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Article

Numerical Analysis of Time-Fractional Whitham-Broer-Kaup Equations with Exponential-Decay Kernel

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Abstract: This paper presents the semi-analytical analysis of the fractional-order non-linear coupled system of Whitham-Broer-Kaup equations. An iterative process is designed to analyze analytical findings to the specified non-linear partial fractional derivatives scheme utilizing the Yang transformation coupled with the Adomian technique. The fractional derivative is considered in the sense of Caputo-Fabrizio. Two numerical problems show the suggested method. Moreover, the results of the suggested technique are compared with the solution of other well-known numerical techniques such as the Homotopy perturbation technique, Adomian decomposition technique, and the Variation iteration technique. Numerical simulation has been carried out to verify that the suggested methodologies are accurate and reliable, and the results are revealed using graphs and tables. Comparing the analytical and actual solutions demonstrates that the proposed approaches effectively solve complicated non-linear problems. Furthermore, the proposed methodologies control and manipulate the achieved numerical solutions in a vast acceptable region in an extreme manner. It will provide us with a simple process to control and adjust the convergence region of the series solution.

Keywords: Adomian decomposition method; system of Whitham-Broer-Kaup equations; Caputo-Fabrizio derivative; Yang transform

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1. Introduction

Fractional calculus (FC) was invented by Newton, but it has recently piqued the interest of many academics. Fascinating breakthroughs in science and engineering applications have been found within the framework of FC over the last 30 years. Due to the complications involved with a heterogeneity issue, the notion of the fractional derivative has been industrialized. The behaviour of complex media with a diffusion mechanism may be captured using non-integer order differential operators [1–4]. It has proven a handy tool, and differential equations of any order may demonstrate various situations more efficiently and precisely. Numerous scholars began to work on calculus and its generalization to express their viewpoints while investigating many complicated events due to the rapid development of mathematical approaches using computer software [5–8].

Differential equations featuring non-linearities are used in science, technology, and engineering to explain a variety of phenomena, ranging from gravity to dynamical systems [9–11]. Non-linear partial differential equations (PDEs) are significant techniques for modeling non-linear dynamical events in a variety of domains, including mathematical biology, fluid mechanics, material science, and fluid dynamics, as shown in [12]. A sufficient set of partial differential equations can represent the bulk of dynamical systems. PDEs are also well-known for being utilized to solve mathematical difficulties like the Poincare and the Calabi conjectures.

It has already been demonstrated that the non-linear development of shallow-water waves may be represented using the technique of the Whitham-Broer-Kaup equation in fluid mechanics (WBKEs) [13]. Whitham, Broer, and Kaup [14–16] developed the integrated framework of the equations as mentioned earlier. The mentioned equations can be written as the shallow water acoustic waves with various diversity connections, as shown [17]. In the classical order, the governing equations for the phenomenon mentioned above are represented by

$$\begin{cases} \mathbb{U}_{\Im} + \mathbb{U}\mathbb{U}_{\varepsilon} + \mathbb{V}_{\varepsilon} + q\mathbb{U}_{\varepsilon\varepsilon} = 0 \\ \mathbb{V}_{\Im} + \mathbb{V}\mathbb{U}_{\varepsilon} + \mathbb{U}\mathbb{V}_{\varepsilon} - q\mathbb{V}_{\varepsilon\varepsilon} + p\mathbb{U}_{\varepsilon\varepsilon\varepsilon} = 0, \end{cases} \tag{1}$$

where $\mathbb{U} = \mathbb{U}(\varepsilon, \Im), \mathbb{V} = \mathbb{V}(\varepsilon, \Im)$ indicates the horizontal velocity and height of the fluids, respectively, which differ greatly from the equilibrium, and q, p are the constants that are composed of various diffusion powers. For the past few decades, investigating the results of non-linear PDEs has been a major focus of research. Several authors have devised numerous mathematical methods to examine approximate results of non-linear PDEs. Mohyud Din et al. [18] investigated the analysis of many integer order PDEs using homotopy perturbation techniques. To solve the coupled set of Burgers and Brusselator equations, Biazar and Aminikhah [19] used the perturbation technique. For the numerical result of many traditional order differential equations by applying other techniques, interested readers can refer to Refs. [20–25]. Numerous strategies have been used to study the solution to the given non-linear coupled scheme (1) of PDEs. To address the classical order coupled systems of the WBK problem, Mohyud-Din et al. [26] employed perturbation methods. As a result, researchers like Xie et al. 2002 (who studied the solution using the hyperbolic technique) have used several powerful and efficient methods to investigate the problem of the WBK coupled equation of classical order PDEs. In the same way, El-Sayed and Kaya used the Adomian decomposition approach to investigate the scheme (1). Moreover, Ahmad et al. [12] used the Adomian decomposition method and He's polynomial to solve the coupled system (1).

Adomian proposed the Adomian decomposition method in 1980, which is a helpful technique for obtaining an explicit and numerical solution to a system of differential equations that represents a physical problem [27–29]. The Laplace transform technique is a vital technique in technology and applied mathematics. Combining the Adomian decomposition method and Yang transformation leads to a well-known technique named the Yang decomposition method. In this study, we convert differential equations to algebraic equations using the Laplace transform, and the non-linear terms are decomposed using Adomain polynomials. This numerical approach is effective for both deterministic and stochastic differential equation systems. It can be applied to a classical and fractional-order ordinary and a PDEs system, both linear and non-linear. There is no need for perturbation or liberalization in this procedure. Furthermore, unlike RK4, it does not require a pre-defined step size. In addition, this technique does not depend upon a parameter, as required for homotopy analysis and homotopy perturbation methods. However, the solutions achieved via this technique are the same as gained by the Adomian decomposition method (for detail, see [30–33]). It must be mentioned that the Yang decomposition method is more effective than the basic Adomian decomposition method.

The rest of this article is organized as follows. In Section 2, we present some basic definitions and properties. Section 3 describes the Yang decomposition method for solving fractional partial differential equations. The conclusion is presented at the end of the article.

2. Preliminaries Concepts

In this section, we provide the fundamental definitions that will be used throughout the article. For the purpose of simplification, we write the exponential decay kernel as, $K(\Im, \varrho) = e^{[-\wp(\Im-\varrho/1-\wp)]}$.

Definition 1. *If the Caputo-Fabrizio derivative is given as follows [34]:*

$$^{CF}D_{\Im}^{\wp}[\mathbb{P}(\Im)] = \frac{N(\wp)}{1-\wp}\int_{0}^{\Im}\mathbb{P}'(\varrho)K(\Im, \varrho)d\varrho, \quad n-1 < \wp \leq n \tag{2}$$

$N(\wp)$ is the normalization function with $N(0) = N(1) = 1$.

$$^{CF}D^{\wp}_{\Im}[\mathbb{P}(\Im)] = \frac{N(\wp)}{1-\wp}\int_0^{\Im}[\mathbb{P}(\Im) - \mathbb{P}(\varrho)]K(\Im, \varrho)d\varrho. \tag{3}$$

Definition 2. *The fractional integral Caputo-Fabrizio is given as [34]*

$$^{CF}I^{\wp}_{\Im}[\mathbb{P}(\Im)] = \frac{1-\wp}{N(\wp)}\mathbb{P}(\Im) + \frac{\wp}{N(\wp)}\int_0^{\Im}\mathbb{P}(\varrho)d\varrho, \;\; \Im \geq 0, \wp \in (0,1]. \tag{4}$$

Definition 3. *For $N(\wp) = 1$, the following result shows the Caputo-Fabrizio derivative of Laplace transformation [34]:*

$$L\left[^{CF}D^{\wp}_{\Im}[\mathbb{P}(\Im)]\right] = \frac{vL[\mathbb{P}(\Im) - \mathbb{P}(0)]}{v + \wp(1-v)}. \tag{5}$$

Definition 4. *The Yang transformation of $\mathbb{P}(\Im)$ is expressed as [35]*

$$\mathbb{Y}[\mathbb{P}(\Im)] = \chi(v) = \int_0^{\infty}\mathbb{P}(\Im)e^{-\frac{\Im}{v}}d\Im. \;\; \Im > 0, \tag{6}$$

Remark 1. *The Yang transformation of a few useful functions is defined as:*

$$\begin{aligned} \mathbb{Y}[1] &= v, \\ \mathbb{Y}[\Im] &= v^2, \\ \mathbb{Y}[\Im^i] &= \Gamma(i+1)v^{i+1}. \end{aligned} \tag{7}$$

Lemma 1. *Let the Laplace transformation of $\mathbb{P}(\Im)$ is $F(v)$, then $\chi(v) = F(1/v)$ [36].*

Proof. From Equation (6), we can achieve another type of the Yang transformation by putting $\Im/v = \zeta$ as

$$L[\mathbb{P}(\Im)] = \chi(v) = v\int_0^{\infty}\mathbb{P}(v\zeta)e^{\zeta}d\zeta. \;\; \zeta > 0, \tag{8}$$

Since $L[\mathbb{P}(\Im)] = F(v)$, this implies that

$$F(v) = L[\mathbb{P}(\Im)] = \int_0^{\infty}\mathbb{P}(\Im)e^{-v\Im}d\Im. \tag{9}$$

Put $\Im = \zeta/v$ in (9), we have

$$F(v) = \frac{1}{v}\int_0^{\infty}\mathbb{P}\left(\frac{\zeta}{v}\right)e^{\zeta}d\zeta. \tag{10}$$

Thus, from Equation (8), we achieve

$$F(v) = \chi\left(\frac{1}{v}\right). \tag{11}$$

Additionally, from Equations (6) and (9), we achieve

$$F\left(\frac{1}{v}\right) = \chi(v). \tag{12}$$

The connections Equations between (11) and (12) represent the duality link between the Laplace and Yang transformation. □

Lemma 2. *Let $\mathbb{P}(\Im)$ be a continuous function; then, the Caputo-Fabrizio derivative Yang transformation of $\mathbb{P}(\Im)$ is defined by [36]*

$$\mathbb{Y}[\mathbb{P}(\Im)] = \frac{\mathbb{Y}[\mathbb{P}(\Im) - v\mathbb{P}(0)]}{1 + \wp(v-1)}. \tag{13}$$

Proof. The Caputo-Fabrizio fractional Laplace transformation is given by

$$L[\mathbb{P}(\Im)] = \frac{L[v\mathbb{P}(\Im) - \mathbb{P}(0)]}{v + \wp(1-v)}, \tag{14}$$

