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# Reverse Minkowski Inequalities Pertaining to New Weighted Generalized Fractional Integral Operators

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**Abstract:** In this paper, we obtain reverse Minkowski inequalities pertaining to new weighted generalized fractional integral operators. Moreover, we derive several important special cases for suitable choices of functions. In order to demonstrate the efficiency of our main results, we offer many concrete examples as applications.

**Keywords:** Minkowski inequalities; weighted generalized fractional integral operators; inequality

## 1. Introduction

Fractional calculus, the study of integrals and derivatives of arbitrary order, is crucial in several problems in mathematics and its related applications (see [1–5]). In addition, it appears in many fields of applied science where integral inequalities are used (see [6–11]).

Moreover, integral inequalities link with other areas such as mathematical analysis, mathematical physics, differential equations, difference equations, discrete fractional calculus and convexity theory (see [12–17]).

**Definition 1** ([18]). Assume that  $\omega$  is a function defined on  $[\zeta_1, \zeta_2]$ . The left and right Riemann-Liouville fractional integrals of order  $\alpha > 0$  are given by

$$\left(\mathcal{I}_{\zeta_1^+}^\alpha \omega\right)(x) = \frac{1}{\Gamma(\alpha)} \int_{\zeta_1}^x (x - \ell)^{\alpha-1} \omega(\ell) d\ell \quad (x > \zeta_1),$$

$$\left(\mathcal{I}_{\zeta_2^-}^\alpha \omega\right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^{\zeta_2} (\ell - x)^{\alpha-1} \omega(\ell) d\ell \quad (x < \zeta_2),$$

respectively.

For further information about fractional integrals and the ways they are defined, see [19–28].

One of the basic types of integral inequalities is the Chebyshev inequality and it is given as follows (see [29–34]):

$$\frac{1}{\zeta_2 - \zeta_1} \int_{\zeta_1}^{\zeta_2} \omega_1(\ell) \omega_2(\ell) d\ell \geq \left( \frac{1}{\zeta_2 - \zeta_1} \int_{\zeta_1}^{\zeta_2} \omega_1(\ell) d\ell \right) \left( \frac{1}{\zeta_2 - \zeta_1} \int_{\zeta_1}^{\zeta_2} \omega_2(\ell) d\ell \right), \quad (1)$$

where  $\omega_1$  and  $\omega_2$  are assumed to be integrable and synchronous functions on  $[\zeta_1, \zeta_2]$ . Therefore, the following inequality holds:

$$(\omega_1(x) - \omega_1(y))(\omega_2(x) - \omega_2(y)) \geq 0$$



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for all  $x, y \in [\zeta_1, \zeta_2]$ .

The Chebyshev inequality (1) is especially useful due to its links with fractional calculus and in the existence of solutions to various fractional-order differential equations (see [35–43]).

Dahmani [44] established reverse Minkowski fractional integral inequalities. In [45], the authors, using Katugampola fractional integral operators, derived several Minkowski inequalities. In [46,47], the authors, via Hadamard fractional integral operators, obtained the reverse Minkowski inequality. Rahman et al. [48] derived Minkowski inequalities via generalized proportional fractional integral operators. Set et al. [49] gave reverse Minkowski inequalities via Riemann-Liouville fractional integrals. In [50], Bougoffa found Hardy's and reverse Minkowski inequalities. Nale et al. [51], using generalized proportional Hadamard fractional integral operators, established Minkowski-type inequalities.

Motivated by the above results and literature, this paper is organized as follows: In Section 2, we recall some basic definitions and introduce the new general family of fractional integral operators. In Section 3, we establish reverse Minkowski inequalities pertaining to this new family of fractional integral operators. In order to demonstrate the significance of our main results, we obtain several important special cases for suitable choices of functions. In Section 4, we derive many concrete examples as applications of our results. Conclusions and future research are given in Section 5.

## 2. Preliminaries

Special functions have many relations with fractional calculus (see [16,52–59]).

We recall the Fox–Wright hypergeometric function  ${}_p\Psi_q(z)$ , which is given by the following series (see [5,60–63]):

$${}_p\Psi_q \left[ \begin{matrix} (t_1, \mathcal{M}_1), \dots, (t_p, \mathcal{M}_p); \\ (J_1, \mathcal{N}_1), \dots, (J_q, \mathcal{M}_q); \end{matrix} z \right] := \sum_{n=0}^{\infty} \frac{\prod_{j=1}^p \Gamma(t_j + \mathcal{M}_j n)}{\prod_{k=1}^q \Gamma(J_k + \mathcal{N}_k n)} \frac{z^n}{n!}, \quad (2)$$

where

$$t_j, J_k \in \mathbb{C} \quad (j = 1, \dots, p; k = 1, \dots, q)$$

with

$$\mathcal{M}_1, \dots, \mathcal{M}_p \in \mathbb{R}^+ \quad \text{and} \quad \mathcal{N}_1, \dots, \mathcal{N}_q \in \mathbb{R}^+$$

satisfies

$$1 + \sum_{k=1}^q \mathcal{N}_k - \sum_{j=1}^p \mathcal{M}_j \geq 0. \quad (3)$$

We turn to a modified version  $\mathcal{R}_{\varrho, \eta}^{\sigma}(z)$  of the Fox-Wright function  ${}_p\Psi_q(z)$  in (2), which was introduced by Wright (see [64], p. 424) as follows:

$$\mathcal{R}_{\varrho, \eta}^{\sigma}(z) = \mathcal{R}_{\varrho, \eta}^{\sigma(0), \sigma(1), \dots}(z) := \sum_{n=0}^{\infty} \frac{\sigma(n)}{\Gamma(\varrho n + \eta)} z^n, \quad (4)$$

where  $\varrho, \eta > 0$ ,  $|z| < R$ , with the bounded sequence  $\{\sigma(n)\}_{n \in \mathbb{N}_0}$  in the real-number set  $\mathbb{R}$ . As already remarked in, for example, [65], this same function  $\mathcal{R}_{\varrho, \eta}^{\sigma}$  was reproduced in [66], but without giving any credit to Wright [64]. In fact, in his recent survey-cum-expository review articles, the above-defined Wright function  $\mathcal{R}_{\varrho, \eta}^{\sigma}$  in (4) as well as its well-motivated companions and extensions were used as the kernels in order to systematically study some general families of fractional calculus operators (fractional integral and fractional derivative) by Srivastava (see, for details, [67]).

**Definition 2** ([65,67,68]). For a given  $\mathcal{L}_1$ -function  $\omega$  on an interval  $[\zeta_1, \zeta_2]$ , the general left-side and right-side fractional integral operators, applied to a prescribed function  $\omega(x)$ , are defined for  $\eta, \varrho > 0$  and  $w \in \mathbb{R}$  by

$$\left(\mathcal{S}_{\varrho, \eta, \zeta_1^+; w}^\sigma \omega\right)(x) = \int_{\zeta_1}^x (x - \zeta)^{\eta-1} \mathcal{R}_{\varrho, \eta}^\sigma [w(x - \zeta)^\varrho] \omega(\zeta) \, d\zeta \quad (x > \zeta_1) \quad (5)$$

and

$$\left(\mathcal{S}_{\varrho, \eta, \zeta_2^-; w}^\sigma \omega\right)(x) = \int_x^{\zeta_2} (\zeta - x)^{\eta-1} \mathcal{R}_{\varrho, \eta}^\sigma [w(\zeta - x)^\varrho] \omega(\zeta) \, d\zeta \quad (x < \zeta_2), \quad (6)$$

where the integrals on the right-hand sides exist and  $\mathcal{R}_{\varrho, \eta}^\sigma$  is the Wright function defined by (4).

**Remark 1** ([69]). The function  $\vartheta : [0, \infty) \rightarrow [0, \infty)$ , fulfills the conditions:

$$\int_0^1 \frac{\vartheta(\zeta)}{\zeta} \, d\zeta < \infty, \quad (7)$$

$$\frac{1}{\mathcal{D}_1} \leq \frac{\vartheta(\zeta_1)}{\vartheta(\zeta_2)} \leq \mathcal{D}_1 \quad \text{for} \quad \frac{1}{2} \leq \frac{\zeta_1}{\zeta_2} \leq 2, \quad (8)$$

$$\frac{\vartheta(\zeta_2)}{\zeta_2^2} \leq \mathcal{D}_2 \frac{\vartheta(\zeta_1)}{\zeta_1^2} \quad \text{for} \quad \zeta_1 \leq \zeta_2 \quad (9)$$

and

$$\left| \frac{\vartheta(\zeta_2)}{\zeta_2^2} - \frac{\vartheta(\zeta_1)}{\zeta_1^2} \right| \leq \mathcal{D}_3 |\zeta_2 - \zeta_1| \frac{\vartheta(\zeta_2)}{\zeta_2^2} \quad \text{for} \quad \frac{1}{2} \leq \frac{\zeta_1}{\zeta_2} \leq 2, \quad (10)$$

where  $\mathcal{D}_1, \mathcal{D}_2$  and  $\mathcal{D}_3 > 0$  are independent of  $\zeta_1, \zeta_2 > 0$ . Furthermore, Sarikaya et al. (see [69]) used the above function  $\vartheta$  in order to define the following definition.

**Definition 3.** The generalized left- and right-side fractional integrals are defined by

$$\zeta_1^+ \mathcal{I}_\vartheta \omega(x) = \int_{\zeta_1}^x \frac{\vartheta(x - \zeta)}{x - \zeta} \omega(\zeta) \, d\zeta \quad (x > \zeta_1) \quad (11)$$

and

$$\zeta_2^- \mathcal{I}_\vartheta \omega(x) = \int_x^{\zeta_2} \frac{\vartheta(\zeta - x)}{\zeta - x} \omega(\zeta) \, d\zeta \quad (x < \zeta_2), \quad (12)$$

respectively.

Sarikaya et al. [69] noticed that the generalized fractional integrals given by Definition 3 may contain different types of fractional integral operators for some special choices of function  $\vartheta$ .

Recently, Srivastava et al. [70] introduced the following general family of fractional integral operators involving the Wright function  $\mathcal{R}_{\varrho, \eta}^\sigma$  defined by (4).

