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Nonlocal Boundary Value Problems for Hilfer Generalized Proportional Fractional Differential Equations

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Abstract: In this paper, we discuss the existence and uniqueness of solutions for boundary value problems for Hilfer generalized proportional fractional differential equations with multi-point boundary conditions. Firstly, we consider the scalar case for which the uniqueness result is proved by using Banach's fixed point theorem and the existence results are established via Krasnosel'skiĭ's fixed point theorem and Leray–Schauder nonlinear alternative. We then establish an existence result in the Banach space case based on Mönch's fixed point theorem and the technique of the measure of noncompactness. Examples are constructed to illustrate the application of the main results. We emphasize that, in this paper, we initiate the study of Hilfer generalized proportional fractional boundary value problems of order in $(1, 2]$.

Keywords: Hilfer proportional fractional derivative; Hilfer proportional fractional integral; fixed point theorems; existence results; measure of noncompactness

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1. Introduction

Fractional calculus, as an extension of usual integer calculus, has been applied to investigate derivatives and integrals of arbitrary orders. Since the derivative and integral operators of integer orders cannot be applied to model all real phenomena, different types of fractional operators have been considered by many authors as a generalization of these operators. The considered equations in fractional calculus are often unable to study complex systems and one can say that the applied methods in fractional calculus have been used to model many phenomena in physics, chemistry, mechanics and other sciences (see [1–7]). For numerical methods applied to fractional differential equations, see [8,9]. In consequence, a diversity of new fractional operators have been introduced by many studies to improve the field of fractional calculus, as can be seen, for example, in [10–15]. Katugampola [16,17] combined the Riemann–Liouville and Hadamard fractional operators by introducing the so-called generalized fractional operator. Jahard et al. [14] modified the generalized derivatives to cover the Caputo and Caputo–Hadamard derivatives [18]. On the other hand, the implication of a conformable derivative was introduced in [19,20] and then studies have researched the nonlocal versions of these operators (see [15]). The conformable derivative has a primary defect so that when this operator with an order of 0 is applied to a function, it does not give the function itself. The deficit was solved in [21,22] by redefining the conformable derivative to obtain the function itself when the order of this operator is zero. The modified definition of the conformable derivative was followed by Jarad et al. [23] so that the fractional version of the mentioned operator was suggested. For some recent results containing Hilfer or proportional fractional differential operators, we refer the reader to [24–29] and the references cited therein. Following the mentioned works, in [30], the notion of fractional derivative of the Hilfer generalized proportional type was defined and the existence and uniqueness of solutions for the nonlinear fractional differential problem of the form were investigated:

$$\begin{cases} D^{p_1,p_2,\eta}u(z) = g(z,u(z)), \ z \in [c,T], \\ I^{1-\gamma,\eta}u(c) = \sum_{i=1}^{m} d_i u(\xi_i), \ \gamma = p_1 + p_2 - p_1p_2, \ \xi_i \in (c,T), \end{cases} \tag{1}$$

where the symbols $D^{p_1,p_2,\eta}$ and $I^{1-\gamma,\eta}$ indicate the Hilfer generalized proportional fractional derivative and integral, respectively, $p_1 \in (0,1)$, $(1-\gamma) \in [p_1,1]$ are the order of fractional derivative and integral, respectively, $p_2 \in [0,1]$ is a parameter of Hilfer type, $\eta \in (0,1]$, $g : [c,T] \times \mathbb{R} \to \mathbb{R}$ is continuous, $d_i \in \mathbb{R}$ and $c < \xi_1 < \xi_2 < \cdots < \xi_m = T$.

To the best of our knowledge, there is no other paper in the literature dealing with Hilfer generalized proportional fractional derivative. Thus, motivated by the above paper, our goal in this paper is to enrich this new research area. Thus, in this paper, we inset and study a nonlocal boundary value problem of Hilfer generalized proportional FDEs given by

$$\begin{cases} \left(D_{c^+}^{\delta,\eta,\sigma} + kD_{c^+}^{\delta-1,\eta,\sigma}\right)w(z) = h(z,w(z)), \ z \in [c,d], \\ w(c) = 0, \qquad w(d) = \sum_{j=1}^{m} \overline{\theta_j} w(\overline{\xi_j}), \end{cases} \tag{2}$$

where $D_{c^+}^{\delta,\eta,\sigma}$ is the fractional derivative of a Hilfer generalized proportional type of order $1 < \delta < 2$, the Hilfer parameter $0 \le \eta \le 1$, $\sigma \in (0,1]$, $k \in \mathbb{R}$, $h : [c,d] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $c \ge 0$, $\overline{\theta_j} \in \mathbb{R}$, $\overline{\xi_j} \in (c,d)$ for $j = 1,2,3,\ldots,m$.

We prove the existence and uniqueness results in the scalar case by applying the classical Banach's and Krasnosel'skiĭ's fixed point theorems, as well as the Leray–Schauder nonlinear alternative. Then, by using the measure of noncompactness and Mönch's fixed point theorem, we established an existence result for Problem (2), when $f : [c,d] \times E \to E$ is a given function, and $(E, \|\cdot\|_\infty)$ is a real Banach space.

Compared with the existing literature, the novelty of this research lies in the fact that we introduce and study a new nonlocal boundary value problem for Hilfer generalized proportional fractional differential equations of order in $(1,2]$. Moreover, we considered sequential fractional derivatives, and studied both cases: the scalar case and the Banach space case. The used methods are standard, but their configuration to Problem (2) is new.

The remaining part of this manuscript is structured as follows: Section 2 contains some basic notations, definitions and basic results of fractional calculus needed in this paper. In Section 3, we prove an auxiliary result which plays a vital role in transforming the Problem (2) into a fixed point problem. In Section 4, based on Banach's contraction mapping principle, we first establish the existence of a unique solution for the Problem (2) and then via Krasnosel'skiĭ's fixed point theorem and Leray–Schauder nonlinear alternative, we prove two existence results. Then, in Section 5, we establish an existence result based on Mönch's theorem and the technique of the measure of noncompactness. Additionally, Section 6 provides examples to illustrate the applicability of the results developed in Sections 4 and 5.

2. Preliminaries

In this section, some notations, definitions and lemmas from the fractional calculus are recalled.

We denote by $C([c,d],E)$ the Banach space of all continuous functions $u : [c,d] \to E$ endowed by

$$\|u\|_\infty = \sup\{\|u(z)\|, z \in [c,d]\}.$$

In the case where $E = \mathbb{R}$, we use the notation

$$\|u\| = \sup\{|u(z)|, z \in [c,d]\}.$$

Definition 1 ([31])**.** *Let $p \in L^1([c,d],\mathbb{R})$. The fractional integral of the Riemann–Liouville type with order $\delta > 0$ is defined by*

$$I^{\delta}_{c^+} p(z) = \frac{1}{\Gamma(\delta)} \int_c^z (z-\tau)^{\delta-1} p(\tau) d\tau,$$

where $\Gamma(\cdot)$ denotes the classical Gamma function.

Definition 2 ([31])**.** *Let $p \in C^n([c,d],\mathbb{R})$. The fractional derivative of Caputo type with order $\delta > 0$ of the function p is defined by*

$${}^{C}D^{\delta}_{c^+} p(z) = \frac{1}{\Gamma(n-\delta)} \int_c^z (z-s)^{n-\delta-1} p^{(n)}(s) ds, \ \delta > 0, \ n-1 < \delta < n, \ n \in \mathbb{N}.$$

Definition 3 ([23])**.** *Let $\sigma \in (0,1]$ and $\delta \in \mathbb{C}$ with $Re(\delta) > 0$. Then, the fractional operator*

$$I^{\delta,\sigma}_{c^+} p(z) = \frac{1}{\sigma^{\delta}\Gamma(\delta)} \int_c^z e^{\frac{\sigma-1}{\sigma}(z-s)} (z-s)^{\delta-1} p(s) ds, \ z > c,$$

is called the left-sided generalized proportional integral of order $\delta > 0$ of the function p.

Definition 4 ([23])**.** *The left generalized proportional fractional derivative of order $\delta > 0$ and $\sigma \in (0,1]$ of the function h is defined by*

$$D^{\delta,\sigma}_{c^+} p(z) = \frac{D^{n,\sigma}}{\sigma^{n-\delta}\Gamma(n-\delta)} \int_c^z e^{\frac{\sigma-1}{\sigma}(z-s)} (z-s)^{n-\delta-1} p(s) ds, \ \ \delta \in \mathbb{C}, \ Re(\delta) > 0,$$

where $\Gamma(\cdot)$ indicates the Gamma function and $n = [\delta] + 1$, $[\delta]$ denotes the integer part of a real number δ.

