fractal and fractional

Article

Robust H_∞ Control of Fractional-Order Switched Systems with Order $0 < \alpha < 1$ and Uncertainty

Bingxin Li [1], **Xiangfei Zhao** [1], **Yaowei Liu** [1,2] **and Xin Zhao** [1,2,*]

[1] Institute of Robotics and Automatic Information System and the Tianjin Key Laboratory of Intelligent Robotics, Nankai University, Tianjin 300071, China; 1120210181@mail.nankai.edu.cn (B.L.); 1120170124@mail.nankai.edu.cn (X.Z.); liuyaowei@mail.nankai.edu.cn (Y.L.)
[2] Institute of Intelligence Technology and Robotic Systems, Shenzhen Research Institute of Nankai University, Shenzhen 518083, China
* Correspondence: zhaoxin@nankai.edu.cn

Abstract: In this paper, robust H_∞ control for fractional-order switched systems (FOSSs) with uncertainty is studied. Firstly, the fractional-order switching law for FOSSs is proposed. Then, H_∞ control for FOSSs is proven based on the switching law and linear matrix inequalities (LMIs). Moreover, H_∞ control for FOSSs with a state feedback controller is extended. Furthermore, the LMI-based condition of robust H_∞ control for FOSSs with uncertainty is proven. Furthermore, the condition of robust H_∞ control is proposed to design the state feedback controller. Finally, four simulation examples verified the effectiveness of the proposed methods.

Keywords: robust H_∞ control; fractional-order switched systems; fractional-order switching law; linear matrix inequalities (LMIs); state feedback controller

Citation: Li, B.; Zhao, X.; Liu, Y.; Zhao, X. Robust H_∞ Control of Fractional-Order Switched Systems with Order $0 < \alpha < 1$ and Uncertainty. *Fractal Fract.* **2022**, *6*, 164. https://doi.org/10.3390/fractalfract6030164

Academic Editors: Xuefeng Zhang, Driss Boutat, Dayan Liu and Norbert Herencsar

Received: 25 February 2022
Accepted: 15 March 2022
Published: 16 March 2022

1. Introduction

Switched systems, as a kind of hybrid system, are composed of multiple subsystems and switching rules [1]. They have attracted the interest of many researchers, not only because they are more complex than other control systems [2], but also because they are widely used in engineering and social sciences [3,4]. A large part of control systems, such as network-based systems [5,6], can be represented by switching systems. Stability analysis is the most fundamental for switched systems. Hence, a large number of results have been published in the field. Adaptive control has been studied for switched systems by using the average dwell time approach in [7,8]. The approach is usually used to judge the stability of switched systems [9]. Finding the Lyapunov function to guarantee the stability for all constituent subsystems is the other typical method for switched systems [10]. Based on the Lyapunov function method, nonlinear and linear switched systems with uncertainties were studied in [11–14]. Tracking control is an effective control technique [15] and was studied for switched systems in [16]. Moreover, the results of finite-time stability and sliding mode control for switched systems were reported in [17,18], respectively.

Fractional-order systems can describe physical phenomena in the real world and are widely considered by the academic community to be more accurate [19,20]. Since the necessary tools for simulation verification are lacking, the development of fractional-order theory is slow. In the past two decades, with the development of advanced computers, many conditions have been published about the stability and stabilization of fractional-order systems. From Figure 1, the stable area of order $0 < \alpha < 1$ is not convex. Hence, the stability analysis of order $0 < \alpha < 1$ is more difficult than order $1 < \alpha < 2$. In [21–23], the results based on linear matrix inequalities (LMI) were proposed to analyze stability for order $0 < \alpha < 1$. In [21], the direct introduction of complex variables brought difficulties to the solution process. Therefore, in [22,23], by using more real variables to replace the complex variables, the results could solve the difficulties of complex variables, but these results introduce more variables. To solve these problems, Reference [24] provided new LMI-based results to solve the stability and stabilization for order $0 < \alpha < 2$ by using two real variables. Although Atangana–Baleanu and Riemann–Liouville definitions of the fractional operator were used in some papers, such as [25], the Caputo definition of the fractional operator is commonly used in the field of control [24,26], since it has a well-understood physical sense and wide applications in engineering [27]. H_∞ control is a great method to counteract the effects of disturbances. The conditions of H_∞ control for fractional-order systems were published in [28,29]. State feedback, dynamic output, and robust H_∞ control for fractional-order systems were studied in [30–32], respectively.

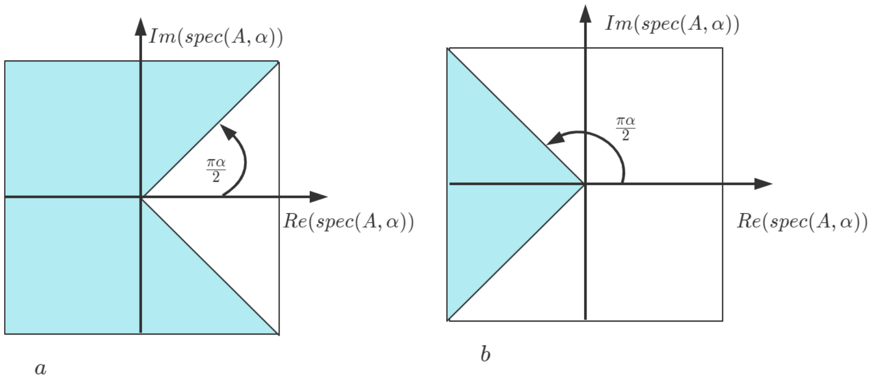

Figure 1. Stable region of fractional-order systems with: (**a**) order $0 < \alpha < 1$; (**b**) order $1 < \alpha < 2$.

Recently, many researchers have tried to introduce fractional calculus into switched systems, to more accurately describe the phenomenon and properties of switched systems. The description is indeed better than the effect of integer-order switched systems, because the physical characteristics of the actual system are more accurately described by the fractional order. Stability and stabilization for fractional switched systems (FOSSs) were studied in [33–35]. State-dependent control and finite-time stability for positive FOSSs were studied in [36–38]. Moreover, References [39–41] studied the stability and robust stabilization for nonlinear and uncertain FOSSs. The results of observer-based and guaranteed cost control for FOSSs were reported in [42,43]. Fault-tolerant control is widely used in the control field [44]. By using the Lyapunov method in [33], fault estimation was investigated for nonlinear fractional-order systems in [45]. Furthermore, decentralized control and state-dependent switching control of nonlinear FOSSs were reported in [46,47].

However, H_∞ control and robust H_∞ for FOSSs with uncertainty have not been reported yet. The main contributions of this paper are as follows:

(1) The fractional-order switching law is proven for FOSSs. From the stable region of order $\alpha \in (0,1)$ in Figure 1, if the fractional-order systems have positive characteristic roots, they may be stable. The characteristic roots in the right stable region were not considered in [45–47]. The fractional-order switching law proposed in this paper overcomes this shortcoming. Hence, it is less conservative;

(2) H_∞ control for FOSSs is proposed under the fractional-order switching law. Then, the controller for closed-loop FOSSs is designed. Furthermore, the conditions based on LMIs are proposed to solve the problem of robust H_∞ control for FOSSs with uncertainty.