**Definition 4.** For a given  $\mathcal{L}_1$ -function  $\omega$  on an interval  $[\zeta_1, \zeta_2]$ , the generalized left- and right-side fractional integral operators, applied to  $\omega(x)$ , are defined for  $\eta, \varrho > 0$ , and  $w \in \mathbb{R}$  by

$$\left(\mathcal{K}_{\sigma, \varrho, \eta, \zeta_1^+; w}^\vartheta \omega\right)(x) = \int_{\zeta_1}^x \frac{\vartheta(x - \zeta)}{x - \zeta} \mathcal{R}_{\varrho, \eta}^\sigma [w(x - \zeta)^\varrho] \omega(\zeta) \, d\zeta \quad (x > \zeta_1) \quad (13)$$

and

$$\left(\mathcal{K}_{\sigma, \varrho, \eta, \zeta_2^-; w}^\vartheta \omega\right)(x) = \int_x^{\zeta_2} \frac{\vartheta(\zeta - x)}{\zeta - x} \mathcal{R}_{\varrho, \eta}^\sigma [w(\zeta - x)^\varrho] \omega(\zeta) \, d\zeta \quad (x < \zeta_2). \quad (14)$$

Inspired by the above Definition 4, we can define the following weighted generalized fractional integral operators.

**Definition 5.** For a given  $\mathcal{L}_1$ -function  $\omega$  and positive function  $\phi$  which has an inverse on an interval  $[\varsigma_1, \varsigma_2]$ , the weighted generalized left- and right-side fractional integral operators, applied to  $\omega(x)$ , are defined for  $\eta, \rho > 0$  and  $w \in \mathbb{R}$  by

$$\begin{aligned} & \left( \mathcal{K}_{\sigma, \rho, \eta, \varsigma_1^+}^{\vartheta, \phi; w} \omega \right) (x) \\ &= \phi^{-1}(x) \int_{\varsigma_1}^x \frac{\vartheta(x - \varsigma)}{x - \varsigma} \phi(\varsigma) \mathcal{R}_{\rho, \eta}^{\sigma} [w(x - \varsigma)^{\rho}] \omega(\varsigma) \, d\varsigma \quad (x > \varsigma_1) \end{aligned} \tag{15}$$

and

$$\begin{aligned} & \left( \mathcal{K}_{\sigma, \rho, \eta, \varsigma_2^-}^{\vartheta, \phi; w} \omega \right) (x) \\ &= \phi^{-1}(x) \int_x^{\varsigma_2} \frac{\vartheta(\varsigma - x)}{\varsigma - x} \phi(\varsigma) \mathcal{R}_{\rho, \eta}^{\sigma} [w(\varsigma - x)^{\rho}] \omega(\varsigma) \, d\varsigma \quad (x < \varsigma_2). \end{aligned} \tag{16}$$

**Remark 2.** Taking  $\phi(\varsigma) \equiv 1$  for all  $\varsigma \in [\varsigma_1, \varsigma_2]$  in Definition 5, we obtain Definition 4.

**Remark 3.** Two important special cases of our Definition 5 are illustrated as follows:

(I) Taking  $\vartheta(\varsigma) = \varsigma(\varsigma_2 - \varsigma)^{\alpha-1}$  for all  $\varsigma \in [\varsigma_1, \varsigma_2]$  and  $\alpha \in (0, 1]$ , we have the weighted conformable left- and right-side fractional integral operators defined by

$$\begin{aligned} & \left( \mathcal{C}_{\sigma, \rho, \eta, \varsigma_1^+}^{\alpha, \phi; w} \omega \right) (x) \\ &= \phi^{-1}(x) \int_{\varsigma_1}^x (\varsigma + \varsigma_2 - x)^{\alpha-1} \phi(\varsigma) \mathcal{R}_{\rho, \eta}^{\sigma} [w(x - \varsigma)^{\rho}] \omega(\varsigma) \, d\varsigma \quad (x > \varsigma_1) \end{aligned} \tag{17}$$

and

$$\begin{aligned} & \left( \mathcal{C}_{\sigma, \rho, \eta, \varsigma_2^-}^{\alpha, \phi; w} \omega \right) (x) \\ &= \phi^{-1}(x) \int_x^{\varsigma_2} (x + \varsigma_2 - \varsigma)^{\alpha-1} \phi(\varsigma) \mathcal{R}_{\rho, \eta}^{\sigma} [w(\varsigma - x)^{\rho}] \omega(\varsigma) \, d\varsigma \quad (x < \varsigma_2). \end{aligned} \tag{18}$$

(II) Choosing

$$\vartheta(\varsigma) = \frac{\varsigma}{\alpha} \exp(-Q\varsigma),$$

where

$$Q = \frac{1 - \alpha}{\alpha}$$

and  $\alpha \in (0, 1]$  for all  $\varsigma \in [\varsigma_1, \varsigma_2]$ , we obtain the weighted exponential left- and right-side fractional integral operators defined by

$$\begin{aligned} & \left( \mathcal{E}_{\sigma, \rho, \eta, \varsigma_1^+}^{\alpha, \phi; w} \omega \right) (x) \\ &= \frac{1}{\alpha} \phi^{-1}(x) \int_{\varsigma_1}^x \exp(-Q(x - \varsigma)) \phi(\varsigma) \mathcal{R}_{\rho, \eta}^{\sigma} [w(x - \varsigma)^{\rho}] \omega(\varsigma) \, d\varsigma \quad (x > \varsigma_1) \end{aligned} \tag{19}$$

and

$$\begin{aligned} & \left( \mathcal{E}_{\sigma, \rho, \eta, \varsigma_2^-}^{\alpha, \phi; w} \omega \right) (x) \\ &= \frac{1}{\alpha} \phi^{-1}(x) \int_x^{\varsigma_2} \exp(-Q(\varsigma - x)) \phi(\varsigma) \mathcal{R}_{\rho, \eta}^{\sigma} [w(\varsigma - x)^{\rho}] \omega(\varsigma) \, d\varsigma \quad (x < \varsigma_2). \end{aligned} \tag{20}$$

### 3. Main Results

In the sequel, we assume that  $\{\sigma(n)\}_{n \in \mathbb{N}_0}$  is a sequence of non-negative real numbers and the function  $\vartheta : [0, \infty) \rightarrow [0, \infty)$  satisfies the conditions (7)–(10). Moreover, we assume that  $p, q, m$  and  $M$  are positive real numbers, with  $m < M$ , and  $\phi$  is a positive function which has an inverse. Our main results are given below.

**Theorem 1.** Let  $p \geq 1$ , with  $\eta, \varrho > 0$  and  $w \in \mathbb{R}$ . Assume that  $\omega_1(\zeta)$  and  $\omega_2(\zeta)$  are positive functions on  $[\zeta_1, \infty)$  such that  $\left(\mathcal{K}_{\sigma, \varrho, \eta, \zeta_1^+; w}^{\vartheta, \phi} \omega_1^p\right)(\zeta) < \infty$  and  $\left(\mathcal{K}_{\sigma, \varrho, \eta, \zeta_1^+; w}^{\vartheta, \phi} \omega_2^p\right)(\zeta) < \infty$  for all  $\zeta > \zeta_1 \geq 0$ . If  $0 < m \leq \frac{\omega_1(r)}{\omega_2(r)} \leq M$ , where  $r \in (\zeta_1, \zeta)$ , we have

$$\begin{aligned} \left[\left(\mathcal{K}_{\sigma, \varrho, \eta, \zeta_1^+; w}^{\vartheta, \phi} \omega_1^p\right)(\zeta)\right]^{\frac{1}{p}} + \left[\left(\mathcal{K}_{\sigma, \varrho, \eta, \zeta_1^+; w}^{\vartheta, \phi} \omega_2^p\right)(\zeta)\right]^{\frac{1}{p}} \\ \leq \frac{1 + M(m + 2)}{(m + 1)(M + 1)} \left[\left(\mathcal{K}_{\sigma, \varrho, \eta, \zeta_1^+; w}^{\vartheta, \phi} (\omega_1 + \omega_2)^p\right)(\zeta)\right]^{\frac{1}{p}}. \end{aligned}$$

**Proof.** Using the condition  $\frac{\omega_1(r)}{\omega_2(r)} \leq M$ , where  $r \in (\zeta_1, \zeta)$ , we have

$$(M + 1)^p \omega_1^p(r) \leq M^p (\omega_1 + \omega_2)^p(r). \tag{21}$$

By multiplying both sides of (21) by

$$\phi^{-1}(\zeta) \frac{\vartheta(\zeta - r)}{\zeta - r} \phi(r) \mathcal{R}_{\varrho, \eta}^\sigma [w(\zeta - r)^\varrho]$$

with  $r \in (\zeta_1, \zeta)$ , we can deduce that

$$\begin{aligned} (M + 1)^p \phi^{-1}(\zeta) \frac{\vartheta(\zeta - r)}{\zeta - r} \phi(r) \mathcal{R}_{\varrho, \eta}^\sigma [w(\zeta - r)^\varrho] \omega_1^p(r) \\ \leq M^p \phi^{-1}(\zeta) \frac{\vartheta(\zeta - r)}{\zeta - r} \phi(r) \mathcal{R}_{\varrho, \eta}^\sigma [w(\zeta - r)^\varrho] (\omega_1 + \omega_2)^p(r), \end{aligned}$$

which, upon integration over  $r \in (\zeta_1, \zeta)$ , yields

$$\left(\mathcal{K}_{\sigma, \varrho, \eta, \zeta_1^+; w}^{\vartheta, \phi} \omega_1^p\right)(\zeta) \leq \left(\frac{M}{M + 1}\right)^p \left(\mathcal{K}_{\sigma, \varrho, \eta, \zeta_1^+; w}^{\vartheta, \phi} (\omega_1 + \omega_2)^p\right)(\zeta).$$

Hence, we obtain

$$\left[\left(\mathcal{K}_{\sigma, \varrho, \eta, \zeta_1^+; w}^{\vartheta, \phi} \omega_1^p\right)(\zeta)\right]^{\frac{1}{p}} \leq \frac{M}{M + 1} \left[\left(\mathcal{K}_{\sigma, \varrho, \eta, \zeta_1^+; w}^{\vartheta, \phi} (\omega_1 + \omega_2)^p\right)(\zeta)\right]^{\frac{1}{p}}. \tag{22}$$

Similarly, using the condition  $m \leq \frac{\omega_1(r)}{\omega_2(r)}$ , where  $r \in (\zeta_1, \zeta)$ , we obtain

$$\left(1 + \frac{1}{m}\right) \omega_2(r) \leq \frac{1}{m} (\omega_1 + \omega_2)(r).$$

Therefore,

$$\left(1 + \frac{1}{m}\right)^p \omega_2^p(r) \leq \left(\frac{1}{m}\right)^p (\omega_1 + \omega_2)^p(r). \tag{23}$$