In addition, we have that the connection among Laplace and Yang property, i.e., $\chi(v) = F(1/v)$. To achieve the necessary result, we substitute v by $1/v$ in Equation (14), and get

$$\begin{aligned}\mathbb{Y}[\mathbb{P}(\Im)] =& \frac{\frac{1}{v}\mathbb{Y}[\mathbb{P}(\Im) - \mathbb{P}(0)]}{\frac{1}{v} + \wp(1-\frac{1}{v})},\\ \mathbb{Y}[\mathbb{P}(\Im)] =& \frac{\mathbb{Y}[\mathbb{P}(\Im) - v\mathbb{P}(0)]}{1 + \wp(v-1)}.\end{aligned} \tag{15}$$

The proof is completed. □

3. The Producer of YDM

In this portion, we discuses the YDM producer for fractional partial differential equations.

$$\begin{aligned}&{}^{CF}D^{\wp}_{\Im}\mathbb{U}(\varepsilon,\Im) + \mathcal{G}_1(\mathbb{U},\mathbb{V}) + \mathcal{L}_1(\mathbb{U},\mathbb{V}) - \mathcal{P}_1(\varepsilon,\Im) = 0,\\ &{}^{CF}D^{\wp}_{\Im}\mathbb{V}(\varepsilon,\Im) + \mathcal{G}_2(\mathbb{U},\mathbb{V}) + \mathcal{L}_2(\mathbb{U},\mathbb{V}) - \mathcal{P}_2(\varepsilon,\Im) = 0, \quad 0 < \wp \leq 1,\end{aligned} \tag{16}$$

with initial condition

$$\mathbb{U}(\varepsilon,0) = g_1(\varepsilon), \quad \mathbb{V}(\varepsilon,0) = g_2(\varepsilon). \tag{17}$$

where $D^{\wp}_{\Im} = \frac{\partial^{\wp}}{\partial\Im^{\wp}}$ is the Caputo fractional derivative of order $\wp$, $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{L}_1$, $\mathcal{L}_2$ are the linear and non-linear functions, respectively, and $\mathcal{P}_1$,$\mathcal{P}_2$ are the source functions.

Using the Yang transformation to Equation (16),

$$\begin{aligned}&\mathbb{Y}[D^{\wp}_{\Im}\mathbb{U}(\varepsilon,\Im)] + \mathbb{Y}[\mathcal{G}_1(\mathbb{U},\mathbb{V}) + \mathcal{L}_1(\mathbb{U},\mathbb{V}) - \mathcal{P}_1(\varepsilon,\Im)] = 0,\\ &\mathbb{Y}[D^{\wp}_{\Im}\mathbb{V}(\varepsilon,\Im)] + \mathbb{Y}[\mathcal{G}_2(\mathbb{U},\mathbb{V}) + \mathcal{L}_2(\mathbb{U},\mathbb{V}) - \mathcal{P}_2(\varepsilon,\Im)] = 0.\end{aligned} \tag{18}$$

Using the Yang transformation differentiation property, we have

$$\begin{aligned}\mathbb{Y}[\mathbb{U}(\varepsilon,\Im)] &= v\mathbb{U}(\varepsilon,0) + (1+\wp(v-1))\mathbb{Y}[\mathcal{P}_1(\varepsilon,\Im)] - (1+\wp(v-1))\mathbb{Y}\{\mathcal{G}_1(\mathbb{U},\mathbb{V}) + \mathcal{L}_1(\mathbb{U},\mathbb{V})\}],\\ \mathbb{Y}[\mathbb{V}(\varepsilon,\Im)] &= v\mathbb{V}(\varepsilon,0) + (1+\wp(v-1))\mathbb{Y}[\mathcal{P}_2(\varepsilon,\Im)] - (1+\wp(v-1))\mathbb{Y}\{\mathcal{G}_2(\mathbb{U},\mathbb{V}) + \mathcal{L}_2(\mathbb{U},\mathbb{V})\}].\end{aligned} \tag{19}$$

YDM describes the solution of infinite series $\mathbb{U}(\varepsilon,\Im)$ and $\mathbb{V}(\varepsilon,\Im)$,

$$\mathbb{U}(\varepsilon,\Im) = \sum_{m=0}^{\infty}\mathbb{U}_m(\varepsilon,\Im), \quad \mathbb{V}(\varepsilon,\Im) = \sum_{m=0}^{\infty}\mathbb{V}_m(\varepsilon,\Im). \tag{20}$$

Adomian polynomials of non-linear terms of $\mathcal{L}_1$ and $\mathcal{L}_2$ are represented as

$$\mathcal{L}_1(\mathbb{U},\mathbb{V}) = \sum_{m=0}^{\infty}\mathcal{A}_m, \quad \mathcal{L}_2(\mathbb{U},\mathbb{V}) = \sum_{m=0}^{\infty}\mathcal{B}_m. \tag{21}$$

The expression for Adomian polynomials is

$$\begin{aligned} \mathcal{A}_m &= \frac{1}{m!}\left[\frac{\partial^m}{\partial\lambda^m}\left\{\sum_{m=0}^{\infty}\lambda^m\mathbb{U}_m, \sum_{m=0}^{\infty}\lambda^m\mathbb{V}_m\right\}\right]_{\lambda=0}, \\ \mathcal{B}_m &= \frac{1}{m!}\left[\frac{\partial^m}{\partial\lambda^m}\left\{\sum_{m=0}^{\infty}\lambda^m\mathbb{U}_m, \sum_{m=0}^{\infty}\lambda^m\mathbb{V}_m\right\}\right]_{\lambda=0}. \end{aligned} \tag{22}$$

Putting Equations (20) and (22) into Equation (19),

$$\begin{aligned} \mathbb{Y}\left[\sum_{m=1}^{\infty}\mathbb{U}_m(\varepsilon,\Im)\right] &= s\mathbb{U}(\varepsilon,0) + (1+\wp(v-1))\mathbb{Y}\{\mathcal{P}_1(\varepsilon,\Im)\} \\ &\quad - (1+\wp(v-1))\mathbb{Y}\left\{\mathcal{G}_1(\sum_{m=0}^{\infty}\mathbb{U}_m, \sum_{m=0}^{\infty}\mathbb{V}_m) + \sum_{m=0}^{\infty}\mathcal{A}_m\right\}, \\ \mathbb{Y}[\sum_{m=1}^{\infty}\mathbb{V}_m(\varepsilon,\Im)] &= v\mathbb{V}(\varepsilon,0) + (1+\wp(v-1))\mathbb{Y}\{\mathcal{P}_2(\varepsilon,\Im)\} \\ &\quad - (1+\wp(v-1))\mathbb{Y}\left\{\mathcal{G}_2(\sum_{m=0}^{\infty}\mathbb{U}_m, \sum_{m=0}^{\infty}\mathbb{V}_m) + \sum_{m=0}^{\infty}\mathcal{B}_m\right\}. \end{aligned} \tag{23}$$

The inverse Yang transformation is implemented on Equation (23),

$$\begin{aligned} \sum_{m=1}^{\infty}\mathbb{U}_m(\varepsilon,\Im) &= \mathbb{Y}^{-1}[v\mathbb{U}(\varepsilon,0) + (1+\wp(v-1))\mathbb{Y}\{\mathcal{P}_1(\varepsilon,\Im)\}] \\ &\quad - \mathbb{Y}^{-1}\left[(1+\wp(v-1))\mathbb{Y}\left\{\mathcal{G}_1(\sum_{m=0}^{\infty}\mathbb{U}_m, \sum_{m=0}^{\infty}\mathbb{V}_m) + \sum_{m=0}^{\infty}\mathcal{A}_m\right\}\right], \\ \sum_{m=1}^{\infty}\mathbb{V}_m(\varepsilon,\Im) &= \mathbb{Y}^{-1}[v\mathbb{V}(\varepsilon,0) + (1+\wp(v-1))\mathbb{Y}\{\mathcal{P}_2(\varepsilon,\Im)\}] \\ &\quad - \mathbb{Y}^{-1}\left[(1+\wp(v-1))\mathbb{Y}\left\{\mathcal{G}_2(\sum_{m=0}^{\infty}\mathbb{U}_m, \sum_{m=0}^{\infty}\mathbb{V}_m) + \sum_{m=0}^{\infty}\mathcal{B}_m\right\}\right]. \end{aligned} \tag{24}$$

Find the $\mathbb{U}_0$ and $\mathbb{V}_0$ using the initial conditions and sources functions. The following terms are expressed:

$$\begin{aligned} \mathbb{U}_0(\varepsilon,\Im) &= \mathbb{Y}^{-1}[v\mathbb{U}(\varepsilon,0) + (1+\wp(v-1))\mathbb{Y}\{\mathcal{P}_1(\varepsilon,\Im)\}], \\ \mathbb{V}_0(\varepsilon,\Im) &= \mathbb{Y}^{-1}[v\mathbb{V}(\varepsilon,0) + (1+\wp(v-1))\mathbb{Y}\{\mathcal{P}_2(\varepsilon,\Im)\}]. \end{aligned} \tag{25}$$

For $m = 1$

$$\begin{aligned} \mathbb{U}_1(\varepsilon,\Im) &= -\mathbb{Y}^{-1}[(1+\wp(v-1))\mathbb{Y}\{\mathcal{G}_1(\mathbb{U}_0,\mathbb{V}_0) + \mathcal{A}_0\}], \\ \mathbb{V}_1(\varepsilon,\Im) &= -\mathbb{Y}^{-1}[(1+\wp(v-1))\mathbb{Y}\{\mathcal{G}_2(\mathbb{U}_0,\mathbb{V}_0) + \mathcal{B}_0\}], \end{aligned}$$

the general for $m \geq 1$, is given by

$$\begin{aligned} \mathbb{U}_{m+1}(\varepsilon,\Im) &= -\mathbb{Y}^{-1}[(1+\wp(v-1))\mathbb{Y}\{\mathcal{G}_1(\mathbb{U}_m,\mathbb{V}_m) + \mathcal{A}_m\}], \\ \mathbb{V}_{m+1}(\varepsilon,\Im) &= -\mathbb{Y}^{-1}[(1+\wp(v-1))\mathbb{Y}\{\mathcal{G}_2(\mathbb{U}_m,\mathbb{V}_m) + \mathcal{B}_m\}], \end{aligned}$$