The outline of this paper is as follows. The preliminaries and problem descriptions are presented in Section 2. In Section 3, the switching law is proven for FOSSs. In Sections 4 and 5, H_∞ control and robust H_∞ control for FOSSs are studied under the switching law, respectively. In Section 6, four numerical examples are shown to prove the effectiveness of the conditions in the paper.

Notations: In this paper, X^T is the transpose of X. $A > 0$ denotes positive definite. $\text{sym}(X) = X + X^T$. $\text{spec}(X)$ denotes the spectrum (set of all eigenvalues) of X. $\min\{N\}$

and $\arg\min\{N\}$ are the minimum value and minimum index of the set N, and $\|G(s)\|_\infty$ denotes the H_∞-norm of $G(s)$. Let $\begin{bmatrix} P & Q \\ * & P \end{bmatrix} = \begin{bmatrix} P & Q \\ Q^T & P \end{bmatrix}$.

2. Preliminaries and Problem Descriptions

2.1. Preliminaries

The Caputo definition [19] is shown as follows:

$$D^\alpha f(t) = \tfrac{1}{\Gamma(n-\alpha)} \int_0^t \tfrac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau \tag{1}$$

where $n-1 < \alpha < n$ and $\Gamma(.)$ is the Gamma function.

Remark 1. *In general, the Caputo derivative has a clear physical meaning at the initial value. Hence, the Caputo derivative is widely used in the control field.*

Consider the following fractional-order system:

$$\begin{aligned} D^\alpha x(t) &= Ax(t) + B_w w(t) \\ z(t) &= Cx(t) + Dw(t) \end{aligned} \tag{2}$$

where $0 < \alpha < 1$. $x(t)$, $w(t)$, and $y(t)$ denote the state, exogenous input, and output vectors. A, B_w, C, and D are appropriate dimension matrices. $G(s) = C(s^\alpha I - A)^{-1} B_w + D$ denotes the transfer matrix.

Then, some lemmas and the H_∞-norm definition is introduced in the following part.

Definition 1 ([29]). *The H_∞-norm is defined as:*

$$\|G(s)\|_\infty = \sup_{Re(s) \geq 0} \sigma(G(s))$$

Lemma 1 ([32]). *For given $\gamma > 0$, System (2) is asymptotically stable and $\|G(s)\|_\infty < \gamma$ iff there exist two matrices $X, Y \in \mathbf{R}^{n\times n}$ such that:*

$$\begin{bmatrix} X & Y \\ -Y & X \end{bmatrix} > 0 \tag{3}$$

$$\begin{bmatrix} sym(aAX + bAY) & (aX - bY)C^T & B_w \\ * & -\gamma I & D \\ * & * & -\gamma I \end{bmatrix} < 0 \tag{4}$$

where $a = \sin(\alpha\frac{\pi}{2})$ and $b = \cos(\alpha\frac{\pi}{2})$.

Remark 2. *Lemma 1 is different from the results in [28,29]; it avoids the trouble of solving the complex matrix. Lemma 1 can be easier to solve by using the LMI toolbox. When $\alpha = 1$, Lemma 1 is equivalent to the results of the integer order. For brevity, a and b denote $\sin(\alpha\frac{\pi}{2})$ and $\cos(\alpha\frac{\pi}{2})$ in this paper.*

Lemma 2 ([19]). *For System (2), if $\alpha < 2$, β is an arbitrary real number, ρ is such that $\pi\alpha/2 < \rho < \min\{\pi, \pi\alpha\}$, and C is a real constant, then:*

$$\|E_{\alpha,\beta}(z)\| \leq \frac{C}{1 + \|z\|}, (\rho \leq \|\arg(z)\| \leq \pi), \|z\| \geq 0.$$

Lemma 3 ([48])**.** *(Schur complement) Matrices W_1, W_2 and W_3 satisfy $W_1 = W_1^T$, $W_3 > 0$.*

$$W_1 + W_2 W_3^{-1} W_2^T < 0$$

if and only if:

$$\begin{bmatrix} W_1 & W_2 \\ W_2^T & -W_3 \end{bmatrix} < 0$$

Lemma 4 ([49])**.** *For matrices H, E, $F^T(t)F(t) \leq I$, and one constant $\epsilon > 0$, then:*

$$HF(t)E + E^T F^T(t) H^T \leq \epsilon H H^T + \epsilon^{-1} E^T E$$

2.2. Problem Descriptions

Consider the following general FOSS:

$$\begin{aligned} D^{\alpha} x(t) &= A_{\sigma} x(t) + B_{\sigma} u_{\sigma}(t) + B_{\sigma w} w(t) \\ z(t) &= C_{\sigma} x(t) \end{aligned} \tag{5}$$

where $0 < \alpha < 1, \sigma \in J = \{1, 2, \ldots, N\}$ is the piecewise constant switching signal. $\sigma = i$ denotes that the i-th subsystem is activated. A_i, B_i, B_{iw}, and C_i $(i \in J)$ are real matrices. The transfer matrix between $w(t)$ and $z(t)$ is $G(s) = C_{\sigma}(s^{\alpha} I - A_{\sigma})^{-1} B_{\sigma w}$.

Then, the necessary lemma is introduced as follows.

Lemma 5 ([41])**.** *System (5) with $u_{\sigma}(t) = 0$ is asymptotically stable iff there exist $X, Y \in \boldsymbol{R}^{n \times n}$ such that:*

$$\begin{bmatrix} X & Y \\ -Y & X \end{bmatrix} > 0$$

$$a A_i X + b A_i Y + a X A_i^T - b Y A_i^T < 0$$

where $i \in J = \{1, 2, \ldots, N\}$.

Our primary aim in this paper was to design a switching law to ensure FOSSs' stability. In addition, H_{∞} and robust H_{∞} performance for FOSSs should be guaranteed.

3. Formal Description of the Switching Law

In this section, the switching signal should be designed. Therefore, the average matrix can be expressed as:

$$\bar{A} = \sum_{i=1}^{N} \lambda_i A_i, i \in J = \{1, 2, \ldots, N\}$$

We can obtain the equation below:

$$\bar{A}(aX + bY) + (aX - bY)\bar{A}^T = -I_n$$

where X, Y satisfy (3). According to Lemma 5, we denote:

$$P_i = A_i(aX + bY) + (aX - bY)A_i^T, \forall i \in J \tag{6}$$

Assume $r_i \in (0, 1), i \in J$, and $x(t_0) = x_0$, and set:

$$\sigma(t_0) = \arg\min\{x_0^T P_1 x_0, \ldots, x_0^T P_N x_0\} \tag{7}$$

Next, let t_1 be:

$$t_1 = \{t > t_0 : x^T(t) P_{\sigma(t_0)} x(t) > -r_{\sigma(t_0)} x^T(t) x(t)\}$$

If the set is empty, $t_1 = \infty$. If the set is not empty, the switching index can be defined as:

$$\sigma(t_1) = \arg\min\{x^T(t_1)P_i x(t_1)\}$$

Therefore, we obtain:

$$\begin{aligned} t_{k+1} &= \{t > t_k : x^T(t)P_{\sigma(t_k)}x(t) > -r_{\sigma(t_k)}x^T(t)x(t)\} \\ \sigma(t_{k+1}) &= \arg\min\{x^T(t_{k+1})P_i x(t_{k+1})\} \quad k, i \in J = \{1, 2, \dots, N\} \end{aligned} \tag{8}$$