Definition 5 ([23])**.** *The left-sided generalized proportional fractional derivative of Caputo type of order $\delta > 0$ and $\sigma \in (0,1]$ of the function $p \in C^n([c,d],\mathbb{R})$ is defined by*

$${}^{C}D^{\delta,\sigma}_{c^+} p(z) = \frac{1}{\sigma^{n-\delta}\Gamma(n-\delta)} \int_c^z e^{\frac{\sigma-1}{\sigma}(z-s)} (z-s)^{n-\delta-1} D^{n,\sigma} p(s) ds, \ \ \delta \in \mathbb{C}, Re(\delta) > 0,$$

provided the right-hand side exists.

Some properties of the generalized proportional fractional integral and derivative are given in the next lemmas.

Lemma 1 ([23])**.** *Assume that $\delta, \overline{\delta} \in \mathbb{C}$ so that $Re(\delta) \geq 0$ and $Re(\overline{\delta}) > 0$. Then, for any $\sigma \in (0,1]$, we have:*

$$(I^{\delta,\sigma}_{c^+} e^{\frac{\sigma-1}{\sigma}s}(s-c)^{\overline{\delta}-1})(z) = \frac{\Gamma(\overline{\delta})}{\sigma^{\delta}\Gamma(\overline{\delta}+\delta)} e^{\frac{\sigma-1}{\sigma}z}(z-c)^{\overline{\delta}+\delta-1},$$
$$(D^{\delta,\sigma}_{c^+} e^{\frac{\sigma-1}{\sigma}s}(s-c)^{\overline{\delta}-1})(z) = \frac{\sigma^{\delta}\Gamma(\overline{\delta})}{\Gamma(\overline{\delta}-\delta)} e^{\frac{\sigma-1}{\sigma}z}(z-c)^{\overline{\delta}-\delta-1},$$
$$(I^{\delta,\sigma}_{c^+} e^{\frac{\sigma-1}{\sigma}(c-s)}(c-s)^{\overline{\delta}-1})(z) = \frac{\Gamma(\overline{\delta})}{\sigma^{\delta}\Gamma(\overline{\delta}+\delta)} e^{\frac{\sigma-1}{\sigma}(c-z)}(c-z)^{\overline{\delta}+\delta-1},$$
$$(D^{\delta,\sigma}_{c^+} e^{\frac{\sigma-1}{\sigma}(c-s)}(s-c)^{\overline{\delta}-1})(z) = \frac{\sigma^{\delta}\Gamma(\overline{\delta})}{\Gamma(\overline{\delta}-\delta)} e^{\frac{\sigma-1}{\sigma}(c-z)}(c-z)^{\overline{\delta}-\delta-1}.$$

Lemma 2 ([23]). *Suppose that $\sigma \in (0,1]$, $Re(\delta_1) > 0$ and $Re(\delta_2) > 0$. If $p \in C([c,d],\mathbb{R})$, then:*

$$I_{c^+}^{\delta,\sigma}(I^{\bar{\delta},\sigma}p)(z) = I_{c^+}^{\bar{\delta},\sigma}(I^{\delta,\sigma}p)(z) = (I_{c^+}^{\delta+\bar{\delta},\sigma}p)(z), \ \ z \geq c. \tag{3}$$

Lemma 3 ([23]). *Let $\sigma \in (0,1]$ and $0 \leq m < [Re(\delta)] + 1$. If $p \in L^1([a,b])$ then:*

$$D_{c^+}^{m,\sigma}(I_{c^+}^{\delta,\sigma}p)(z) = (I_{c^+}^{\delta-m}p)(z), \ \ z > c.$$

Now the Hilfer generalized proportional fractional derivative is introduced.

Definition 6 ([30]). *Let $n-1 < \delta < n$, $n \in \mathbb{N}$, $\sigma \in (0,1]$ and $0 \leq \eta \leq 1$. Then, the fractional proportional derivative of Hilfer type with order δ, parameter η and proportional number σ of the function p is praised by*

$$(D_{c^+}^{\delta,\eta,\sigma}p)(z) = I_{c^+}^{\eta(n-\delta),\sigma}[D^{\sigma}(I_{c^+}^{(1-\eta)(n-\delta),\sigma}p)](z),$$

in which $D^{\sigma}p(z) = (1-\sigma)p(z) + \sigma p'(z)$ and $I^{(\cdot),\sigma}$ is the generalized proportional fractional integral defined in Definition 3.

The Hilfer generalized proportional fractional derivative is equivalent to:

$$(D_{c^+}^{\delta,\eta,\sigma}p)(z) = I_{c^+}^{\eta(n-\delta),\sigma}[D^{n,\sigma}](I_{c^+}^{(1-\eta)(n-\delta),\sigma}p)(z) = (I_{c^+}^{\eta(n-\delta),\sigma}D^{\gamma,\sigma}p)(z),$$

where $\gamma = \delta + \eta(n-\delta)$. Thus, the operator $D_{c^+}^{\delta,\eta,\sigma}$ can be represented in terms of the operators given in Definition 1 and Definition 2. The parameter γ satisfies:

$$1 < \gamma \leq 2, \ \ \gamma \geq \delta, \ \ \gamma > \eta, \ \ n-\gamma < n - \eta(n-\delta).$$

Lemma 4 ([30]). *Let $n-1 < \delta < n$, $\sigma \in (0,1]$, $0 \leq \eta \leq 1$ and $\gamma = \delta + \eta(n-\delta) \in [\delta, n]$. If $p \in L^1(c,d)$ and $I_{c^+}^{n-\gamma,\sigma}p \in C^n([c,d],\mathbb{R})$, then:*

$$I_{c^+}^{\delta,\sigma}D_{c^+}^{\delta,\eta,\sigma}p(z) = p(z) - \sum_{j=1}^{n} e^{\frac{\sigma-1}{\sigma}(z-c)}\frac{(z-c)^{\gamma-j}}{\sigma^{\gamma-j}\Gamma(\gamma+1-j)}(I^{j-\gamma,\sigma}p)(c^+).$$

3. An Auxiliary Result

The following lemma dealing with a linear variant of the Problem (2) is the basic tool for transforming the Problem (2) into a fixed point problem.

Lemma 5. *Let $1 < \delta < 2$, $0 \leq \eta \leq 1$, $\gamma = \delta + \eta(2-\delta) \in [\delta, 2]$, $\sigma \in (0,1]$, $g \in C([c,d],\mathbb{R})$ and:*

$$\Delta = \frac{(d-c)^{\gamma-1}}{\Gamma(\gamma)}e^{\frac{\sigma-1}{\sigma}(d-c)} - \sum_{j=1}^{m}\theta_j\frac{(\xi_j-c)^{\gamma-1}}{\Gamma(\gamma)}e^{\frac{\sigma-1}{\sigma}(\xi_j-c)} \neq 0.$$

Then, u is the solution of the linear problem:

$$\begin{cases} \left(D_{c_+}^{\delta,\eta,\sigma} + kD_{c_+}^{\delta-1,\eta,\sigma}\right)u(z) = g(z), \ \ z \in [c,d], \\ u(c) = 0, \quad u(d) = \sum_{j=1}^{m}\theta_j u(\xi_j), \end{cases} \tag{4}$$

if and only if:

$$\begin{aligned} u(z) &= I_{c^+}^{\delta,\sigma} g(z) + \frac{(z-c)^{\gamma-1}}{\Delta\Gamma(\gamma)} \left[\frac{1}{\sigma^\delta \Gamma(\delta)} \sum_{j=1}^{m} \theta_j \int_c^{\xi_j} e^{\frac{\sigma-1}{\sigma}(\xi_j - s)} (\xi_j - s)^{\delta-1} g(s) ds \right. \\ &\quad - \frac{1}{\sigma^\delta \Gamma(\delta)} \int_c^d e^{\frac{\sigma-1}{\sigma}(d-s)} (d-s)^{\delta-1} g(s) ds \\ &\quad \left. - \frac{k}{\sigma} \sum_{j=1}^{m} \theta_j \int_c^{\xi_j} u(s) e^{\frac{\sigma-1}{\sigma}(\xi_j - s)} ds + \frac{k}{\sigma} \int_c^d u(s) e^{\frac{\sigma-1}{\sigma}(d-s)} ds \right] e^{\frac{\sigma-1}{\sigma}(z-c)} \\ &\quad - \frac{k}{\sigma} \int_c^z u(s) e^{\frac{\sigma-1}{\sigma}(z-s)} ds. \end{aligned} \tag{5}$$

Proof. Let u be a solution of the Problem (4). Then, in view of Lemma 4, we conclude that

$$\begin{aligned} u(z) &= I_{c^+}^{\delta,\sigma} g(z) + c_0 \frac{(z-c)^{\gamma-2}}{\Gamma(\gamma-1)} e^{\frac{\sigma-1}{\sigma}(z-c)} \\ &\quad + c_1 \frac{(z-c)^{\gamma-1}}{\Gamma(\gamma)} e^{\frac{\sigma-1}{\sigma}(z-c)} - \frac{k}{\sigma} \int_c^z u(s) e^{\frac{\sigma-1}{\sigma}(z-s)} ds. \end{aligned} \tag{6}$$