4. Numerical Results

Example 1. *Consider the fractional-order system of WBKEs [11]*

$$
\begin{aligned}
&{}^{CF}D^{\wp}_{\Im}\mathbb{U}(\varepsilon,\Im)+\mathbb{U}(\varepsilon,\Im)\frac{\partial\mathbb{U}(\varepsilon,\Im)}{\partial\varepsilon}+\frac{\partial\mathbb{U}(\varepsilon,\Im)}{\partial\varepsilon}+\frac{\partial\mathbb{V}(\varepsilon,\Im)}{\partial\varepsilon}=0,\\
&{}^{CF}D^{\wp}_{\Im}\mathbb{V}(\varepsilon,\Im)+\mathbb{U}(\varepsilon,\Im)\frac{\partial\mathbb{V}(\varepsilon,\Im)}{\partial\varepsilon}+\mathbb{V}(\varepsilon,\Im)\frac{\partial\mathbb{U}(\varepsilon,\Im)}{\partial\varepsilon}+3\frac{\partial^3\mathbb{U}(\varepsilon,\Im)}{\partial\varepsilon^3}-\frac{\partial^2\mathbb{V}(\varepsilon,\Im)}{\partial\varepsilon^2}=0,\\
&0<\wp\le 1,\quad -1<\Im\le 1,\quad -10\le\varepsilon\le 10,
\end{aligned}
\tag{26}
$$

with the initial conditions

$$
\begin{aligned}
\mathbb{U}(\varepsilon,0)&=\frac{1}{2}-8\tanh(-2\varepsilon),\\
\mathbb{V}(\varepsilon,0)&=16-16\tanh^2(-2\varepsilon).
\end{aligned}
\tag{27}
$$

Applying the Yang transformation of Equation (26), we have

$$
\begin{aligned}
&\mathbb{Y}\left\{\frac{\partial^{\wp}\mathbb{U}(\varepsilon,\Im)}{\partial\Im^{\wp}}\right\}=-\mathbb{Y}\left[\mathbb{U}(\varepsilon,\Im)\frac{\partial\mathbb{U}(\varepsilon,\Im)}{\partial\varepsilon}+\frac{\partial\mathbb{U}(\varepsilon,\Im)}{\partial\varepsilon}+\frac{\partial\mathbb{V}(\varepsilon,\Im)}{\partial\varepsilon}\right],\\
&\mathbb{Y}\left\{\frac{\partial^{\wp}\mathbb{V}(\varepsilon,\Im)}{\partial\Im^{\wp}}\right\}=-\mathbb{Y}\left[\mathbb{U}(\varepsilon,\Im)\frac{\partial\mathbb{V}(\varepsilon,\Im)}{\partial\varepsilon}+\mathbb{V}(\varepsilon,\Im)\frac{\partial\mathbb{U}(\varepsilon,\Im)}{\partial\varepsilon}+3\frac{\partial^3\mathbb{U}(\varepsilon,\Im)}{\partial\varepsilon^3}-\frac{\partial^2\mathbb{V}(\varepsilon,\Im)}{\partial\varepsilon^2}\right],
\end{aligned}
$$

$$
\begin{aligned}
&\frac{1}{(1+\wp(v-1))}\mathbb{Y}\{\mathbb{U}(\varepsilon,\Im)\}-v\mathbb{U}(\varepsilon,0)=-\mathbb{Y}\left[\mathbb{U}(\varepsilon,\Im)\frac{\partial\mathbb{U}(\varepsilon,\Im)}{\partial\varepsilon}+\frac{\partial\mathbb{U}(\varepsilon,\Im)}{\partial\varepsilon}+\frac{\partial\mathbb{V}(\varepsilon,\Im)}{\partial\varepsilon}\right]\\
&\frac{1}{(1+\wp(v-1))}\mathbb{Y}\{\mathbb{V}(\varepsilon,\Im)\}-v\mathbb{V}(\varepsilon,0)=-\mathbb{Y}\left[\mathbb{U}(\varepsilon,\Im)\frac{\partial\mathbb{V}(\varepsilon,\Im)}{\partial\varepsilon}+\mathbb{V}(\varepsilon,\Im)\frac{\partial\mathbb{U}(\varepsilon,\Im)}{\partial\varepsilon}+3\frac{\partial^3\mathbb{U}(\varepsilon,\Im)}{\partial\varepsilon^3}-\frac{\partial^2\mathbb{V}(\varepsilon,\Im)}{\partial\varepsilon^2}\right].
\end{aligned}
$$

The above equation is simplified

$$
\begin{aligned}
&\mathbb{Y}\{\mathbb{U}(\varepsilon,\Im)\}=v\{\mathbb{U}(\varepsilon,0)\}-(1+\wp(v-1))\mathbb{Y}\left[\mathbb{U}(\varepsilon,\Im)\frac{\partial\mathbb{U}(\varepsilon,\Im)}{\partial\varepsilon}+\frac{\partial\mathbb{U}(\varepsilon,\Im)}{\partial\varepsilon}+\frac{\partial\mathbb{V}(\varepsilon,\Im)}{\partial\varepsilon}\right],\\
&\mathbb{Y}\{\mathbb{V}(\varepsilon,\Im)\}=v\{\mathbb{V}(\varepsilon,0)\}-(1+\wp(v-1))\mathbb{Y}\left[\mathbb{U}(\varepsilon,\Im)\frac{\partial\mathbb{V}(\varepsilon,\Im)}{\partial\varepsilon}+\mathbb{V}(\varepsilon,\Im)\frac{\partial\mathbb{U}(\varepsilon,\Im)}{\partial\varepsilon}+3\frac{\partial^3\mathbb{U}(\varepsilon,\Im)}{\partial\varepsilon^3}-\frac{\partial^2\mathbb{V}(\varepsilon,\Im)}{\partial\varepsilon^2}\right].
\end{aligned}
\tag{28}
$$

Using inverse Yang transform, we have

$$
\begin{aligned}
&\mathbb{U}(\varepsilon,\Im)=\mathbb{U}(\varepsilon,0)-\mathbb{Y}^{-1}\left[(1+\wp(v-1))\mathbb{Y}\left\{\mathbb{U}(\varepsilon,\Im)\frac{\partial\mathbb{U}(\varepsilon,\Im)}{\partial\varepsilon}+\frac{\partial\mathbb{U}(\varepsilon,\Im)}{\partial\varepsilon}+\frac{\partial\mathbb{V}(\varepsilon,\Im)}{\partial\varepsilon}\right\}\right],\\
&\mathbb{V}(\varepsilon,\Im)=\mathbb{V}(\varepsilon,0)-\mathbb{Y}^{-1}\left[(1+\wp(v-1))\mathbb{Y}\left\{\mathbb{U}(\varepsilon,\Im)\frac{\partial\mathbb{V}(\varepsilon,\Im)}{\partial\varepsilon}+\mathbb{V}(\varepsilon,\Im)\frac{\partial\mathbb{U}(\varepsilon,\Im)}{\partial\varepsilon}+3\frac{\partial^3\mathbb{U}(\varepsilon,\Im)}{\partial\varepsilon^3}-\frac{\partial^2\mathbb{V}(\varepsilon,\Im)}{\partial\varepsilon^2}\right\}\right].
\end{aligned}
\tag{29}
$$

Assume that the $\mathbb{U}(\varepsilon,\Im)$ and the $\mathbb{V}(\varepsilon,\Im)$ infinite series solution functions as follows:

$$
\mathbb{U}(\varepsilon,\Im)=\sum_{m=0}^{\infty}\mathbb{U}_m(\varepsilon,\Im)\quad \textit{and}\quad \mathbb{V}(\varepsilon,\Im)=\sum_{m=0}^{\infty}\mathbb{V}_m(\varepsilon,\Im).
$$

Remember that the Adomian polynomials are given as $\mathbb{U}\mathbb{U}_\varepsilon=\sum_{m=0}^{\infty}\mathcal{A}_m$, $\mathbb{U}\mathbb{V}_\varepsilon=\sum_{m=0}^{\infty}\mathcal{B}_m$ and $\mathbb{V}\mathbb{U}_\varepsilon=\sum_{m=0}^{\infty}\mathcal{C}_m$

$$
\begin{aligned}
&\sum_{m=0}^{\infty}\mathbb{U}_m(\varepsilon,\Im)=\mathbb{U}(\varepsilon,0)-\mathbb{Y}^{-1}\left[(1+\wp(v-1))\mathbb{Y}\left\{\sum_{m=0}^{\infty}\mathcal{A}_m+\frac{\partial\mathbb{U}(\varepsilon,\Im)}{\partial\varepsilon}+\frac{\partial\mathbb{V}(\varepsilon,\Im)}{\partial\varepsilon}\right\}\right],\\
&\sum_{m=0}^{\infty}\mathbb{V}_m(\varepsilon,\Im)=\mathbb{V}(\varepsilon,0)-\mathbb{Y}^{-1}\left[(1+\wp(v-1))\mathbb{Y}\left\{\sum_{m=0}^{\infty}\mathcal{B}_m+\sum_{m=0}^{\infty}\mathcal{C}_m+3\frac{\partial^3\mathbb{U}(\varepsilon,\Im)}{\partial\varepsilon^3}-\frac{\partial^2\mathbb{V}(\varepsilon,\Im)}{\partial\varepsilon^2}\right\}\right],
\end{aligned}
$$

$$\sum_{m=0}^{\infty}\mathbb{U}_m(\varepsilon,\Im)=\frac{1}{2}-8\tanh(-2\varepsilon)-\mathbb{Y}^{-1}\left[(1+\wp(v-1))\mathbb{Y}\left\{\sum_{m=0}^{\infty}\mathcal{A}_m+\frac{\partial\mathbb{U}(\varepsilon,\Im)}{\partial\varepsilon}+\frac{\partial\mathbb{V}(\varepsilon,\Im)}{\partial\varepsilon}\right\}\right],$$
$$\sum_{m=0}^{\infty}\mathbb{V}_m(\varepsilon,\Im)=16-16\tanh^2(-2\varepsilon)-\mathbb{Y}^{-1}\left[(1+\wp(v-1))\mathbb{Y}\left\{\sum_{m=0}^{\infty}\mathcal{B}_m+\sum_{m=0}^{\infty}\mathcal{C}_m+3\frac{\partial^3\mathbb{U}(\varepsilon,\Im)}{\partial\varepsilon^3}-\frac{\partial^2\mathbb{V}(\varepsilon,\Im)}{\partial\varepsilon^2}\right\}\right]. \quad (30)$$

