{J}^{\alpha,k}_{\left(\frac{\eta_1+\eta_2}{2}\right)+}\Upsilon(\eta_2)\right]\right|$$
$$\leq \frac{(\eta_2-\eta_1)}{4}\left[A_4^{\frac{\alpha}{k}}(1;s)\left(\frac{3|\Upsilon'(\eta_1)|^r + |\Upsilon'(\eta_2)|^r}{4}\right)^{\frac{1}{r}} + B_4^{\frac{\alpha}{k}}(1;s)\left(\frac{|\Upsilon'(\eta_1)|^r + 3|\Upsilon'(\eta_2)|^r}{4}\right)^{\frac{1}{r}}\right].$$

(ii) *If we set* $\lambda = \mu = 0$, *then we obtain the following midpoint type inequality:*

$$\left|\Upsilon\left(\frac{\eta_1+\eta_2}{2}\right) - \frac{\Gamma_k(\alpha+k)}{2^{\frac{k-\alpha}{k}}(\eta_2-\eta_1)^{\frac{\alpha}{k}}}\left[\mathcal{J}^{\alpha,k}_{\eta_1+}\Upsilon\left(\frac{\eta_1+\eta_2}{2}\right) + \mathcal{J}^{\alpha,k}_{\left(\frac{\eta_1+\eta_2}{2}\right)+}\Upsilon(\eta_2)\right]\right|$$
$$\leq \frac{(\eta_2-\eta_1)}{4}\left[A_4^{\frac{\alpha}{k}}(0;s)\left(\frac{3|\Upsilon'(\eta_1)|^r + |\Upsilon'(\eta_2)|^r}{4}\right)^{\frac{1}{r}} + B_4^{\frac{\alpha}{k}}(0;s)\left(\frac{|\Upsilon'(\eta_1)|^r + 3|\Upsilon'(\eta_2)|^r}{4}\right)^{\frac{1}{r}}\right].$$

(iii) *If we* $\lambda = \mu = \frac{1}{3}$, *then we obtain the following Simpson's type inequality:*

$$\left|\frac{1}{6}\left[\Upsilon(\eta_1) + 4\Upsilon\left(\frac{\eta_1+\eta_2}{2}\right) + \Upsilon(\eta_2)\right] - \frac{\Gamma_k(\alpha+k)}{2^{\frac{k-\alpha}{k}}(\eta_2-\eta_1)^{\frac{\alpha}{k}}}\left[\mathcal{J}^{\alpha,k}_{\eta_1+}\Upsilon\left(\frac{\eta_1+\eta_2}{2}\right) + \mathcal{J}^{\alpha,k}_{\left(\frac{\eta_1+\eta_2}{2}\right)+}\Upsilon(\eta_2)\right]\right|$$
$$\leq \frac{(\eta_2-\eta_1)}{4}\left[A_4^{\frac{\alpha}{k}}\left(\frac{1}{3};s\right)\left(\frac{3|\Upsilon'(\eta_1)|^r + |\Upsilon'(\eta_2)|^r}{4}\right)^{\frac{1}{r}} + B_4^{\frac{\alpha}{k}}\left(\frac{1}{3};s\right)\left(\frac{|\Upsilon'(\eta_1)|^r + 3|\Upsilon'(\eta_2)|^r}{4}\right)^{\frac{1}{r}}\right].$$

Remark 14. *With different choices of* φ *like* $\varphi(t) = t$, $\varphi(t) = \frac{t^\alpha}{\Gamma(\alpha)}$, $\varphi(t) = \frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$, $\varphi(t) = \frac{1}{\Gamma(\alpha)}t(x-t)^s\left(x^{s+1}-t^{s+1}\right)^{\alpha-1}$, $\varphi(t) = \frac{1}{\Gamma(\alpha)}\frac{[\log(x)-\log(x-t)]^{\alpha-1}}{x-t}$ *and* $\varphi(t) = \frac{t}{\alpha}\exp\left(-\frac{1-\alpha}{\alpha}t\right)$ *in Theorems 2–4 and Corollaries 3, 5 and 7, we can establish more midpoint inequalities, trapezoidal inequalities and Simpson's inequalities through classical Riemann integral, fractional Riemann–Liouville fractional integrals, k-fractional integrals, Hadamard fractional integrals and fractional integrals with exponential kernel.*