Now, due to $u(c) = 0$, we have $c_0 = 0$. Consequently:

$$u(z) = I_{c^+}^{\delta,\sigma} g(z) + c_1 \frac{(z-c)^{\gamma-1}}{\Gamma(\gamma)} e^{\frac{\sigma-1}{\sigma}(z-c)} - \frac{k}{\sigma} \int_c^z u(s) e^{\frac{\sigma-1}{\sigma}(z-s)} ds. \tag{7}$$

From $u(d) = \sum_{j=1}^{m} \theta_j u(\xi_j)$, we obtain:

$$\begin{aligned} c_1 &= \frac{1}{\Delta} \left[\frac{1}{\sigma^\delta \Gamma(\delta)} \sum_{j=1}^{m} \theta_j \int_c^{\xi_j} e^{\frac{\sigma-1}{\sigma}(\xi_j - s)} (\xi_j - s)^{\delta-1} g(s) ds \right. \\ &\quad - \frac{1}{\sigma^\delta \Gamma(\delta)} \int_c^d e^{\frac{\sigma-1}{\sigma}(d-s)} (d-s)^{\delta-1} g(s) ds \\ &\quad \left. - \frac{k}{\sigma} \sum_{j=1}^{m} \theta_j \int_c^{\xi_j} u(s) e^{\frac{\sigma-1}{\sigma}(\xi_j - s)} ds + \frac{k}{\sigma} \int_c^d u(s) e^{\frac{\sigma-1}{\sigma}(d-s)} ds \right]. \end{aligned} \tag{8}$$

By inserting c_1 in (7), we obtain (5).

Conversely, by taking the operators $D_{c^+}^{\delta,\eta,\sigma}$ and $D_{c^+}^{\delta-1,\eta,\sigma}$ to (5) and putting them to the left hand side of the first equation in (4), we can obtain the right hand side of (4). It is obvious that $u(c) = 0$. By substituting $z = d$ and $z = \xi_j$ in (4) and direct computation, we have the second condition of (4). The proof is completed. □

4. Existence and Uniqueness Results in the Scalar Case

By Lemma 5, we define an operator $F : C([c,d], \mathbb{R}) \longrightarrow C([c,d], \mathbb{R})$ associated with the Problem (2) as

$$\begin{aligned} (Fu)(z) &= I_{c^+}^{\delta,\sigma} h(z, u(z)) \\ &\quad + \frac{(z-c)^{\gamma-1}}{\Delta\Gamma(\gamma)} \left[\frac{1}{\sigma^\delta \Gamma(\delta)} \sum_{j=1}^{m} \theta_j \int_c^{\xi_j} e^{\frac{\sigma-1}{\sigma}(\xi_j - s)} (\xi_j - s)^{\delta-1} h(s, u(s)) ds \right. \\ &\quad - \frac{1}{\sigma^\delta \Gamma(\delta)} \int_c^d e^{\frac{\sigma-1}{\sigma}(d-s)} (d-s)^{\delta-1} h(s, u(s)) ds \\ &\quad \left. - \frac{k}{\sigma} \sum_{j=1}^{m} \theta_j \int_c^{\xi_j} u(s) e^{\frac{\sigma-1}{\sigma}(\xi_j - s)} ds + \frac{k}{\sigma} \int_c^d u(s) e^{\frac{\sigma-1}{\sigma}(d-s)} ds \right] e^{\frac{\sigma-1}{\sigma}(z-c)} \\ &\quad - \frac{k}{\sigma} \int_c^z u(s) e^{\frac{\sigma-1}{\sigma}(z-s)} ds. \end{aligned} \tag{9}$$

Notice that the existence of fixed points of the operator F implies the existence of solutions for Problem (2).

For the computational convenience, we set:

$$\Phi_1 = \frac{(d-c)^{\delta}}{\sigma^{\delta}\Gamma(\delta+1)} + \frac{(d-c)^{\gamma-1}}{|\Delta|\Gamma(\gamma)} \frac{1}{\sigma^{\delta}\Gamma(\delta+1)} \left[\sum_{j=1}^{m} \theta_j (\xi_j - c)^{\delta} + (d-c)^{\delta} \right], \tag{10}$$

$$\Phi_2 = \frac{(d-c)^{\gamma-1}}{|\Delta|\Gamma(\gamma)} \frac{k}{\sigma} \left[\sum_{j=1}^{m} \theta_j (\xi_j - c) + (d-c) \right] + \frac{k}{\sigma}(d-c). \tag{11}$$

In this section, fixed point theorems are applied to present the existence and uniqueness results concerning the Problem (2). First, Banach's theorem is applied to establish the uniqueness result.

Lemma 6 (Banach fixed point theorem [32])**.** *Let G be a closed set in X and $H : G \to G$ satisfies:*

$$|Hu_1 - Hu_2| \leq \lambda |u_1 - u_2|, \ \text{for some } \lambda \in (0,1), \text{ and for all } u_1, u_2 \in G.$$

Then, H admits a unique fixed point in G.

Theorem 1. *Let $1 < \delta < 2$, $0 \leq \eta \leq 1$, $\gamma = \delta + \eta(2-\delta) \in [\delta, 2]$, $\sigma \in (0,1]$. Assume that:*
(D_1) There exists $L > 0$ such that:

$$|h(z,u_1) - h(z,u_2)| \leq L|u_1 - u_2|, \ \ \forall z \in [c,d], \ u_1, u_2 \in \mathbb{R}.$$

Then, Problem (2) has a unique solution on $[c,d]$, provided that:

$$L\Phi_1 + \Phi_2 < 1, \tag{12}$$

where Φ_1 and Φ_2 are defined by (10) and (11), respectively.

Proof. First, we show that F defined by (9) satisfies $FB_r \subset B_r$, where $B_r = \{u \in C([c,d],\mathbb{R}) : \|u\| \leq r\}$ with $r > \dfrac{M\Phi_1}{1 - L\Phi_1 - \Phi_2}$ and $\sup_{z\in[c,d]} |h(z,0)| = M < \infty$. For any $u \in B_r$, and using the condition (D_1), we have

$$|h(z,u)| = |h(z,u) - h(z,0) + h(z,0)| \leq L\|u\| + M \leq Lr + M.$$

Thus, for any $u \in B_r$, and using the fact that $|e^{\frac{\sigma-1}{\sigma}z}| \leq 1$, we have:

$$\begin{aligned}
|(Fu)(z)| \leq\ & \frac{1}{\sigma^{\delta}\Gamma(\delta)} \int_c^z e^{\frac{\sigma-1}{\sigma}(z-s)} (z-s)^{\delta-1} |h(s,u(s))| ds \\
& + \frac{(z-c)^{\gamma-1}}{|\Delta|\Gamma(\gamma)} \left[\frac{1}{\sigma^{\delta}\Gamma(\delta)} \sum_{j=1}^{m} \theta_j \int_c^{\xi_j} e^{\frac{\sigma-1}{\sigma}(z-s)} (\xi_j - s)^{\delta-1} |h(s,u(s))| ds \right. \\
& + \frac{1}{\sigma^{\delta}\Gamma(\delta)} \int_c^d e^{\frac{\sigma-1}{\sigma}(d-s)} (d-s)^{\delta-1} |h(s,u(s))| ds \\
& \left. + \frac{k}{\sigma} \sum_{j=1}^{m} \theta_j \int_c^{\xi_j} |u(s)| e^{\frac{\sigma-1}{\sigma}(\xi_j - s)} ds + \frac{k}{\sigma} \int_c^d |u(s)| e^{\frac{\sigma-1}{\sigma}(\xi_j - s)} ds \right] e^{\frac{\sigma-1}{\sigma}(z-c)} \\
& + \frac{k}{\sigma} \int_c^z |u(s)| e^{\frac{\sigma-1}{\sigma}(z-s)} ds
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{(Lr+M)}{\sigma^{\delta}\Gamma(\delta+1)}(d-c)^{\delta} \\
&\quad +\frac{(Lr+M)(d-c)^{\gamma-1}}{|\Delta|\Gamma(\gamma)}\left[\frac{1}{\sigma^{\delta}\Gamma(\delta+1)}\sum_{j=1}^{m}\theta_j(\xi_j-c)^{\delta}+\frac{1}{\sigma^{\delta}\Gamma(\delta+1)}(d-c)^{\delta}\right] \\
&\quad +\frac{(d-c)^{\gamma-1}r}{|\Delta|\Gamma(\gamma)}\frac{k}{\sigma}\left[\sum_{j=1}^{m}\theta_j(\xi_j-c)+(d-c)\right]+\frac{k}{\sigma}(d-c)r \\
&= Lr\Phi_1+r\Phi_2+M\Phi_1<r,
\end{aligned}
$$

which implies that $\|Fu\| \leq r$ and consequently $Fu \in B_r$, for any $u \in B_r$. Therefore, $FB_r \subset B_r$.