Theorem 1. *Under the fractional-order switching law (8), System (5) is asymptotically stable and well-posed.*

Proof. Firstly, let $i = \sigma_{t_k+}$. Based on the switching signal, we can obtain:

$$\begin{gathered} (1)x^T(t_k)P_i x(t_k) = \min_{j\in J}\{x^T(t_k)P_j x(t_k)\} \\ (2)x^T(t_k+1)P_i x(t_k+1) \geq -r_i x^T(t_k+1)x(t_k+1) \end{gathered}$$

According to $\sum_{j\in J}\lambda_j P_j = -I_n$, $\sum_{j\in J}\lambda_j = 1$, and (1), we obtain:

$$x^T(t_k)P_i x(t_k) \leq -x^T(t_k)x(t_k) \tag{9}$$

Let $\mu > 1$, $x_k = x(t_k)$, and $x_{k+1} = x(t_{k+1})$. Then,

$$\mu\|x_{k+1}\| \geq \|x(t)\| \tag{10}$$

where $t \in [t_k, t_{k+1}]$.

From the above condition, we can define:

$$f(t) = x^T(t)(P_i + I_n)x(t) \tag{11}$$

Based on (9) and (2), we obtain:

$$f(t_k) \leq 0, f(t_{k+1}) \geq (1 - r_i)x_{k+1}^T x_{k+1} \tag{12}$$

Hence, we have:

$$\dot{f}(t) = x^T(t)(A_i^T(P_i + I_n) + (P_i + I_n)^T A_i)x(t) \tag{13}$$

Denote $\rho_i = \|A_i^T(P_i + I_n) + (P_i + I_n)^T A_i\|$; from (10) and Lemma 3, we have:

$$|\dot{f}(t)| \leq \mu^2\rho_i x_{k+1}^T x_{k+1}$$

From (12), we can obtain:

$$\mu^2\rho_i(t_{k+1} - t_k) \geq (1 - r_i) \tag{14}$$

that is to say,

$$(1 - r_i)/(\mu^2\rho_i) \leq t_{k+1} - t_k \tag{15}$$

Assume (10) does not hold. Then, $t^* \in [t_k, t_{k+1}]$ can be found to satisfy:

$$\mu\|x_{k+1}\| < \|x(t^*)\| \tag{16}$$

From the Mittag–Leffler function [19], we can obtain:

$$x(t^*) = E_\alpha(A_i(t^* - t_{k+1})^\alpha)x_{k+1}$$

According to (16) and Lemma 2, we can obtain $C > \mu$ and:

$$\mu < \|E_\alpha(A_i(t^* - t_{k+1})^\alpha)\| \leq \frac{C}{1 + \|A_i(t^* - t_{k+1})^\alpha\|}$$

Then,

$$D^\alpha \frac{C-\mu}{\mu\|A_i\|} < t_{k+1} - t^* \leq t_{k+1} - t_k \geq$$

In summary,

$$\phi = \sup_{\mu>1} \min_{i\in J}\left(\frac{(1-r_i}{\mu^2\rho_i}, D^\alpha \frac{C-\mu}{\mu\|A_i\|}\right) \leq t_{k+1} - t_k$$

Therefore, $\phi > 0$, and the switching signal is well-defined.
From (6), define $V(x) = x^T(aX + bY)x$. Then, we obtain:

$$\dot{V} = x^T(t)P_\sigma x(t) \leq -r_\sigma x^T(t)x(t) \leq -rx^T(t)x(t)$$

where $r = \min\{r_1, \ldots, r_N\}$. From Lemma 5, we can prove that System (5) is asymptotically stable. Hence, we complete the proof. □

4. *H*$_\infty$ Control

In this section, H_∞ control for FOSSs is studied. According to Theorem 1 and Lemma 1 proposed in the paper, the following theorems are derived.

Theorem 2. *Given any constant* $\gamma > 0$, *N, matrices X, Y, and scalars* $\lambda_i \geq 0(i \in J = \{1, 2, \ldots, N\})$, $\sum_{i=1}^N \lambda_i = 1$, *System (5) is asymptotically stable, and* $\|G(s)\|_\infty < \gamma$, *if:*

$$\begin{bmatrix} X & Y \\ -Y & X \end{bmatrix} > 0 \tag{17}$$

$$\begin{bmatrix} sym(a\bar{A}X + bX\bar{A}^T) & (aX - bY)\bar{C}^T & \bar{B}_w \\ * & -\gamma I & 0 \\ * & * & -\gamma I \end{bmatrix} < 0 \tag{18}$$

where:

$$\bar{A} = \sum_{i=1}^N \lambda_i A_i, \bar{B}_w = \sum_{i=1}^N \sqrt{\lambda_i} B_{iw}, \bar{C} = \sum_{i=1}^N \sqrt{\lambda_i} C_i$$
$$i \in J = \{1, 2, \ldots, N\}.$$

Then, the switching law is:

$$\begin{aligned} \sigma(t) \quad = \quad & \arg\min_{i\in J}\{x^T(A_i(aX + bY) \\ + \quad & (aX - bY)A_i^T + \gamma^{-1}B_{iw}B_{iw}^T \\ + \quad & \gamma^{-1}(aX - bY)C_i^T C_i(aX + bY))x\} \end{aligned} \tag{19}$$

Proof. From Theorem 1, (19) can be proven easily. Then, we prove that System (5) is asymptotically stable, and $\|G(s)\|_\infty < \gamma$.

Suppose that $\{(t_k, r_k)|r_k \in J, k = 1, 2, \ldots, N\}$ is the switching sequence in $[0, T)$. Then, from Lemma 3, (18) is equivalent to the following equation:

$$\begin{bmatrix} \mathrm{sym}(a\bar{A}X + bX\bar{A}^T) & (aX - bY)\bar{C}^T \\ * & -\gamma I \end{bmatrix} + \frac{1}{\gamma}\begin{bmatrix} \bar{B}_w \\ 0 \end{bmatrix}\begin{bmatrix} \bar{B}_w^T & 0 \end{bmatrix} < 0 \tag{20}$$

Owing to the existing matrices X and Y, Inequality (20) is equivalent to Inequality (21), when Inequality (20) is multiplied by γ.

$$\gamma\begin{bmatrix} \mathrm{sym}(a\bar{A}X + bX\bar{A}^T) & (aX - bY)\bar{C}^T \\ * & 0 \end{bmatrix} + \begin{bmatrix} \bar{B}_w\bar{B}_w^T & 0 \\ 0 & -\gamma^2 I \end{bmatrix} < 0 \tag{21}$$

Let $P = X\gamma$ and $Q = Y\gamma$, to make the proof simple. From (21), we have:

$$\begin{bmatrix} \bar{A}^T & \bar{C}^T \\ I_n & 0 \end{bmatrix}^T \left(\begin{bmatrix} 0 & aP + bQ \\ aP - bQ & 0 \end{bmatrix} \otimes I_m\right) \begin{bmatrix} \bar{A}^T & \bar{C}^T \\ I_n & 0 \end{bmatrix} + \begin{bmatrix} \bar{B}_w\bar{B}_w^T & 0 \\ 0 & -\gamma^2 I \end{bmatrix} < 0 \tag{22}$$

where $a = \sin(\alpha\frac{\pi}{2})$ and $b = \cos(\alpha\frac{\pi}{2})$.