6. Conclusions

In this work, we proved some parameterized integral inequalities for differentiable convex functions via generalized fractional integrals. The advantage of parameters involved in integral inequalities is that these inequalities could be converted into midpoint inequalities, trapezoidal inequalities and Simpson's inequalities for differentiable convex functions. Moreover, the advantage of fractional integrals used in these inequalities is that these inequalities could be converted into classical integral inequalities, Riemann–Liouville fractional integrals inequalities and k-fractional integrals inequalities without proving them separately. It is an interesting and new problem that the upcoming researchers can obtain similar inequalities for functions of two variables.

Author Contributions: Conceptualization, D.Z., M.A.A., C.P. and T.S.; methodology, D.Z., M.A.A., C.P. and T.S.; software, D.Z., M.A.A., C.P. and T.S.; validation, D.Z., M.A.A., C.P. and T.S.; investigation, D.Z., M.A.A., C.P. and T.S.; writing—original draft preparation, D.Z., M.A.A., C.P. and T.S.; writing—review and editing, D.Z., M.A.A., C.P. and T.S.; visualization, D.Z., M.A.A., C.P. and T.S.; supervision, D.Z.; funding acquisition, D.Z. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by Open Fund of National Cryosphere Desert Data Center of China (2021kf03), and Key Projects of the Educational Commission of Hubei Province of China (D20192501). This work was also supported by National Science, Research and Innovation Fund (NSRF) and King Mongkut's University of Technology North Bangkok. Contract no. KMUTNB-FF-65-49.

Data Availability Statement: Not applicable.

Acknowledgments: We thanks to worthy referees and editor for their valuable comments.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Sarikaya, M.Z.; Set, E.; Yaldiz, H.; Başak, N. Hermite–Hadamard's inequalities for fractional integrals and relatedfractional inequalities. *Math. Comput. Model.* **2013**, *57*, 2403–2407. [CrossRef]
2. Set, E. New inequalities of Ostrowski type for mappings whose derivatives are *s*-convex in the second sense via fractional integrals. *Comput. Math. Appl.* **2012**, *63*, 1147–1154. [CrossRef]
3. İşcan, İ.; Wu, S. Hermite–Hadamard type inequalities for harmonically convex functions via fractional integrals. *Appl. Math. Comput.* **2014**, *238*, 237–244. [CrossRef]
4. Sarikaya, M.Z.; Yildrim, H. On Hermite-Hadamard type inequalities for Riemann-Liouville fractional integrals. *Miskolc Math. Notes* **2016**, *17*, 1049–1059. [CrossRef]
5. Peng, C.; Zhou, C.; Du, T.S. Riemann-Liouville fractional Simpson's inequalities through generalized (m, h_1, h_2)-preinvexity. *Ital. J. Pure Appl. Math.* **2017**, *38*, 345–367.
6. Mubeen, S.; Habibullah, G.M. *k*-fractional integrals and application. *Int. J. Contemp. Math. Sci.* **2012**, *7*, 89–94.
7. Agarwal, P.; Jleli, M.; Tamor, M. Certain Hermite-Hadamard type inequalities via generalized *k*-fractional integrals. *J. Inequal. Appl.* **2017**, *2017*, 55. [CrossRef]
8. Farid, G.; Usman, M. Ostrowski type *k*-fractional integral inequalities for MT-convex and *h*-convex functions. *Nonlinear Funct. Anal. Appl.* **2017**, *22*, 627–639.
9. Zhang, Y.; Du, T.S.; Wang, H.; Shen, Y.J.; Kashuri, A. Extensions of different type parameterized inequalities for generalized (m, h)-preinvex mappings via *k*-fractional integrals. *J. Inequal. Appl.* **2018**, *2018*, 49. [CrossRef]
10. Sarikaya, M.Z.; Ertugral, F. On the generalized Hermite-Hadamard inequalities. *Ann. Univ. Craiova-Math. Comput. Sci. Ser.* **2020**, *47*, 193–213.
11. Zhao, D.F.; Ali, M.A.; Kashuri, A.; Budak, H. Generalized fractional integral inequalities of Hermite–Hadamard type for harmonically convex functions. *Adv. Differ. Equ.* **2020**, *2020*, 1–14. [CrossRef]
12. Budak, H.; Hezenci, F.; Kara, H. On parameterized inequalities of Ostrowski and Simpson type for convex functions via generalized fractional integral. *Math. Methods Appl. Sci.* **2021**, *44*, 12522–12536. [CrossRef]
13. Akdemir, A.O.; Butt, S.I.; Nadeem, M.; Ragusa, M.A. New general variants of Chebyshev type inequalities via generalized fractional integral operators. *Mathematics* **2021**, *9*, 122. [CrossRef]
14. Awan, M.U.; Talib, S.; Chu, Y.M.; Noor, M.A.; Noor, K.I. Some new refinements of Hermite–Hadamard-type inequalities involving-Riemann–Liouville fractional integrals and applications. *Math. Probl. Eng.* **2020**, *2020*, 3051920. [CrossRef]
15. Butt, S.I.; Nadeem, M.; Farid, G. On Caputo fractional derivatives via exponential *s*-convex functions. *Turk. J. Sci.* **2020**, *5*, 140–146.
16. Ekincia, A.; Özdemir, M.E.; Set, E. New Integral Inequalities of Ostrowski Type for Quasi-Convex Functions with Applications. *Turk. J. Sci.* **2020**, *5*, 290–304.
17. Ekincia, A.; Özdemir, M.E. Some New Integral Inequalities Via Riemann-Liouville Integral Operators. *Appl. Comput. Math.* **2019**, *3*, 288–295.
18. Kashuri, A.; Liko, R. Generalized trapezoidal type integral inequalities and their applications. *J. Anal.* **2020**, *28*, 1023–1043. [CrossRef]
19. Khan, M.A.; Iqbal, A.; Suleman, M.; Chu, Y.M. Hermite–Hadamard type inequalities for fractional integrals via Green's function. *J. Inequal. Appl.* **2018**, *2018*, 161. [CrossRef]
20. Khan, M.A.; Ali, T.; Dragomir, S.S.; Sarikaya, M.Z. Hermite–Hadamard type inequalities for conformable fractional integrals. *Rev. Real Acad. Cienc. Exactas Fís. Nat. A Mat.* **2018**, *112*, 1033–1048. [CrossRef]