We then show that F is a contraction. For all $u_1, u_2 \in C([c,d],\mathbb{R})$ and $z \in [c,d]$, we have:

$$
\begin{aligned}
&|(Fu_1)(z)-(Fu_2)(z)| \\
\leq\ & \frac{1}{\sigma^{\delta}\Gamma(\delta)}\int_c^z e^{\frac{\sigma-1}{\sigma}(z-s)}(z-s)^{\delta-1}|h(s,u_1(s))-h(s,u_2(s))|ds \\
&+\frac{(z-c)^{\gamma-1}}{|\Delta|\Gamma(\gamma)}\left[\frac{1}{\sigma^{\delta}\Gamma(\delta)}\sum_{j=1}^{m}\theta_j\int_c^{\xi_j}e^{\frac{\sigma-1}{\sigma}(z-s)}(\xi_j-s)^{\delta-1}|h(s,u_1(s))-h(s,u_2(s))|ds\right. \\
&+\frac{1}{\sigma^{\delta}\Gamma(\delta)}\int_c^d e^{\frac{\sigma-1}{\sigma}(d-s)}(d-s)^{\delta-1}|h(s,u_1(s))-h(s,u_2(s))|ds \\
&+\frac{k}{\sigma}\sum_{j=1}^{m}\theta_j\int_c^{\xi_j}|u_1(s)-u_2(s)|e^{\frac{\sigma-1}{\sigma}(\xi_j-s)}ds \\
&\left.+\frac{k}{\sigma}\int_c^d|u_1(s)-u_2(s)|e^{\frac{\sigma-1}{\sigma}(\xi_j-s)}ds\right]e^{\frac{\sigma-1}{\sigma}(z-c)} \\
&+\frac{k}{\sigma}\int_c^z|u_1(s)-u_2(s)|e^{\frac{\sigma-1}{\sigma}(z-s)}ds \\
\leq\ & \frac{L\|u_1-u_2\|}{\sigma^{\delta}\Gamma(\delta+1)}(d-c)^{\delta} \\
&+\frac{L(d-c)^{\gamma-1}\|u_1-u_2\|}{|\Delta|\Gamma(\gamma)}\left[\frac{1}{\sigma^{\delta}\Gamma(\delta+1)}\sum_{j=1}^{m}\theta_j(\xi_j-c)^{\delta}+\frac{1}{\sigma^{\delta}\Gamma(\delta+1)}(d-c)^{\delta}\right] \\
&+\frac{(d-c)^{\gamma-1}\|u_1-u_2\|}{|\Delta|\Gamma(\gamma)}\frac{k}{\sigma}\left[\sum_{j=1}^{m}\theta_j(\xi_j-c)+(d-c)\right]+\frac{k}{\sigma}(d-c)\|u_1-u_2\|.
\end{aligned}
$$

Hence:

$$
\begin{aligned}
&\|Fu_1-Fu_2\| \\
\leq\ & \left\{L\left[\frac{(d-c)^{\delta}}{\sigma^{\delta}\Gamma(\delta+1)}+\frac{(d-c)^{\gamma-1}}{|\Delta|\Gamma(\gamma)}\frac{1}{\sigma^{\delta}\Gamma(\delta+1)}\left[\sum_{j=1}^{m}\theta_j(\xi_j-c)^{\delta}+(d-c)^{\delta}\right]\right.\right. \\
&\left.+\frac{(d-c)^{\gamma-1}}{|\Delta|\Gamma(\gamma)}\frac{k}{\sigma}\left[\sum_{j=1}^{m}\theta_j(\xi_j-c)+(d-c)\right]+\frac{k}{\sigma}(d-c)\right\}\|u_1-u_2\| \\
=\ & (L\Phi_1+\Phi_2)\|u_1-u_2\|.
\end{aligned}
$$

Consequently, by (12), F is a contraction and by applying Banach's fixed point theorem, Problem (2) has a unique solution. The proof is completed. □

Now, by applying Krasnosel'skiı̆'s fixed point theorem, we prove the existence of the result of Problem (2).

Lemma 7 (Krasnosel'skiĭ fixed point theorem [33]). *Let N indicate a closed, bounded, convex and nonempty subset of a Banach space Y and C, D be operators such that (i) $Cx + Dy \in N$ where $x, y \in N$, (ii) C is compact and continuous and (iii) D is a contraction mapping. Then, there exists $z \in N$ such that $z = Cz + Dz$.*

Theorem 2. *Let $1 < \delta < 2$, $0 \leq \eta \leq 1$, $\gamma = \delta + \eta(2-\delta) \in [\delta, 2]$, $\sigma \in (0,1]$. Assume that:*
(D_2) $h : [c,d] \times \mathbb{R} \to \mathbb{R}$ *is a continuous function such that:*

$$|h(z,u(z))| \leq \phi(z), \quad \forall (z,u) \in [c,d] \times \mathbb{R} \text{ with } \phi \in C([c,d] \times \mathbb{R}^+).$$

(D_3) $\Phi_2 < 1$, *where Φ_2 is defined by (11).*
Then, the Problem (2) has at least one solution on $[c,d]$.

Proof. We verify that the assumptions of Krasnosel'skiĭ's fixed point theorem (Lemma 7) are satisfied by the operator F. To do this we split the operator F defined by (9) into the sum of two operators F_1 and F_2 on the closed ball $B_\rho = \{u \in C([c,d], \mathbb{R}) : \|u\| \leq \rho\}$ with $\rho \geq \dfrac{\|\phi\|\Phi_1}{1-\Phi_2}$, $\sup_{t\in[c,d]} \phi(t) = \|\phi\|$, where:

$$\begin{aligned}
(F_1u)(z) &= I_{c^+}^{\delta,\sigma} h(z,u(z)) \\
&\quad + \frac{(z-c)^{\gamma-1}}{\Delta\Gamma(\gamma)} \left[\frac{1}{\sigma^\delta \Gamma(\delta)} \sum_{j=1}^m \theta_j \int_c^{\xi_j} e^{\frac{\sigma-1}{\sigma}(\xi_j - s)} (\xi_j - s)^{\delta-1} h(s,u(s)) ds \right. \\
&\quad \left. - \frac{1}{\sigma^\delta \Gamma(\delta)} \int_c^d e^{\frac{\sigma-1}{\sigma}(d-s)} (d-s)^{\delta-1} h(s,u(s)) ds \right] e^{\frac{\sigma-1}{\sigma}(z-c)},
\end{aligned}$$

and:

$$\begin{aligned}
(F_2u)(z) &= \frac{(z-c)^{\gamma-1}}{\Delta\Gamma(\gamma)} \left[-\frac{k}{\sigma} \sum_{j=1}^m \theta_j \int_c^{\xi_j} u(s) e^{\frac{\sigma-1}{\sigma}(\xi_j - s)} ds \right. \\
&\quad \left. + \frac{k}{\sigma} \int_c^d u(s) e^{\frac{\sigma-1}{\sigma}(d-s)} ds \right] e^{\frac{\sigma-1}{\sigma}(z-c)} - \frac{k}{\sigma} \int_c^z u(s) e^{\frac{\sigma-1}{\sigma}(z-s)} ds.
\end{aligned}$$

For all $z \in [c,d]$ and $u, v \in B_\rho$, we have:

$$\begin{aligned}
& |(F_1u)(z) + (F_2)v(z)| \\
\leq\ & \frac{\|\phi\|}{\sigma^\delta \Gamma(\delta+1)} (d-c)^\delta \\
& + \frac{\|\phi\|(d-c)^{\gamma-1}}{|\Delta|\Gamma(\gamma)} \left[\frac{1}{\sigma^\delta \Gamma(\delta+1)} \sum_{j=1}^m \theta_j (\xi_j - c)^\delta + \frac{1}{\sigma^\delta \Gamma(\delta+1)} (d-c)^\delta \right] \\
& + \frac{(d-c)^{\gamma-1}\rho}{|\Delta|\Gamma(\gamma)} \frac{k}{\sigma} \left[\sum_{j=1}^m \theta_j (\xi_j - c) + (d-c) \right] + \frac{k}{\sigma}(d-c)\rho \\
=\ & \|\phi\|\Phi_1 + \rho\Phi_2 < \rho,
\end{aligned}$$

and consequently, $\|F_1u + F_2v\| \leq \rho$ which means that we have $F_1u + F_2v \in B_\rho$. We can easily show that F_2 is a contraction using assumption (D_3).