Multiplying both sides of (23) by

$$\phi^{-1}(\zeta) \frac{\vartheta(\zeta - r)}{\zeta - r} \phi(r) \mathcal{R}_{\varrho, \eta}^\sigma [w(\zeta - r)^\varrho]$$

with  $r \in (\zeta_1, \zeta)$ , and integrating over  $r \in (\zeta_1, \zeta)$  yields

$$\left[ \left( \mathcal{K}_{\sigma, \rho, \eta, \zeta_1^+; w}^{\vartheta, \phi} \omega_2^p \right) (\zeta) \right]^{\frac{1}{p}} \leq \frac{1}{m+1} \left[ \left( \mathcal{K}_{\sigma, \rho, \eta, \zeta_1^+; w}^{\vartheta, \phi} (\omega_1 + \omega_2)^p \right) (\zeta) \right]^{\frac{1}{p}}. \tag{24}$$

Adding inequalities (22) and (24), we complete the proof of Theorem 1.  $\square$

**Corollary 1.** Under the hypotheses of Theorem 1, if we take

$$\vartheta(\zeta) = \zeta(\zeta_2 - \zeta)^{\alpha-1} \quad (\forall \zeta \in [\zeta_1, \zeta_2]; \alpha \in (0, 1]),$$

we have

$$\begin{aligned} \left[ \left( \mathcal{C}_{\sigma, \rho, \eta, \zeta_1^+; w}^{\alpha, \phi} \omega_1^p \right) (\zeta) \right]^{\frac{1}{p}} + \left[ \left( \mathcal{C}_{\sigma, \rho, \eta, \zeta_1^+; w}^{\alpha, \phi} \omega_2^p \right) (\zeta) \right]^{\frac{1}{p}} \\ \leq \frac{1 + M(m+2)}{(m+1)(M+1)} \left[ \left( \mathcal{C}_{\sigma, \rho, \eta, \zeta_1^+; w}^{\alpha, \phi} (\omega_1 + \omega_2)^p \right) (\zeta) \right]^{\frac{1}{p}}. \end{aligned}$$

**Corollary 2.** Under the hypotheses of Theorem 1, if we choose

$$\vartheta(\zeta) = \frac{\zeta}{\alpha} \exp(-Q\zeta),$$

where

$$Q = \frac{1 - \alpha}{\alpha}$$

and  $\alpha \in (0, 1]$  for all  $\zeta \in [\zeta_1, \zeta_2]$ , we obtain

$$\begin{aligned} \left[ \left( \mathcal{E}_{\sigma, \rho, \eta, \zeta_1^+; w}^{\alpha, \phi} \omega_1^p \right) (\zeta) \right]^{\frac{1}{p}} + \left[ \left( \mathcal{E}_{\sigma, \rho, \eta, \zeta_1^+; w}^{\alpha, \phi} \omega_2^p \right) (\zeta) \right]^{\frac{1}{p}} \\ \leq \frac{1 + M(m+2)}{(m+1)(M+1)} \left[ \left( \mathcal{E}_{\sigma, \rho, \eta, \zeta_1^+; w}^{\alpha, \phi} (\omega_1 + \omega_2)^p \right) (\zeta) \right]^{\frac{1}{p}}. \end{aligned}$$

**Theorem 2.** Let  $p \geq 1$ , with  $\eta, \rho > 0$  and  $w \in \mathbb{R}$ . Assume that  $\omega_1(\zeta)$  and  $\omega_2(\zeta)$  are positive functions on  $[\zeta_1, \infty)$  such that  $\left( \mathcal{K}_{\sigma, \rho, \eta, \zeta_1^+; w}^{\vartheta, \phi} \omega_1^p \right) (\zeta) < \infty$  and  $\left( \mathcal{K}_{\sigma, \rho, \eta, \zeta_1^+; w}^{\vartheta, \phi} \omega_2^p \right) (\zeta) < \infty$  for all  $\zeta > \zeta_1 \geq 0$ . If  $0 < m \leq \frac{\omega_1(r)}{\omega_2(r)} \leq M$ , where  $r \in (\zeta_1, \zeta)$ , we have

$$\begin{aligned} \left[ \left( \mathcal{K}_{\sigma, \rho, \eta, \zeta_1^+; w}^{\vartheta, \phi} \omega_1^p \right) (\zeta) \right]^{\frac{2}{p}} + \left[ \left( \mathcal{K}_{\sigma, \rho, \eta, \zeta_1^+; w}^{\vartheta, \phi} \omega_2^p \right) (\zeta) \right]^{\frac{2}{p}} \\ \geq \left( \frac{(m+1)(M+1)}{M} - 2 \right) \left[ \left( \mathcal{K}_{\sigma, \rho, \eta, \zeta_1^+; w}^{\vartheta, \phi} \omega_1^p \right) (\zeta) \right]^{\frac{1}{p}} \left[ \left( \mathcal{K}_{\sigma, \rho, \eta, \zeta_1^+; w}^{\vartheta, \phi} \omega_2^p \right) (\zeta) \right]^{\frac{1}{p}}. \end{aligned}$$

**Proof.** Multiplying inequalities (22) and (24), we have

$$\begin{aligned} \frac{(m+1)(M+1)}{M} \left[ \left( \mathcal{K}_{\sigma, \rho, \eta, \zeta_1^+; w}^{\vartheta, \phi} \omega_1^p \right) (\zeta) \right]^{\frac{1}{p}} \left[ \left( \mathcal{K}_{\sigma, \rho, \eta, \zeta_1^+; w}^{\vartheta, \phi} \omega_2^p \right) (\zeta) \right]^{\frac{1}{p}} \\ \leq \left( \left[ \left( \mathcal{K}_{\sigma, \rho, \eta, \zeta_1^+; w}^{\vartheta, \phi} (\omega_1 + \omega_2)^p \right) (\zeta) \right]^{\frac{1}{p}} \right)^2. \tag{25} \end{aligned}$$

Applying the Minkowski inequality to the right-hand side of (25), we obtain

$$\begin{aligned} & \left( \left[ \left( \mathcal{K}_{\sigma, \varrho, \eta, \varsigma_1^+}^{\vartheta, \phi} (\omega_1 + \omega_2)^p \right) (\varsigma) \right]^{\frac{1}{p}} \right)^2 \\ & \leq \left( \left[ \left( \mathcal{K}_{\sigma, \varrho, \eta, \varsigma_1^+}^{\vartheta, \phi} \omega_1^p \right) (\varsigma) \right]^{\frac{1}{p}} + \left[ \left( \mathcal{K}_{\sigma, \varrho, \eta, \varsigma_1^+}^{\vartheta, \phi} \omega_2^p \right) (\varsigma) \right]^{\frac{1}{p}} \right)^2, \end{aligned}$$

which implies that

$$\begin{aligned} & \left( \left[ \left( \mathcal{K}_{\sigma, \varrho, \eta, \varsigma_1^+}^{\vartheta, \phi} (\omega_1 + \omega_2)^p \right) (\varsigma) \right]^{\frac{1}{p}} \right)^2 \\ & \leq \left[ \left( \mathcal{K}_{\sigma, \varrho, \eta, \varsigma_1^+}^{\vartheta, \phi} \omega_1^p \right) (\varsigma) \right]^{\frac{2}{p}} + \left[ \left( \mathcal{K}_{\sigma, \varrho, \eta, \varsigma_1^+}^{\vartheta, \phi} \omega_2^p \right) (\varsigma) \right]^{\frac{2}{p}} \\ & \quad + 2 \left[ \left( \mathcal{K}_{\sigma, \varrho, \eta, \varsigma_1^+}^{\vartheta, \phi} \omega_1^p \right) (\varsigma) \right]^{\frac{1}{p}} \left[ \left( \mathcal{K}_{\sigma, \varrho, \eta, \varsigma_1^+}^{\vartheta, \phi} \omega_2^p \right) (\varsigma) \right]^{\frac{1}{p}}. \quad (26) \end{aligned}$$

Using inequalities (25) and (26), we complete the proof of Theorem 2.  $\square$

**Corollary 3.** Under the hypotheses of Theorem 2, if we take

$$\vartheta(\varsigma) = \varsigma(\varsigma_2 - \varsigma)^{\alpha-1} \quad (\forall \varsigma \in [\varsigma_1, \varsigma_2]; \alpha \in (0, 1]),$$

we have

$$\begin{aligned} & \left[ \left( \mathcal{C}_{\sigma, \varrho, \eta, \varsigma_1^+}^{\alpha, \phi} \omega_1^p \right) (\varsigma) \right]^{\frac{2}{p}} + \left[ \left( \mathcal{C}_{\sigma, \varrho, \eta, \varsigma_1^+}^{\alpha, \phi} \omega_2^p \right) (\varsigma) \right]^{\frac{2}{p}} \\ & \geq \left( \frac{(m+1)(M+1)}{M} - 2 \right) \left[ \left( \mathcal{C}_{\sigma, \varrho, \eta, \varsigma_1^+}^{\alpha, \phi} \omega_1^p \right) (\varsigma) \right]^{\frac{1}{p}} \left[ \left( \mathcal{C}_{\sigma, \varrho, \eta, \varsigma_1^+}^{\alpha, \phi} \omega_2^p \right) (\varsigma) \right]^{\frac{1}{p}}. \end{aligned}$$

**Corollary 4.** Under the hypotheses of Theorem 2, if we choose

$$\vartheta(\varsigma) = \frac{\varsigma}{\alpha} \exp(-Q\varsigma),$$

where

$$Q = \frac{1 - \alpha}{\alpha}$$

and  $\alpha \in (0, 1]$  for all  $\varsigma \in [\varsigma_1, \varsigma_2]$ , we obtain

$$\begin{aligned} & \left[ \left( \mathcal{E}_{\sigma, \varrho, \eta, \varsigma_1^+}^{\alpha, \phi} \omega_1^p \right) (\varsigma) \right]^{\frac{2}{p}} + \left[ \left( \mathcal{E}_{\sigma, \varrho, \eta, \varsigma_1^+}^{\alpha, \phi} \omega_2^p \right) (\varsigma) \right]^{\frac{2}{p}} \\ & \geq \left( \frac{(m+1)(M+1)}{M} - 2 \right) \left[ \left( \mathcal{E}_{\sigma, \varrho, \eta, \varsigma_1^+}^{\alpha, \phi} \omega_1^p \right) (\varsigma) \right]^{\frac{1}{p}} \left[ \left( \mathcal{E}_{\sigma, \varrho, \eta, \varsigma_1^+}^{\alpha, \phi} \omega_2^p \right) (\varsigma) \right]^{\frac{1}{p}}. \end{aligned}$$