21. Set, E.; Choi, J.; Gözpinar, A. Hermite-Hadamard type inequalities for the generalized k-fractional integral operators. *J. Inequal. Appl.* **2017**, *2017*, 206. [CrossRef] [PubMed]
22. Tunc, M. On new inequalities for *h*-convex functions via Riemann-Liouville fractional integration. *Filomat* **2013**, *27*, 559–565. [CrossRef]
23. Vivas-Cortez, M.; Ali, M.A.; Kashuri, A.; Budak, H. Generalizations of fractional Hermite-Hadamard-Mercer like inequalities for convex functions. *AIMS Math.* **2021**, *6*, 9397–9421. [CrossRef]
24. Zhao, D.F.; Ali, M.A.; Kashuri, A.; Budak, H.; Sarikaya, M.Z. Hermite–Hadamard-type inequalities for the interval-valued approximately *h*-convex functions via generalized fractional integrals. *J. Inequal. Appl.* **2020**, *2020*, 222. [CrossRef]
25. Gorenflo, R.; Mainardi, F. *Fractional Calculus: Integral and Differential Equations of Fractional Order*; Springer: Wien, Austria, 1997.
26. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*; Elsevier: Amsterdam, The Netherlands, 2006.
27. Pećarixcx, J.E.; Proschan, F.; Tong, Y.L. *Convex Functions, Partial Orderings and Statistical Applications*; Academic Press: Boston, MA, USA, 1992.
28. Farid, G.; Rehman, A.U.; Zahra, M. On Hadamard inequalities for *k*-fractional integrals. *Nonlinear Funct. Appl.* **2016**, *21*, 463–478.
29. Xi, B.Y.; Qi, F. Some integral inequalities of Hermite-Hadamard type for convex functions with applications to means. *J. Funct. Spaces* **2012**, *2012*, 980438. [CrossRef]