In the final step, it is shown that F_1 is compact and continuous. Since h is continuous, we conclude that F_1 is also continuous. Furthermore, F_1 is uniformly bounded on B_r since

$$\|F_1u\| \leq \Phi_1\|\phi\|.$$

Now, we prove that F_1 is compact. If $\sup_{(z,u)\in[c,d]\times B_\rho} |h(z,u(z))| = \bar{h}$, then for all $c < z_1 < z_2 < T$, we obtain:

$$\begin{aligned}
& |(F_1u)(z_2) - (F_1u)(z_1)| \\
\leq\; & \frac{1}{\sigma^\delta\Gamma(\delta)} \int_c^{z_2} [(z_2-s)^{\delta-1} - (z_1-s)^{\delta-1}]|h(s,u(s))|ds \\
& + \frac{1}{\sigma^\delta\Gamma(\delta)} \int_{z_1}^{z_2} (z_2-s)^{\delta-1}|h(s,u(s))|ds \\
& + \frac{(z_2-c)^{\gamma-1} - (z_1-c)^{\gamma-1}}{|\Delta|\Gamma(\gamma)} \Bigg[\frac{1}{\sigma^\delta\Gamma(\delta)} \sum_{j=1}^m \theta_j \int_c^{\xi_j} e^{\frac{\sigma-1}{\sigma}(\xi_j-s)} (\xi_j-s)^{\delta-1}|h(s,u(s))|ds \\
& + \frac{1}{\sigma^\delta\Gamma(\delta)} \int_c^d e^{\frac{\sigma-1}{\sigma}(d-s)} (d-s)^{\delta-1}|h(s,u(s))|ds \Bigg] \\
\leq\; & \frac{\bar{h}}{\sigma^\delta\Gamma(\delta+1)} \Big[|(z_2-c)^\delta - (z_1-c)^\delta| + 2(z_2-z_1)^\delta \Big] \\
& + \frac{(z_2-c)^{\gamma-1} - (z_1-c)^{\gamma-1}}{|\Delta|\Gamma(\gamma)} \bar{h} \frac{1}{\sigma^\delta\Gamma(\delta+1)} \left[\sum_{j=1}^m \theta_j(\xi_j-c)^\delta + (d-c)^\delta \right],
\end{aligned}$$

which tends towards zero as $z_2 \to z_1$, independently of $u \in B_\rho$. Thus, F_1 is equicontinuous. According to Arzelá–Asccoli theorem, we conclude that F_1 is compact on B_ρ. Hence, the hypotheses of Krasnosel'skiĭ's fixed point theorem hold true, and consequently, the operator $F_1u + F_2u = Fu$ has a fixed point, which implies that the Problem (2) has at least one solution on $[c,d]$. The proof is finished. □

Now, we apply Leray–Schauder's nonlinear alternative to present the second existence result.

Lemma 8 (Leray–Schauder nonlinear alternative [34])**.** *Let the set Ω be a closed bounded convex in X and O an open set contained in Ω with $0 \in O$. Then, for the continuous and compact $T : \bar{U} \to \Omega$, either:*

($\mathfrak{a}$) *T admits a fixed–point in $\bar{U}$; or*
($\mathfrak{aa}$) *$\exists\, \mathfrak{u} \in \partial U$ and $\mu \in (0,1)$ with $\mathfrak{u} = \mu T(\mathfrak{u})$.*

Theorem 3. *Let (D_3) holds. In addition, we assume that:*
(D_4) there exist $\psi \in C([0,\infty),(0,\infty))$ and $p \in C([c,d],\mathbb{R}^+)$ such that:

$$|h(z,u)| \leq p(t)\psi(|u|) \quad \text{for each} \quad (z,u) \in [c,d]\times\mathbb{R};$$

(D_5) A constant $K > 0$ exists, such that:

$$\frac{(1-\Phi_2)K}{L_1\psi(K)\|p\|\Phi_1} > 1,$$

where Φ_1 and Φ_2 are defined by (10) and (11), respectively.
Then, the Problem (2) has at least one solution on $[c,d]$.

Proof. Let the operator F be defined by (9). Firstly, we shall show that the operator F maps bounded sets into bounded sets in $C([c,d],\mathbb{R})$. Let $B_r = \{x \in C([c,d],\mathbb{R}) : \|u\| \leq r\}$ $r > 0$, be a bounded ball in $C([c,d],\mathbb{R})$. Then, for $t \in [a,b]$, we have:

$$
\begin{aligned}
|(Fu)(z)| \leq\; & \frac{1}{\sigma^{\delta}\Gamma(\delta)}\int_c^z e^{\frac{\sigma-1}{\sigma}(z-s)}(z-s)^{\delta-1}|h(s,u(s))|ds \\
& + \frac{(z-c)^{\gamma-1}}{|\Delta|\Gamma(\gamma)}\Bigg[\frac{1}{\sigma^{\delta}\Gamma(\delta)}\sum_{j=1}^{m}\theta_j\int_c^{\xi_j} e^{\frac{\sigma-1}{\sigma}(z-s)}(\xi_j-s)^{\delta-1}|h(s,u(s))|ds \\
& + \frac{1}{\sigma^{\delta}\Gamma(\delta)}\int_c^d e^{\frac{\sigma-1}{\sigma}(d-s)}(d-s)^{\delta-1}|h(s,u(s))|ds \\
& + \frac{k}{\sigma}\sum_{j=1}^{m}\theta_j\int_c^{\xi_j}|u(s)|e^{\frac{\sigma-1}{\sigma}(\xi_j-s)}ds + \frac{k}{\sigma}\int_c^d |u(s)|e^{\frac{\sigma-1}{\sigma}(\xi_j-s)}ds\Bigg] e^{\frac{\sigma-1}{\sigma}(z-c)} \\
& + \frac{k}{\sigma}\int_c^z |u(s)|e^{\frac{\sigma-1}{\sigma}(z-s)}ds \\
\leq\; & \frac{\|p\|\psi(\|u\|)}{\sigma^{\delta}\Gamma(\delta+1)}(d-c)^{\delta} \\
& + \frac{\|p\|\psi(\|u\|)(d-c)^{\gamma-1}}{|\Delta|\Gamma(\gamma)}\Bigg[\frac{1}{\sigma^{\delta}\Gamma(\delta+1)}\sum_{j=1}^{m}\theta_j(\xi_j-c)^{\delta} + \frac{1}{\sigma^{\delta}\Gamma(\delta+1)}(d-c)^{\delta}\Bigg] \\
& + \frac{(d-c)^{\gamma-1}\|u\|}{|\Delta|\Gamma(\gamma)}\frac{k}{\sigma}\Bigg[\sum_{j=1}^{m}\theta_j(\xi_j-c)+(d-c)\Bigg] + \frac{k}{\sigma}(d-c)\|u\| \\
\leq\; & \|p\|\psi(r)\Phi_1 + r\Phi_2,
\end{aligned}
$$

and consequently,

$$\|Fu\| \leq \|p\|\psi(r)\Phi_1 + r\Phi_2.$$

Secondly, we will show that the operator F maps bounded sets into equicontinuous sets of $C([c,d],\mathbb{R})$. Let $z_1, z_2 \in [c,d]$ with $z_1 < z_2$ and $u \in B_r$. Then, we have:

$$
\begin{aligned}
& |(Fu)(z_2) - (Fu)(z_1)| \\
\leq\; & \frac{1}{\sigma^{\delta}\Gamma(\delta)}\int_c^{z_2}[(z_2-s)^{\delta-1}-(z_1-s)^{\delta-1}]|h(s,u(s))|ds \\
& + \frac{1}{\sigma^{\delta}\Gamma(\delta)}\int_{z_1}^{z_2}(z_2-s)^{\delta-1}|h(s,u(s))|ds \\
& + \frac{(z_2-c)^{\gamma-1}-(z_1-c)^{\gamma-1}}{|\Delta|\Gamma(\gamma)}\Bigg[\frac{1}{\sigma^{\delta}\Gamma(\delta)}\sum_{j=1}^{m}\theta_j\int_c^{\xi_j} e^{\frac{\sigma-1}{\sigma}(\xi_j-s)}(\xi_j-s)^{\delta-1}|h(s,u(s))|ds \\
& + \frac{1}{\sigma^{\delta}\Gamma(\delta)}\int_c^d e^{\frac{\sigma-1}{\sigma}(d-s)}(d-s)^{\delta-1}|h(s,u(s))|ds\Bigg] \\
& + \frac{(z_2-c)^{\gamma-1}-(z_1-c)^{\gamma-1}}{|\Delta|\Gamma(\gamma)}\frac{k}{\sigma}\Bigg[\sum_{j=1}^{m}\theta_j\int_c^{\xi_j}|u(s)|e^{\frac{\sigma-1}{\sigma}(\xi_j-s)}ds \\
& + \int_c^d |u(s)|e^{\frac{\sigma-1}{\sigma}(\xi_j-s)}ds\Bigg] \\
& + \frac{k}{\sigma}\Bigg[\int_c^{z_1}|u(s)|\Big[e^{\frac{\sigma-1}{\sigma}(z_1-s)} - e^{\frac{\sigma-1}{\sigma}(z_2-s)}\Big]ds + \int_{z_1}^{z_2}|u(s)|\Big[e^{\frac{\sigma-1}{\sigma}(z_2-s)}\Big]ds\Bigg] \\
\leq\; & \frac{\|p\|\psi(r)}{\sigma^{\delta}\Gamma(\delta+1)}\Big[|(z_2-c)^{\delta}-(z_1-c)^{\delta}| + 2(z_2-z_1)^{\delta}\Big]
\end{aligned}
$$