According to Lemmas 1 and 2 in [28], when $0 < \alpha < 1$, let $\|G(s)\|_\infty < \gamma$, and consider the curve:

$$\Gamma_{11} = (\begin{bmatrix} 0 & aP + bQ \\ aP - bQ & 0 \end{bmatrix}, 0).$$

Since $\Gamma_{11}(s) \subset \{s^\alpha : Re(s) \geq 0\}$, we can obtain:

$$\bar{H}(\lambda) = (\lambda I - \bar{A}^T)^{-1}\bar{C}^T$$

and we have:

$$\begin{bmatrix} \bar{H}(\lambda) \\ I_n \end{bmatrix}^T \begin{bmatrix} \bar{B}_w\bar{B}_w^T & 0 \\ 0 & -\gamma^2 I \end{bmatrix} \begin{bmatrix} \bar{H}(\lambda) \\ I_n \end{bmatrix} < 0 \tag{23}$$

Unfolding Equality (23), we can obtain:

$$\bar{H}^T(\lambda)\bar{B}_w\bar{B}_w^T\bar{H}(\lambda) - \gamma^2 I < 0 \tag{24}$$

From Equality (24):

$$[\bar{C}(\lambda I - \bar{A})^{-1}\bar{B}_w][\bar{C}(\lambda I - \bar{A})^{-1}\bar{B}_w]^T - \gamma^2 I < 0 \tag{25}$$

According to Equality (25):

$$\begin{aligned} \gamma > \|G(s)\|_{H_\infty} &= \sup_{Re(s)\geq 0} [\bar{C}(s^\alpha I - \bar{A})^{-1}\bar{B}_w] \\ &\geq \sup_{\lambda \in \Gamma_{11}} [\bar{C}(s^\alpha I - \bar{A})^{-1}\bar{B}_w] \end{aligned} \tag{26}$$

The proof is complete. □

Based on Theorem 2, H_∞ control of FOSSs with the state feedback controller is given as follows.

Theorem 3. *Given any constant* $\gamma > 0$*,* N*, matrices* X*,* Y*,* Z*, and scalars* $\lambda_i \geq 0 (i \in J = \{1, 2, \ldots, N\})$*,* $\sum_{i=1}^{N} \lambda_i = 1$*, System (5) with state feedback* $u_\sigma(t) = Kx(t)$ *is asymptotically stabilizable, and* $\|G(s)\|_\infty < \gamma$*, if:*

$$\begin{bmatrix} X & Y \\ -Y & X \end{bmatrix} > 0 \tag{27}$$

$$\begin{bmatrix} sym(a\bar{A}X + bX\bar{A}^T + \bar{B}Z) & (aX - bY)\bar{C}^T & \bar{B}_w \\ * & -\gamma I & 0 \\ * & * & -\gamma I \end{bmatrix} < 0 \tag{28}$$

where:

$$\bar{A} = \sum_{i=1}^{N} \lambda_i A_i, \bar{B}_w = \sum_{i=1}^{N} \sqrt{\lambda_i} B_{iw}, \bar{C} = \sum_{i=1}^{N} \sqrt{\lambda_i} C_i,$$
$$i \in J = \{1, 2, \ldots, N\}.$$

Then, the gain matrix is:

$$K = Z(aX + bY)^{-1} \tag{29}$$

The switching law is:

$$\begin{aligned} \sigma(t) &= \arg\min_{i \in J} \{x^T((A_i + B_iK)(aX + bY) \\ &+ (aX - bY)(A_i + B_iK)^T \\ &+ \gamma^{-1}(aX - bY)C_i^T C_i(aX + bY) \\ &+ \gamma^{-1} B_{iw} B_{iw}^T)x\} \end{aligned} \tag{30}$$

Proof. Let $\hat{A} = \bar{A} + \bar{B}K$; the proof of Theorem 3 is directly derived from Theorem 2. □

5. Robust H_∞ Control

In this section, robust H_∞ control of FOSSs with uncertainty is studied.

Consider the following FOSS with uncertainty:

$$\begin{aligned} D^\alpha x(t) &= (A_\sigma + \Delta A)x(t) + (B_\sigma + \Delta B)u_\sigma(t) \\ &+ B_{\sigma w} w(t) \\ z(t) &= C_\sigma x(t) \end{aligned} \tag{31}$$

where $G(s) = C_\sigma(s^\alpha I - A_\sigma - \Delta A)^{-1} B_{\sigma w}$. ΔA and ΔB are the norm-bounded uncertainties, and:

$$\begin{bmatrix} \Delta A & \Delta B \end{bmatrix} = MF(t) \begin{bmatrix} N_1 & N_2 \end{bmatrix} \tag{32}$$

where M, N_1 and N_2 are constant matrices of appropriate dimensions. $F(t)$ satisfies $F^T(t)F(t) \leq I$.

According to Theorem 1, Theorem 2, and Lemma 5, the following theorem is derived.

Theorem 4. *Given any constant* $\gamma > 0$, N, *matrices* X, Y, *and scalars* $\lambda_i \geq 0 (i \in J = \{1, 2, \ldots, N\})$, $\sum_{i=1}^{N} \lambda_i = 1$, ϵ, *System (31) is quadratically stable, and* $\|G(s)\|_\infty < \gamma$, *if:*

$$\begin{bmatrix} X & Y \\ -Y & X \end{bmatrix} > 0 \tag{33}$$

$$\begin{bmatrix} \Pi_{11} & (aX - bY)\bar{C}^T & \bar{B}_w & N_1(aX + bY) \\ * & -\gamma I & 0 & 0 \\ * & * & -\gamma I & 0 \\ * & * & * & -\epsilon I \end{bmatrix} < 0 \tag{34}$$

where:

$$\Pi_{11} = sym(a\bar{A}X + bX\bar{A}^T) + \epsilon MM^T,$$

$$\bar{A} = \sum_{i=1}^{N} \lambda_i A_i, \bar{B}_w = \sum_{i=1}^{N} \sqrt{\lambda_i} B_{iw}, \bar{C} = \sum_{i=1}^{N} \sqrt{\lambda_i} C_i,$$

$$i \in J = \{1, 2, \ldots, N\}.$$

Then, the switching law is:

$$\begin{aligned} \sigma(t) &= \arg\min_{i \in J} \{x^T (A_i(aX + bY) \\ &+ (aX - bY)A_i^T + \gamma^{-1} B_{iw} B_{iw}^T \\ &+ \gamma^{-1}(aX - bY)C_i^T C_i(aX + bY))x\} \end{aligned} \tag{35}$$

Proof. Let $\bar{A} = A_\sigma + \Delta A$, and substitute $\bar{A}$ into Theorem 2. We can obtain:

$$\begin{bmatrix} \Phi_{11} & (aX - bY)\bar{C}^T & \bar{B}_w \\ * & -\gamma I & 0 \\ * & * & -\gamma I \end{bmatrix} < 0$$

where $\Phi_{11} = a(\bar{A} + \Delta A)X + bX(\bar{A} + \Delta A)^T + b(\bar{A} + \Delta A)Y - bY(\bar{A} + \Delta A)^T$.