**Theorem 3.** Let  $p > 1$ ,  $\frac{1}{p} + \frac{1}{q} = 1$  with  $\eta, \varrho > 0$  and  $w \in \mathbb{R}$ . Assume that  $\omega_1(\zeta)$  and  $\omega_2(\zeta)$  are positive functions on  $[\zeta_1, \infty)$  such that  $\left(\mathcal{K}_{\sigma, \varrho, \eta, \zeta_1^+}^{\vartheta, \phi; w} \omega_1\right)(\zeta) < \infty$  and  $\left(\mathcal{K}_{\sigma, \varrho, \eta, \zeta_1^+}^{\vartheta, \phi; w} \omega_2\right)(\zeta) < \infty$  for all  $\zeta > \zeta_1 \geq 0$ . If  $0 < m \leq \frac{\omega_1(r)}{\omega_2(r)} \leq M$ , where  $r \in (\zeta_1, \zeta)$ , we have

$$\left[\left(\mathcal{K}_{\sigma, \varrho, \eta, \zeta_1^+}^{\vartheta, \phi; w} \omega_1\right)(\zeta)\right]^{\frac{1}{p}} \left[\left(\mathcal{K}_{\sigma, \varrho, \eta, \zeta_1^+}^{\vartheta, \phi; w} \omega_2\right)(\zeta)\right]^{\frac{1}{q}} \leq \left(\frac{M}{m}\right)^{\frac{1}{pq}} \left(\mathcal{K}_{\sigma, \varrho, \eta, \zeta_1^+}^{\vartheta, \phi; w} \omega_1^{\frac{1}{p}} \omega_2^{\frac{1}{q}}\right)(\zeta).$$

**Proof.** Since  $\frac{\omega_1(r)}{\omega_2(r)} \leq M$ , where  $r \in (\zeta_1, \zeta)$ , we have

$$[\omega_2(r)]^{\frac{1}{q}} \geq M^{-\frac{1}{q}} [\omega_1(r)]^{\frac{1}{q}}.$$

It follows that

$$[\omega_1(r)]^{\frac{1}{p}} [\omega_2(r)]^{\frac{1}{q}} \geq M^{-\frac{1}{q}} [\omega_1(r)]^{\frac{1}{p}} [\omega_1(r)]^{\frac{1}{q}} = M^{-\frac{1}{q}} \omega_1(r). \quad (27)$$

Multiplying both sides of (27) by

$$\phi^{-1}(\zeta) \frac{\vartheta(\zeta - r)}{\zeta - r} \phi(r) \mathcal{R}_{\varrho, \eta}^{\sigma} [w(\zeta - r)^{\varrho}]$$

with  $r \in (\zeta_1, \zeta)$ , and integrating over  $r \in (\zeta_1, \zeta)$  yields

$$\left(\mathcal{K}_{\sigma, \varrho, \eta, \zeta_1^+}^{\vartheta, \phi; w} \omega_1^{\frac{1}{p}} \omega_2^{\frac{1}{q}}\right)(\zeta) \geq M^{-\frac{1}{q}} \left(\mathcal{K}_{\sigma, \varrho, \eta, \zeta_1^+}^{\vartheta, \phi; w} \omega_1\right)(\zeta).$$

Consequently, we obtain

$$\left[\left(\mathcal{K}_{\sigma, \varrho, \eta, \zeta_1^+}^{\vartheta, \phi; w} \omega_1^{\frac{1}{p}} \omega_2^{\frac{1}{q}}\right)(\zeta)\right]^{\frac{1}{p}} \geq M^{-\frac{1}{pq}} \left[\left(\mathcal{K}_{\sigma, \varrho, \eta, \zeta_1^+}^{\vartheta, \phi; w} \omega_1\right)(\zeta)\right]^{\frac{1}{p}}. \quad (28)$$

On the other hand, if  $m\omega_2(r) \leq \omega_1(r)$ , where  $r \in (\zeta_1, \zeta)$ , then we obtain

$$[\omega_1(r)]^{\frac{1}{p}} \geq m^{\frac{1}{p}} [\omega_2(r)]^{\frac{1}{p}}.$$

Hence, we have

$$[\omega_1(r)]^{\frac{1}{p}} [\omega_2(r)]^{\frac{1}{q}} \geq m^{\frac{1}{p}} [\omega_2(r)]^{\frac{1}{p}} [\omega_2(r)]^{\frac{1}{q}} = m^{\frac{1}{p}} \omega_2(r). \quad (29)$$

Multiplying both sides of (29) by

$$\phi^{-1}(\zeta) \frac{\vartheta(\zeta - r)}{\zeta - r} \phi(r) \mathcal{R}_{\varrho, \eta}^{\sigma} [w(\zeta - r)^{\varrho}]$$

with  $r \in (\zeta_1, \zeta)$ , and integrating over  $r \in (\zeta_1, \zeta)$  yields

$$\left[\left(\mathcal{K}_{\sigma, \varrho, \eta, \zeta_1^+}^{\vartheta, \phi; w} \omega_1^{\frac{1}{p}} \omega_2^{\frac{1}{q}}\right)(\zeta)\right]^{\frac{1}{q}} \geq m^{\frac{1}{pq}} \left[\left(\mathcal{K}_{\sigma, \varrho, \eta, \zeta_1^+}^{\vartheta, \phi; w} \omega_2\right)(\zeta)\right]^{\frac{1}{q}}. \quad (30)$$

Multiplying inequalities (28) and (30), we complete the proof of Theorem 3.  $\square$



**Corollary 5.** Under the hypotheses of Theorem 3, if we replace  $\omega_1$  and  $\omega_2$  by  $\omega_1^p$  and  $\omega_2^q$ , respectively, we have

$$\left[ \left( \mathcal{K}_{\sigma, \varrho, \eta, \varsigma_1^+; w}^{\vartheta, \phi} \omega_1^p \right) (\varsigma) \right]^{\frac{1}{p}} \left[ \left( \mathcal{K}_{\sigma, \varrho, \eta, \varsigma_1^+; w}^{\vartheta, \phi} \omega_2^q \right) (\varsigma) \right]^{\frac{1}{q}} \leq \left( \frac{M}{m} \right)^{\frac{1}{pq}} \left( \mathcal{K}_{\sigma, \varrho, \eta, \varsigma_1^+; w}^{\vartheta, \phi} \omega_1 \omega_2 \right) (\varsigma).$$

**Corollary 6.** Under the hypotheses of Theorem 3, if we take

$$\vartheta(\varsigma) = \varsigma(\varsigma_2 - \varsigma)^{\alpha-1} \quad (\forall \varsigma \in [\varsigma_1, \varsigma_2]; \alpha \in (0, 1)),$$

we obtain

$$\left[ \left( \mathcal{C}_{\sigma, \varrho, \eta, \varsigma_1^+; w}^{\alpha, \phi} \omega_1 \right) (\varsigma) \right]^{\frac{1}{p}} \left[ \left( \mathcal{C}_{\sigma, \varrho, \eta, \varsigma_1^+; w}^{\alpha, \phi} \omega_2 \right) (\varsigma) \right]^{\frac{1}{q}} \leq \left( \frac{M}{m} \right)^{\frac{1}{pq}} \left( \mathcal{C}_{\sigma, \varrho, \eta, \varsigma_1^+; w}^{\alpha, \phi} \omega_1^{\frac{1}{p}} \omega_2^{\frac{1}{q}} \right) (\varsigma).$$

**Corollary 7.** Under the hypotheses of Theorem 3, if we choose

$$\vartheta(\varsigma) = \frac{\varsigma}{\alpha} \exp(-Q\varsigma),$$

where

$$Q = \frac{1 - \alpha}{\alpha}$$

and  $\alpha \in (0, 1]$  for all  $\varsigma \in [\varsigma_1, \varsigma_2]$ , we obtain

$$\left[ \left( \mathcal{E}_{\sigma, \varrho, \eta, \varsigma_1^+; w}^{\alpha, \phi} \omega_1 \right) (\varsigma) \right]^{\frac{1}{p}} \left[ \left( \mathcal{E}_{\sigma, \varrho, \eta, \varsigma_1^+; w}^{\alpha, \phi} \omega_2 \right) (\varsigma) \right]^{\frac{1}{q}} \leq \left( \frac{M}{m} \right)^{\frac{1}{pq}} \left( \mathcal{E}_{\sigma, \varrho, \eta, \varsigma_1^+; w}^{\alpha, \phi} \omega_1^{\frac{1}{p}} \omega_2^{\frac{1}{q}} \right) (\varsigma).$$

**Theorem 4.** Let  $p > 1, \frac{1}{p} + \frac{1}{q} = 1$  with  $\eta, \varrho > 0$  and  $w \in \mathbb{R}$ . Assume that  $\omega_1(\varsigma)$  and  $\omega_2(\varsigma)$  are positive functions on  $[\varsigma_1, \infty)$  such that  $\left( \mathcal{K}_{\sigma, \varrho, \eta, \varsigma_1^+; w}^{\vartheta, \phi} \omega_1^p \right) (\varsigma) < \infty$  and  $\left( \mathcal{K}_{\sigma, \varrho, \eta, \varsigma_1^+; w}^{\vartheta, \phi} \omega_2^q \right) (\varsigma) < \infty$  for all  $\varsigma > \varsigma_1 \geq 0$ . If  $0 < m \leq \frac{\omega_1(r)}{\omega_2(r)} \leq M$ , where  $r \in (\varsigma_1, \varsigma)$ , we have

$$\begin{aligned} \left( \mathcal{K}_{\sigma, \varrho, \eta, \varsigma_1^+; w}^{\vartheta, \phi} \omega_1 \omega_2 \right) (\varsigma) &\leq \frac{2^{p-1} M^p}{p(M+1)^p} \left( \mathcal{K}_{\sigma, \varrho, \eta, \varsigma_1^+; w}^{\vartheta, \phi} (\omega_1^p + \omega_2^p) \right) (\varsigma) \\ &\quad + \frac{2^{q-1}}{q(m+1)^q} \left( \mathcal{K}_{\sigma, \varrho, \eta, \varsigma_1^+; w}^{\vartheta, \phi} (\omega_1^q + \omega_2^q) \right) (\varsigma). \end{aligned}$$

**Proof.** Since  $\frac{\omega_1(r)}{\omega_2(r)} \leq M$ , where  $r \in (\varsigma_1, \varsigma)$ , we have

$$(M+1)^p [\omega_1(r)]^p \leq M^p [\omega_1(r) + \omega_2(r)]^p. \tag{31}$$

Multiplying both sides of (31) by

$$\phi^{-1}(\varsigma) \frac{\vartheta(\varsigma-r)}{\varsigma-r} \phi(r) \mathcal{R}_{\varrho, \eta}^{\sigma} [w(\varsigma-r)^{\varrho}]$$

with  $r \in (\varsigma_1, \varsigma)$ , and integrating over  $r \in (\varsigma_1, \varsigma)$  yields

$$\left( \mathcal{K}_{\sigma, \varrho, \eta, \varsigma_1^+; w}^{\vartheta, \phi} \omega_1^p \right) (\varsigma) \leq \frac{M^p}{(M+1)^p} \left( \mathcal{K}_{\sigma, \varrho, \eta, \varsigma_1^+; w}^{\vartheta, \phi} (\omega_1 + \omega_2)^p \right) (\varsigma). \tag{32}$$