$$+\frac{(z_2-c)^{\gamma-1}-(z_1-c)^{\gamma-1}}{|\Delta|\Gamma(\gamma)}\|p\|\psi(r)\frac{1}{\sigma^{\delta}\Gamma(\delta+1)}\left[\sum_{j=1}^{m}\theta_j(\xi_j-c)^{\delta}+(d-c)^{\delta}\right]$$
$$+\frac{(z_2-c)^{\gamma-1}-(z_1-c)^{\gamma-1}}{|\Delta|\Gamma(\gamma)}\frac{kr}{\sigma}\left[\sum_{j=1}^{m}\theta_j(\xi_j-c)+(d-c)\right]$$
$$+r\frac{k}{\sigma}\left[\int_c^{z_1}\left(e^{\frac{\sigma-1}{\sigma}(z_1-s)}-e^{\frac{\sigma-1}{\sigma}(z_2-s)}\right)ds+\int_{z_1}^{z_2}\left(e^{\frac{\sigma-1}{\sigma}(z_2-s)}\right)ds\right].$$

The right-hand side in the above inequality is independent of $u\in B_r$ and tends towards zero as $z_2-z_1\to 0$. Hence, according to the Arzelá–Ascoli theorem, the operator $F:C([c,d],\mathbb{R})\to C([c,d],\mathbb{R})$ is completely continuous.

Finally, we will prove that the set of all solutions to equation $u=\lambda Fu$ for $\lambda\in(0,1)$ is bounded.

Let u be a solution. Then, we have for $z\in[c,d]$, as in the first step,

$$|u(t)|\le\|p\|\psi(\|x\|)\Phi_1+\|u\|\Phi_2,$$

and consequently,

$$\frac{(1-\Phi_2)\|u\|}{\|p\|\psi(\|x\|)\Phi_1}\le 1.$$

By (D_5), there exists K such that $\|u\|\neq K$. Consider the set:

$$U=\{u\in C([c,d],\mathbb{R}):\|u\|<K\}.$$

The operator $F:\bar{U}\to C([c,d],\mathbb{R})$ is completely continuous. There is no $u\in\partial U$ such that $u=\lambda Fu$ for some $\lambda\in(0,1)$, by the choice of U. Thus, F has a fixed point $u\in\bar{U}$, by Lemma 8, which is a solution of the Problem (2). The proof is completed. □

Corollary 1. *Setting $A,B>0$, we give two special cases of a function $\psi(u)$ as follows:*

(i) *If $\psi(u)=A|u|+B$ and if $A\|p\|\Phi_1+\Phi_2<1$, then there exists a constant $K>\frac{\|p\|\Phi_1}{1-(A\|p\|\Phi_1+\Phi_2)}$ satisfying (D_5).*

(ii) *If $\psi(u)=Au^2+B$ and if $\frac{4AB\|p\|^2\Phi_1^2}{(1-\Phi_2)^2}<1$ then from (D_5), there exists a constant K such that:*

$$K\in\left(\frac{1-\sqrt{1-\frac{4AB\|p\|^2\Phi_1^2}{(1-\Phi_2)^2}}}{2\frac{A\|p\|\Phi_1}{(1-\Phi_2)}},\frac{1+\sqrt{1-\frac{4AB\|p\|^2\Phi_1^2}{(1-\Phi_2)^2}}}{2\frac{A\|p\|\Phi_1}{(1-\Phi_2)}}\right).$$

5. Existence Results in Banach Space

In this section, the technique of measuring noncompactness is applied to construct an existence result concerning the Problem (2). First, some elementary concepts about the notion of the measure of noncompactness are recalled.

Definition 7 ([35]). *Assume that E is a Banach space and M_E indicates the set of all bounded subsets of E. The mapping $\Omega:M_E\longrightarrow[0,\infty)$ is defined via:*

$$\Omega(N)=\inf\left\{\varepsilon>0:N\subseteq\cup_{i=1}^{m}N_i,\ \mathrm{diam}(N_i)\le\varepsilon\right\},$$

which is called the Kuratowski measure of noncompactness.

The measure of noncompactness Ω has the following properties [35]:

(1) $\Omega(N)=0\Leftrightarrow\overline{N}$ is compact;
(2) $\Omega(N)=\Omega(\overline{N})$;

(3) $N_1 \subset N_2 \Rightarrow \Omega(N_1) \leq \Omega(N_2)$;
(4) $\Omega(N_1 + N_2) \leq \Omega(N_1) + \Omega(N_2)$;
(5) $\Omega(\lambda N) = |\lambda|\Omega(N), \lambda \in \mathbb{R}$; and
(6) $\Omega(\text{conv}N) = \Omega(N)$.

Lemma 9 ([36]). *Assume that $G \subseteq C([c,d], E)$ is a bounded and equicontinuous subset. Then, the function $z \longrightarrow \Omega(G(z))$ is continuous on $[c,d]$:*

$$\Omega_C(G) = \max_{z\in[c,d]} \Omega(G(z)),$$

and:

$$\Omega\left(\int_c^d u(s)ds\right) \leq \int_c^d \Omega(G(s))ds,$$

where $G(s) = \{u(s) : u \in G\}, s \in [c,d]$.

Theorem 4 (Mönch's fixed point theorem [37]). *Let the set V be a closed, bounded and convex subset in a Banach space Y such that $0 \in Y$ and let $T : V \longrightarrow V$ be a continuous mapping satisfying:*

$$\overline{V} = \overline{conv}T(\overline{V}), \text{ or } \overline{V} = T(\overline{V}) \cup \{0\} \Rightarrow \Omega(\overline{V}) = 0, \tag{13}$$

for all subset $\overline{V}$ of V. Then, T has a fixed point.

Definition 8 ([38]). *The function $h : [c,d] \times E \longrightarrow E$ satisfies the Carathéodory conditions if:*
(i) $h(z,u)$ is measurable with respect to z for all $u \in E$;
(ii) $h(z,u)$ is continuous with respect to $u \in E$ for $z \in [c,d]$.

Theorem 5. *Let (D_3) holds. In addition, assume that:*
(G_1) The Carathéodory conditions are satisfied by the function $h : [c,d] \times E \longrightarrow E$;
(G_2) There exist $\Omega_h \in C([c,d], \mathbb{R}_+)$ and $\varphi \in C(\mathbb{R}_+, \mathbb{R}_+)$ with φ being nondecreasing such that:

$$\|h(z,u)\| \leq \Omega_h(z)\varphi(\|u\|), \text{ for a.e. } z \in [c,d] \text{ and } u,v \in E.$$

(G_3) For each bounded set $G \subseteq E$ and for all $z \in [c,d]$, we have:

$$\Omega(h(z,G)) \leq \Omega_h(z)\Omega(G).$$

If:

$$\Omega_h^*\Phi_1 + \Phi_2 < 1, \tag{14}$$

where $\Omega_h^* = \sup_{z\in[c,d]} \Omega_h(z)$, then the boundary value Problem (2) has at least one solution on $[c,d]$:

Proof. Consider the operator $F : C([c,d], E) \longrightarrow C([c,d], E)$ defined by (9). Define:

$$B_r = \left\{u \in C([c,d], E) : \|u\| \leq r\right\},$$

where:

$$r \geq \frac{\Omega_h^*\varphi(r)\Phi_1}{1 - \Phi_2}.$$

Step 1. *F maps B_r into itself.*

For all $u \in B_r$ and $z \in [c, d]$, we obtain:

$$\begin{aligned} \|(Fu)(z)\| \leq & \frac{1}{\sigma^\delta \Gamma(\delta)} \int_c^z (z-c)^{\delta-1} e^{\frac{\sigma-1}{\sigma}(z-s)} \|h(s, u(s))\| ds \\ & + \frac{(z-c)^{\gamma-1}}{|\Delta| \Gamma(\gamma)} \left[\frac{1}{\sigma^\delta \Gamma(\delta)} \sum_{j=1}^m \theta_j \int_c^{\xi_j} e^{\frac{\sigma-1}{\sigma}(\xi_j - s)} (\xi_j - s)^{\delta-1} \|h(s, u(s))\| ds \right. \\ & + \frac{1}{\sigma^\delta \Gamma(\delta)} \int_c^d e^{\frac{\sigma-1}{\sigma}(d-s)} (d-s)^{\delta-1} \|h(s, u(s))\| ds \\ & \left. + \frac{k}{\sigma} \sum_{j=1}^m \theta_j \int_c^{\xi_j} \|u(s)\| e^{\frac{\sigma-1}{\sigma}(\xi_j - s)} ds + \frac{k}{\sigma} \int_c^d \|u(s)\| e^{\frac{\sigma-1}{\sigma}(d-s)} ds \right] e^{\frac{\sigma-1}{\sigma}(z-c)} \\ & + \frac{k}{\sigma} \int_c^z \|u(s)\| e^{\frac{\sigma-1}{\sigma}(z-s)} ds \\ \leq & \frac{\Omega_h^* \varphi(r)}{\sigma^\delta \Gamma(\delta+1)} (d-c)^\delta \\ & + \frac{\Omega_h^* \varphi(r)(d-c)^{\gamma-1}}{|\Delta| \Gamma(\gamma)} \left[\frac{1}{\sigma^\delta \Gamma(\delta+1)} \sum_{j=1}^m \theta_j (\xi_j - c)^\delta + \frac{1}{\sigma^\delta \Gamma(\delta+1)} (d-c)^\delta \right] \\ & + \frac{(d-c)^{\gamma-1} r}{|\Delta| \Gamma(\gamma)} \frac{k}{\sigma} \left[\sum_{j=1}^m \theta_j (\xi_j - c) + (d-c) \right] + \frac{k}{\sigma} (d-c) r \\ = & \Omega_h^* \varphi(r) \Phi_1 + r \Phi_2 \leq r. \end{aligned}$$