Then, by applying Lemma 4, the above inequality can be expressed as (34). Theorem 4 can be easily proven. □

Based on Theorems 2 and 4, robust H_∞ control of FOSSs with the state feedback controller and uncertainty is given as follows.

Theorem 5. *Given any constant* $\gamma > 0$, N, *matrices* X, Y, Z, W, *and scalars* $\lambda_i \geq 0 (i \in J = \{1, 2, \ldots, N\})$, $\sum_{i=1}^{N} \lambda_i = 1$, ϵ, *System (31) with state feedback* $u_\sigma(t) = Kx(t)$ *is quadratically stabilizable, and* $\|G(s)\|_\infty < \gamma$, *if:*

$$\begin{bmatrix} X & Y \\ -Y & X \end{bmatrix} > 0 \tag{36}$$

$$\begin{bmatrix} \Pi_{11} & (aX - bY)\bar{C}^T & \bar{B}_w & N_1(aX + bY) + N_2 W \\ * & -\gamma I & 0 & 0 \\ * & * & -\gamma I & 0 \\ * & * & * & -\epsilon I \end{bmatrix} < 0 \tag{37}$$

where:

$$\Pi_{11} = sym(a\bar{A}X + b\bar{A}Y + \bar{B}Z) + \epsilon MM^T,$$

$$\bar{A} = \sum_{i=1}^{N} \lambda_i A_i, \bar{B}_w = \sum_{i=1}^{N} \sqrt{\lambda_i} B_{iw}, \bar{C} = \sum_{i=1}^{N} \sqrt{\lambda_i} C_i$$

$$i \in J = \{1, 2, \ldots, N\}.$$

Then, the gain matrix is:

$$K = Z(aX + bY)^{-1} \tag{38}$$

The switching law is:

$$\begin{aligned} \sigma(t) &= \arg\min_{i\in J}\{x^T((A_i + B_iK)(aX + bY) \\ &+ (aX - bY)(A_i + B_iK)^T \\ &+ \gamma^{-1}(aX - bY)C_i^TC_i(aX + bY) \\ &+ \gamma^{-1}B_{iw}B_{iw}^T)x\} \end{aligned} \tag{39}$$

Proof. Let $\hat{A} = (A_\sigma + \Delta A) + (B_\sigma + \Delta B)K$; substitute $\hat{A}$ into Theorem 2, and set $W = K(aX + bY)$. Theorem 5 is directly derived from Theorems 2 and 5. ☐

6. Examples

6.1. Example 1

Consider System (5) with order $\alpha = 0.5$, $N = 2$, and:

$$A_1 = \begin{bmatrix} -1.9 & 1 \\ 2 & 1 \end{bmatrix}, A_2 = \begin{bmatrix} -3 & -4.3 \\ -1 & -0.5 \end{bmatrix}, B_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$B_2 = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}, B_{1w} = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}, B_{2w} = \begin{bmatrix} 1 \\ 0.1 \end{bmatrix}$$

$$C_1 = \begin{bmatrix} -1 & 2 \end{bmatrix}, C_2 = \begin{bmatrix} 2 & 1 \end{bmatrix}$$

Set $\lambda_1 = 0.7, \lambda_2 = 0.3$ and the disturbance attenuation level $\gamma = 0.8$, according to Theorem 3; we can obtain:

$$X = \begin{bmatrix} 0.7717 & 0.0430 \\ 0.0430 & 0.0024 \end{bmatrix}, Y = \begin{bmatrix} 0 & -0.0024 \\ 0.0024 & 0 \end{bmatrix}$$

$$K = \begin{bmatrix} 2.5946 & -3.1100 \\ -2.5777 & -0.9836 \end{bmatrix}$$

Let:

$$\begin{aligned} P_i &= (A_i + B_iK)(aX + bY) \\ &+ (aX - bY)(A_i + B_iK)^T \\ &+ \gamma^{-1}(aX - bY)C_i^TC_i(aX + bY) \\ &+ \gamma^{-1}B_{iw}B_{iw}^T \end{aligned}$$

Then, obtain the switching law:

$$\sigma(t) = i = \begin{cases} 1, & x^T(P_1 - P_2)x < 0; \\ 2, & x^T(P_1 - P_2)x \geq 0 \end{cases}$$

Figure 2 shows that the state trajectory converges to zero, and the designed controllers make the associated subsystems asymptotically stable by the switching strategy (30).

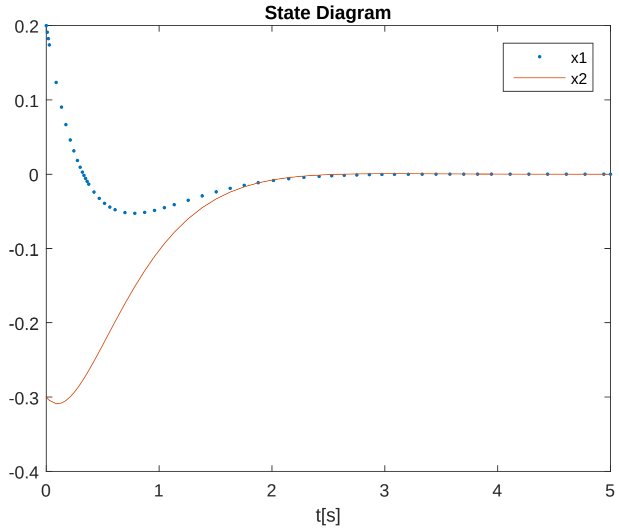

Figure 2. State trajectories of Example 1.

6.2. Example 2

Consider System (5) with order $\alpha = 0.23$, $N = 2$, and:

$$A_1 = \begin{bmatrix} -1 & 1 & -2 \\ 2 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix}, A_2 = \begin{bmatrix} -3 & -2 & -2 \\ -1 & -2 & 1 \\ -1 & -1 & 2 \end{bmatrix}, B_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$B_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, B_{1w} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, B_{2w} = \begin{bmatrix} 1 \\ 0.5 \\ 1 \end{bmatrix}$$

$$C_1 = \begin{bmatrix} -1 & 2 & 2 \end{bmatrix}, C_2 = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix}$$

Set $\lambda_1 = 0.4, \lambda_2 = 0.6$ and the disturbance attenuation level $\gamma = 1.2$; from Theorem 3, we can obtain:

$$X = \begin{bmatrix} 13.6532 & 3.6714 & -0.0721 \\ 3.6714 & 3.5815 & -2.0306 \\ -0.0721 & -2.0306 & -1.0467 \end{bmatrix}, Y = \begin{bmatrix} 0 & 0.0721 & 2.0306 \\ -0.0721 & 0 & 1.0467 \\ -2.0306 & -1.0467 & 0 \end{bmatrix}$$

$$K = \begin{bmatrix} -4.4841 & -96.8927 & -77.1058 \end{bmatrix}$$

Let:

$$\begin{aligned} P_i &= (A_i + B_iK)(aX + bY) \\ &+ (aX - bY)(A_i + B_iK)^T \\ &+ \gamma^{-1}(aX - bY)C_i^T C_i(aX + bY) \\ &+ \gamma^{-1}B_{iw}B_{iw}^T \end{aligned}$$

Then, we obtain the switching law:

$$\sigma(t) = i = \begin{cases} 1, & x^T(P_1 - P_2)x < 0; \\ 2, & x^T(P_1 - P_2)x \geq 0 \end{cases}$$

Figure 3 shows that the state trajectory converges to zero, and the designed controllers make the associated subsystems asymptotically stable by the switching strategy (30).