On the other hand, if  $m\omega_2(r) \leq \omega_1(r)$ , where  $r \in (\varsigma_1, \varsigma)$ , then we obtain

$$(m+1)^q [\omega_2(r)]^q \leq [\omega_1(r) + \omega_2(r)]^q. \tag{33}$$

Multiplying both sides of (33) by

$$\phi^{-1}(\zeta) \frac{\vartheta(\zeta - r)}{\zeta - r} \phi(r) \mathcal{R}_{\varrho, \eta}^{\sigma} [w(\zeta - r)^{\varrho}]$$

with  $r \in (\zeta_1, \zeta)$ , and integrating over  $r \in (\zeta_1, \zeta)$  yields

$$\left( \mathcal{K}_{\sigma, \varrho, \eta, \zeta_1^+; w}^{\vartheta, \phi} \omega_2^{\varrho} \right) (\zeta) \leq \frac{1}{(m + 1)^{\varrho}} \left( \mathcal{K}_{\sigma, \varrho, \eta, \zeta_1^+; w}^{\vartheta, \phi} (\omega_1 + \omega_2)^{\varrho} \right) (\zeta). \tag{34}$$

Now, using Young’s inequality, we have

$$\omega_1(r)\omega_2(r) \leq \frac{[\omega_1(r)]^p}{p} + \frac{[\omega_2(r)]^q}{q}. \tag{35}$$

Multiplying both sides of (35) by

$$\phi^{-1}(\zeta) \frac{\vartheta(\zeta - r)}{\zeta - r} \phi(r) \mathcal{R}_{\varrho, \eta}^{\sigma} [w(\zeta - r)^{\varrho}]$$

with  $r \in (\zeta_1, \zeta)$ , and integrating over  $r \in (\zeta_1, \zeta)$  yields

$$\left( \mathcal{K}_{\sigma, \varrho, \eta, \zeta_1^+; w}^{\vartheta, \phi} \omega_1 \omega_2 \right) (\zeta) \leq \frac{1}{p} \left( \mathcal{K}_{\sigma, \varrho, \eta, \zeta_1^+; w}^{\vartheta, \phi} \omega_1^p \right) (\zeta) + \frac{1}{q} \left( \mathcal{K}_{\sigma, \varrho, \eta, \zeta_1^+; w}^{\vartheta, \phi} \omega_2^q \right) (\zeta). \tag{36}$$

Applying inequalities (32) and (34) in (36), we obtain

$$\begin{aligned} \left( \mathcal{K}_{\sigma, \varrho, \eta, \zeta_1^+; w}^{\vartheta, \phi} \omega_1 \omega_2 \right) (\zeta) &\leq \frac{M^p}{p(M + 1)^p} \left( \mathcal{K}_{\sigma, \varrho, \eta, \zeta_1^+; w}^{\vartheta, \phi} (\omega_1 + \omega_2)^p \right) (\zeta) \\ &\quad + \frac{1}{q(m + 1)^q} \left( \mathcal{K}_{\sigma, \varrho, \eta, \zeta_1^+; w}^{\vartheta, \phi} (\omega_1 + \omega_2)^q \right) (\zeta). \end{aligned} \tag{37}$$

Now, using the inequality  $(\mu_1 + \mu_2)^{\theta} \leq 2^{\theta-1}(\mu_1^{\theta} + \mu_2^{\theta})$ ,  $\theta > 1$  with  $\mu_1, \mu_2 > 0$  in (37), we complete the proof of Theorem 4.  $\square$

**Corollary 8.** Under the hypotheses of Theorem 4, if we take

$$\vartheta(\zeta) = \zeta(\zeta_2 - \zeta)^{\alpha-1} \quad (\forall \zeta \in [\zeta_1, \zeta_2]; \alpha \in (0, 1]),$$

we have

$$\begin{aligned} \left( \mathcal{C}_{\sigma, \varrho, \eta, \zeta_1^+; w}^{\alpha, \phi} \omega_1 \omega_2 \right) (\zeta) &\leq \frac{2^{p-1}M^p}{p(M + 1)^p} \left( \mathcal{C}_{\sigma, \varrho, \eta, \zeta_1^+; w}^{\alpha, \phi} (\omega_1^p + \omega_2^p) \right) (\zeta) \\ &\quad + \frac{2^{q-1}}{q(m + 1)^q} \left( \mathcal{C}_{\sigma, \varrho, \eta, \zeta_1^+; w}^{\alpha, \phi} (\omega_1^q + \omega_2^q) \right) (\zeta). \end{aligned}$$

**Corollary 9.** Under the hypotheses of Theorem 4, if we choose

$$\vartheta(\zeta) = \frac{\zeta}{\alpha} \exp(-\mathcal{Q}\zeta),$$

where

$$\mathcal{Q} = \frac{1 - \alpha}{\alpha}$$

and  $\alpha \in (0, 1]$  for all  $\zeta \in [\zeta_1, \zeta_2]$ , we obtain

$$\begin{aligned} \left( \mathcal{E}_{\sigma, \varrho, \eta, \zeta_1^+}^{\alpha, \phi; w} \omega_1 \omega_2 \right) (\zeta) &\leq \frac{2^{p-1} M^p}{p(M+1)^p} \left( \mathcal{E}_{\sigma, \varrho, \eta, \zeta_1^+}^{\alpha, \phi; w} (\omega_1^p + \omega_2^p) \right) (\zeta) \\ &\quad + \frac{2^{q-1}}{q(m+1)^q} \left( \mathcal{E}_{\sigma, \varrho, \eta, \zeta_1^+}^{\alpha, \phi; w} (\omega_1^q + \omega_2^q) \right) (\zeta). \end{aligned}$$

**Theorem 5.** Let  $p \geq 1$  with  $\eta, \varrho > 0$  and  $w \in \mathbb{R}$ . Assume that  $\omega_1(\zeta)$  and  $\omega_2(\zeta)$  are positive functions on  $[\zeta_1, \infty)$  such that  $\left( \mathcal{K}_{\sigma, \varrho, \eta, \zeta_1^+}^{\vartheta, \phi; w} \omega_1^p \right) (\zeta) < \infty$  and  $\left( \mathcal{K}_{\sigma, \varrho, \eta, \zeta_1^+}^{\vartheta, \phi; w} \omega_2^p \right) (\zeta) < \infty$  for all  $\zeta > \zeta_1 \geq 0$ . If  $0 < k < m \leq \frac{\omega_1(r)}{\omega_2(r)} \leq M$ , where  $r \in (\zeta_1, \zeta)$ , we have

$$\begin{aligned} \left( \frac{M+1}{M-k} \right) \left[ \left( \mathcal{K}_{\sigma, \varrho, \eta, \zeta_1^+}^{\vartheta, \phi; w} (\omega_1 - k\omega_2)^p \right) (\zeta) \right]^{\frac{1}{p}} \\ \leq \left[ \left( \mathcal{K}_{\sigma, \varrho, \eta, \zeta_1^+}^{\vartheta, \phi; w} \omega_1^p \right) (\zeta) \right]^{\frac{1}{p}} + \left[ \left( \mathcal{K}_{\sigma, \varrho, \eta, \zeta_1^+}^{\vartheta, \phi; w} \omega_2^p \right) (\zeta) \right]^{\frac{1}{p}} \\ \leq \left( \frac{m+1}{m-k} \right) \left[ \left( \mathcal{K}_{\sigma, \varrho, \eta, \zeta_1^+}^{\vartheta, \phi; w} (\omega_1 - k\omega_2)^p \right) (\zeta) \right]^{\frac{1}{p}}. \end{aligned}$$

**Proof.** Since  $0 < k < m \leq \frac{\omega_1(r)}{\omega_2(r)} \leq M$ , where  $r \in (\zeta_1, \zeta)$ , we have

$$km \leq kM, \tag{38}$$

which implies that

$$\frac{M+1}{M-k} \leq \frac{m+1}{m-k}.$$

Furthermore, we obtain

$$m - k \leq \frac{\omega_1(r) - k\omega_2(r)}{\omega_2(r)} \leq M - k.$$

Hence, we obtain

$$\frac{(\omega_1(r) - k\omega_2(r))^p}{(M - k)^p} \leq [\omega_2(r)]^p \leq \frac{(\omega_1(r) - k\omega_2(r))^p}{(m - k)^p}. \tag{39}$$

Moreover, we have

$$\frac{1}{M} \leq \frac{\omega_2(r)}{\omega_1(r)} \leq \frac{1}{m},$$

which implies that

$$\left( \frac{M}{M-k} \right)^p (\omega_1(r) - k\omega_2(r))^p \leq [\omega_1(r)]^p \leq \left( \frac{m}{m-k} \right)^p (\omega_1(r) - k\omega_2(r))^p. \tag{40}$$

Multiplying both sides of (39) by

$$\phi^{-1}(\zeta) \frac{\vartheta(\zeta - r)}{\zeta - r} \phi(r) \mathcal{R}_{\varrho, \eta}^{\sigma} [w(\zeta - r)^{\varrho}]$$

with  $r \in (\zeta_1, \zeta)$ , and integrating over  $r \in (\zeta_1, \zeta)$  yields

$$\begin{aligned} \frac{1}{M-k} \left[ \left( \mathcal{K}_{\sigma, \varrho, \eta, \zeta_1^+; w}^{\vartheta, \phi} (\omega_1 - k\omega_2)^p \right) (\zeta) \right]^{\frac{1}{p}} &\leq \left[ \left( \mathcal{K}_{\sigma, \varrho, \eta, \zeta_1^+; w}^{\vartheta, \phi} \omega_2^p \right) (\zeta) \right]^{\frac{1}{p}} \\ &\leq \frac{1}{m-k} \left[ \left( \mathcal{K}_{\sigma, \varrho, \eta, \zeta_1^+; w}^{\vartheta, \phi} (\omega_1 - k\omega_2)^p \right) (\zeta) \right]^{\frac{1}{p}}. \end{aligned} \tag{41}$$