Hence, the ball B_r is transformed into itself.

Step 2. *The operator F is continuous.*

Let $\{u_n\} \in B_r$ such that $u_n \longrightarrow u$ as $n \longrightarrow \infty$. We indicate that $\|Fu_n - Fu\| \to 0$ as $n \to \infty$. Since h satisfies the Carathéodory conditions, we conclude that $h(s, u_n(s)) \to h(s, u(s))$, as $n \to \infty$. Hence, from (G_2) and the Lebesgue dominated convergence theorem, we have $\|Fu_n - Fu\| \to 0$ as $n \to \infty$, which implies that F is continuous on B_r.

Step 3. *The operator F is equicontinuous (with respect to z).*

Let $z_1, z_2 \in [c, d]$ with $z_1 < z_2$ and $u \in B_r$. Then, we have:

$$\begin{aligned} & \|(Fu)(z_2) - (Fu)(z_1)\| \\ \leq & \frac{1}{\sigma^\delta \Gamma(\delta)} \int_c^{z_2} [(z_2 - s)^{\delta-1} - (z_1 - s)^{\delta-1}] \|h(s, u(s))\| ds \\ & + \frac{1}{\sigma^\delta \Gamma(\delta)} \int_{z_1}^{z_2} (z_2 - s)^{\delta-1} \|h(s, u(s))\| ds \\ & + \frac{(z_2-c)^{\gamma-1} - (z_1-c)^{\gamma-1}}{|\Delta| \Gamma(\gamma)} \left[\frac{1}{\sigma^\delta \Gamma(\delta)} \sum_{j=1}^m \theta_j \int_c^{\xi_j} e^{\frac{\sigma-1}{\sigma}(\xi_j - s)} (\xi_j - s)^{\delta-1} \|h(s, u(s))\| ds \right. \\ & \left. + \frac{1}{\sigma^\delta \Gamma(\delta)} \int_c^d e^{\frac{\sigma-1}{\sigma}(d-s)} (d-s)^{\delta-1} \|h(s, u(s))\| ds \right] \\ & + \frac{(z_2-c)^{\gamma-1} - (z_1-c)^{\gamma-1}}{|\Delta| \Gamma(\gamma)} \frac{k}{\sigma} \left[\sum_{j=1}^m \theta_j \int_c^{\xi_j} \|u(s)\| e^{\frac{\sigma-1}{\sigma}(\xi_j - s)} ds \right. \\ & \left. + \int_c^d \|u(s)\| e^{\frac{\sigma-1}{\sigma}(\xi_j - s)} ds \right] \end{aligned}$$

$$
\begin{aligned}
& +\frac{k}{\sigma}\left[\int_c^{z_1}\|u(s)\|\left[e^{\frac{\sigma-1}{\sigma}(z_1-s)}-e^{\frac{\sigma-1}{\sigma}(z_2-s)}\right]ds+\int_{z_1}^{z_2}\|u(s)\|\left[e^{\frac{\sigma-1}{\sigma}(z_2-s)}\right]ds\right] \\
\leq\; & \frac{\Omega_h^*\varphi(r)}{\sigma^\delta\Gamma(\delta+1)}\left[|(z_2-c)^\delta-(z_1-c)^\delta|+2(z_2-z_1)^\delta\right] \\
& +\frac{(z_2-c)^{\gamma-1}-(z_1-c)^{\gamma-1}}{|\Delta|\Gamma(\gamma)}\frac{\Omega_h^*\varphi(r)}{\sigma^\delta\Gamma(\delta+1)}\left[\sum_{j=1}^m\theta_j(\xi_j-c)^\delta+(d-c)^\delta\right] \\
& +\frac{(z_2-c)^{\gamma-1}-(z_1-c)^{\gamma-1}}{|\Delta|\Gamma(\gamma)}\frac{kr}{\sigma}\left[\sum_{j=1}^m\theta_j(\xi_j-c)+(d-c)\right] \\
& +r\frac{k}{\sigma}\left[\int_c^{z_1}\left(e^{\frac{\sigma-1}{\sigma}(z_1-s)}-e^{\frac{\sigma-1}{\sigma}(z_2-s)}\right)ds+\int_{z_1}^{z_2}\left(e^{\frac{\sigma-1}{\sigma}(z_2-s)}\right)ds\right],
\end{aligned}
$$

which tends towards zero as $z_2 \longrightarrow z_1$, independently of $u \in B_r$. Hence, F is equicontinuous.

Step 4. *The Condition (13) of Theorem 4 is satisfied.*

Let $V \subseteq \overline{conv}(F(V)\cup\{0\}$ be a bounded and equicontinuous subset. Hence, the function $T(z)=\Omega(V(z))$ is continuous on $[c,d]$. Now, in view of Lemma 9 and (G_3), we have:

$$
\begin{aligned}
T(z) \;=\; & K(V(z)) \leq \Omega(\overline{conv}(F(V)\cup\{0\})) \leq \Omega(F(V)(z)) \\
\leq\; & \Omega\left\{\frac{1}{\sigma^\delta\Gamma(\delta)}\int_c^z e^{\frac{\sigma-1}{\sigma}(z-s)}(z-s)^{\delta-1}h(s,u(s))ds : u\in V\right\} \\
& +\frac{(z-c)^{\gamma-1}}{\Delta\Gamma(\gamma)}\left[\Omega\left\{\frac{1}{\sigma^\delta\Gamma(\delta)}\sum_{j=1}^m\theta_j\int_c^{\xi_j}e^{\frac{\sigma-1}{\sigma}(\xi_j-s)}(\xi_j-s)^{\delta-1}h(s,u(s))ds : u\in V\right\}\right. \\
& +\Omega\left\{\frac{1}{\sigma^\delta\Gamma(\delta)}\int_c^d e^{\frac{\sigma-1}{\sigma}(d-s)}(d-s)^{\delta-1}h(s,u(s))ds : u\in V\right\} \\
& +\Omega\left\{\frac{k}{\sigma}\sum_{j=1}^m\theta_j\int_c^{\xi_j}u(s)e^{\frac{\sigma-1}{\sigma}(\xi_j-s)}ds : u\in V\right\} \\
& \left.+\Omega\left\{\frac{k}{\sigma}\int_c^d u(s)e^{\frac{\sigma-1}{\sigma}(d-s)}ds : u\in V\right\}\right]e^{\frac{\sigma-1}{\sigma}(z-c)} \\
& +\Omega\left\{\frac{k}{\sigma}\int_c^z u(s)e^{\frac{\sigma-1}{\sigma}(z-s)}ds : u\in V\right\} \\
\leq\; & \frac{1}{\sigma^\delta\Gamma(\delta)}\int_c^z e^{\frac{\sigma-1}{\sigma}(z-s)}(z-s)^{\delta-1}\Omega(h(s,V(s)))ds \\
& +\frac{(z-c)^{\gamma-1}}{\Delta\Gamma(\gamma)}\left[\frac{1}{\sigma^\delta\Gamma(\delta)}\sum_{j=1}^m\theta_j\int_c^{\xi_j}e^{\frac{\sigma-1}{\sigma}(\xi_j-s)}(\xi_j-s)^{\delta-1}\Omega(h(s,V(s)))ds\right. \\
& +\frac{1}{\sigma^\delta\Gamma(\delta)}\int_c^d e^{\frac{\sigma-1}{\sigma}(d-s)}(d-s)^{\delta-1}\Omega(h(s,u(s)))ds \\
& \left.+\frac{k}{\sigma}\sum_{j=1}^m\theta_j\int_c^{\xi_j}\Omega(V(s))ds+\frac{k}{\sigma}\int_c^d\Omega(V(s))ds\right]+\frac{k}{\sigma}\int_c^z\Omega(V(s))ds \\
\leq\; & \|T\|\left\{\frac{(d-c)^\delta\Omega_h^*}{\sigma^\delta\Gamma(\delta+1)}+\frac{(d-c)^{\gamma-1}}{|\Delta|\Gamma(\gamma)}\frac{\Omega_h^*}{\sigma^\delta\Gamma(\delta+1)}\left[\sum_{j=1}^m\theta_j(\xi_j-c)^\delta+(d-c)^\delta\right]\right. \\
& \left.+\frac{(d-c)^{\gamma-1}}{|\Delta|\Gamma(\gamma)}\frac{k}{\sigma}\left[\sum_{j=1}^m\theta_j(\xi_j-c)+(d-c)\right]+\frac{k}{\sigma}(d-c)\right\}.
\end{aligned}
$$