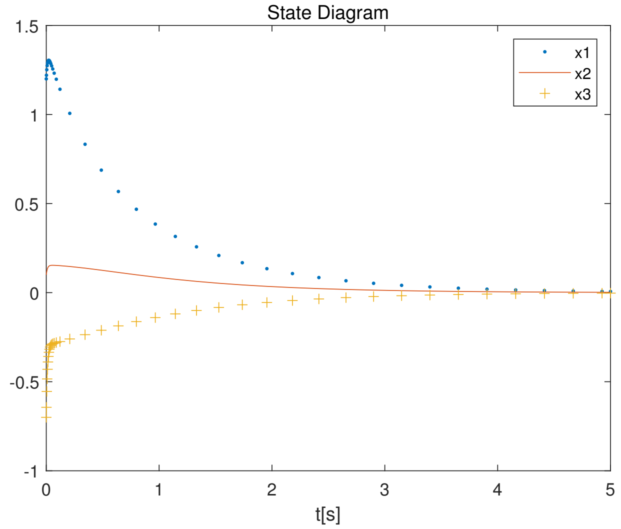

Figure 3. State trajectories of Example 2.

Remark 3. *The characteristic roots of* $A_2 + B_2K$ *are* $\{-3.0638, 9.9254 + 20.4436i, 9.9254 - 20.4436i\}$ *and* $|arg(9.9254 + 20.4436i)| = 1.1188 > \frac{0.23\pi}{2} = 0.3613$. *Although there are positive real roots, the second subsystem is stable under our results. Hence, compared with the results in [45–47], the results in this paper are less conservative.*

6.3. Example 3

Consider System (31) with order $\alpha = 0.3$, $N = 2$, and:

$$A_1 = \begin{bmatrix} -1 & 0 & 1 \\ 2 & 1 & 2 \\ 2 & 1 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 2 & -1 & 1 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}, B_1 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$B_2 = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}, B_{1w} = \begin{bmatrix} 0.5 \\ 1 \\ 0.75 \end{bmatrix}, B_{2w} = \begin{bmatrix} 0.2 \\ 0.5 \\ 0.1 \end{bmatrix}$$

$$C_1 = \begin{bmatrix} -1 & 0 & 1 \end{bmatrix}, C_2 = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}$$

$$M = \begin{bmatrix} -2 & 2 & 2 \\ -3 & -2 & 2 \\ -2 & -2 & -4 \end{bmatrix}, N_1 = I_3, N_2 = \begin{bmatrix} 0.1 & 0.15 & 0.2 \\ 0.2 & 0.1 & 0.1 \\ 0.15 & 0.25 & 0.3 \end{bmatrix}$$

Set $\lambda_1 = 0.8, \lambda_2 = 0.2$ and the disturbance attenuation level $\gamma = 1.2$; according to Theorem 5, we can obtain:

$$X = \begin{bmatrix} 1.0667 & 0.0976 & 0.2477 \\ 0.0976 & -0.1708 & 0.1370 \\ 0.2477 & 0.1370 & -0.1728 \end{bmatrix}$$

$$Y = \begin{bmatrix} 0 & -0.2477 & -0.1370 \\ 0.2477 & 0 & 0.1728 \\ 0.1370 & -0.1728 & 0 \end{bmatrix}$$

$$K = \begin{bmatrix} -9.3285 & 108.9177 & 240.5395 \\ -255.9823 & 317.2618 & 521.5005 \\ 149.9447 & -279.5623 & -481.4994 \end{bmatrix}$$

Let:

$$\begin{aligned} W_i &= (A_i + B_i K)(aX + bY) \\ &+ (aX - bY)(A_i + B_i K)^T \\ &+ \gamma^{-1}(aX - bY)C_i^T C_i(aX + bY) \\ &+ \gamma^{-1} B_{iw} B_{iw}^T \end{aligned}$$

Then, we obtain the switching law:

$$\sigma(t) = i = \begin{cases} 1, & x^T(W_1 - W_2)x < 0; \\ 2, & x^T(W_1 - W_2)x \geq 0 \end{cases}$$

Figure 4 shows that System (31) with gain K is quadratically stable by switching strategy (39), when the system initializes at $x(0) = \begin{bmatrix} 0.7 & 0.4 & -0.5 \end{bmatrix}^T$.

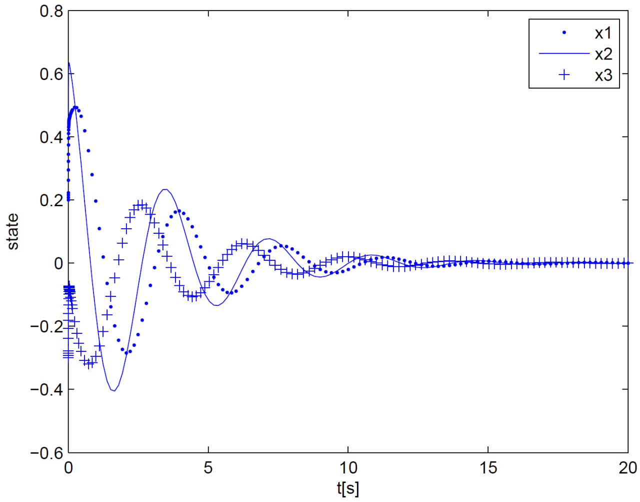

Figure 4. State trajectories of Example 3.

6.4. Example 4

Consider System (31) with order $\alpha = 0.63$, $N = 2$, and:

$$A_1 = \begin{bmatrix} -2 & 2 & 1.5 \\ 1.2 & 0.1 & 2.1 \\ 2 & 1 & 1 \end{bmatrix}, A_2 = \begin{bmatrix} 1.2 & -2.1 & 0.2 \\ 0.12 & 2.2 & 1.1 \\ 0.3 & 1.2 & 1.5 \end{bmatrix}, B_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$B_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, B_{1w} = \begin{bmatrix} 1.5 \\ 1 \\ 2.75 \end{bmatrix}, B_{2w} = \begin{bmatrix} 1 \\ 2.5 \\ 1 \end{bmatrix}$$

$$C_1 = \begin{bmatrix} -1 & 2 & 1 \end{bmatrix}, C_2 = \begin{bmatrix} 3 & 1 & 1 \end{bmatrix}$$

$$M = \begin{bmatrix} 1.2 & 0.2 & 2.2 \\ 1.3 & 0.2 & 1.2 \\ 1.2 & 0.2 & 2.4 \end{bmatrix}, N_1 = \begin{bmatrix} 0.1 & 0.2 & 0.1 \\ 0.2 & 0.1 & 0.3 \\ 0.1 & 0.2 & 0.3 \end{bmatrix}, N_2 = \begin{bmatrix} 0.21 & 0.13 & 0.12 \\ 0.23 & 0.21 & 0.11 \\ 0.35 & 0.22 & 0.31 \end{bmatrix}$$