With the aid of the Adomian polynomial, according to Equation (22), all forms of non-linear may be stated as

$$\mathcal{A}_0=\mathbb{U}_0\frac{\partial\mathbb{U}_0}{\partial\varepsilon},\quad \mathcal{A}_1=\mathbb{U}_0\frac{\partial\mathbb{U}_1}{\partial\varepsilon}+\mathbb{U}_1\frac{\partial\mathbb{U}_0}{\partial\varepsilon},\quad \mathcal{B}_0=\mathbb{U}_0\frac{\partial\mathbb{V}_0}{\partial\beta},\quad \mathcal{B}_1=\mathbb{U}_0\frac{\partial\mathbb{V}_1}{\partial\beta}+\mathbb{U}_1\frac{\partial\mathbb{V}_0}{\partial\beta},$$
$$\mathcal{C}_0=\mathbb{V}_0\frac{\partial\mathbb{U}_0}{\partial\varepsilon},\quad \mathcal{C}_1=\mathbb{V}_0\frac{\partial\mathbb{U}_1}{\partial\varepsilon}+\mathbb{V}_1\frac{\partial\mathbb{U}_0}{\partial\varepsilon}.$$

Therefore we can easily obtain

$$\mathbb{U}_0(\varepsilon,\Im)=\frac{1}{2}-8\tanh(-2\varepsilon),\quad \mathbb{V}_0(\varepsilon,\Im)=16-16\tanh^2(-2\varepsilon).$$

For $m=0$

$$\mathbb{U}_1(\varepsilon,\Im)=-8\sec h^2(-2\varepsilon)\Big\{1+\wp\Im-\wp\Big\},\quad \mathbb{V}_1(\varepsilon,\Im)=-32\sec h^2(-2\varepsilon)\tanh(-2\varepsilon)\Big\{1+\wp\Im-\wp\Big\}.$$

For $m=1$

$$\mathbb{U}_2(\varepsilon,\Im)=-16\sec h^2(-2\varepsilon)\Big(4\sec h^2(-2\varepsilon)-8\tanh^2(-2\varepsilon)+3\tanh(-2\varepsilon)\Big)\Big\{(1-\wp)2\wp\Im+(1-\wp)^2+\frac{\wp^2\Im^2}{2}\Big\},$$
$$\mathbb{V}_2(\varepsilon,\Im)=-32\sec h^2(-2\varepsilon)\{40\sec h^2(-2\varepsilon)\tanh(-2\varepsilon)+96\tanh(-2\varepsilon)-2\tanh^2(-2\varepsilon)$$
$$-32\tanh^3(-2\varepsilon)-25\sec h^2(-2\varepsilon)\}\Big\{(1-\wp)2\wp\Im+(1-\wp)^2+\frac{\wp^2\Im^2}{2}\Big\}.$$

The remaining steps of the YDM outcomes may be conveniently gathered from $\mathbb{U}m$ and $\mathbb{V}m$ ($m\geq 2$) using the same methods. Then, we assess the sequence of possibilities as follows:

$$\mathbb{U}(\varepsilon,\Im)=\sum_{m=0}^{\infty}\mathbb{U}_m(\varepsilon,\Im)=\mathbb{U}_0(\varepsilon,\Im)+\mathbb{U}_1(\varepsilon,\Im)+\mathbb{U}_2(\varepsilon,\Im)+\mathbb{U}_3(\varepsilon,\Im)+\cdots.$$
$$\mathbb{V}(\varepsilon,\Im)=\sum_{m=0}^{\infty}\mathbb{V}_m(\varepsilon,\Im)=\mathbb{V}_0(\varepsilon,\Im)+\mathbb{V}_1(\varepsilon,\Im)+\mathbb{V}_2(\varepsilon,\Im)+\mathbb{V}_3(\varepsilon,\Im)+\cdots.$$

$$\mathbb{U}(\varepsilon,\Im)=\frac{1}{2}-8\tanh(-2\varepsilon)-8\sec h^2(-2\varepsilon)\Big\{1+\wp\Im-\wp\Big\}-16\sec h^2(-2\varepsilon)\Big(4\sec h^2(-2\varepsilon)$$
$$-8\tanh^2(-2\varepsilon)+3\tanh(-2\varepsilon)\Big)\Big\{(1-\wp)2\wp\Im+(1-\wp)^2+\frac{\wp^2\Im^2}{2}\Big\}-\cdots.$$
$$\mathbb{V}(\varepsilon,\Im)=16-16\tanh^2(-2\varepsilon)-32\sec h^2(-2\varepsilon)\tanh(-2\varepsilon)\Big\{1+\wp\Im-\wp\Big\}-32\sec h^2(-2\varepsilon)$$
$$\{40\sec h^2(-2\varepsilon)\tanh(-2\varepsilon)+96\tanh(-2\varepsilon)-2\tanh^2(-2\varepsilon)-32\tanh^3(-2\varepsilon)$$
$$-25\sec h^2(-2\varepsilon)\}\Big\{(1-\wp)2\wp\Im+(1-\wp)^2+\frac{\wp^2\Im^2}{2}\Big\}-\cdots.$$

At integer order $\wp=1$, the following series form of solution is achieved:

$$\begin{aligned}\mathbb{U}(\varepsilon,\Im) =& \frac{1}{2} - 8\tanh(-2\varepsilon) - 8\,\mathrm{sech}(-2\varepsilon)^2\Im + 8\,\mathrm{sech}^2(-2\varepsilon)\\ &\times \Big\{3\tanh(-2\varepsilon) + 8\tanh(-2\varepsilon)^2 + 4\,\mathrm{sech}^2(-2\varepsilon)\Big\}\Im^2 + \cdots .\\ \mathbb{V}(\varepsilon,\Im) =& 16 - 16\tanh^2(-2\varepsilon) - 32\,\mathrm{sech}^2(-2\varepsilon)\tanh(-2\varepsilon)\Im\\ &- 16\sec h^2(-2\varepsilon)\Big\{96\tanh(-2\varepsilon) - 32\tanh^3(-2\varepsilon)\\ &+ 40\sec h^2(-2\varepsilon)\tanh(-2\varepsilon) - 2\tanh^2(-2\varepsilon) - 25\sec h^2(-2\varepsilon)\Big\}\Im^2 + \cdots .\end{aligned}$$

The exact solution of Equation (26) at $\wp = 1$,

$$\begin{aligned}\mathbb{U}(\varepsilon,\Im) &= \frac{1}{2} - 8\tanh\left\{-2\left(\varepsilon - \frac{\Im}{2}\right)\right\},\\ \mathbb{V}(\varepsilon,\Im) &= 16 - 16\tanh^2\left\{-2\left(\varepsilon - \frac{\Im}{2}\right)\right\}.\end{aligned} \tag{31}$$

In Figures 1 and 2, the actual and Yang decomposition method solutions at an integer-order $\wp = 1$ *are represented for both* $\mathbb{U}(\varepsilon,\Im)$ *and* $\mathbb{V}(\varepsilon,\Im)$ *of Example 1. It is observed that Yang decomposition method results are in good contact with the actual result of the models. In Figures 3 and 4, various fractional-order solutions of Example 1, at different fractional-orders,* $\wp = 1, 0.8, 0.6, 0.4$ *are plotted. It is investigated that for Example 1, the fractional-order solutions are convergent to an integer-order solution for both* $\mathbb{U}(\varepsilon,\Im)$ *and* $\mathbb{V}(\varepsilon,\Im)$*. In Tables 1 and 2 show that yang decomposition method of different fractional order* $\wp$ *of Example 1. In Tables 3 and 4 compassion of different analytical and numerical methods of Example 1.*

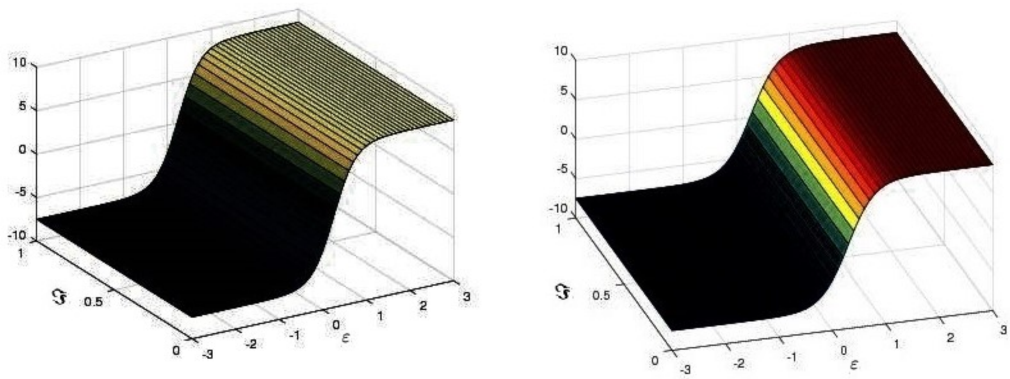

Figure 1. The actual and YDM solution of $\mathbb{U}(\varepsilon,\Im)$ at $\wp = 1$ of Example 1.

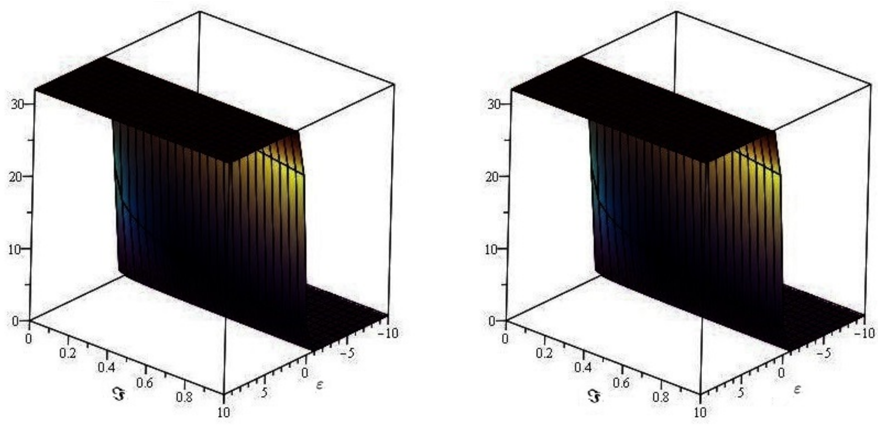

Figure 2. The actual and YDM solution of $\mathbb{V}(\varepsilon,\Im)$ at $\wp = 1$ of Example 1.