Again, multiplying both sides of (40) by

$$\phi^{-1}(\zeta) \frac{\vartheta(\zeta-r)}{\zeta-r} \phi(r) \mathcal{R}_{\varrho, \eta}^{\sigma} [w(\zeta-r)^{\varrho}]$$

with  $r \in (\zeta_1, \zeta)$ , and integrating over  $r \in (\zeta_1, \zeta)$  yields

$$\begin{aligned} \frac{M}{M-k} \left[ \left( \mathcal{K}_{\sigma, \varrho, \eta, \zeta_1^+; w}^{\vartheta, \phi} (\omega_1 - k\omega_2)^p \right) (\zeta) \right]^{\frac{1}{p}} &\leq \left[ \left( \mathcal{K}_{\sigma, \varrho, \eta, \zeta_1^+; w}^{\vartheta, \phi} \omega_1^p \right) (\zeta) \right]^{\frac{1}{p}} \\ &\leq \frac{m}{m-k} \left[ \left( \mathcal{K}_{\sigma, \varrho, \eta, \zeta_1^+; w}^{\vartheta, \phi} (\omega_1 - k\omega_2)^p \right) (\zeta) \right]^{\frac{1}{p}}. \end{aligned} \tag{42}$$

Finally, by adding inequalities (41) and (42), we complete the proof of Theorem 5.  $\square$

**Corollary 10.** Under the hypotheses of Theorem 5, if we take

$$\vartheta(\zeta) = \zeta(\zeta_2 - \zeta)^{\alpha-1} \quad (\forall \zeta \in [\zeta_1, \zeta_2]; \alpha \in (0, 1]),$$

we have

$$\begin{aligned} \left( \frac{M+1}{M-k} \right) \left[ \left( \mathcal{C}_{\sigma, \varrho, \eta, \zeta_1^+; w}^{\alpha, \phi} (\omega_1 - k\omega_2)^p \right) (\zeta) \right]^{\frac{1}{p}} \\ \leq \left[ \left( \mathcal{C}_{\sigma, \varrho, \eta, \zeta_1^+; w}^{\alpha, \phi} \omega_1^p \right) (\zeta) \right]^{\frac{1}{p}} + \left[ \left( \mathcal{C}_{\sigma, \varrho, \eta, \zeta_1^+; w}^{\alpha, \phi} \omega_2^p \right) (\zeta) \right]^{\frac{1}{p}} \\ \leq \left( \frac{m+1}{m-k} \right) \left[ \left( \mathcal{C}_{\sigma, \varrho, \eta, \zeta_1^+; w}^{\alpha, \phi} (\omega_1 - k\omega_2)^p \right) (\zeta) \right]^{\frac{1}{p}}. \end{aligned}$$

**Corollary 11.** Under the hypotheses of Theorem 5, if we choose

$$\vartheta(\zeta) = \frac{\zeta}{\alpha} \exp(-\mathcal{Q}\zeta),$$

where

$$\mathcal{Q} = \frac{1-\alpha}{\alpha}$$

and  $\alpha \in (0, 1]$  for all  $\zeta \in [\zeta_1, \zeta_2]$ , we obtain

$$\begin{aligned} \left( \frac{M+1}{M-k} \right) \left[ \left( \mathcal{E}_{\sigma, \varrho, \eta, \zeta_1^+; w}^{\alpha, \phi} (\omega_1 - k\omega_2)^p \right) (\zeta) \right]^{\frac{1}{p}} \\ \leq \left[ \left( \mathcal{E}_{\sigma, \varrho, \eta, \zeta_1^+; w}^{\alpha, \phi} \omega_1^p \right) (\zeta) \right]^{\frac{1}{p}} + \left[ \left( \mathcal{E}_{\sigma, \varrho, \eta, \zeta_1^+; w}^{\alpha, \phi} \omega_2^p \right) (\zeta) \right]^{\frac{1}{p}} \\ \leq \left( \frac{m+1}{m-k} \right) \left[ \left( \mathcal{E}_{\sigma, \varrho, \eta, \zeta_1^+; w}^{\alpha, \phi} (\omega_1 - k\omega_2)^p \right) (\zeta) \right]^{\frac{1}{p}}. \end{aligned}$$

**Theorem 6.** Let  $p \geq 1$  with  $\eta, \rho > 0$  and  $w \in \mathbb{R}$ . Assume that  $\omega_1(\zeta)$  and  $\omega_2(\zeta)$  are positive functions on  $[\zeta_1, \infty)$  such that  $\left(\mathcal{K}_{\sigma, \rho, \eta, \zeta_1^+}^{\theta, \phi; w} \omega_1^p\right)(\zeta) < \infty$  and  $\left(\mathcal{K}_{\sigma, \rho, \eta, \zeta_1^+}^{\theta, \phi; w} \omega_2^p\right)(\zeta) < \infty$  for all  $\zeta > \zeta_1 \geq 0$ . If  $0 \leq \mu_1 \leq \omega_1(r) \leq \mu_2$  and  $0 \leq \nu_1 \leq \omega_2(r) \leq \nu_2$ , where  $r \in (\zeta_1, \zeta)$ , we have

$$\begin{aligned} & \left[\left(\mathcal{K}_{\sigma, \rho, \eta, \zeta_1^+}^{\theta, \phi; w} \omega_1^p\right)(\zeta)\right]^{\frac{1}{p}} + \left[\left(\mathcal{K}_{\sigma, \rho, \eta, \zeta_1^+}^{\theta, \phi; w} \omega_2^p\right)(\zeta)\right]^{\frac{1}{p}} \\ & \leq \left(\frac{\mu_2(\mu_1 + \nu_2) + \nu_2(\nu_1 + \mu_2)}{(\mu_2 + \nu_1)(\nu_2 + \mu_1)}\right) \left[\left(\mathcal{K}_{\sigma, \rho, \eta, \zeta_1^+}^{\theta, \phi; w} (\omega_1 + \omega_2)^p\right)(\zeta)\right]^{\frac{1}{p}}. \end{aligned}$$

**Proof.** Under the given hypothesis, we have

$$\frac{1}{\nu_2} \leq \frac{1}{\omega_2(r)} \leq \frac{1}{\nu_1}. \tag{43}$$

The product of (43) with  $0 \leq \mu_1 \leq \omega_1(r) \leq \mu_2$  yields

$$\frac{\mu_1}{\nu_2} \leq \frac{\omega_1(r)}{\omega_2(r)} \leq \frac{\mu_2}{\nu_1}. \tag{44}$$

From (44), we obtain

$$[\omega_2(r)]^p \leq \left(\frac{\nu_2}{\mu_1 + \nu_2}\right)^p (\omega_1(r) + \omega_2(r))^p \tag{45}$$

and

$$[\omega_1(r)]^p \leq \left(\frac{\mu_2}{\nu_1 + \mu_2}\right)^p (\omega_1(r) + \omega_2(r))^p. \tag{46}$$

Multiplying both sides of inequalities (45) and (46) by

$$\phi^{-1}(\zeta) \frac{\vartheta(\zeta - r)}{\zeta - r} \phi(r) \mathcal{R}_{\rho, \eta}^{\sigma} [w(\zeta - r)^{\rho}]$$

with  $r \in (\zeta_1, \zeta)$ , and integrating them over  $r \in (\zeta_1, \zeta)$  yields

$$\left[\left(\mathcal{K}_{\sigma, \rho, \eta, \zeta_1^+}^{\theta, \phi; w} \omega_2^p\right)(\zeta)\right]^{\frac{1}{p}} \leq \frac{\nu_2}{\mu_1 + \nu_2} \left[\left(\mathcal{K}_{\sigma, \rho, \eta, \zeta_1^+}^{\theta, \phi; w} (\omega_1 + \omega_2)^p\right)(\zeta)\right]^{\frac{1}{p}} \tag{47}$$

and

$$\left[\left(\mathcal{K}_{\sigma, \rho, \eta, \zeta_1^+}^{\theta, \phi; w} \omega_1^p\right)(\zeta)\right]^{\frac{1}{p}} \leq \frac{\mu_2}{\nu_1 + \mu_2} \left[\left(\mathcal{K}_{\sigma, \rho, \eta, \zeta_1^+}^{\theta, \phi; w} (\omega_1 + \omega_2)^p\right)(\zeta)\right]^{\frac{1}{p}}. \tag{48}$$

By adding inequalities (47) and (48), we complete the proof of Theorem 6.  $\square$

**Corollary 12.** Under the hypotheses of Theorem 6, if we take

$$\vartheta(\zeta) = \zeta(\zeta_2 - \zeta)^{\alpha-1} \quad (\forall \zeta \in [\zeta_1, \zeta_2]; \alpha \in (0, 1]),$$

we have

$$\begin{aligned} & \left[\left(\mathcal{C}_{\sigma, \rho, \eta, \zeta_1^+}^{\alpha, \phi; w} \omega_1^p\right)(\zeta)\right]^{\frac{1}{p}} + \left[\left(\mathcal{C}_{\sigma, \rho, \eta, \zeta_1^+}^{\alpha, \phi; w} \omega_2^p\right)(\zeta)\right]^{\frac{1}{p}} \\ & \leq \left(\frac{\mu_2(\mu_1 + \nu_2) + \nu_2(\nu_1 + \mu_2)}{(\mu_2 + \nu_1)(\nu_2 + \mu_1)}\right) \left[\left(\mathcal{C}_{\sigma, \rho, \eta, \zeta_1^+}^{\alpha, \phi; w} (\omega_1 + \omega_2)^p\right)(\zeta)\right]^{\frac{1}{p}}. \end{aligned}$$

**Corollary 13.** Under the hypotheses of Theorem 6, if we choose

$$\vartheta(\zeta) = \frac{\zeta}{\alpha} \exp(-Q\zeta),$$

where

$$Q = \frac{1 - \alpha}{\alpha}$$

and  $\alpha \in (0, 1]$  for all  $\zeta \in [\zeta_1, \zeta_2]$ , we obtain

$$\begin{aligned} & \left[ \left( \mathcal{E}_{\sigma, \varrho, \eta, \zeta_1^+}^{\alpha, \phi; w} \omega_1^p \right) (\zeta) \right]^{\frac{1}{p}} + \left[ \left( \mathcal{E}_{\sigma, \varrho, \eta, \zeta_1^+}^{\alpha, \phi; w} \omega_2^p \right) (\zeta) \right]^{\frac{1}{p}} \\ & \leq \left( \frac{\mu_2(\mu_1 + \nu_2) + \nu_2(\nu_1 + \mu_2)}{(\mu_2 + \nu_1)(\nu_2 + \mu_1)} \right) \left[ \left( \mathcal{E}_{\sigma, \varrho, \eta, \zeta_1^+}^{\alpha, \phi; w} (\omega_1 + \omega_2)^p \right) (\zeta) \right]^{\frac{1}{p}}. \end{aligned}$$