Set $\lambda_1 = 0.25, \lambda_2 = 0.75$ and the disturbance attenuation level $\gamma = 2.5$; according to Theorem 5, we can obtain:

$$X = \begin{bmatrix} 2.4811 & 2.3268 & 1.2724 \\ 2.3268 & 17.5209 & -16.7539 \\ 1.2724 & -16.7539 & -10.0580 \end{bmatrix}$$

$$Y = \begin{bmatrix} 0 & -1.2724 & 16.7539 \\ 1.2724 & 0 & 10.0580 \\ -16.7539 & -10.0580 & 0 \end{bmatrix}$$

$$K = \begin{bmatrix} 41.8007 & 27.9848 & 22.9541 \end{bmatrix}$$

Let:

$$\begin{aligned} W_i &= (A_i + B_iK)(aX + bY) \\ &+ (aX - bY)(A_i + B_iK)^T \\ &+ \gamma^{-1}(aX - bY)C_i^T C_i(aX + bY) \\ &+ \gamma^{-1}B_{iw}B_{iw}^T \end{aligned}$$

Then, we obtain the switching law:

$$\sigma(t) = i = \begin{cases} 1, & x^T(W_1 - W_2)x < 0; \\ 2, & x^T(W_1 - W_2)x \geq 0 \end{cases}$$

Figure 5 shows that System (31) with gain K is quadratically stable by the switching strategy (39).

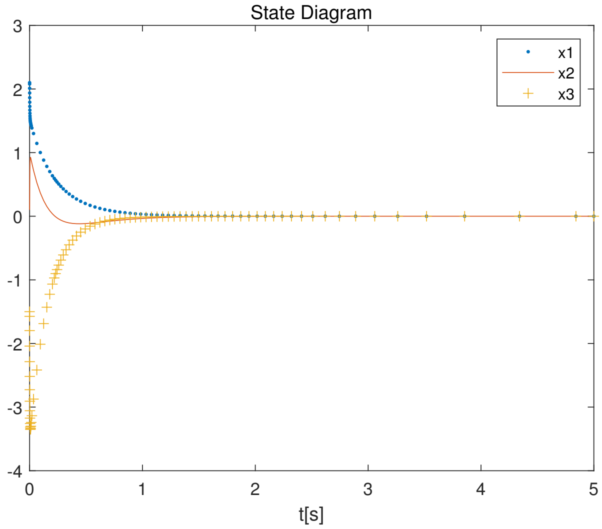

Figure 5. State trajectories of Example 4.

7. Conclusions

In this paper, the fractional-order switching law was derived for fractional-order switched systems (FOSSs) with order $0 < \alpha < 1$. Under the above switching law, stability and well-posedness can be proven for FOSSs. Then, the conditions of H_∞ control and controller design for FOSSs were proposed based on linear matrix inequalities (LMIs) in the paper, which can ensure the H_∞ performance for closed-loop FOSSs. Furthermore, the LMI-based conditions of robust H_∞ control and performance analysis were proven for FOSSs with uncertainty. Four the numerical simulation, results were given to verify the validity of the results proposed in this paper.

In the future, output feedback H_∞ control for FOSSs and robust H_∞ control for FOSSs with poly-topic uncertainty are the desired research directions.

Author Contributions: Conceptualization, B.L. and X.Z. (Xiangfei Zhao); methodology, B.L. and X.Z. (Xin Zhao); investigation, X.Z. (Xiangfei Zhao) and X.Z. (Xin Zhao); resources, X.Z. (Xin Zhao); data curation, X.Z. (Xin Zhao); writing—original draft preparation, B.L.; writing—review and editing, Y.L. and X.Z. (Xin Zhao); supervision, X.Z. (Xin Zhao); project administration, X.Z. (Xin Zhao); funding acquisition, Y.L. and X.Z. (Xin Zhao). All authors read and agreed to the published version of the manuscript.

Funding: This research was jointly supported by the National Key R&D Program of China (2018YFB1304905) and the National Natural Science Foundation of China (62027812, U1813210).

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declareno conflict of interest.

References

1. Liberzon, D. *Switching in Systems and Control*; Birkäuser: Basel, Switzerland, 2003.
2. Daafouz, J.; Riedinger, P.; Lung, C. Stability analysis and control synthesis for switched systems: A switched Lyapunov function approach. *IEEE Trans. Autom. Contr.* **2002**, *47*, 1883–1887. [CrossRef]
3. Sun, Z.; Ge, S.S.; Lee, T.H. Controllability and reachability criteria for switched linear systems. *Automatica* **2002**, *38*, 775–786. [CrossRef]
4. Lin, H.; Antsaklis, P.J. Stability and stabilizability of switched linear systems: A short survey of recent results. *IEEE Trans. Autom. Contr.* **2009**, *54*, 24–29. [CrossRef]
5. Bai, J.; Su, H.; Gao, J.; Sun, T.; Wu, Z. Modeling and stabilization of a wireless network control system with packet loss and time dela. *J. Frankl. Inst.* **2012**, *349*, 2420–2430. [CrossRef]
6. Donkers, M.; Heemels, W.; Bernardini, D.; Bemporad, A.; Shneer, V. Stability analysis of stochastic networked control systems. *Automatica* **2012**, *48*, 917–925. [CrossRef]
7. Niu, B.; Zhao, P.; Liu, J.D.; Ma, H.J.; Liu, Y.J. Global adaptive control of switched uncertain nonlinear systems: An improved MDADT method. *Automatica* **2020**, *115*, 1–10. [CrossRef]
8. Niu, B.; Wang, D.; Alotaibi, N.D.; Alsaadi, F.E. Adaptive neural state-feedback tracking control of stochastic nonlinear switched systems: An average dwell-time method. *IEEE Trans. Neural Netw. Learn. Syst.* **2019**, *30*, 1076–1087. [CrossRef]
9. Zhai, G.S.; Hu, B.; Yasuda, K.; Michel, A.N. Stability analysis of switched systems with stable and unstable subsystems: An average dwell time approach. *Int. J. Syst. Sci.* **2001**, *32*, 1055–1061. [CrossRef]
10. Wang, Y.E.; Sun, X.M.; Mazenc, F. Stability of switched nonlinear systems with delay and disturbance. *Automaitca* **2016**, *69*, 78–86. [CrossRef]
11. Zhai, G.; Lin, H.; Antsaklis, P.J. Quadratic stabilizability of switched linear systems with polytopic uncertainties. *Int. J. Control* **2003**, *76*, 747–753. [CrossRef]
12. Wu, C.Y.; Zhao, X.D.; Xu, N.; Han, X.M. Stabilization of hybrid systems under state constraints. *Nonlinear Anal. Hybrid Syst.* **2021**, *40*, 101015. [CrossRef]
13. Li, C.S.; Guo, H.; Fu, J.; Pang, H.B. H_∞ control for switched non-linear systems with structural uncertainty by using robust passivity. *Nonlinear Anal. Hybrid Syst.* **2021**, *40*, 101017. [CrossRef]
14. Akar, M.; Paul, A.; Michael, G.; Mitra, U. Conditions on the stability of a class of second-order switched systems. *IEEE Trans. Autom. Contr.* **2006**, *51*, 338–340. [CrossRef]
15. Zhang, J.X.; Yang, G.H. Low-complexity tracking control of strict-feedback systems with unknown control directions. *IEEE Trans. Autom. Contr.* **2019**, *64*, 5175–5182. [CrossRef]
16. Zhao, X.D.; Zhang, L.X.; Niu, B.; Liu, L. Adaptive tracking control for a class of uncertain switched nonlinear systems. *Automatica* **2015**, *52*, 185–191. [CrossRef]
17. Zong, G.D.; Ren, H.L.; Ho, L.L. Finite-time stability of interconnected impulsive switched systems. *IET Control Theory Appl.* **2016**, *10*, 648–654. [CrossRef]
18. Mobayen, S.; Tchier, F. Composite nonlinear feedback integral sliding mode tracker design for uncertain switched systems with input saturation. *Commun. Nonlinear Sci. Numer. Simul.* **2018**, *65*, 173–184. [CrossRef]
19. Podlubny, I. *Fractional Differential Equations*; Academic Press: New York, NY, USA, 1999.
20. Valerio D.; Costa J. Time-domain implementation of fractional-order controllers. *IEE Proc. Control Theory Appl.* **2005**, *152*, 539–552. [CrossRef]