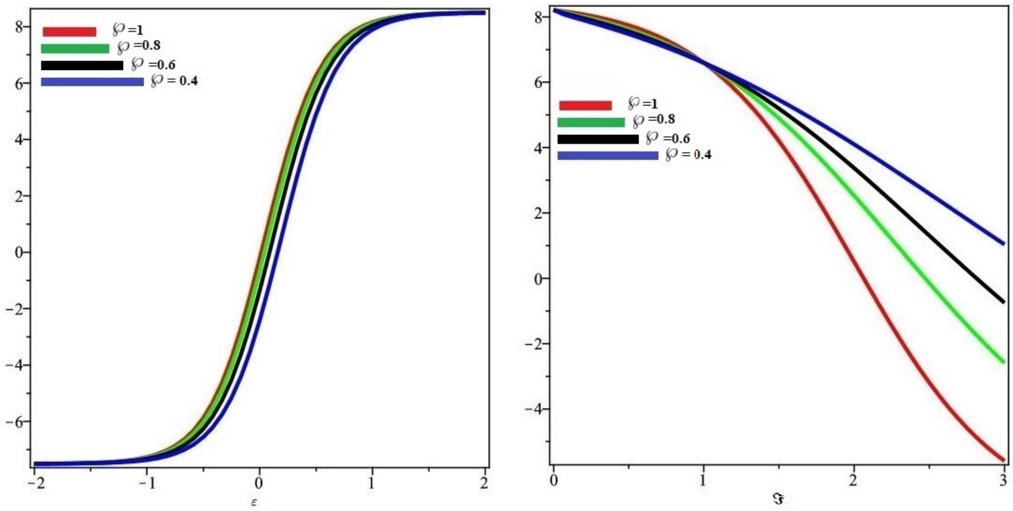

Figure 3. The fractional-order solutions of $\mathbb{V}(\varepsilon, \Im)$ at $\wp$ of Example 1.

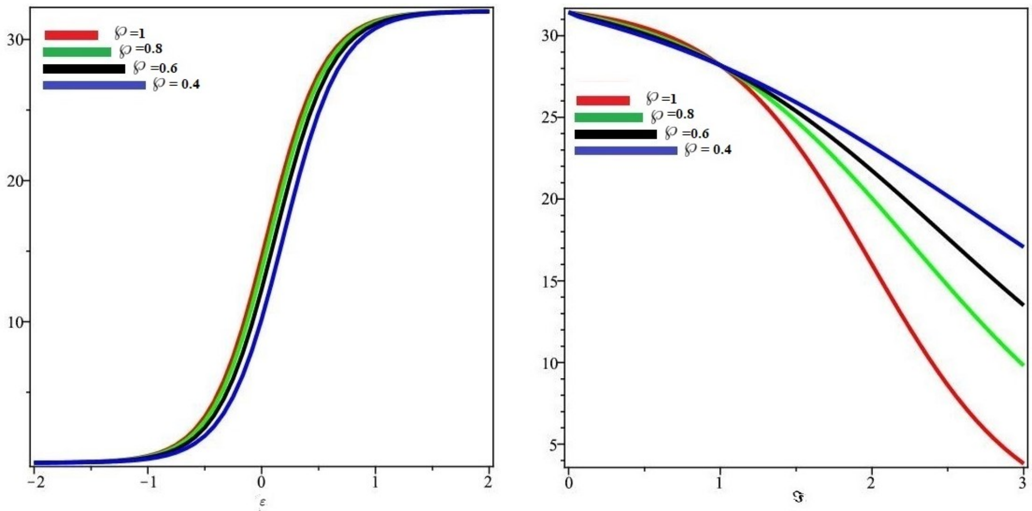

Figure 4. The fractional-order solutions of $\mathbb{V}(\varepsilon, \Im)$ at $\wp$ of Example 1.

Table 1. YDM solution of $\mathbb{U}(\varepsilon, \Im)$ at various fractional-order $\wp$ of Example 1.

$(\varepsilon; \Im)$	$\mathbb{U}(\varepsilon, \Im)$ at $\wp = 0.5$	$\mathbb{U}(\varepsilon, \Im)$ at $\wp = 0.75$	$\mathbb{U}(\varepsilon, \Im)$ at $\wp = 1$	**Exact Result**
(0.1, 0.2)	0.501928	0.501886	0.501893	0.501893
(0.1, 0.4)	0.501964	0.501938	0.501920	0.501920
(0.1, 0.6)	0.501989	0.501968	0.501858	0.501948
(0.2, 0.2)	0.499230	0.497189	0.499196	0.498090
(0.2, 0.4)	0.499265	0.497242	0.499223	0.498223
(0.2, 0.6)	0.499389	0.499269	0.499248	0.499148
(0.3, 0.2)	0.496582	0.496570	0.496569	0.494569
(0.3, 0.4)	0.496636	0.496413	0.496595	0.496595
(0.3, 0.6)	0.496659	0.496638	0.496620	0.496620
(0.4, 0.2)	0.49384	0.493818	0.493988	0.493988
(0.4, 0.4)	0.493874	0.493830	0.493833	0.493833
(0.4, 0.6)	0.493896	0.493877	0.493859	0.493859
(0.5, 0.2)	0.491544	0.491324	0.491512	0.491512
(0.5, 0.4)	0.491576	0.491354	0.491537	0.491327
(0.5, 0.6)	0.491598	0.491578	0.491562	0.491442

Table 2. YDM solution of $\mathbb{V}(\varepsilon, \Im)$ at various fractional-order $\wp$ of Example 1.

$(\varepsilon; \Im)$	$\mathbb{V}(\varepsilon, \Im)$ at $\wp = 0.5$	$\mathbb{V}(\varepsilon, \Im)$ at $\wp = 0.75$	$\mathbb{V}(\varepsilon, \Im)$ at $\wp = 1$	Exact Result
(0.1, 0.2)	0.0828104	0.0828124	0.0827800	0.0828900
(0.1, 0.4)	0.0828425	0.0828208	0.0828235	0.0839235
(0.1, 0.6)	0.0828646	0.0828460	0.0828280	0.0828391
(0.2, 0.2)	0.0804153	0.0803760	0.0803648	0.0803648
(0.2, 0.4)	0.0804264	0.0804054	0.0803886	0.0803886
(0.2, 0.6)	0.0804478	0.0804318	0.0804124	0.0804124
(0.3, 0.2)	0.0780546	0.0782358	0.0780250	0.0782472
(0.3, 0.4)	0.0780847	0.0780843	0.0780481	0.0782481
(0.3, 0.6)	0.0781055	0.0780881	0.0780711	0.0782711
(0.4, 0.2)	0.0757854	0.0757671	0.0757567	0.0757567
(0.4, 0.4)	0.0758148	0.0758148	0.0757810	0.0757780
(0.4, 0.6)	0.0758347	0.0758178	0.0758014	0.0758014
(0.5, 0.2)	0.0735850	0.0735673	0.0735572	0.0735578
(0.5, 0.4)	0.0736133	0.0736141	0.0735788	0.0735788
(0.5, 0.6)	0.0736328	0.0736164	0.0736225	0.0738005

Table 3. Comparison of absolute error (AE) of $\mathbb{U}(\varepsilon, \Im)$ at $\wp = 1$ obtained by various methods.

$(\varepsilon, \Im)$	AE Of ADM [37]	AE Of VIM [38]	AE Of OHAM [39]	AE of YDM
(0.1, 0.2)	1.05983×10^{-5}	1.34144×10^{-6}	1.18169×10^{-7}	1.56432×10^{-11}
(0.1, 0.4)	9.75585×10^{-6}	3.78688×10^{-6}	3.15656×10^{-7}	4.42375×10^{-10}
(0.1, 0.6)	8.77423×10^{-6}	6.27984×10^{-6}	4.92412×10^{-7}	2.18645×10^{-9}
(0.2, 0.2)	4.37319×10^{-5}	1.38978×10^{-6}	1.12395×10^{-7}	1.46768×10^{-11}
(0.2, 0.4)	3.82189×10^{-5}	3.51189×10^{-6}	2.86457×10^{-6}	4.35336×10^{-10}
(0.2, 0.6)	3.51272×10^{-5}	6.10117×10^{-5}	4.51245×10^{-6}	1.86439×10^{-09}
(0.3, 0.2)	9.62833×10^{-5}	1.25698×10^{-5}	1.13664×10^{-6}	1.38262×10^{-11}
(0.3, 0.4)	8.84418×10^{-5}	3.61977×10^{-5}	2.62353×10^{-6}	4.13675×10^{-10}
(0.3, 0.6)	8.33563×10^{-5}	5.96721×10^{-5}	4.46642×10^{-6}	1.46354×10^{-09}
(0.4, 0.2)	1.86687×10^{-4}	1.24938×10^{-5}	9.24537×10^{-5}	1.84245×10^{-11}
(0.4, 0.4)	1.72542×10^{-4}	3.52859×10^{-5}	2.63564×10^{-5}	3.60624×10^{-10}
(0.4, 0.6)	1.58687×10^{-4}	5.81821×10^{-5}	4.65446×10^{-5}	1.56784×10^{-09}
(0.5, 0.2)	2.88628×10^{-4}	1.21847×10^{-5}	9.72736×10^{-5}	1.42355×10^{-11}
(0.5, 0.4)	2.47825×10^{-4}	3.44373×10^{-5}	2.33457×10^{-5}	3.52237×10^{-10}
(0.5, 0.6)	2.47295×10^{-4}	5.47346×10^{-5}	4.38895×10^{-5}	1.66734×10^{-09}

Table 4. Comparison of absolute error of $\mathbb{V}(\varepsilon, \Im)$ at $\wp = 1$ obtained by various methods.