This give that:

$$\|T\|_\infty \leq (\Omega_h^*\Phi_1+\Phi_2)\|T\|_\infty.$$

Due to (14), we conclude that $\|T\|_\infty = 0$. Consequently, for all $z \in [c, d]$, we have $T(z) = 0$ which implies that $\Omega(V(z)) = 0$. Thus, $V(z)$ is relatively compact in E and according to the Arzelá–Ascoli theorem, V is relatively compact in B_r. Now, by Theorem 13, F has a fixed point on B_r which is a solution of the Problem (2). This completes the proof. □

6. Illustrative Examples

Example 1. *Consider the problem:*

$$\begin{cases} \left(D_{\frac{1}{8}}^{\frac{3}{2},\frac{2}{3},\frac{4}{5}} + \frac{1}{12}D_{\frac{1}{8}}^{\frac{1}{2},\frac{2}{3},\frac{4}{5}}\right)u(z) = h(z, u(z)), \qquad z \in \left[\frac{1}{8}, \frac{11}{8}\right], \\ u\left(\frac{1}{8}\right) = 0, \quad u\left(\frac{11}{8}\right) = \frac{1}{11}u\left(\frac{3}{8}\right) + \frac{2}{21}u\left(\frac{5}{8}\right) + \frac{3}{31}u\left(\frac{7}{8}\right) + \frac{4}{41}u\left(\frac{9}{8}\right). \end{cases} \tag{15}$$

Here $\delta = 3/2$, $\eta = 2/3$, $\sigma = 4/5$, $k = 1/12$, $c = 1/8$, $d = 11/8$, $m = 4$, $\theta_1 = 1/11$, $\theta_2 = 2/21$, $\theta_3 = 3/31$, $\theta_4 = 4/41$, $\xi_1 = 3/8$, $\xi_2 = 5/8$, $\xi_3 = 7/8$ and $\xi_4 = 9/8$. We can then find that $\gamma = 11/6$, $\Delta \approx 0.7455888545$, $\Phi_1 \approx 4.363182248$ and $\Phi_2 \approx 0.3968238407$.

(i) Let the nonlinear function $h(z, u)$ be given by

$$h(z, u) = \frac{e^{-\sin^2 \pi t}}{16}\left(\frac{u^2 + 2|u|}{1 + |u|}\right) + \frac{1}{3}. \tag{16}$$

We can then find that $h(z, u)$ satisfies the condition (D_1) $|h(z, u) - h(z, v)| \le (1/8)|u - v|$ in Theorem 1 by setting $L = 1/8$. Therefore, we have $L\Phi_1 + \Phi_2 \approx 0.9422216217 < 1$. Hence, (12) holds and by applying Theorem 1, the Problem (15) with h given by (16) has a unique solution $u(z)$ on $[1/8, 11/8]$ such that $\|u\| \le r$, where $r \ge ((1/3)\Phi_1)/(1 - L\Phi_1 + \Phi_2) \approx 25.17194365$.

(ii) We now consider the nonlinear function $h(z, u)$ as

$$h(z, u) = \frac{1}{2}\left(\frac{|u|}{1 + |u|}\cos^2 t\right) + \frac{1}{4}. \tag{17}$$

It is obvious that $h(z, u)$ satisfies the Lipchitz condition with a constant $L = 1/2$. In addition, we obtain $|h(z, u)| \le (1/2)\cos^2 t + (1/4) := \phi(z)$. Since $\Phi_2 \approx 0.3968238407 < 1$, then (D_3) is true. The conclusion of Theorem 2 can be applied and thus the Problem (15) with h given by (17) has at least one solution on $[1/8, 11/8]$. Now, we remark that the uniqueness result cannot be obtained in this situation since $L\Phi_1 + \Phi_2 \approx 2.578414965 > 1$.

(iii) If $h(z, u)$ is defined by

$$h(z, u) = \frac{1}{5}\left(\frac{8}{8t + 15}\right)\left(\frac{|u|^{33}}{u^{32} + 1} + \frac{5}{11}\right), \tag{18}$$

then setting $p(t) = (8/(8t + 15))$ and $\psi(|u|) = (1/5)|u| + (1/11)$, we obtain $|h(z, u)| \le p(t)\psi(|u|)$. Choosing $\|p\| = 1/2$, $A = 1/5$ and $B = 1/11$, we obtain $A\|p\|\Phi_1 + \Phi_2 \approx 0.8331420655 < 1$. Therefore, by Corollary 1, the Problem (15) with (18) has at least one solution on $[1/8, 11/8]$.

(iv) If the term $|u|^{33}$ of $h(z, u)$ in (18) is replaced by u^{34} as

$$h(z, u) = \frac{1}{5}\left(\frac{8}{8t + 15}\right)\left(\frac{u^{34}}{u^{32} + 1} + \frac{5}{11}\right). \tag{19}$$

We then have:

$$|h(z, u)| \le \left(\frac{8}{8t + 15}\right)\left(\frac{1}{5}u^2 + \frac{1}{11}\right).$$

Consequently setting constants as in (iii), we obtain $\frac{4AB\|p\|^2\Phi_1^2}{(1-\Phi_2)^2} \approx 0.9513836480 < 1$. The benefit of Corollary 1 implies that Problem (15) with h given by (19) has at least one solution on $[1/8, 11/8]$.

Example 2. *Let:*

$$E = c_0 = \{u = (u_1, u_2, \dots, u_n, \dots) : u_n \to 0\}$$

be the Banach space of real sequences converging to zero, endowed with the norm:

$$\|u\|_\infty = \sup_{n\geq 1} |u_n|.$$

Consider the problem given in Example 15.

Let $h : [1/8, 11/8] \times c_0 \longrightarrow c_0$ be defined by

$$h(z,u) = \left\{ \frac{1}{z+10}\left(\frac{1}{3^n} + \ln\left(1 + |u_n|\right)\right)\right\}_{n\geq 1}, \; u = \{u_n\} \in c_0. \tag{20}$$

Obviously, the hypothesis (G_1) holds true. Furthermore, for all $z \in [1/8, 11/8]$, we obtain:

$$\|h(u,z)\|_\infty \leq \left\|\frac{1}{z+1}\left(\frac{1}{3^n} + |u_n|\right)\right\|_\infty \leq \frac{1}{z+10}(\|u\| + 1) = \Omega_h(z)\varphi(\|u\|).$$

Hence, the assumption (G_2) is satisfied with $\Omega_h(z) = \frac{1}{z+10}$ and $\varphi(u) = 1 + u$. On the other hand, if $D \subseteq c_0$ be a bounded set, then:

$$\Omega(h(z,D)) \leq \Omega_h(z)\Omega(D).$$

We have $\Omega_h^* = \frac{8}{81}$ and from the given data, we obtain:

$$\Omega_h^*\Phi_1 + \Phi_2 \approx 0.8277554207 < 1.$$

Consequently, by Theorem 5, the Problem (15) with h given by (20) has at least one solution on $[1/8, 11/8]$.

7. Conclusions

In this paper, we presented the existence and uniqueness criteria for solutions of Hilfer generalized proportional fractional differential equations supplemented with nonlocal boundary conditions. First, the nonlinear boundary value problem at hand is converted into a fixed point problem by proving an auxiliary result concerning a linear variant of the given problem. We then studied two cases. The scalar case in which we proved the existence of a unique solution via Banach fixed point theorem and two existence results by using Krasnosel'skiĭ's fixed point theorem and Leray–Schauder nonlinear alternative. Then, in the Banach space case, we established an existence result based on Mönch's fixed point theorem and the technique of the measure of noncompactness. All results obtained for scalar and Banach space cases are well illustrated by numerical examples. We emphasize that, in this paper, we initiated the study of Hilfer generalized proportional fractional boundary value problems of order in $(1,2]$. Our results are new in the given configuration and enrich the literature on boundary value problems for Hilfer generalized proportional fractional differential equations. We plan to apply the methods of this paper in future studies to obtain similar results for different types of boundary conditions or different kinds of sequential fractional derivatives.

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