21. Sabatier, J.; Mathieu, M.; Farges, C. LMI stability conditions for fractional-order systems. *Comput. Math. Appl.* **2010**, *59*, 1594–1609. [CrossRef]
22. Lu, J.G.; Chen, Y.Q. Robust stability and stabilization of fractionalorder interval systems with the fractional-order α: The $0 < \alpha < 1$ case. *IEEE Trans. Autom. Cont.* **2010**, *55*, 152–158. [CrossRef]
23. Ahn, H.S.; Chen, Y.Q. Necessary and sufficient stability condition of fractional-order interval linear systems. *Automatica* **2008**, *44*, 2985–2988. [CrossRef]
24. Zhang, X.F.; Lin, C.; Chen, Y.Q. A Unified Framework of Stability Theorems for LTI Fractional Order Systems With $0 < \alpha < 2$. *IEEE Trans. Circuits II* **2020**, *67*, 3237–3241. [CrossRef]
25. Bas, E.; Ozarslan, R. Real world applications of fractional models by Atangana–Baleanu fractional derivative. *Chaos Solit. Fract.* **2018**, *116*, 121–125. [CrossRef]
26. Chen, L.P.; Wu, R.C.; He, Y.G.; Yin, L.S. Robust stability and stabilization of fractional-order linear systems with polytopic uncertainties. *Appl. Math. Comput.* **2015**, *257*, 274–284. [CrossRef]
27. Wei, Y.H.; Chen, Y.Q.; Liang, S.; Wang, Y. A novel algorithm on adaptive backstepping control of fractional-order systems. *Neurocomputing* **2015**, *165*, 395–402. [CrossRef]
28. Farges, C.; Fadiga, L.; Sabatier, J. H_∞ analysis and control of commensurate fractional-order systems. *Mechatronics* **2013**, *23*, 772–780. [CrossRef]
29. Liang, S.; Wei, Y.H.; Pan, J.W.; Gao, Q.; Wang, Y. Bounded real lemmas for fractional-order systems. *Int. J. Autom. Comput.* **2015**, *15*, 192–198. [CrossRef]
30. Shen, J.; Lam, J. State feedback H_∞ control of commensurate fractional-order systems. *Int. J. Syst. Sci.* **2014**, *45*, 363–372. [CrossRef]
31. Li, H.; Yang, G.H. Dynamic output feedback H_∞ control for fractional-order linear uncertain systems with actuator faults. *J. Frankl. Inst.* **2019**, *356*, 4442–4466. [CrossRef]
32. Li, B.X.; Liu, Y.W.; Zhao, X. Robust H_∞ control for fractional-order systems with order α $(0 < \alpha < 1)$. *Frac. Fract.* **2022**, *6*, 86. [CrossRef]
33. Saeed, B.; Sedigh, A.K. Sufficient condition for stabilization of linear time invariant fractional-order switched systems and variable structure control stabilizers. *ISA Trans.* **2012**, *51*, 65–73. [CrossRef]
34. HosseinNia, S.H.; Tejado, I.; Vinagre, B.M. Stability of fractional-order switching systems. *Comput. Math. Appl.* **2013**, *66*, 585–596. [CrossRef]
35. Feng, T.; Guo, L.H.; Wu, B.W.; Chen, Y.Q. Stability analysis of switched fractional-order continuous-time systems. *Nonlinear Dynam.* **2020**, *102*, 2467–2478 [CrossRef]
36. Zhao, X.D.; Yin, Y.F.; Zheng, X.L. State-dependent switching control of switched positive fractional-order systems. *ISA Trans.* **2016**, *62*, 103–108. [CrossRef]
37. Zhang, J.F.; Zhao, X.D.; Chen, Y. Finite-time stability and stabilization of fractional-order positive switched systems. *Circuits Syst. Signal Process* **2016**, *35*, 2450–2470. [CrossRef]
38. Yang, Y.; Chen, G.P. Finite-time stability of fractional-order impulsive switched systems. *Int. J. Robust Nonlinear Control* **2015**, *25*, 2207–2222. [CrossRef]
39. Wu, C.; Liu, X.Z. Lyapunov and external stability of Caputo fractional-order switching systems. *Nonlinear Anal. Hybrid Syst.* **2019**, *34* 131–146. [CrossRef]
40. Yang, H.; Jiang, B. Stability of fractional-order switched non-linear systems. *IET Control Theory Appl.* **2016**, *20*, 965–970. [CrossRef]
41. Zhang, X.F.; Wang, Z. Stability and robust stabilization of uncertain switched fractional-order systems. *ISA Trans.* **2020**, *103*, 1–9. [CrossRef]
42. Wang, Z.; Xue, D.Y.; Pan, F. Observer-based robust control for singular switched fractional-order systems subject to actuator saturation. *Appl. Math. Comput.* **2021**, *411*, 126538. [CrossRef]
43. Liu, L.P.; Di Y.F.; Shang, Y.L.; Fu, Z.M.; Fan, B. Guaranteed cost and finite-time non-fragile control of fractional-order positive switched systems with asynchronous switching and impulsive moments. *Circ. Syst. Sign. Proc.* **2021**, *40*, 3143—3160. [CrossRef]
44. Zhang, J.X.; Yang, G.H. Robust adaptive fault-tolerant control for a class of unknown nonlinear systems. *IEEE Trans. Ind. Elect.* **2017**, *64*, 585–594. [CrossRef]
45. Zhang, C.; Yang, H.; Jiang, B. Fault Estimation and Accommodation of Fractional-Order Nonlinear, Switched, and Interconnected Systems. *IEEE Trans. Cybern.* **2022**, *52*, 1443–1453. [CrossRef]
46. Bi, W.; Wang, T.; Yu, X. Fuzzy Adaptive Decentralized Control for Nonstrict-Feedback Large-Scale Switched Fractional-Order Nonlinear Systems. *IEEE Trans. Cybern.* 2021, *Early Access*. [CrossRef]
47. Peng, X.; Wang, Y.L.; Zuo, Z.Q. Co-design of state-dependent switching law and control scheme for variable-order fractional nonlinear switched systems. *Appl. Math. Comput.* **2022**, *415*, 126725. [CrossRef]
48. MacDuffee, C.C. *The Theory of Matrices*; Chelsea: New York, NY, USA, 1946.
49. Xie, L. Output feedback H_∞ control of systems with parameter uncertainty. *Int. J. Cont.* **1996**, *63*, 741–750. [CrossRef]