$(\zeta, \Im)$	AE Of ADM [37]	AE Of VIM [38]	AE Of OHAM [39]	AE of YDM
(0.1, 0.2)	6.52318×10^{-4}	1.23581×10^{-5}	5.72451×10^{-6}	3.262182×10^{-11}
(0.1, 0.4)	5.87694×10^{-4}	3.53456×10^{-5}	3.24632×10^{-6}	8.94623×10^{-10}
(0.1, 0.6)	5.72618×10^{-4}	5.63261×10^{-5}	3.38923×10^{-6}	4.23455×10^{-09}
(0.2, 0.2)	1.44292×10^{-3}	1.18127×10^{-5}	5.45771×10^{-6}	3.18974×10^{-11}
(0.2, 0.4)	1.33452×10^{-3}	3.34512×10^{-5}	2.86341×10^{-5}	8.21855×10^{-10}
(0.2, 0.6)	1.25527×10^{-3}	5.47838×10^{-5}	2.82545×10^{-5}	3.72424×10^{-09}
(0.3, 0.2)	2.14563×10^{-3}	1.14848×10^{-5}	5.36746×10^{-5}	2.45694×10^{-11}
(0.3, 0.4)	1.84963×10^{-3}	3.22828×10^{-5}	2.74231×10^{-5}	7.67817×10^{-10}
(0.3, 0.6)	1.72318×10^{-3}	5.32558×10^{-4}	2.66463×10^{-5}	3.4356×10^{-11}
(0.4, 0.2)	2.98211×10^{-3}	1.11468×10^{-4}	5.23838×10^{-5}	2.71232×10^{-11}
(0.4, 0.4)	2.59845×10^{-3}	3.13456×10^{-4}	2.72338×10^{-3}	7.24545×10^{-10}
(0.4, 0.6)	2.61896×10^{-3}	5.15382×10^{-3}	2.54328×10^{-3}	3.25166×10^{-09}
(0.5, 0.2)	3.84384×10^{-3}	9.86396×10^{-3}	4.83832×10^{-3}	2.13536×10^{-11}
(0.5, 0.4)	3.58728×10^{-3}	2.84228×10^{-3}	2.84563×10^{-3}	6.19148×10^{-10}
(0.5, 0.6)	3.35348×10^{-3}	4.72446×10^{-3}	2.52741×10^{-3}	3.24436×10^{-09}

Example 2. *Consider the fractional-order system of WBKEs [11]*

$$\begin{aligned}
&{}^{CF}D^{\wp}_{\Im}\mathbb{U}(\varepsilon,\Im)+\mathbb{U}(\varepsilon,\Im)\frac{\partial\mathbb{U}(\varepsilon,\Im)}{\partial\varepsilon}+\frac{1}{2}\frac{\partial\mathbb{U}(\varepsilon,\Im)}{\partial\varepsilon}+\frac{\partial\mathbb{V}(\varepsilon,\Im)}{\partial\varepsilon}=0,\\
&{}^{CF}D^{\wp}_{\Im}\mathbb{V}(\varepsilon,\Im)+\mathbb{U}(\varepsilon,\Im)\frac{\partial\mathbb{V}(\varepsilon,\Im)}{\partial\varepsilon}+\mathbb{V}(\varepsilon,\Im)\frac{\partial\mathbb{U}(\varepsilon,\Im)}{\partial\varepsilon}-\frac{1}{2}\frac{\partial^2\mathbb{V}(\varepsilon,\Im)}{\partial\varepsilon^2}=0,\\
&0<\wp\leq 1,\quad 0<\Im\leq 1,\quad -100\leq\varepsilon\leq 100,
\end{aligned}\tag{32}$$

with the initial conditions

$$\begin{aligned}
\mathbb{U}(\varepsilon,0)&=\xi-\kappa\coth[\kappa(\varepsilon+\theta)],\\
\mathbb{V}(\varepsilon,0)&=-\kappa^2\text{cosech}^2[\kappa(\varepsilon+\theta)].
\end{aligned}\tag{33}$$

Applying the Yang transformation of Equation (32), we have

$$\begin{aligned}
\mathbb{Y}\left\{\frac{\partial^{\wp}\mathbb{U}(\varepsilon,\Im)}{\partial\Im^{\wp}}\right\}&=-\mathbb{Y}\left[\mathbb{U}(\varepsilon,\Im)\frac{\partial\mathbb{U}(\varepsilon,\Im)}{\partial\varepsilon}+\frac{1}{2}\frac{\partial\mathbb{U}(\varepsilon,\Im)}{\partial\varepsilon}+\frac{\partial\mathbb{V}(\varepsilon,\Im)}{\partial\varepsilon}\right],\\
\mathbb{Y}\left\{\frac{\partial^{\wp}\mathbb{V}(\varepsilon,\Im)}{\partial\Im^{\wp}}\right\}&=-\mathbb{Y}\left[\mathbb{U}(\varepsilon,\Im)\frac{\partial\mathbb{V}(\varepsilon,\Im)}{\partial\varepsilon}+\mathbb{V}(\varepsilon,\Im)\frac{\partial\mathbb{U}(\varepsilon,\Im)}{\partial\varepsilon}-\frac{1}{2}\frac{\partial^2\mathbb{V}(\varepsilon,\Im)}{\partial\varepsilon^2}\right],
\end{aligned}$$

$$\begin{aligned}
\frac{1}{(1+\wp(v-1))}\mathbb{Y}\{\mathbb{U}(\varepsilon,\Im)\}-v\mathbb{U}(\varepsilon,0)&=-\mathbb{Y}\left[\mathbb{U}(\varepsilon,\Im)\frac{\partial\mathbb{U}(\varepsilon,\Im)}{\partial\varepsilon}+\frac{1}{2}\frac{\partial\mathbb{U}(\varepsilon,\Im)}{\partial\varepsilon}+\frac{\partial\mathbb{V}(\varepsilon,\Im)}{\partial\varepsilon}\right],\\
\frac{1}{(1+\wp(v-1))}\mathbb{Y}\{\mathbb{V}(\varepsilon,\Im)\}-v\mathbb{V}(\varepsilon,0)&=-\mathbb{Y}\left[\mathbb{U}(\varepsilon,\Im)\frac{\partial\mathbb{V}(\varepsilon,\Im)}{\partial\varepsilon}+\mathbb{V}(\varepsilon,\Im)\frac{\partial\mathbb{U}(\varepsilon,\Im)}{\partial\varepsilon}-\frac{1}{2}\frac{\partial^2\mathbb{V}(\varepsilon,\Im)}{\partial\varepsilon^2}\right].
\end{aligned}$$

The above equation is simplified

$$\begin{aligned}
\mathbb{Y}\{\mathbb{U}(\varepsilon,\Im)\}&=v\{\mathbb{U}(\varepsilon,0)\}-(1+\wp(v-1))\mathbb{Y}\left[\mathbb{U}(\varepsilon,\Im)\frac{\partial\mathbb{U}(\varepsilon,\Im)}{\partial\varepsilon}+\frac{1}{2}\frac{\partial\mathbb{U}(\varepsilon,\Im)}{\partial\varepsilon}+\frac{\partial\mathbb{V}(\varepsilon,\Im)}{\partial\varepsilon}\right],\\
\mathbb{Y}\{\mathbb{V}(\varepsilon,\Im)\}&=v\{\mathbb{V}(\varepsilon,0)\}-(1+\wp(v-1))\mathbb{Y}\left[\mathbb{U}(\varepsilon,\Im)\frac{\partial\mathbb{V}(\varepsilon,\Im)}{\partial\varepsilon}+\mathbb{V}(\varepsilon,\Im)\frac{\partial\mathbb{U}(\varepsilon,\Im)}{\partial\varepsilon}-\frac{1}{2}\frac{\partial^2\mathbb{V}(\varepsilon,\Im)}{\partial\varepsilon^2}\right].
\end{aligned}\tag{34}$$

Using inverse Yang transform, we have

$$\begin{aligned}
\mathbb{U}(\varepsilon,\Im)&=\mathbb{U}(\varepsilon,0)-\mathbb{Y}^{-1}\left[(1+\wp(v-1))\mathbb{Y}\left\{\mathbb{U}(\varepsilon,\Im)\frac{\partial\mathbb{U}(\varepsilon,\Im)}{\partial\varepsilon}+\frac{1}{2}\frac{\partial\mathbb{U}(\varepsilon,\Im)}{\partial\varepsilon}+\frac{\partial\mathbb{V}(\varepsilon,\Im)}{\partial\varepsilon}\right\}\right],\\
\mathbb{V}(\varepsilon,\Im)&=\mathbb{V}(\varepsilon,0)-\mathbb{Y}^{-1}\left[(1+\wp(v-1))\mathbb{Y}\left\{\mathbb{U}(\varepsilon,\Im)\frac{\partial\mathbb{V}(\varepsilon,\Im)}{\partial\varepsilon}+\mathbb{V}(\varepsilon,\Im)\frac{\partial\mathbb{U}(\varepsilon,\Im)}{\partial\varepsilon}-\frac{1}{2}\frac{\partial^2\mathbb{V}(\varepsilon,\Im)}{\partial\varepsilon^2}\right\}\right].
\end{aligned}\tag{35}$$

Assume that the infinite series solution functions $\mathbb{U}(\varepsilon,\Im)$ and $\mathbb{V}(\varepsilon,\Im)$ are as follows:

$$\mathbb{U}(\varepsilon,\Im)=\sum_{m=0}^{\infty}\mathbb{U}_m(\varepsilon,\Im),\quad \textit{and}\quad \mathbb{V}(\varepsilon,\Im)=\sum_{m=0}^{\infty}\mathbb{V}_m(\varepsilon,\Im).$$

Remember that $\mathbb{U}\mathbb{U}_\varepsilon=\sum_{m=0}^{\infty}\mathcal{A}_m$, $\mathbb{U}\mathbb{V}_\varepsilon=\sum_{m=0}^{\infty}\mathcal{B}_m$ and $\mathbb{V}\mathbb{U}_\varepsilon=\sum_{m=0}^{\infty}\mathcal{C}_m$ are the Adomian polynomials

$$\begin{aligned}
\sum_{m=0}^{\infty}\mathbb{U}_m(\varepsilon,\Im)&=\mathbb{U}(\varepsilon,0)-\mathbb{Y}^{-1}\left[(1+\wp(v-1))\mathbb{Y}\left\{\sum_{m=0}^{\infty}\mathcal{A}_m+\frac{1}{2}\frac{\partial\mathbb{U}(\varepsilon,\Im)}{\partial\varepsilon}+\frac{\partial\mathbb{V}(\varepsilon,\Im)}{\partial\varepsilon}\right\}\right],\\
\sum_{m=0}^{\infty}\mathbb{V}_m(\varepsilon,\Im)&=\mathbb{V}(\varepsilon,0)-\mathbb{Y}^{-1}\left[(1+\wp(v-1))\mathbb{Y}\left\{\sum_{m=0}^{\infty}\mathcal{B}_m+\sum_{m=0}^{\infty}\mathcal{C}_m-\frac{1}{2}\frac{\partial^2\mathbb{V}(\varepsilon,\Im)}{\partial\varepsilon^2}\right\}\right],
\end{aligned}$$

$$\begin{aligned}\sum_{m=0}^{\infty}\mathbb{U}_m(\varepsilon,\Im) &= \xi-\kappa\coth[\kappa(\varepsilon+\theta)]-\mathbb{Y}^{-1}\left[(1+\wp(v-1))\mathbb{Y}\left\{\sum_{m=0}^{\infty}\mathcal{A}_m+\frac{1}{2}\frac{\partial\mathbb{U}(\varepsilon,\Im)}{\partial\varepsilon}+\frac{\partial\mathbb{V}(\varepsilon,\Im)}{\partial\varepsilon}\right\}\right],\\ \sum_{m=0}^{\infty}\mathbb{V}_m(\varepsilon,\Im) &= -\kappa^2\mathrm{cosech}^2[\kappa(\varepsilon+\theta)]-\mathbb{Y}^{-1}\left[(1+\wp(v-1))\mathbb{Y}\left\{\sum_{m=0}^{\infty}\mathcal{B}_m+\sum_{m=0}^{\infty}\mathcal{C}_m-\frac{1}{2}\frac{\partial^2\mathbb{V}(\varepsilon,\Im)}{\partial\varepsilon^2}\right\}\right].\end{aligned} \tag{36}$$