**Theorem 7.** Let  $\eta, \varrho > 0$  and  $w \in \mathbb{R}$ . Assume that  $\omega_1(\zeta)$  and  $\omega_2(\zeta)$  be two positive functions on  $[\zeta_1, \infty)$  such that  $\left( \mathcal{K}_{\sigma, \varrho, \eta, \zeta_1^+}^{\vartheta, \phi; w} \omega_1 \right) (\zeta) < \infty$  and  $\left( \mathcal{K}_{\sigma, \varrho, \eta, \zeta_1^+}^{\vartheta, \phi; w} \omega_2 \right) (\zeta) < \infty$  for all  $\zeta > \zeta_1 \geq 0$ . If  $0 < m \leq \frac{\omega_1(r)}{\omega_2(r)} \leq M$ , where  $r \in (\zeta_1, \zeta)$ , we have

$$\begin{aligned} \frac{1}{M} \left( \mathcal{K}_{\sigma, \varrho, \eta, \zeta_1^+}^{\vartheta, \phi; w} \omega_1 \omega_2 \right) (\zeta) & \leq \frac{1}{(m+1)(M+1)} \left( \mathcal{K}_{\sigma, \varrho, \eta, \zeta_1^+}^{\vartheta, \phi; w} (\omega_1 + \omega_2)^2 \right) (\zeta) \\ & \leq \frac{1}{m} \left( \mathcal{K}_{\sigma, \varrho, \eta, \zeta_1^+}^{\vartheta, \phi; w} \omega_1 \omega_2 \right) (\zeta). \end{aligned}$$

**Proof.** Since  $0 < m \leq \frac{\omega_1(r)}{\omega_2(r)} \leq M$ , where  $r \in (\zeta_1, \zeta)$ , we have

$$(m+1)\omega_2(r) \leq \omega_1(r) + \omega_2(r) \leq (M+1)\omega_2(r) \tag{49}$$

and

$$\left( \frac{M+1}{M} \right) \omega_1(r) \leq \omega_1(r) + \omega_2(r) \leq \left( \frac{m+1}{m} \right) \omega_1(r). \tag{50}$$

After multiplying both sides of inequalities (49) and (50), we obtain

$$\frac{\omega_1(r)\omega_2(r)}{M} \leq \frac{(\omega_1(r) + \omega_2(r))^2}{(m+1)(M+1)} \leq \frac{\omega_1(r)\omega_2(r)}{m}. \tag{51}$$

Multiplying both sides of inequality (51) by

$$\phi^{-1}(\zeta) \frac{\vartheta(\zeta - r)}{\zeta - r} \phi(r) \mathcal{R}_{\varrho, \eta}^{\sigma} [w(\zeta - r)^{\varrho}]$$

with  $r \in (\zeta_1, \zeta)$ , and integrating them over  $r \in (\zeta_1, \zeta)$ , we complete the proof of Theorem 7.  $\square$

**Corollary 14.** Under the hypotheses of Theorem 7, if we take

$$\vartheta(\zeta) = \zeta(\zeta_2 - \zeta)^{\alpha-1} \quad (\forall \zeta \in [\zeta_1, \zeta_2]; \alpha \in (0, 1]),$$

we have

$$\begin{aligned} \frac{1}{M} \left( \mathcal{C}_{\sigma, \varrho, \eta, \zeta_1^+}^{\alpha, \phi; w} \omega_1 \omega_2 \right) (\zeta) & \leq \frac{1}{(m+1)(M+1)} \left( \mathcal{C}_{\sigma, \varrho, \eta, \zeta_1^+}^{\alpha, \phi; w} (\omega_1 + \omega_2)^2 \right) (\zeta) \\ & \leq \frac{1}{m} \left( \mathcal{C}_{\sigma, \varrho, \eta, \zeta_1^+}^{\alpha, \phi; w} \omega_1 \omega_2 \right) (\zeta). \end{aligned}$$

**Corollary 15.** Under the hypotheses of Theorem 7, if we choose

$$\vartheta(\zeta) = \frac{\zeta}{\alpha} \exp(-Q\zeta),$$

where

$$Q = \frac{1 - \alpha}{\alpha}$$

and  $\alpha \in (0, 1]$  for all  $\zeta \in [\zeta_1, \zeta_2]$ , we obtain

$$\begin{aligned} \frac{1}{M} \left( \mathcal{E}_{\sigma, \rho, \eta, \zeta_1^+; w}^{\alpha, \phi} \omega_1 \omega_2 \right) (\zeta) &\leq \frac{1}{(m+1)(M+1)} \left( \mathcal{E}_{\sigma, \rho, \eta, \zeta_1^+; w}^{\alpha, \phi} (\omega_1 + \omega_2)^2 \right) (\zeta) \\ &\leq \frac{1}{m} \left( \mathcal{E}_{\sigma, \rho, \eta, \zeta_1^+; w}^{\alpha, \phi} \omega_1 \omega_2 \right) (\zeta). \end{aligned}$$

**Theorem 8.** Let  $p \geq 1$  with  $\eta, \rho > 0$  and  $w \in \mathbb{R}$ . Assume that  $\omega_1(\zeta)$  and  $\omega_2(\zeta)$  are positive functions on  $[\zeta_1, \infty)$  such that  $\left( \mathcal{K}_{\sigma, \rho, \eta, \zeta_1^+; w}^{\vartheta, \phi} \omega_1^p \right) (\zeta) < \infty$  and  $\left( \mathcal{K}_{\sigma, \rho, \eta, \zeta_1^+; w}^{\vartheta, \phi} \omega_2^p \right) (\zeta) < \infty$  for all  $\zeta > \zeta_1 \geq 0$ . If  $0 < m \leq \frac{\omega_1(r)}{\omega_2(r)} \leq M$ , where  $r \in (\zeta_1, \zeta)$ , we have

$$\begin{aligned} \left[ \left( \mathcal{K}_{\sigma, \rho, \eta, \zeta_1^+; w}^{\vartheta, \phi} \omega_1^p \right) (\zeta) \right]^{\frac{1}{p}} + \left[ \left( \mathcal{K}_{\sigma, \rho, \eta, \zeta_1^+; w}^{\vartheta, \phi} \omega_2^p \right) (\zeta) \right]^{\frac{1}{p}} \\ \leq 2 \left[ \left( \mathcal{K}_{\sigma, \rho, \eta, \zeta_1^+; w}^{\vartheta, \phi} Y^p(\omega_1, \omega_2) \right) (\zeta) \right]^{\frac{1}{p}}, \end{aligned}$$

where

$$Y(\omega_1(r), \omega_2(r)) := \max \left\{ \left( \frac{M}{m} + 1 \right) \omega_1(r) - M\omega_2(r), \frac{(M+m)\omega_2(r) - \omega_1(r)}{m} \right\}.$$

**Proof.** Since  $0 < m \leq \frac{\omega_1(r)}{\omega_2(r)} \leq M$ , where  $r \in (\zeta_1, \zeta)$ , we have

$$0 < m \leq M + m - \frac{\omega_1(r)}{\omega_2(r)} \leq M. \tag{52}$$

From inequality (52), we obtain

$$\omega_2(r) \leq \frac{(M+m)\omega_2(r) - \omega_1(r)}{m} \leq Y(\omega_1(r), \omega_2(r)). \tag{53}$$

Similarly,

$$\frac{1}{M} \leq \frac{1}{M} + \frac{1}{m} - \frac{\omega_2(r)}{\omega_1(r)} \leq \frac{1}{m}. \tag{54}$$

From inequality (54), we obtain

$$\frac{1}{M} \leq \frac{\left( \frac{1}{M} + \frac{1}{m} \right) \omega_1(r) - \omega_2(r)}{\omega_1(r)} \leq \frac{1}{m}. \tag{55}$$

It is clear from inequality (55) that  $\omega_1(r) \leq Y(\omega_1(r), \omega_2(r))$ , which means

$$[\omega_1(r)]^p \leq Y^p(\omega_1(r), \omega_2(r)). \tag{56}$$

Similarly, from inequality (53), we obtain

$$[\omega_2(r)]^p \leq Y^p(\omega_1(r), \omega_2(r)). \tag{57}$$

Multiplying both sides of inequalities (56) and (57) by

$$\phi^{-1}(\zeta) \frac{\vartheta(\zeta - r)}{\zeta - r} \phi(r) \mathcal{R}_{\varrho, \eta}^{\sigma} [w(\zeta - r)^{\varrho}]$$

with  $r \in (\zeta_1, \zeta)$ , integrating them over  $r \in (\zeta_1, \zeta)$ , and adding them together, we complete the proof of Theorem 8.  $\square$

**Corollary 16.** Under the hypotheses of Theorem 8, if we take

$$\vartheta(\zeta) = \zeta(\zeta_2 - \zeta)^{\alpha-1} \quad (\forall \zeta \in [\zeta_1, \zeta_2]; \alpha \in (0, 1]),$$

we have

$$\left[ \left( \mathcal{E}_{\sigma, \varrho, \eta, \zeta_1^+}^{\alpha, \phi; w} \omega_1^p \right) (\zeta) \right]^{\frac{1}{p}} + \left[ \left( \mathcal{E}_{\sigma, \varrho, \eta, \zeta_1^+}^{\alpha, \phi; w} \omega_2^p \right) (\zeta) \right]^{\frac{1}{p}} \leq 2 \left[ \left( \mathcal{E}_{\sigma, \varrho, \eta, \zeta_1^+}^{\alpha, \phi; w} Y^p(\omega_1, \omega_2) \right) (\zeta) \right]^{\frac{1}{p}}.$$

**Corollary 17.** Under the hypotheses of Theorem 8, if we choose

$$\vartheta(\zeta) = \frac{\zeta}{\alpha} \exp(-Q\zeta),$$

where

$$Q = \frac{1 - \alpha}{\alpha}$$

and  $\alpha \in (0, 1]$  for all  $\zeta \in [\zeta_1, \zeta_2]$ , we obtain

$$\left[ \left( \mathcal{E}_{\sigma, \varrho, \eta, \zeta_1^+}^{\alpha, \phi; w} \omega_1^p \right) (\zeta) \right]^{\frac{1}{p}} + \left[ \left( \mathcal{E}_{\sigma, \varrho, \eta, \zeta_1^+}^{\alpha, \phi; w} \omega_2^p \right) (\zeta) \right]^{\frac{1}{p}} \leq 2 \left[ \left( \mathcal{E}_{\sigma, \varrho, \eta, \zeta_1^+}^{\alpha, \phi; w} Y^p(\omega_1, \omega_2) \right) (\zeta) \right]^{\frac{1}{p}}.$$

#### 4. Examples

**Example 1.** Assume that  $p \geq 1$ , with  $\eta, \varrho, a > 0$  and  $w \in \mathbb{R}$ . Then, for all  $\zeta > \zeta_1 \geq 1$  and  $r \in (\zeta_1, \zeta)$ , the following inequality holds:

$$\begin{aligned} \left[ \left( \mathcal{K}_{\sigma, \varrho, \eta, \zeta_1^+}^{\vartheta, \phi; w} r^p \right) (\zeta) \right]^{\frac{1}{p}} + \left[ \left( \mathcal{K}_{\sigma, \varrho, \eta, \zeta_1^+}^{\vartheta, \phi; w} (r+a)^p \right) (\zeta) \right]^{\frac{1}{p}} \\ \leq \frac{3a+4}{2(a+2)} \left[ \left( \mathcal{K}_{\sigma, \varrho, \eta, \zeta_1^+}^{\vartheta, \phi; w} (2r+a)^p \right) (\zeta) \right]^{\frac{1}{p}}. \end{aligned}$$

**Proof.** Taking  $\omega_1(r) = r+a$  and  $\omega_2(r) = r$ , we have, respectively,  $m = 1$  and  $M = a+1$ . Applying Theorem 1, we get the desired result.  $\square$

**Example 2.** Suppose that  $p \geq 1$ , with  $\eta, \varrho, a > 0$  and  $w \in \mathbb{R}$ . Then, for all  $\zeta > \zeta_1 \geq 1$  and  $r \in (\zeta_1, \zeta)$ , the following inequality holds:

$$\begin{aligned} \left[ \left( \mathcal{K}_{\sigma, \varrho, \eta, \zeta_1^+}^{\vartheta, \phi; w} r^p \right) (\zeta) \right]^{\frac{2}{p}} + \left[ \left( \mathcal{K}_{\sigma, \varrho, \eta, \zeta_1^+}^{\vartheta, \phi; w} (r+a)^p \right) (\zeta) \right]^{\frac{2}{p}} \\ \geq \frac{2}{a+1} \left[ \left( \mathcal{K}_{\sigma, \varrho, \eta, \zeta_1^+}^{\vartheta, \phi; w} r^p \right) (\zeta) \right]^{\frac{1}{p}} \left[ \left( \mathcal{K}_{\sigma, \varrho, \eta, \zeta_1^+}^{\vartheta, \phi; w} (r+a)^p \right) (\zeta) \right]^{\frac{1}{p}}. \end{aligned}$$

**Proof.** Choosing  $\omega_1(r) = r+a$  and  $\omega_2(r) = r$ , we obtain, respectively,  $m = 1$  and  $M = a+1$ . Using Theorem 2, we have the desired result.  $\square$

**Example 3.** Assume that  $p > 1$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ , with  $\eta, \varrho, a > 0$  and  $w \in \mathbb{R}$ . Then, for all  $\zeta > \zeta_1 \geq 1$  and  $r \in (\zeta_1, \zeta)$ , the following inequality holds:



$$\left[ \left( \mathcal{K}_{\sigma, \varrho, \eta, \varsigma_1^+; w}^{\vartheta, \phi}(r+a) \right) (\varsigma) \right]^{\frac{1}{p}} \left[ \left( \mathcal{K}_{\sigma, \varrho, \eta, \varsigma_1^+; w}^{\vartheta, \phi}(r) \right) (\varsigma) \right]^{\frac{1}{q}} \leq (a+1)^{\frac{1}{pq}} \left( \mathcal{K}_{\sigma, \varrho, \eta, \varsigma_1^+; w}^{\vartheta, \phi}(r+a)^{\frac{1}{p}} r^{\frac{1}{q}} \right) (\varsigma).$$

**Proof.** Taking  $\omega_1(r) = r + a$  and  $\omega_2(r) = r$ , we have, respectively,  $m = 1$  and  $M = a + 1$ . Applying Theorem 3, we get the desired result.  $\square$

**Example 4.** Suppose that  $p > 1, \frac{1}{p} + \frac{1}{q} = 1$ , with  $\eta, \varrho, a > 0$  and  $w \in \mathbb{R}$ . Then, for all  $\varsigma > \varsigma_1 \geq 1$  and  $r \in (\varsigma_1, \varsigma)$ , the following inequality holds:

$$\begin{aligned} \left( \mathcal{K}_{\sigma, \varrho, \eta, \varsigma_1^+; w}^{\vartheta, \phi}(r(r+a)) \right) (\varsigma) &\leq \frac{2^{p-1}(a+1)^p}{p(a+2)^p} \left( \mathcal{K}_{\sigma, \varrho, \eta, \varsigma_1^+; w}^{\vartheta, \phi}(r^p + (r+a)^p) \right) (\varsigma) \\ &\quad + \frac{1}{2q} \left( \mathcal{K}_{\sigma, \varrho, \eta, \varsigma_1^+; w}^{\vartheta, \phi}(r^q + (r+a)^q) \right) (\varsigma). \end{aligned}$$

**Proof.** Choosing  $\omega_1(r) = r + a$  and  $\omega_2(r) = r$ , we obtain, respectively,  $m = 1$  and  $M = a + 1$ . Using Theorem 4, we have the desired result.  $\square$

**Example 5.** Assume that  $p \geq 1$ , with  $\eta, \varrho, a > 0$  and  $w \in \mathbb{R}$ . Then, for all  $\varsigma > \varsigma_1 \geq 1$  with  $k \in (0, 1)$  and  $r \in (\varsigma_1, \varsigma)$ , the following inequality holds:

$$\begin{aligned} \left( \frac{a+2}{a+1-k} \right) \left[ \left( \mathcal{K}_{\sigma, \varrho, \eta, \varsigma_1^+; w}^{\vartheta, \phi}(r(1-k)+a)^p \right) (\varsigma) \right]^{\frac{1}{p}} \\ \leq \left[ \left( \mathcal{K}_{\sigma, \varrho, \eta, \varsigma_1^+; w}^{\vartheta, \phi}(r^p) \right) (\varsigma) \right]^{\frac{1}{p}} + \left[ \left( \mathcal{K}_{\sigma, \varrho, \eta, \varsigma_1^+; w}^{\vartheta, \phi}(r+a)^p \right) (\varsigma) \right]^{\frac{1}{p}} \\ \leq \left( \frac{2}{1-k} \right) \left[ \left( \mathcal{K}_{\sigma, \varrho, \eta, \varsigma_1^+; w}^{\vartheta, \phi}(r(1-k)+a)^p \right) (\varsigma) \right]^{\frac{1}{p}}. \end{aligned}$$

**Proof.** Taking  $\omega_1(r) = r + a$  and  $\omega_2(r) = r$ , we have, respectively,  $m = 1$  and  $M = a + 1$ . Applying Theorem 5, we get the desired result.  $\square$

**Example 6.** Suppose that  $p \geq 1$ , with  $\eta, \varrho > 0$  and  $w \in \mathbb{R}$ . Then, for all  $\varsigma > \varsigma_1 \geq 0$  and  $r \in (\varsigma_1, \varsigma)$ , the following inequality holds:

$$\left[ \left( \mathcal{K}_{\sigma, \varrho, \eta, \varsigma_1^+; w}^{\vartheta, \phi}(\sin^2 r) \right) (\varsigma) \right]^{\frac{1}{p}} + \left[ \left( \mathcal{K}_{\sigma, \varrho, \eta, \varsigma_1^+; w}^{\vartheta, \phi}(\cos^2 r) \right) (\varsigma) \right]^{\frac{1}{p}} \leq 2 \left[ \left( \mathcal{K}_{\sigma, \varrho, \eta, \varsigma_1^+; w}^{\vartheta, \phi}(1) \right) (\varsigma) \right]^{\frac{1}{p}}.$$

**Proof.** Choosing  $\omega_1(r) = \sin^2 r$  and  $\omega_2(r) = \cos^2 r$ , we obtain, respectively,  $\mu_1 = \nu_1 = 0$  and  $\mu_2 = \nu_2 = 1$ . Using Theorem 6, we have the desired result.  $\square$

**Example 7.** Assume that  $\eta, \varrho, a > 0$  and  $w \in \mathbb{R}$ . Then, for all  $\varsigma > \varsigma_1 \geq 1$  and  $r \in (\varsigma_1, \varsigma)$ , the following inequality holds:

$$\begin{aligned} \frac{1}{a+1} \left( \mathcal{K}_{\sigma, \varrho, \eta, \varsigma_1^+; w}^{\vartheta, \phi}(r(r+a)) \right) (\varsigma) &\leq \frac{1}{2(a+2)} \left( \mathcal{K}_{\sigma, \varrho, \eta, \varsigma_1^+; w}^{\vartheta, \phi}(2r+a)^2 \right) (\varsigma) \\ &\leq \left( \mathcal{K}_{\sigma, \varrho, \eta, \varsigma_1^+; w}^{\vartheta, \phi}(r(r+a)) \right) (\varsigma). \end{aligned}$$

**Proof.** Taking  $\omega_1(r) = r + a$  and  $\omega_2(r) = r$ , we have, respectively,  $m = 1$  and  $M = a + 1$ . Applying Theorem 7, we get the desired result.  $\square$

**Example 8.** Suppose that  $p \geq 1$ , with  $\eta, \varrho, a > 0$  and  $w \in \mathbb{R}$ . Then, for all  $\zeta > \zeta_1 \geq 1$  and  $r \in (\zeta_1, \zeta)$ , the following inequality holds:

$$\left[ \left( \mathcal{K}_{\sigma, \varrho, \eta, \zeta_1^+; w}^{\theta, \phi} r^p \right) (\zeta) \right]^{\frac{1}{p}} + \left[ \left( \mathcal{K}_{\sigma, \varrho, \eta, \zeta_1^+; w}^{\theta, \phi} (r+a)^p \right) (\zeta) \right]^{\frac{1}{p}} \leq 2 \left[ \left( \mathcal{K}_{\sigma, \varrho, \eta, \zeta_1^+; w}^{\theta, \phi} [Y_a(r)]^p \right) (\zeta) \right]^{\frac{1}{p}},$$

where

$$Y_a(r) := \max \{ a(a+2) + r, r(a+1) - a \}.$$

**Proof.** Choosing  $\omega_1(r) = r + a$  and  $\omega_2(r) = r$ , we obtain, respectively,  $m = 1$  and  $M = a + 1$ . Using Theorem 8, we have the desired result.  $\square$

## 5. Conclusions

In this paper, we obtained reverse Minkowski inequalities pertaining to new weighted generalized fractional integral operators. In order to demonstrate the significance of our main results, several special cases for suitable choices of functions are given. Finally, some concrete examples demonstrated the significance of our results. For future research, using our ideas and techniques, we will define a new general family of fractional integral operators, the so-called weighted generalized fractional integral operators associated with positive, increasing, measurable and monotone functions. We will derive several new interesting inequalities using Chebyshev, Markov and Minkowski inequalities. Moreover, we will find many important inequalities using finite products of functions. We hope that our results may stimulate further research in different areas of pure and applied sciences.

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