With the aid of the Adomian polynomial according to Equation (22), all forms of non-linear may be stated as

$$\mathcal{A}_0=\mathbb{U}_0\frac{\partial\mathbb{U}_0}{\partial\varepsilon},\quad \mathcal{A}_1=\mathbb{U}_0\frac{\partial\mathbb{U}_1}{\partial\varepsilon}+\mathbb{U}_1\frac{\partial\mathbb{U}_0}{\partial\varepsilon},\quad \mathcal{B}_0=\mathbb{U}_0\frac{\partial\mathbb{V}_0}{\partial\beta},\quad \mathcal{B}_1=\mathbb{U}_0\frac{\partial\mathbb{V}_1}{\partial\beta}+\mathbb{U}_1\frac{\partial\mathbb{V}_0}{\partial\beta},$$
$$\mathcal{C}_0=\mathbb{V}_0\frac{\partial\mathbb{U}_0}{\partial\varepsilon},\quad \mathcal{C}_1=\mathbb{V}_0\frac{\partial\mathbb{U}_1}{\partial\varepsilon}+\mathbb{V}_1\frac{\partial\mathbb{U}_0}{\partial\varepsilon}.$$

Hence, one can easily obtain

$$\mathbb{U}_0(\varepsilon,\Im)=\xi-\kappa\coth[\kappa(\varepsilon+\theta)],\quad \mathbb{V}_0(\varepsilon,\Im)=-\kappa^2\mathrm{cosech}^2[\kappa(\varepsilon+\theta)].$$

For $m=0$

$$\begin{aligned}\mathbb{U}_1(\varepsilon,\Im)&=-\xi\kappa^2\mathrm{cosech}^2[\kappa(\varepsilon+\theta)]\left\{1+\wp\Im-\wp\right\},\\ \mathbb{V}_1(\varepsilon,\Im)&=-\xi\kappa^2\mathrm{cosech}^2[\kappa(\varepsilon+\theta)]\coth[\kappa(\varepsilon+\theta)]\left\{1+\wp\Im-\wp\right\}.\end{aligned}$$

For $m=1$

$$\begin{aligned}&\mathbb{U}_2(\varepsilon,\Im)=\xi\kappa^4\mathrm{cosech}^2[\kappa(\varepsilon+\theta)]\Big\{2\xi\kappa\Big\{(1-\wp)^2 3\wp\Im+(1-\wp)^3+\frac{3\wp^2(1-\wp)\Im^2}{2}+\frac{\wp^3\Im^3}{3!}\Big\}\\ &-(3\coth^2([\kappa(\varepsilon+\theta)]-1))\Big\{(1-\wp)2\wp\Im+(1-\wp)^2+\frac{\wp^2\Im^2}{2}\Big\}\Big\},\\ &\mathbb{V}_2(\varepsilon,\Im)=[2\xi\kappa^5\mathrm{cosech}^2[\kappa(\varepsilon+\theta)]]\Big[\xi\kappa\mathrm{cosech}^2(3\coth^2([\kappa(\varepsilon+\theta)]-1))\Big\{(1-\wp)^2 3\wp\Im+(1-\wp)^3+\frac{3\wp^2(1-\wp)\Im^2}{2}\\ &+\frac{\wp^3\Im^3}{3!}\Big\}+\frac{2\xi\kappa\mathrm{cosech}^2\coth^2([\kappa(\varepsilon+\theta)])\Im^{3\wp}}{\Gamma(\wp+1)\Gamma(3\wp+1)}-2\xi\coth(3\mathrm{cosech}^2([\kappa(\varepsilon+\theta)]-1))\Big\{(1-\wp)2\wp\Im+(1-\wp)^2+\frac{\wp^2\Im^2}{2}\Big\}\Big].\end{aligned}$$

The remaining steps of the YDM results may be conveniently gathered from $\mathbb{U}_m$ *and* $\mathbb{V}_m$ $(m\geq 2)$ *using the same procedure. The alternative series can then be assessed as follows:*

$$\begin{aligned}\mathbb{U}(\varepsilon,\Im)&=\sum_{m=0}^{\infty}\mathbb{U}_m(\varepsilon,\Im)=\mathbb{U}_0(\varepsilon,\Im)+\mathbb{U}_1(\varepsilon,\Im)+\mathbb{U}_2(\varepsilon,\Im)+\mathbb{U}_3(\varepsilon,\Im)+\cdots.\\ \mathbb{V}(\varepsilon,\Im)&=\sum_{m=0}^{\infty}\mathbb{V}_m(\varepsilon,\Im)=\mathbb{V}_0(\varepsilon,\Im)+\mathbb{V}_1(\varepsilon,\Im)+\mathbb{V}_2(\varepsilon,\Im)+\mathbb{V}_3(\varepsilon,\Im)+\cdots.\end{aligned}$$

$$\begin{aligned}&\mathbb{U}(\varepsilon,\Im)=\xi-\kappa\coth[\kappa(\varepsilon+\theta)]-\xi\kappa^2\mathrm{cosech}^2[\kappa(\varepsilon+\theta)]\Big\{1+\wp\Im-\wp\Big\}\\ &+\xi\kappa^4\mathrm{cosech}^2[\kappa(\varepsilon+\theta)]\Big\{2\xi\kappa\Gamma(2\wp+1)\Big\{(1-\wp)^2 3\wp\Im+(1-\wp)^3+\frac{3\wp^2(1-\wp)\Im^2}{2}+\frac{\wp^3\Im^3}{3!}\Big\}\\ &-(3\coth^2([\kappa(\varepsilon+\theta)]-1))\Big\{(1-\wp)2\wp\Im+(1-\wp)^2+\frac{\wp^2\Im^2}{2}\Big\}\Big\}-\cdots.\\ &\mathbb{V}(\varepsilon,\Im)=-\kappa^2\mathrm{cosech}^2[\kappa(\varepsilon+\theta)]-\xi\kappa^2\mathrm{cosech}^2[\kappa(\varepsilon+\theta)]\coth[\kappa(\varepsilon+\theta)]\Big\{1+\wp\Im-\wp\Big\}\\ &+[2\xi\kappa^5\mathrm{cosech}^2[\kappa(\varepsilon+\theta)]]\Big[\xi\kappa\mathrm{cosech}^2(3\coth^2([\kappa(\varepsilon+\theta)]-1))\Big\{(1-\wp)^2 3\wp\Im+(1-\wp)^3+\frac{3\wp^2(1-\wp)\Im^2}{2}+\frac{\wp^3\Im^3}{3!}\Big\}\\ &+\frac{2\xi\kappa\mathrm{cosech}^2\coth^2([\kappa(\varepsilon+\theta)])\Im^{3\wp}}{\Gamma(\wp+1)\Gamma(3\wp+1)}-2\xi\coth(3\mathrm{cosech}^2([\kappa(\varepsilon+\theta)]-1))\Big\{(1-\wp)2\wp\Im+(1-\wp)^2+\frac{\wp^2\Im^2}{2}\Big\}\Big]-\cdots.\end{aligned}$$

We achieve the following series solution at integer order $\wp = 1, \kappa = 0.1, \zeta = 0.005$, $\theta = 10$, defined by

$$\begin{aligned}
\mathbb{U}(\varepsilon,\Im) =& 0.005 - 0.1\coth(0.1\varepsilon+10) - 0.0005\text{cosech}^2(0.1\varepsilon+10)\Im + 5\times10^{-7}\text{cosech}^2(0.1\varepsilon+10)0.003\Im^3 \\
&-0.5\Big(3\coth^2(0.1\varepsilon+10)-1.\Big)\Big)\Im^2, \\
\mathbb{V}(\varepsilon,\Im) =& -0.01\text{cosech}^2(0.1\varepsilon+10) - 0.000010\text{cosech}^2(0.1\varepsilon+10)\times\coth(0.1\varepsilon+10)\Im + 1.0\times10^{-7}\text{cosech}^2(0.1\varepsilon+10) \\
&\times\Big[8.3\times10^{-5}\Im^3\text{cosech}^2(0.1\varepsilon+10)(3\coth(0.1\varepsilon+10)-1) - \Im^2\coth(0.1\varepsilon+10)\Big(3\text{cosech}^2(0.1\varepsilon+10)-1\Big) \\
&+1.6\times10^{-4}\Im^3\text{cosech}^2(0.1\varepsilon+10)\coth(0.1\varepsilon+10)\Big].
\end{aligned}$$

The exact result of Equation (32) at $\wp = 1$ and taking $\xi = 0.005$, $\theta = 10$ and $\kappa = 0.1$.

$$\begin{aligned}
\mathbb{U}(\varepsilon,\Im) &== \xi - \kappa\coth[\kappa(\varepsilon+\theta-\xi\Im)], \\
\mathbb{V}(\varepsilon,\Im) &= -\kappa^2\text{cosech}^2[\kappa(\varepsilon+\theta-\xi\Im)].
\end{aligned} \tag{37}$$

In Figures 5 and 6, the actual and Yang decomposition method solutions at an integer-order $\wp = 1$ are represented for both $\mathbb{U}(\varepsilon,\Im)$ and $\mathbb{V}(\varepsilon,\Im)$ of Example 1. It is observed that Yang decomposition method results are in good contact with the actual result of the models. In Figures 7 and 8, various fractional-order solutions of Example 2, at different fractional-orders, $\wp = 1, 0.8, 0.6, 0.4$ are plotted. It is investigated that for Example 2, the fractional-order solutions are convergent to an integer-order solution for both $\mathbb{U}(\varepsilon,\Im)$ and $\mathbb{V}(\varepsilon,\Im)$. In Tables 5 and 6 show that yang decomposition method of different fractional order $\wp$ of Example 2.

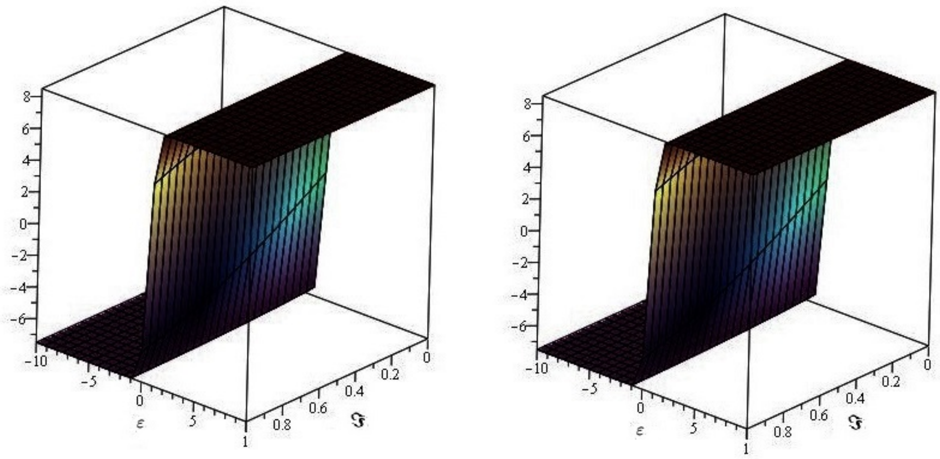

Figure 5. The actual and YDM solution of $\mathbb{U}(\varepsilon,\Im)$ at $\wp = 1$ of Example 2.

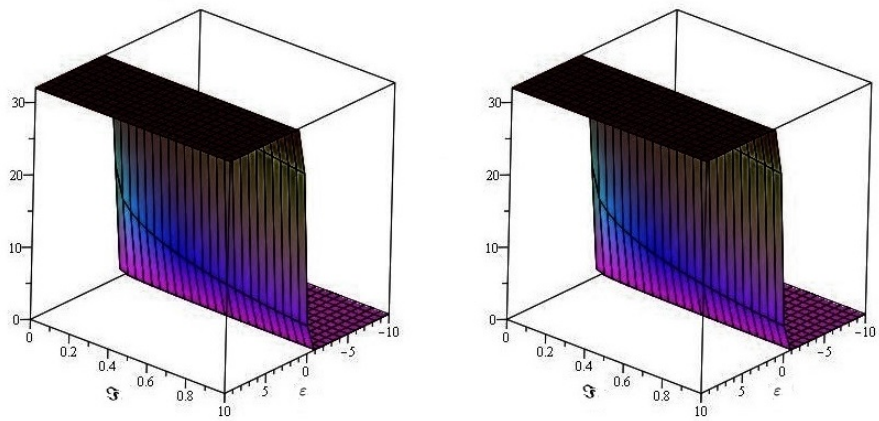

Figure 6. The actual and YDM solution of $\mathbb{V}(\varepsilon,\Im)$ at $\wp = 1$ of Example 2.

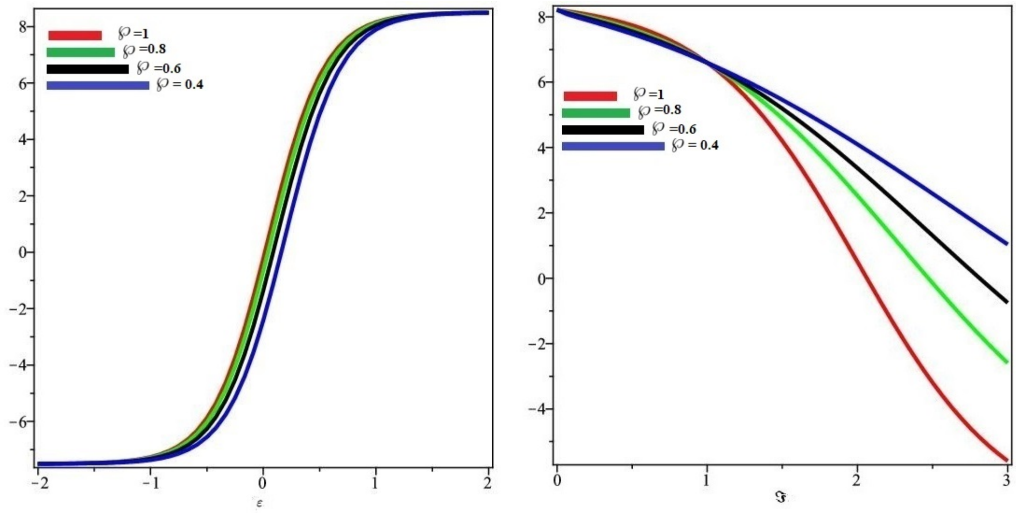

Figure 7. The fractional-order solutions of $\mathbb{V}(\varepsilon, \Im)$ at $\wp$ of Example 2.

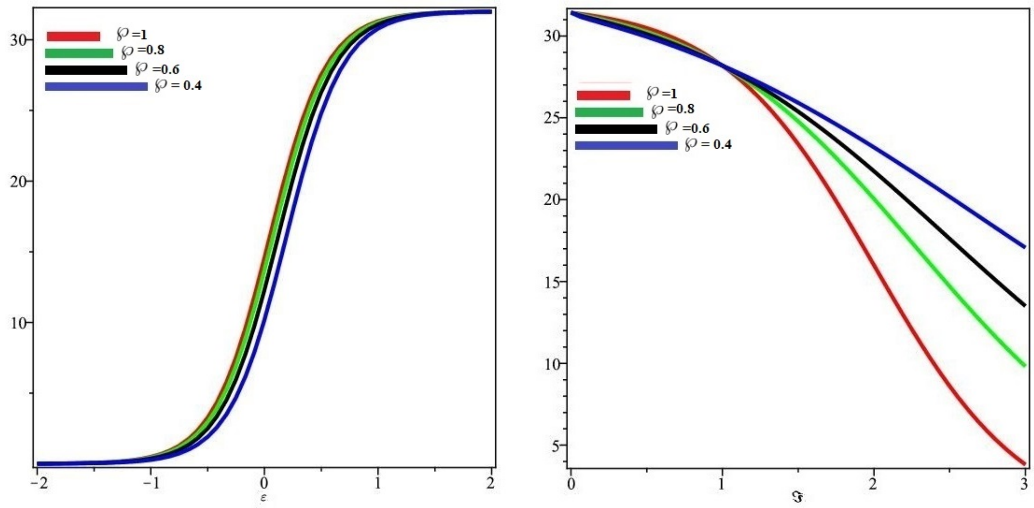

Figure 8. The fractional-order solutions of $\mathbb{V}(\varepsilon, \Im)$ at $\wp$ of Example 2.

Table 5. YDM solution of $\mathbb{U}(\varepsilon, \Im)$ at different fractional-order $\wp$ of Example 2.

$(\varepsilon; \Im)$	$\mathbb{U}(\varepsilon, \Im)$ at $\wp = 0.5$	$\mathbb{U}(\varepsilon, \Im)$ at $\wp = 0.75$	$\mathbb{U}(\varepsilon, \Im)$ at $\wp = 1$	**Exact Result**
(0.1, 0.2)	0.500726	0.500684	0.500671	0.500761
(0.1, 0.4)	0.500742	0.500738	0.500720	0.500720
(0.1, 0.6)	0.500767	0.500746	0.500726	0.500826
(0.2, 0.2)	0.497230	0.497187	0.498174	0.498074
(0.2, 0.4)	0.497243	0.497221	0.497453	0.498121
(0.2, 0.6)	0.496267	0.497047	0.497248	0.498128
(0.3, 0.2)	0.494382	0.494360	0.494347	0.495447
(0.3, 0.4)	0.494414	0.494411	0.494373	0.495473
(0.3, 0.6)	0.494437	0.494418	0.494400	0.49540
(0.4, 0.2)	0.491920	0.491818	0.492786	0.492886
(0.4, 0.4)	0.491852	0.491831	0.492810	0.492911
(0.4, 0.6)	0.491874	0.491855	0.491993	0.492937
(0.5, 0.2)	0.491322	0.491322	0.490312	0.490410
(0.5, 0.4)	0.491354	0.491332	0.490315	0.490415
(0.5, 0.6)	0.491278	0.491358	0.490342	0.490440

Table 6. YDM solution of $\mathbb{V}(\varepsilon, \Im)$ at different fractional-order $\wp$ of Example 2.

$(\varepsilon; \Im)$	$\mathbb{V}(\varepsilon, \Im)$ at $\wp = 0.5$	$\mathbb{V}(\varepsilon, \Im)$ at $\wp = 0.75$	$\mathbb{V}(\varepsilon, \Im)$ at $\wp = 1$	Exact Result
(0.1, 0.2)	0.0939215	0.0939015	0.09389	0.09389
(0.1, 0.4)	0.0939536	0.0939319	0.0939146	0.0939146
(0.1, 0.6)	0.0939757	0.0939571	0.0939391	0.0939391
(0.2, 0.2)	0.0915064	0.091487	0.0914759	0.0914759
(0.2, 0.4)	0.0915375	0.0915165	0.0914997	0.0914997
(0.2, 0.6)	0.0915589	0.0915409	0.0915235	0.0915235
(0.3, 0.2)	0.0891657	0.0891469	0.0891361	0.0891361
(0.3, 0.4)	0.0891958	0.0891754	0.0891592	0.0891592
(0.3, 0.6)	0.0892166	0.0891992	0.0891822	0.0891822
(0.4, 0.2)	0.0868965	0.0868782	0.0868678	0.0868678
(0.4, 0.4)	0.0869257	0.0869059	0.0868901	0.08688901
(0.4, 0.6)	0.0869458	0.0869289	0.0869125	0.0869125
(0.5, 0.2)	0.0846961	0.0846784	0.0846683	0.0846683
(0.5, 0.4)	0.0847244	0.0847052	0.0846899	0.0846899
(0.5, 0.6)	0.0847439	0.0847275	0.0847116	0.0847116

5. Conclusions

This research applies the Yang decomposition method to a fractional-order non-linear Whitham-Broer-Kaup equations system. The suggested approach has been thoroughly researched for fractional-order systems of linear and non-linear differential equations. The numerical results show that the approach is accurate and effective in achieving numerical solutions for non-linear fractional partial differential equations. The proposed methodology is an effective and convenient tool for evaluating numerical solutions to non-linear coupled systems of fractional PDEs compared to previous analytical techniques. Furthermore, the proposed scheme is easy and intuitive, requiring less computing time to solve additional fractional-order partial differential equations.

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